Interface Problems using the Fokas Method

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Abstract

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Interface problems for partial differential equations are initial boundary value problems for which the solution of an equation in one domain prescribes boundary conditions for the equations in adjacent domains. These types of problems occur widely in applications including heat transfer, quantum mechanics, and mathematical biology. These problems, though linear, are often not solvable analytically using classical approaches. In this dissertation I present an extension of the Fokas Method appropriate for solving these types of problems. I consider problems with both dissipative and dispersive behavior and consider general boundary and interface conditions. An analog for the Dirichlet to Neumann map for interface problems is also constructed.

TABLE OF CONTENTS

LIST OF FIGURES

Figure Number Page

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To my partner Scott.

NOTATION AND ABBREVIATIONS

$1_{m\times n}$	:	$m \times n$ matrix with every entry equal to 1.
BVP	:	boundary-value problem
$\mathbb{C}$	:	The complex plane
$\mathbb{C}^+$	:	The upper half plane: $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$
$\mathbb{C}^-$	:	The lower half plane: $\{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}$
D^+	:	$\{k \in \mathbb{C} : k \in D \cap \mathbb{C}^+\}$
D^-	:	$\{k \in \mathbb{C} : k \in D \cap \mathbb{C}^-\}$
$D_R^\pm$	:	$\{k \in D^\pm : \lvert k\rvert > R\}$
∂D	:	the boundary of the region D traversed in the clockwise direction
$\mathrm{erf}(\cdot)$	:	the error function, $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-y^2)\,\mathrm{d}y$
$I_{m\times n}$	:	$m \times n$ identity matrix
IVP	:	initial-value problem
KdV	:	Korteweg-de Vries equation
$\mathcal{L}_{D^{(j)}}$	:	$\partial D^{(j)} \cap \{k : \lvert k\rvert < C\}$
$\mathcal{L}_C^{(j)}$	:	$\{k \in D^{(j)} : \lvert k\rvert = C\}$
$\mathcal{L}^{(j)}$	:	$\mathcal{L}_{D^{(j)}} \cup \mathcal{L}_C^{(j)}$
LS	:	linear Schrödinger equation
$\mathbb{N}$	:	the set of natural numbers, $\{1, 2, \ldots\}$
NLS	:	nonlinear Schrödinger equation
PDE	:	partial differential equation
$\mathbb{R}^+$	:	the set of positive real numbers: $\{x \in \mathbb{R} : x \geq 0\}$
$\hat{u}(k,t)$	:	$\int_{x_{j-1}}^{x_j} e^{-ikx} u(x,t)\,\mathrm{d}x$ for $x_{j-1} < x < x_j$ and $t > 0$
$\hat{u}_0(k)$	:	$\int_{x_{j-1}}^{x_j} e^{-ikx} u(x,0)\,\mathrm{d}x$ for $x_{j-1} < x < x_j$

Chapter 1

Introduction

1.1 Interface problems

Interface problems for partial differential equations (PDEs) are initial boundary value problems for which the solution of an equation in one domain prescribes boundary conditions for the equations in adjacent domains. In applications, precise interface conditions often follow from conservations laws [38]. Interface problems occur widely in applications. Examples include heat flowing through a composite rod [10, 33], the time-dependent linear Schrödinger equation with a piecewise constant potential [19, 41, 48], and shock waves as a method of healing fractured bones [18]. Finding solutions to equations modeling water waves can also be considered an interface problem where the interface is between air and water [43, 66].

To derive the "boundary conditions" that must be imposed at the interface we consider differential equations that are valid on either side of the interface. The integral form of the equation or important integral relations are often the best ways to infer the necessary conditions on the unknowns at the interface. As an example, we derive the boundary conditions relating to heat conduction used in Chapter 2. We follow closely what is outlined in [38].

Consider a thin rod of some heat-conducting material with density $\rho(x)$ and unit cross-sectional area. Assume that the surface of the rod is perfectly insulated so no heat is lost or gained through this surface. This problem is one-dimensional in the sense that all material properties depend only on the distance x along the rod. If we consider an infinitesimal section of length $\mathrm{d}x$ we know that $\mathrm{d}Q$, the heat content in the section, is proportional to the mass and temperature $\theta(x,t)$. That is

$$\mathrm{d}Q = c(x)\rho(x)\theta(x,t)\,\mathrm{d}x,$$

where $c(x)$ is the specific heat. Thus, the total heat content in the interval $x_1 \le x \le x_2$ is

$$Q(t) = \int_{x_1}^{x_2} c(x)\rho(x)\theta(x,t)\,\mathrm{d}x.$$

Fourier's Law for heat conduction [30] states that the rate of heat flowing into a body is proportional to the area of that element and to the outward normal derivative of the temperature at that location. The constant of proportionality is $k(x)$, the thermal conductivity. In our example, the net inflow of heat through the boundaries x_1 and x_2 is

$$R(t) = k(x)\theta_x(x_2,t) - k(x)\theta_x(x_1,t).$$

Conservation of heat implies that $Q_t(t) = R(t)$. That is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} c(x)\rho(x)\theta(x,t)\,\mathrm{d}x = k(x_2)\theta_x(x_2,t) - k(x_1)\theta_x(x_1,t). \tag{1.1}$$

This is a typical conservation law. Define $\kappa^2(x) = k(x)/(c(x)\rho(x))$ as the thermal diffusivity. Assume there are two rods of different materials which are put in perfect thermal contact at $x = 0$ such that

$$\rho(x) = \begin{cases} \rho_1(x), & x < 0, \\ \rho_2(x), & x > 0, \end{cases}$$

$$\theta(x) = \begin{cases} \theta^{(1)}, (x,t) & x < 0, \\ \theta^{(2)}(x,t), & x > 0, \end{cases}$$

$$k(x) = \begin{cases} k_1, & x < 0, \\ k_2, & x > 0, \end{cases},$$

$$\kappa^2(x) = \begin{cases} \kappa_1^2, & x < 0, \\ \kappa_2^2, & x > 0. \end{cases}$$

"Perfect thermal contact" means that the temperatures of the two surfaces are equal [10]. Thus, we have our first interface condition, $\theta(0^-) = \theta(0^+)$. Since κ_1^2, κ_2^2, k_1, and k_2 are constants, then for $x < 0$, we have $c(x)\rho_1(x) = c\rho_1$, a constant, and for $x > 0$, $c(x)\rho_2(x) = c\rho_2$, a constant. Evaluating (1.1) across the boundary $x = 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(c\rho_1 \lim_{\epsilon\to 0}\int_{-\epsilon}^{0}\theta^{(1)}(x,t)\,\mathrm{d}x + c\rho_2 \lim_{\epsilon\to 0}\int_{0}^{\epsilon}\theta^{(2)}(x,t)\,\mathrm{d}x\right) = \lim_{\epsilon\to 0}(k_2\theta_x^{(2)}(\epsilon,t) - k_1\theta_x^{(1)}(-\epsilon,t)). \quad (1.2)$$

The left-hand-side of (1.2) is zero since temperature is continuous across the boundary. This implies $k_1\theta_x^{(1)}(0,t) = k_2\theta_x^{(2)}(0,t)$. Thus heat flux is continuous across the interface.

Maxwell's equations are another typical example of conservation laws that are used to define boundary conditions for problems related to electromagnetism [19]. In general, this procedure can be used to construct interface conditions for any situation where differential equations are satisfied on either side of a sharp boundary where some property changes.

In this thesis we study PDEs with piecewise-constant coefficients by posing them as interface problems. Although not undertaken here, this could be a path toward the study of PDEs with continuous coefficients by letting the number of interfaces tend toward infinity. We consider interface problems where there are different PDEs on either side of the interface in Appendix A. In Appendix A we consider only the most simple case of the transport and heat equations. Although the results presented here are preliminary, interface problems with different equations is an area of current interest and active research [13].

1.2 The Fokas Method

The Fokas Method, alternatively called the Unified Transform Method (UTM), is a relatively new method for solving initial-boundary-value problems for linear and integrable PDEs with constant coefficients [15, 25, 27]. This method allows for the explicit solution of some problems for which no classical approach exists. For problems where classical solutions do exist, they are found as special cases of the Fokas Method. The chapters that follow rely heavily on the use of the Fokas Method. Thus we begin with a simple example.

Consider the heat equation defined on the positive half line:

$$u_t = u_{xx}, \qquad x \geq 0,\ t > 0, \tag{1.3a}$$

$$u(x,0) = u_0(x), \qquad x \geq 0, \tag{1.3b}$$

$$\lim_{x\to\infty} u(x,t) = 0. \tag{1.3c}$$

We will assume Dirichlet boundary data is given. That is $u(0,t)$ is known. Of course, this problem is easily solved using classical methods (*e.g.*, the method of images [38]). However, to the point of understanding the Fokas Method for a "simple" problem, we begin by rewriting (1.3a) as a one-parameter family of PDEs in divergence form:

$$(e^{-ikx+\omega(k)t}u)_t = (e^{-ikx+\omega(k)t}(u_x + iku))_x, \tag{1.4}$$

where the dispersion relation is given by $\omega(k) = k^2$. This is the *local relation*. Applying Green's Theorem [1] in the strip $(0,\infty)\times(0,t)$ in the right-half plane (see Figure 1.1) one finds

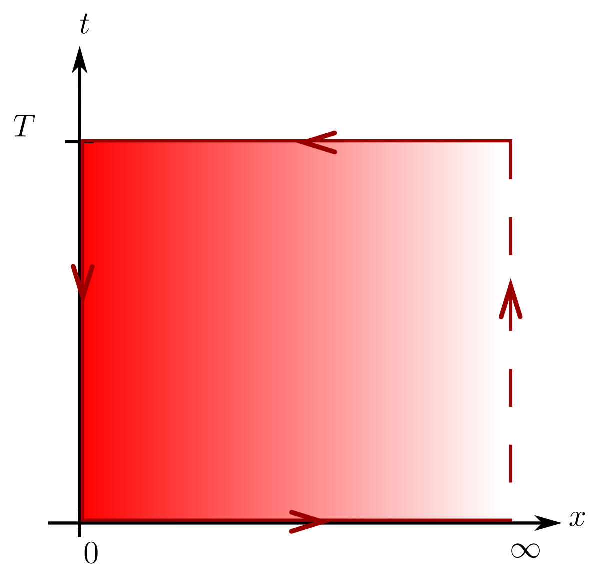

Figure 1.1: Domain for the application of Green's Theorem in the case of one semi-infinite rod.

$$\begin{aligned}&\int_0^t \int_0^\infty (e^{-ikx+k^2s}u)_s - (e^{-ikx+k^2s}(u_x+iku))_x \,\mathrm{d}x\,\mathrm{d}s = 0\\ \Rightarrow &\int_0^\infty e^{-ikx}u_0(x)\,\mathrm{d}x - \int_0^\infty e^{-ikx+k^2t}u(x,t)\,\mathrm{d}x - \int_0^t e^{k^2s}(u_x(0,s)+iku(0,s))\,\mathrm{d}s = 0.\end{aligned} \tag{1.5}$$

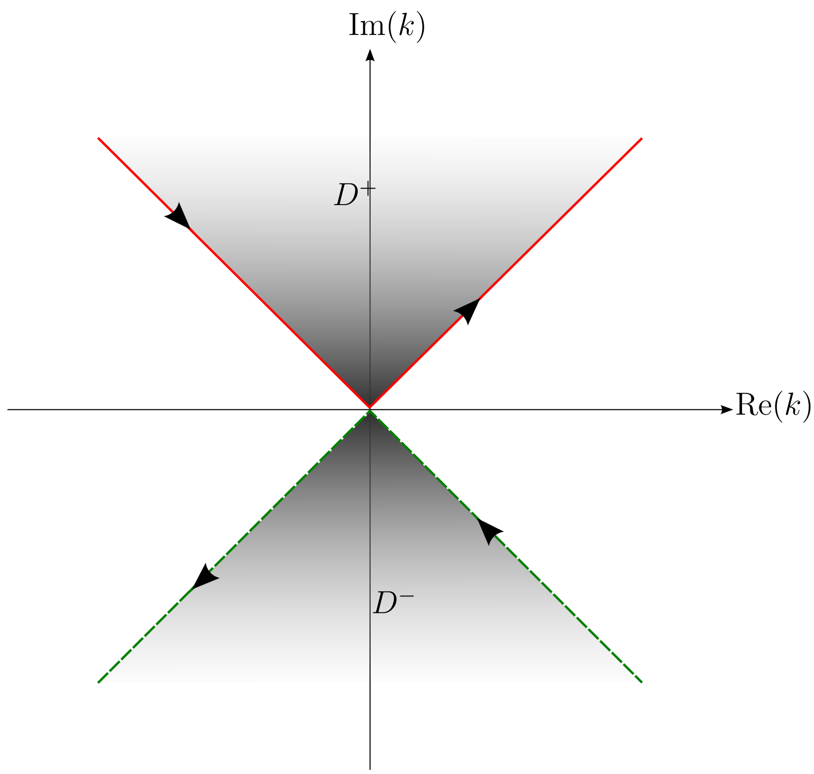

Figure 1.2: The domains D^+ and D^- for the heat equation.

Let $\mathbb{C}$ denote the complex numbers and $\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. Similarly, let $\mathbb{C}^- = \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}$. Since x can become arbitrarily large, we require the imaginary part of k to be negative, $k \in \mathbb{C}^-$, in order to guarantee that the integrals above are defined. Let $D = \{k \in \mathbb{C} : \mathrm{Re}(k^2) < 0\} = D^+ \cup D^-$. The region D is shown in Figure 1.2. For $k \in \mathbb{C}$

define the following:

$$
\begin{aligned}
\hat{u}_0(k) &= \int_0^\infty e^{-ikx} u_0(x)\,\mathrm{d}x, \\
\hat{u}(k,t) &= \int_0^\infty e^{-ikx} u(x,t)\,\mathrm{d}x, \\
g_0(\omega,t) &= \int_0^t e^{\omega s} u(0,s)\,\mathrm{d}s, \\
g_1(\omega,t) &= \int_0^t e^{\omega s} u_x(0,s)\,\mathrm{d}s.
\end{aligned}
$$

Using these definitions, the global relation (1.5) is

$$\hat{u}_0(k) - e^{k^2 s}\hat{u}(k,t) - g_1(\omega,t) - ikg_0(\omega,t) = 0. \tag{1.6}$$

Since the dispersion relation is invariant under the reflection $k \to -k$, so are $g_0(\omega,t)$ and $g_1(\omega,t)$. Thus one can supplement (1.6) with its evaluation at $-k$, namely

$$\hat{u}_0(-k) - e^{k^2 t}\hat{u}(-k,t) - g_1(\omega,t) + ikg_0(\omega,t) = 0. \tag{1.7}$$

Equation (1.7) is valid for $k \in \mathbb{C}^+$. Inverting the Fourier transforms in (1.6) we have

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2 t}\hat{u}_0(k)\,\mathrm{d}k - \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2 s}(g_1(\omega,t) + ikg_0(\omega,t))\,\mathrm{d}k, \tag{1.8}$$

for $x > 0$ and $t > 0$. Everything about the first integral in (1.8) is known. The integrand of the second integral is entire and decays as $k \to \infty$ for $k \in \mathbb{C}^+ \backslash D^+$. Thus, the integral can be deformed up to ∂D^+, the boundary of D^+, to obtain

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2 t}\hat{u}_0(k)\,\mathrm{d}k - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2 t}(g_1(\omega,t) + ikg_0(\omega,t))\,\mathrm{d}k. \tag{1.9}$$

Equation (1.9) depends on unprescribed boundary data, namely $g_1(\omega,t)$. To resolve this we solve (1.7) for $g_1(\omega,t)$. Substituting this into (1.9) we have

$$
\begin{aligned}
u(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2 t}\hat{u}_0(k)\,\mathrm{d}k - \frac{1}{\pi}\int_{\partial D^+} e^{ikx-k^2 t} ikg_0(\omega,t)\,\mathrm{d}k \\
&+ \frac{1}{2\pi}\int_{\partial D^+} e^{ikx}\hat{u}(-k,t)\,\mathrm{d}k - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2 t}\hat{u}_0(-k)\,\mathrm{d}k.
\end{aligned} \tag{1.10}
$$

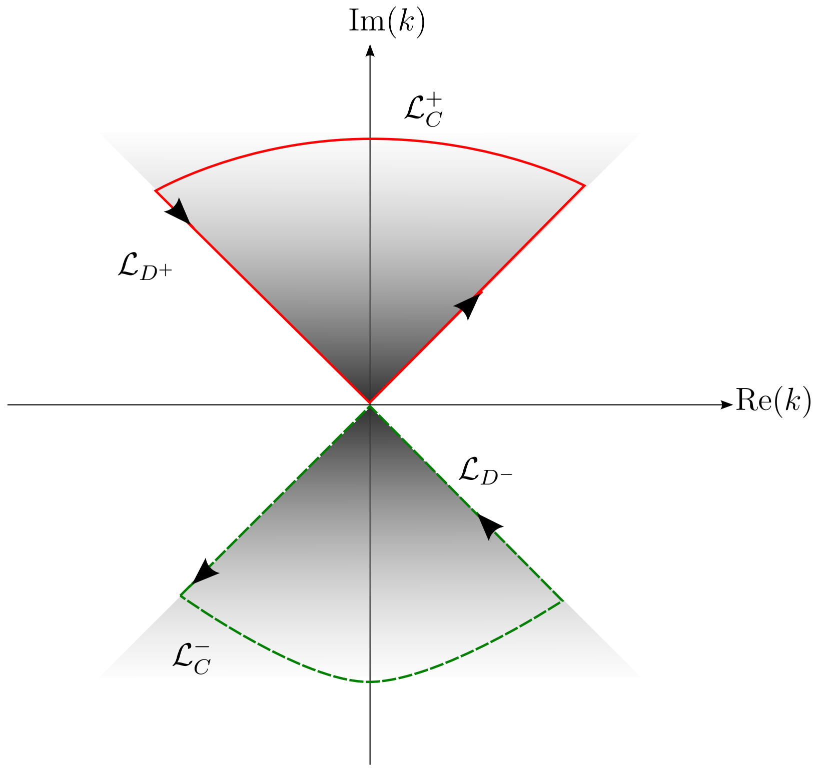

Figure 1.3: The contour $\mathcal{L}^-$ is shown as green dashed line. An application of Cauchy's Integral Theorem [1] using this contour allows elimination of the contribution of terms involving the Fourier transform of the solution.

This expression contains the solution we seek in the third integral on the right-hand side. However, $e^{ikx}\hat{u}(-k,t)$ is an analytic function that decays in the upper-half plane. Thus, by Jordan's Lemma [1], the integral of $\exp(ikx)\hat{u}(-k,t)$ along a closed, bounded curve in $\mathbb{C}^+$ must vanish. In particular we consider the closed curve $\mathcal{L}^+ = \mathcal{L}_{D^+} \cup \mathcal{L}_C^+$ where $\mathcal{L}_{D^+} = \partial D^+ \cap \{k : |k| < C\}$ and $\mathcal{L}_C^+ = \{k \in D^+ : |k| = C\}$, see Figure 1.3.

Since the integral along $\mathcal{L}_C^+$ vanishes for large C, the third integral on the right-hand side of (1.10) must vanish since the contour $\mathcal{L}_{D^+}$ becomes ∂D^+ as $C \to \infty$. The uniform decay of $\hat{u}(-k,t)$ for large k is exactly the condition required for the integral to vanish, using Jordan's Lemma. Our solution in terms of only initial and boundary conditions is now

$$
\begin{aligned}
u(x,t) = &\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{u}_0(k)\,\mathrm{d}k - \frac{1}{\pi}\int_{\partial D^+} e^{ikx-k^2t} ik g_0(\omega,t)\,\mathrm{d}k \\
&- \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2t}\hat{u}_0(-k)\,\mathrm{d}k,
\end{aligned}
$$

where $g_0(\omega, t)$ is the time transform of the given Dirichlet data as defined earlier. The Fokas Method is applicable to the general constant-coefficient linear evolution PDE

$$u_t + \omega(-i\partial_x)q = 0, \qquad x > 0, \qquad 0 < t \leq T, \tag{1.11}$$

where $\omega(k)$ is a polynomial of degree n. Equation (1.11) admits a one-parameter family of solutions $e^{ikx-\omega(k)t}$. To ensure the solutions are not exponentially growing in time, we require $\mathrm{Re}(\omega(k)) \geq 0$ for real k. Let

$$\omega(k) = \sum_{j=0}^{n} \alpha_j k^j.$$

In the limit as $|k| \to \infty$, the condition $\mathrm{Re}(\omega(k)) \geq 0$, $k \in \mathbb{R}$, implies that if n is odd then $\alpha_n = \pm i$ and if n is even $\mathrm{Re}(\alpha_n) \geq 0$ [25].

Define the regions $D = \{k : \mathrm{Re}(\omega(k)) < 0\}$, $D^+ = D \cap \mathbb{C}^+$, and $D^- = D \cap \mathbb{C}^-$. The local relation is given by [15, 25]

$$\partial_t(e^{-ikx+\omega(k)t}u(x,t)) - \partial_x\left(e^{-ikx+\omega(k)t}\sum_{j=0}^{n-1} c_j \partial_x^j u(x,t)\right) = 0,$$

where

$$\sum_{j=0}^{n-1} c_j(k)\partial_x^j u(x,t) = i\left(\frac{\omega(k)-\omega(m)}{k-m}\right)\Bigg|_{m=-i\partial_x} u(x,t),$$

and the global relation is

$$e^{\omega(k)T}\hat{u}(k,T) = \hat{u}_0(k) - \sum_{j=0}^{n-1} c_j(k) g_j(\omega(k),T),$$

with $\mathrm{Im}(k) \leq 0$ and

$$g_j(\omega,T) = \int_0^T e^{\omega s}\partial_x^j u(0,s)\,\mathrm{d}s.$$

Applying the inverse Fourier transform to the global relation we have the integral expression

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx-\omega(k)t}\hat{u}_0(k)\,\mathrm{d}k - \frac{1}{2\pi}\int_{\partial D^+}\left(e^{-ikx+\omega(k)t}\sum_{j=0}^{n-1} c_j(k)g_j(\omega(k),t)\right)\mathrm{d}k. \quad (1.12)$$

In order to eliminate any unknown boundary conditions from (1.12) we use the mappings between roots of the $\omega(k)$ ($k \to \nu(k)$) to find new versions of the global relation. These global relations allow us to solve for the unknown boundary data but introduce terms that depend on $\hat{u}(\nu(k),t)$. However, the transform $\hat{u}(\nu(k),t)$ is analytic and bounded in the region D^+. We use the Cauchy Integral Theorem to eliminate that contribution. In the finite interval case, the classical series solutions (when they exist) can be found by deforming the integral ∂D^+ to circles around any isolated roots of the integrand. The general method for finite-interval problems and more examples can be found in the book on this method by Fokas [25] and in a review paper by Deconinck, Trogdon, and Vasan [15].

1.3 Overview

As the title suggests this dissertation is an overview of methods for solving interface problems of the type described in Section 1.1 using the Fokas Method. The Fokas Method for linear constant-coefficient problem has many advantages over the standard methods. For instance, in addition to producing an explicit formula for the solution, the method allows one to determine, in a straightforward way, how many and which boundary conditions result in a well-posed problem. The method, which produces solution formulas for many problems which classical methods cannot, is not a collection of situation specific approaches tailored to given equations and boundary conditions. Instead, it is a unified method and the differences for different equations and boundary conditions appear only computationally. It is our goal to generalize this method to the case of interface problems.

Standard results from complex analysis are used heavily throughout. For proofs, theorem statements, and more information see [1, 67]. We also rely heavily on the use of the Fokas

Method which was introduced by A.S. Fokas [21, 22, 23, 25, 27] and has continued to be expanded on by himself and collaborators in recent years. A review paper on the use of the method for linear PDEs [15] is a good place to begin to better understand this method. The repository [59] which is regularly updated with books and papers on the subject is also useful and contains further applications of the Fokas Method.

In Chapter 2 the problem of heat conduction in one-dimensional piecewise homogeneous composite materials is examined by providing an explicit solution of the heat equation in each domain. The location of the interfaces is known, but neither temperature nor heat flux are prescribed there. Instead, the physical assumptions of their continuity at the interfaces are the only conditions imposed. We examine finite and infinite domains and allow the possibility of periodic boundary conditions. We also include the solution to Burgers' equation through the Cole-Hopf transformation.

We extend the results of the previous chapter to examine heat conduction on networks of multiply connected rods in Chapter 3. Again, we provide an explicit solution of the one-dimensional heat equation in each domain. The size and connectivity of the rods is given, but neither temperature nor heat flux are prescribed at the interface.

In Chapter 4 we study an interface problem for the linear Schrödinger equation in one-dimensional piecewise homogeneous domains. The location of the interfaces is known and the continuity of the wave function and a jump in the derivative at the interface are the only conditions imposed. The methods we use here are similar to those used in Chapter 2 but the dispersive nature of the problem presents additional difficulties that we address.

Chapter 5 generalizes the work in Chapter 4 to include a piecewise-constant potential. This problem is well studied in textbooks [19, 41, 48]. It is one of only a few solvable models in quantum mechanics and shares many qualitative features with physically important models. In examples such as "particle in a box" and tunneling, attention is restricted to the time-independent Schrödinger equation. In this chapter we present fully explicit solutions for the time-dependent problem for a general piecewise-constant potential.

The interface problem for the linear Korteweg-de Vries (KdV) equation in one-dimensional piecewise homogeneous domains is examined in Chapter 6 by constructing an explicit solution in each domain. The location of the interface is known and a number of compatibility conditions at the boundary are imposed. We provide an explicit characterization of necessary interface conditions for the construction of a solution. This work is the first known exploration into interface problems with higher than second-order derivatives. One of the great strengths of the Fokas Method is that the method for solving equations of any order is essentially the same. In Chapter 6 we show this extends to interface problems and find a surprising result on the number of interface conditions necessary for a well-posed problem.

Chapter 7 provides an analog to the well known Dirichlet to Neumann map for interface problems. We develop a map from the initial conditions to the value of the function at the interface. This map provides an alternative approach to solving the problem in each domain simultaneously as suggested in Chapters 2-6. With the initial to interface map one could use the initial conditions to solve for the necessary interface values. At that point, the problem could be solved as a regular boundary-value problem (BVP) using the Fokas Method or any other appropriate solution method.

In Chapter 8 we generalize the interface problems considered to include those that have a phase boundary that moves with time. We study the classical Stefan problem which describes the temperature distribution in a homogenous medium undergoing a phase change, for example, ice passing to water. The discussion in Chapter 8 is mostly restricted to the one-phase case, that is the heat equation is prescribed only in one domain while the temperature in the second domain is assumed to remain constant. The results presented in this chapter are not new but the methods used are suggestive of a more general method that could be used for the two-phase problem (the heat equation imposed on both domains). The work in this chapter is ongoing.

Chapter 2

Non-steady state heat conduction

In this chapter, the problem of heat conduction in one-dimensional piecewise homogeneous composite materials is examined by providing an explicit solution of the one-dimensional heat equation in each domain. The location of the interfaces is known, but neither temperature nor heat flux are prescribed there. Instead, the physical assumptions of continuity at the interfaces is the only condition imposed. The problem of two semi-infinite domains and that of two finite-sized domains are examined in detail. We indicate how to extend the solution method to the setting of one finite-sized domain surrounded on both sides by semi-infinite domains, and on that of three finite-sized domains. In the final section we examine the case of periodic boundary conditions. Parts of this chapter were first published in [14, 55].

The problem of heat conduction in a composite wall is a classical problem in design and construction. It is usual to restrict attention to the case of walls whose constitutive parts are in perfect thermal contact. We also assume the walls have physical properties that are constant throughout the material and are considered to be of infinite extent in the directions parallel to the wall. Further, we assume that temperature and heat flux do not vary in these directions. In that case, the mathematical model for heat conduction in each wall layer is given by [33, Chapter 10]:

$$\begin{aligned} u_t^{(j)} &= \kappa_j u_{xx}^{(j)}, & x_{j-1} < x < x_j, \\ u^{(j)}(x,0) &= u_0^{(j)}(x), & x_{j-1} < x < x_j, \end{aligned} \tag{2.1}$$

where $u^{(j)}(x,t)$ denotes the temperature in the wall layer indexed by j, $\kappa_j > 0$ is the heat-conduction coefficient of the j-th layer (the inverse of its thermal diffusivity), $x = x_{j-1}$ is the left extent of the layer, and $x = x_j$ is its right extent. The subscripts x and t denote derivatives with respect to the one-dimensional spatial variable x and the temporal variable t. The function $u_0^{(j)}(x)$ is the prescribed initial condition of the system. The continuity of the temperature $u^{(j)}(x,t)$ and of its associated heat flux $\kappa_j u_x^{(j)}(x,t)$ are imposed across the interface between layers. In what follows it is convenient to use the quantity σ_j, defined as the positive square root of κ_j: $\sigma_j = \sqrt{\kappa_j}$.

If the layer is either at the far left or far right of the wall, Dirichlet, Neumann, Robin, or periodic boundary conditions can be imposed on its far left or right boundary respectively, corresponding to prescribing the "outside" temperature, heat flux, or a combination of these. A derivation of the interface conditions is found in [33, Chapter 1] and repeated in Section 1.1.

In what follows, we use the Fokas Method to provide explicit solution formulae for different heat transport interface problems of the type described above. We investigate problems in both finite and infinite domains and we compare our method with classical solution approaches that can be found in the literature. Throughout, our emphasis is on non-steady state solutions. Even for the simplest of the problems we consider (Section 2.2, two finite walls in thermal contact), the classical approach using separation of variables [33] can provide an answer only implicitly. Indeed, the solution obtained in [33] depends on certain eigenvalues defined through a transcendental equation that can be solved only numerically. In contrast, the Fokas Method produces an explicit solution formula involving only known quantities. For other problems we consider, no solution has been derived using classical methods to our knowledge.

The representation formulae for the solution can be evaluated numerically, hence the problem can be solved in practice using hybrid analytical-numerical approaches [20]. Asymptotic approximations for the solutions may be obtained using standard techniques [25]. The result of such a numerical calculation is shown at the end of Section 2.1.

The problem of heat conduction through composite walls is discussed in many excellent texts, see for instance [10, 33]. References to the treatment of specific problems are given in the sections below where these problems are investigated.

2.1 Two semi-infinite domains

In this section, we consider the problem of heat flow through two walls of semi-infinite width, or of two semi-infinite rods as shown in Figure 2.1. We seek two functions

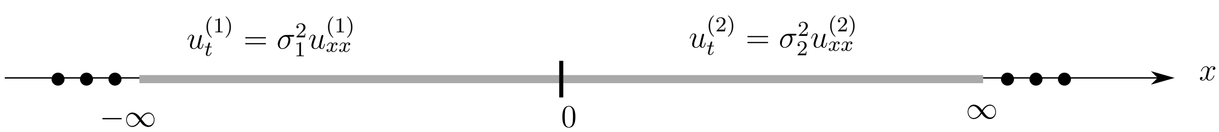

Figure 2.1: The heat equation for two semi-infinite domains.

$$u^{(1)}(x,t), \quad x<0, \quad t \geq 0,$$

and

$$u^{(2)}(x,t), \quad x>0, \quad t \geq 0,$$

satisfying the equations

$$u_t^{(1)}(x,t) = \sigma_1^2 u_{xx}^{(1)}(x,t), \qquad x<0,\ t>0, \tag{2.2a}$$

$$u_t^{(2)}(x,t) = \sigma_2^2 u_{xx}^{(2)}(x,t), \qquad x>0,\ t>0, \tag{2.2b}$$

the initial conditions

$$u^{(1)}(x,0) = u_0^{(1)}(x), \qquad x<0, \tag{2.3a}$$

$$u^{(2)}(x,0) = u_0^{(2)}(x), \qquad x>0, \tag{2.3b}$$

the asymptotic conditions

$$\lim_{x\to-\infty} u^{(1)}(x,t) = \gamma^{(1)}, \qquad t \geq 0, \tag{2.4a}$$

$$\lim_{x\to\infty} u^{(2)}(x,t) = \gamma^{(2)}, \qquad t \geq 0, \tag{2.4b}$$

and the continuity interface conditions

$$u^{(1)}(0,t) = u^{(2)}(0,t), \qquad t > 0, \tag{2.5a}$$

$$\sigma_1^2 u_x^{(1)}(0,t) = \sigma_2^2 u_x^{(2)}(0,t), \qquad t > 0. \tag{2.5b}$$

The sub- and super-indices 1 and 2 denote the left and right rod, respectively. A special case of this problem is discussed in Chapter 10 of [33], but only for a specific initial condition. Further, for the problem treated there both $\lim_{x\to\infty} u^{(2)}(x,t)$ and $\lim_{x\to-\infty} u^{(1)}(x,t)$ are assumed to be zero. This assumption is made for mathematical convenience and no physical reason exists to impose it. If constant (in time) limit values are assumed, a simple translation allows one of the limit values to be equated to zero, but not both. Since no great advantage is obtained by assuming a zero limit using our approach, we make the more general assumption (2.4).

We define $v^{(1)}(x,t) = u^{(1)}(x,t) - \gamma^{(1)}$ and $v^{(2)}(x,t) = u^{(2)}(x,t) - \gamma^{(2)}$. Then $v^{(1)}(x,t)$ and $v^{(2)}(x,t)$ satisfy

$$v_t^{(1)}(x,t) = \sigma_1^2 v_{xx}^{(1)}(x,t), \qquad x < 0 \qquad t \geq 0, \tag{2.6a}$$

$$v_t^{(2)}(x,t) = \sigma_2^2 v_{xx}^{(2)}(x,t), \qquad x > 0 \qquad t \geq 0, \tag{2.6b}$$

$$\lim_{x\to-\infty} v^{(1)}(x,t) = 0, \qquad t \geq 0, \tag{2.6c}$$

$$\lim_{x\to\infty} v^{(2)}(x,t) = 0, \qquad t \geq 0, \tag{2.6d}$$

$$v^{(1)}(0,t) + \gamma^{(1)} = v^{(2)}(0,t) + \gamma^{(2)}, \qquad t \geq 0, \tag{2.6e}$$

$$\sigma_1^2 v_x^{(1)}(0,t) = \sigma_2^2 v_x^{(2)}(0,t), \qquad t \geq 0. \tag{2.6f}$$

At this point, we start by following the standard steps in the application of the Fokas Method. We begin with the so-called “local relations”:

$$
\begin{aligned}
(e^{-ikx+\omega_1 t}v^{(1)}(x,t))_t =& (\sigma_1^2 e^{-ikx+\omega_1(k)t}(v_x^{(1)}(x,t)+ikv^{(1)}(x,t)))_x,\\
(e^{-ikx+\omega_2 t}v^{(2)}(x,t))_t =& (\sigma_2^2 e^{-ikx+\omega_2(k)t}(v_x^{(2)}(x,t)+ikv^{(2)}(x,t)))_x,
\end{aligned}
$$

where $\omega_j(k) = (\sigma_j k)^2$. These relations form a one-parameter family obtained by rewriting (2.6a) and (2.6b).

Integrating around the domain and applying Green's Theorem in the strip $(-\infty,0)\times(0,t)$ in the left-half plane (see Figure 2.2) we find

$$
\begin{aligned}
0 =& \int_{-\infty}^{0} e^{-ikx} v_0^{(1)}(x)\,\mathrm{d}x - \int_{-\infty}^{0} e^{-ikx+\omega_1 t} v^{(1)}(x,t)\,\mathrm{d}x\\
&+ \int_0^t \sigma_1^2 e^{\omega_1 s}(v_x^{(1)}(0,s)+ikv^{(1)}(0,s))\,\mathrm{d}s.
\end{aligned}
\tag{2.7}
$$

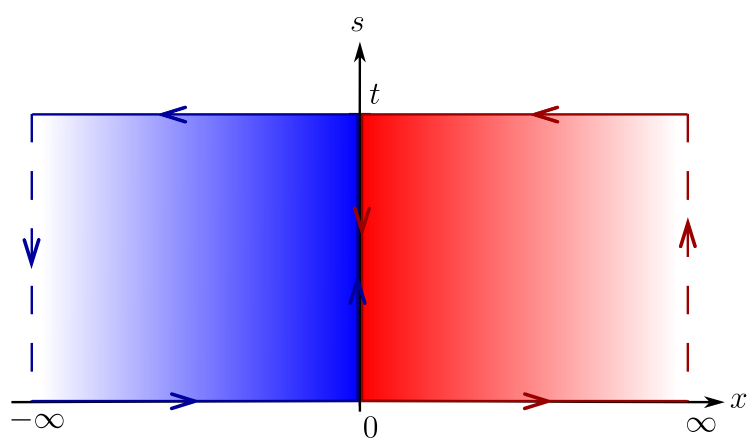

Figure 2.2: Domains for the application of Green's Theorem in the case of two semi-infinite rods.

Since $|x|$ can become arbitrarily large, we require $k \in \mathbb{C}^+$ in (2.7) in order to guarantee that the first two integrals are well defined. Let $D = \{k \in \mathbb{C} : \mathrm{Re}(\omega_j(k)) < 0\} = D^+ \cup D^-$. The region D is shown in Figure 1.2.

For $k \in \mathbb{C}$ we define the following transforms:

$$\begin{aligned}
g_0(\omega,t) &= \int_0^t e^{\omega s} v^{(1)}(0,s)\,\mathrm{d}s = \int_0^t e^{\omega s}(v^{(2)}(0,s) + \gamma^{(2)} - \gamma^{(1)})\,\mathrm{d}s \\
&= \frac{(\gamma^{(2)} - \gamma^{(1)})(e^{\omega t} - 1)}{\omega} + \int_0^t e^{\omega s} v^{(2)}(0,s)\,\mathrm{d}s, \\
g_1(\omega,t) &= \int_0^t e^{\omega s} v_x^{(1)}(0,s)\,\mathrm{d}s = \frac{\sigma_2^2}{\sigma_1^2}\int_0^t e^{\omega s} v_x^{(2)}(0,s)\,\mathrm{d}s, \\
\hat{v}^{(1)}(k,t) &= \int_{-\infty}^0 e^{-ikx} v^{(1)}(x,t)\,\mathrm{d}x, \\
\hat{v}_0^{(1)}(k) &= \int_{-\infty}^0 e^{-ikx} v_0^{(1)}(x)\,\mathrm{d}x, \\
\hat{v}^{(2)}(k,t) &= \int_0^\infty e^{-ikx} v^{(2)}(x,t)\,\mathrm{d}x, \\
\hat{v}_0^{(2)}(k) &= \int_0^\infty e^{-ikx} v_0^{(2)}(x)\,\mathrm{d}x.
\end{aligned}$$

Using these definitions, the global relation (2.7) is rewritten as

$$\hat{v}_0^{(1)}(k) - e^{\omega_1 t}\hat{v}^{(1)}(k,t) + ik\sigma_1^2 g_0(\omega_1,t) + \sigma_1^2 g_1(\omega_1,t) = 0, \quad k \in \mathbb{C}^+. \tag{2.8}$$

Since the dispersion relation $\omega_1(k) = (\sigma_1 k)^2$ is invariant under $k \to -k$, so are $g_0(\omega_1,t)$ and $g_1(\omega_1,t)$. Thus we can supplement (2.8) with its evaluation at $-k$, namely

$$\hat{v}_0^{(1)}(-k) - e^{\omega_1 t}\hat{v}^{(1)}(-k,t) - ik\sigma_1^2 g_0(\omega_1,t) + \sigma_1^2 g_1(\omega_1,t) = 0. \tag{2.9}$$

This relation is valid for $k \in \mathbb{C}^-$. Using Green's Formula on $(0,\infty)\times(0,t)$ (see Figure 2.2), the global relation for $v^{(2)}(x,t)$ is

$$0 = \hat{v}_0^{(2)}(k) - e^{\omega_2 t}\hat{v}^{(2)}(k,t) - ik\sigma_2^2 g_0(\omega_2,t) + \frac{i}{k}(\gamma^{(1)} - \gamma^{(2)})(e^{\omega_2 t} - 1) - \sigma_1^2 g_1(\omega_2,t), \tag{2.10}$$

valid for $k \in \mathbb{C}^-$. As above, using the invariance of $\omega_2(k) = (\sigma_2 k)^2$, $g_0(\omega_2, t)$, and $g_1(\omega_2, t)$ under $k \to -k$, we supplement (2.10) with

$$0 = \hat{v}_0^{(2)}(-k) - e^{\omega_2 t}\hat{v}^{(2)}(-k,t) + ik\sigma_2^2 g_0(\omega_2,t) - \frac{i}{k}(\gamma^{(1)} - \gamma^{(2)})(e^{\omega_2 t} - 1) - \sigma_1^2 g_1(\omega_2,t), \quad (2.11)$$

for $k \in \mathbb{C}^+$.

Inverting the Fourier transforms in (2.8) we have

$$v^{(1)}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{v}_0^{(1)}(k)\,\mathrm{d}k + \frac{\sigma_1^2}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}(ikg_0(\omega_1,t) + g_1(\omega_1,t))\,\mathrm{d}k, \quad (2.12)$$

for $x < 0$ and $t > 0$. The integrand of the second integral in (2.12) is entire and decays as $k \to \infty$ for $k \in \mathbb{C}^- \setminus D^-$. Using the analyticity of the integrand and applying Jordan's Lemma we can replace the contour of integration of the second integral by $-\int_{\partial D^-}$:

$$v^{(1)}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{v}_0^{(1)}(k)\,\mathrm{d}k - \frac{\sigma_1^2}{2\pi}\int_{\partial D^-} e^{ikx-\omega_1 t}(ikg_0(\omega_1,t) + g_1(\omega_1,t))\,\mathrm{d}k. \quad (2.13)$$

Proceeding similarly on the right, starting from (2.10), we have

$$\begin{aligned} v^{(2)}(x,t) =& \frac{\gamma^{(1)} - \gamma^{(2)}}{2}\left(1 - \operatorname{erf}\left(\frac{x}{2\sqrt{\sigma_2^2 t}}\right)\right) + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{v}_0^{(2)}(k)\,\mathrm{d}k \\ &- \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}(ik\sigma_2^2 g_0(\omega_2,t) + \sigma_1^2 g_1(\omega_2,t))\,\mathrm{d}k, \\ =& \frac{\gamma^{(1)} - \gamma^{(2)}}{2}\left(1 - \operatorname{erf}\left(\frac{x}{2\sqrt{\sigma_2^2 t}}\right)\right) + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{v}_0^{(2)}(k)\,\mathrm{d}k \\ &- \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-\omega_2 t}(ik\sigma_2^2 g_0(\omega_2,t) + \sigma_1^2 g_1(\omega_2,t))\,\mathrm{d}k. \end{aligned} \quad (2.14)$$

for $x > 0$ and $t > 0$. Here $\operatorname{erf}(\cdot)$ denotes the error function: $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-y^2)\,\mathrm{d}y$. To obtain the second equality above we integrated the terms that are explicit.

The expressions (2.13) and (2.14) for $v^{(1)}(x,t)$ and $v^{(2)}(x,t)$ depend on the unknown functions g_0 and g_1, evaluated at different arguments. These functions need to be expressed in terms of known quantities. To obtain a system of two equations for the two unknown functions we use (2.9) and (2.10) for $g_0(\omega_1, t)$, and $g_1(\omega_1, t)$. This requires the transformation $k \to -\sigma_1 k/\sigma_2$ in (2.10). The $-$ sign is required to ensure that both equations are valid on $\mathbb{C}^-$, allowing for their simultaneous solution. We find

$$\begin{aligned} ik\sigma_1^2 g_0(\omega_1, t) =& \frac{-\sigma_1\left(e^{\omega_1 t}(\hat{v}^{(1)}(-k,t) + \hat{v}^{(2)}\left(k\frac{\sigma_1}{\sigma_2}, t\right)) - \hat{v}_0^{(1)}(-k) - \hat{v}_0^{(2)}\left(k\frac{\sigma_1}{\sigma_2}\right)\right)}{\sigma_1+\sigma_2} \\ &+ \frac{i(\gamma^{(1)} - \gamma^{(2)})\left(1-e^{\omega_1 t}\right)}{k(\sigma_1+\sigma_2)}, \end{aligned} \tag{2.15a}$$

$$\begin{aligned} \sigma_1^2 g_1(\omega_1, t) =& \frac{e^{\omega_1 t}\left(\sigma_2 \hat{v}^{(1)}(-k,t) - \sigma_1 \hat{v}^{(2)}\left(k\frac{\sigma_1}{\sigma_2}, t\right)\right) + \sigma_1 \hat{v}_0^{(1)}\left(k\frac{\sigma_1}{\sigma_2}\right) - \sigma_2 \hat{v}_0^{(1)}(-k)}{\sigma_1+\sigma_2} \\ &+ \frac{i(\gamma^{(1)} - \gamma^{(2)})(1-e^{\omega_1 t})}{k(\sigma_1+\sigma_2)}, \end{aligned} \tag{2.15b}$$

valid for $k \in \mathbb{C}^-$. These expressions are substituted into (2.13). This results in an expression for $v^{(1)}(x,t)$ that appears to depend on $v^{(1)}(x,t)$ and $v^{(2)}(x,t)$ themselves. We examine the contribution of the terms involving $\hat{v}^{(1)}(\cdot, t)$ and $\hat{v}^{(2)}(\cdot, t)$. We obtain for $v^{(1)}(x,t)$ the following expression:

$$\begin{aligned} v^{(1)}(x,t) =& \frac{\sigma_2(\gamma^{(2)} - \gamma^{(1)})}{\sigma_1+\sigma_2}\left(1 + \operatorname{erf}\left(\frac{x}{2\sqrt{\sigma_1^2 t}}\right)\right) \\ &+ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{v}_0^{(1)}(k)\,\mathrm{d}k \\ &+ \int_{\partial D^-} \frac{\sigma_2-\sigma_1}{2\pi(\sigma_1+\sigma_2)} e^{ikx-\omega_1 t}\hat{v}_0^{(1)}(-k)\,\mathrm{d}k \\ &- \int_{\partial D^-} \frac{\sigma_1}{\pi(\sigma_1+\sigma_2)} e^{ikx-\omega_1 t}\hat{v}_0^{(2)}\left(k\frac{\sigma_1}{\sigma_2}\right)\mathrm{d}k \\ &+ \int_{\partial D^-} \frac{\sigma_1-\sigma_2}{2\pi(\sigma_1+\sigma_2)} e^{ikx}\hat{v}^{(1)}(-k,t)\,\mathrm{d}k \\ &+ \int_{\partial D^-} \frac{\sigma_1}{\pi(\sigma_1+\sigma_2)} e^{ikx}\hat{v}^{(3)}\left(k\frac{\sigma_1}{\sigma_2}, t\right)\mathrm{d}k, \end{aligned} \tag{2.16}$$

for $x < 0$, $t > 0$. The first four terms depend only on known functions. In the second-to-last term, notice that the integrand is analytic for all $k \in \mathbb{C}^-$ and that $\hat{v}^{(1)}(-k,t)$ decays for $k \to \infty$ for $k \in \mathbb{C}^-$. Thus, by Jordan's Lemma, the integral of $\exp(ikx)\hat{v}^{(1)}(-k,t)$ along a closed, bounded curve in $\mathbb{C}^-$ vanishes. In particular we consider the closed curve $\mathcal{L}^- = \mathcal{L}_{D^-} \cup \mathcal{L}_C^-$ where $\mathcal{L}_{D^-} = \partial D^- \cap \{k : |k| < C\}$ and $\mathcal{L}_C^- = \{k \in D^- : |k| = C\}$, see Figure 1.3.

Since the integral along $\mathcal{L}_C^-$ vanishes for large C, the fourth integral on the right-hand side of (2.16) must vanish since the contour $\mathcal{L}_{D^-}$ becomes ∂D^- as $C \to \infty$. The uniform decay of $\hat{v}^{(1)}(-k,t)$ for large k is exactly the condition required for the integral to vanish, using Jordan's Lemma. For the final integral in Equation (2.16) we use the fact that $\hat{v}^{(2)}(k\frac{\sigma_1}{\sigma_2},t)$ is analytic and bounded for $k \in \mathbb{C}^-$. Using the same argument as above, the fifth integral in (2.16) vanishes and we have an explicit representation for $v^{(1)}(x,t)$ in terms of initial conditions:

$$
\begin{aligned}
v^{(1)}(x,t) =& \frac{\sigma_2(\gamma^{(2)} - \gamma^{(1)})}{\sigma_1 + \sigma_2}\left(1 + \operatorname{erf}\left(\frac{x}{2\sqrt{\sigma_1^2 t}}\right)\right) \\
&+ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx - \omega_1 t}\hat{v}_0^{(1)}(k)\,\mathrm{d}k \\
&+ \int_{\partial D^-} \frac{\sigma_2 - \sigma_1}{2\pi(\sigma_1 + \sigma_2)} e^{ikx - \omega_1 t}\hat{v}_0^{(1)}(-k)\,\mathrm{d}k \\
&- \int_{\partial D^-} \frac{\sigma_1}{\pi(\sigma_1 + \sigma_2)} e^{ikx - \omega_1 t}\hat{v}_0^{(2)}\left(k\frac{\sigma_1}{\sigma_2}\right)\,\mathrm{d}k.
\end{aligned}
\tag{2.17}
$$

To find an explicit expression for $v^{(2)}(x,t)$ we need to evaluate g_0 and g_1 at different arguments, ensuring that the expressions are valid for $k \in \mathbb{C}^+ \setminus D^+$. From (2.15a) and (2.15b), we find

$$
\begin{aligned}
ik\sigma_2^2 g_0(\omega_2,t) = \quad & \frac{\sigma_2\left(e^{\omega_2 t}(\hat{v}^{(1)}(k\frac{\sigma_2}{\sigma_1},t)+\hat{v}^{(2)}(-k,t))-\hat{v}_0^{(1)}(k\frac{\sigma_2}{\sigma_1})-\hat{v}_0^{(2)}(-k)\right)}{\sigma_1+\sigma_2} \\
& +\frac{i\sigma_2(\gamma^{(1)}-\gamma^{(2)})(1-e^{\omega_2 t})}{k(\sigma_1+\sigma_2)}, \\
\sigma_1^2 g_1(\omega_2,t) = \quad & \frac{e^{\omega_2 t}\left(\sigma_2\hat{v}^{(1)}(k\frac{\sigma_2}{\sigma_1},t)-\sigma_1\hat{v}^{(2)}(-k,t)\right)+\sigma_1\hat{v}_0^{(2)}(-k)-\sigma_2\hat{v}_0^{(1)}(k\frac{\sigma_2}{\sigma_1})}{\sigma_1+\sigma_2} \\
& -\frac{i(\gamma^{(1)}-\gamma^{(2)})\,(1-e^{\omega_2 t})}{k(\sigma_1+\sigma_2)}.
\end{aligned}
$$

Substituting these into equation (2.14), we obtain

$$
\begin{aligned}
v^{(2)}(x,t) = & \frac{\sigma_1(\gamma^{(1)}-\gamma^{(2)})}{\sigma_1+\sigma_2}\left(1-\operatorname{erf}\left(\frac{x}{2\sqrt{\sigma_2^2 t}}\right)\right)+\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{v}_0^{(2)}(k)\,\mathrm{d}k \\
& +\int_{\partial D^+}\frac{\sigma_2-\sigma_1}{2\pi(\sigma_1+\sigma_2)}e^{ikx-\omega_2 t}\hat{v}_0^{(2)}(-k)\,\mathrm{d}k \\
& +\int_{\partial D^+}\frac{\sigma_2}{\pi(\sigma_1+\sigma_2)}e^{ikx-\omega_2 t}\hat{v}_0^{(1)}\left(k\frac{\sigma_2}{\sigma_1},t\right)\,\mathrm{d}k \\
& +\int_{\partial D^+}\frac{\sigma_1-\sigma_2}{2\pi(\sigma_1+\sigma_2)}e^{ikx}\hat{v}^{(2)}(-k,t)\,\mathrm{d}k \\
& -\int_{\partial D^+}\frac{\sigma_2}{\pi(\sigma_1+\sigma_2)}e^{ikx}\hat{v}^{(1)}\left(k\frac{\sigma_2}{\sigma_1},t\right)\,\mathrm{d}k.
\end{aligned}
\tag{2.18}
$$

for $x>0$, $t>0$. As before, everything about the first three integrals is known. To compute the fourth and fifth integral we proceed as we did for $v^{(1)}(x,t)$ and eliminate integrals that decay in the regions over which they are integrated. The final solution is

$$
\begin{aligned}
v^{(2)}(x,t) = & \frac{\sigma_1(\gamma^{(1)}-\gamma^{(2)})}{\sigma_1+\sigma_2}\left(1-\operatorname{erf}\left(\frac{x}{2\sqrt{\sigma_2^2 t}}\right)\right)+\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{v}_0^{(2)}(k)\,\mathrm{d}k \\
& +\int_{\partial D^+}\frac{\sigma_2-\sigma_1}{2\pi(\sigma_1+\sigma_2)}e^{ikx-\omega_2 t}v_0^{(2)}(-k)\,\mathrm{d}k \\
& +\int_{\partial D^+}\frac{\sigma_2}{\pi(\sigma_1+\sigma_2)}e^{ikx-\omega_2 t}v_0^{(1)}\left(k\frac{\sigma_2}{\sigma_1}\right)\,\mathrm{d}k.
\end{aligned}
\tag{2.19}
$$

Returning to the original variables we have the following proposition which determines $u^{(2)}(x,t)$ and $u^{(1)}(x,t)$ fully explicitly in terms of the given initial conditions and the prescribed asymptotic conditions as $|x| \to \infty$.

Proposition 2.1. *The solution of the heat transfer problem (2.2)-(2.5) is given by*

$$\begin{aligned} u^{(1)}(x,t) =& \gamma^{(1)} + \frac{\sigma_2(\gamma^{(2)}-\gamma^{(1)})}{\sigma_1+\sigma_2}\left(1-\operatorname{erf}\left(\frac{x}{2\sqrt{\sigma_1^2 t}}\right)\right) + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{v}_0^{(1)}(k)\,\mathrm{d}k \\ &+ \int_{\partial D^-} \frac{\sigma_2-\sigma_1}{2\pi(\sigma_1+\sigma_2)} e^{ikx-\omega_1 t}\hat{v}_0^{(1)}(-k)\,\mathrm{d}k \\ &- \int_{\partial D^-} \frac{\sigma_1}{\pi(\sigma_1+\sigma_2)} e^{ikx-\omega_1 t}\hat{v}_0^{(2)}\left(k\frac{\sigma_1}{\sigma_2}\right)\,\mathrm{d}k, \end{aligned} \tag{2.20a}$$

$$\begin{aligned} u^{(2)}(x,t) =& \gamma^{(2)} + \frac{\sigma_1(\gamma^{(1)}-\gamma^{(2)})}{\sigma_1+\sigma_2}\left(1-\operatorname{erf}\left(\frac{x}{2\sqrt{\sigma_2^2 t}}\right)\right) + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{v}_0^{(2)}(k)\,\mathrm{d}k \\ &+ \int_{\partial D^+} \frac{\sigma_2-\sigma_1}{2\pi(\sigma_1+\sigma_2)} e^{ikx-\omega_2 t}v_0^{(2)}(-k)\,\mathrm{d}k \\ &+ \int_{\partial D^+} \frac{\sigma_2}{\pi(\sigma_1+\sigma_2)} e^{ikx-\omega_2 t}v_0^{(1)}(k\frac{\sigma_2}{\sigma_1})\,\mathrm{d}k. \end{aligned} \tag{2.20b}$$

Remarks:

- The use of the discrete symmetries of the dispersion relation is an important aspect of the Fokas Method. When solving the heat equation in a single medium, the only discrete symmetry required is $k \to -k$, which was used here as well to obtain (2.9) and (2.11). Due to the two media, there are two dispersion relations in the present problem: $\omega_1(k) = (\sigma_1 k)^2$ and $\omega_2(k) = (\sigma_2 k)^2$. The collection of both dispersion relations $\{\omega_1(k), \omega_2(k)\}$ retains the discrete symmetry $k \to -k$, but admits two additional ones, namely: $k \to (\frac{\sigma_2}{\sigma_1})k$ and $k \to (\frac{\sigma_1}{\sigma_2})k$, which transform the two dispersion relations to each other. All nontrivial discrete symmetries of $\{\omega_1(k), \omega_2(k)\}$ are needed to derive the final solution representation, and indeed they are used *e.g.* to obtain the relations (2.15a) and (2.15b).

- With $\sigma_1 = \sigma_2$ and $\gamma^{(1)} = \gamma^{(2)} = 0$, the solution formulae (2.20) in their proper x-domains of definition reduce to the solution of the whole line problem as given in [25].

- Classical approaches to the problem presented in this section can be found in the literature, for the case $\gamma^{(1)} = 0 = \gamma^{(2)}$. For instance, for one special pair of initial conditions, a solution is presented in [33]. No explicit solution formulae using classical methods with general initial conditions exist to our knowledge. At best, one is left with having to find the solution of an equation involving inverse Laplace transforms, where the unknowns are embedded within these inverse transforms.

- The steady-state solution to (2.2) with initial conditions which decay sufficiently fast to the boundary values (2.4) at $\pm\infty$ is easily obtained by letting $t \to \infty$ in (2.20). This gives $\lim_{t\to\infty} u^{(2)}(x,t) = \lim_{t\to\infty} u^{(1)}(x,t) = (\gamma^{(1)}\sigma_1 + \gamma^{(2)}\sigma_2)/(\sigma_1 + \sigma_2)$. This is the weighted average of the boundary conditions at infinity with weights given by σ_1 and σ_2. This is consistent with the steady state limit $(\gamma^{(1)} + \gamma^{(2)})/2$ for the whole-line problem with initial conditions that limit to different values $\gamma^{(1)}$ and $\gamma^{(2)}$ as $x \to \pm\infty$. This result is easily obtained from the solution of the heat equation defined on the whole line using the Fokas Method, but it can also be observed by employing piecewise-constant initial data in the classical Green's function solution, as described in Theorem 4-1 on page 171 (and comments thereafter) of [32]. It should be emphasized that the steady state problem for (2.2)-(2.5) (or even for the heat equation defined on the whole line with different boundary conditions at $+\infty$ and $-\infty$) is ill posed in the sense that the steady solution cannot satisfy the boundary conditions.

Using a slight variation on the method presented in [20] one can compute the solutions (2.20) numerically with specified initial conditions. We plot solutions for the case of vanishing boundary conditions ($\gamma^{(1)} = \gamma^{(2)} = 0$) with

$$\begin{aligned} u_0^{(1)}(x) &= \quad x^2 e^{c_1^2 x}, \\ u_0^{(2)}(x) &= \quad x^2 e^{-c_2^2 x}, \end{aligned}$$

where $c_1 = 25$ and $c_2 = 30$ in Figure 2.3. The Fourier transforms of these initial conditions may be computed explicitly. We choose $\sigma_1 = .02$ and $\sigma_2 = .06$. The initial conditions are chosen so as to satisfy the interface boundary conditions (2.5) at $t = 0$. The results clearly illustrate the discontinuity in the first derivative of the temperature at the interface $x = 0$.

2.2 Two finite domains

Next, we consider the problem of heat conduction through two walls of finite width (or of two finite rods) with Robin boundary conditions. We seek two functions:

$$u^{(1)}(x,t), \quad -x_0 < x < x_1, \quad t \geq 0,$$

and

$$u^{(2)}(x,t), \quad x_1 < x < x_2, \quad t \geq 0,$$

satisfying the equations

$$u_t^{(1)}(x,t) = \sigma_1^2 u_{xx}^{(1)}(x,t), \qquad -x_0 < x < x_1,\ t > 0, \tag{2.21a}$$

$$u_t^{(2)}(x,t) = \sigma_2^2 u_{xx}^{(2)}(x,t), \qquad x_1 < x < x_2,\ t > 0, \tag{2.21b}$$

the initial conditions

$$u^{(1)}(x,0) = u_0^{(1)}(x), \qquad -x_0 < x < x_1, \tag{2.22a}$$

$$u^{(2)}(x,0) = u_0^{(2)}(x), \qquad x_1 < x < x_2, \tag{2.22b}$$

the boundary conditions

$$f_1(t) = \beta_1 u^{(1)}(-x_0,t) + \beta_2 u_x^{(1)}(-x_0,t), \qquad t > 0, \tag{2.23a}$$

$$f_2(t) = \beta_3 u^{(2)}(x_2,t) + \beta_4 u_x^{(2)}(x_2,t), \qquad t > 0, \tag{2.23b}$$

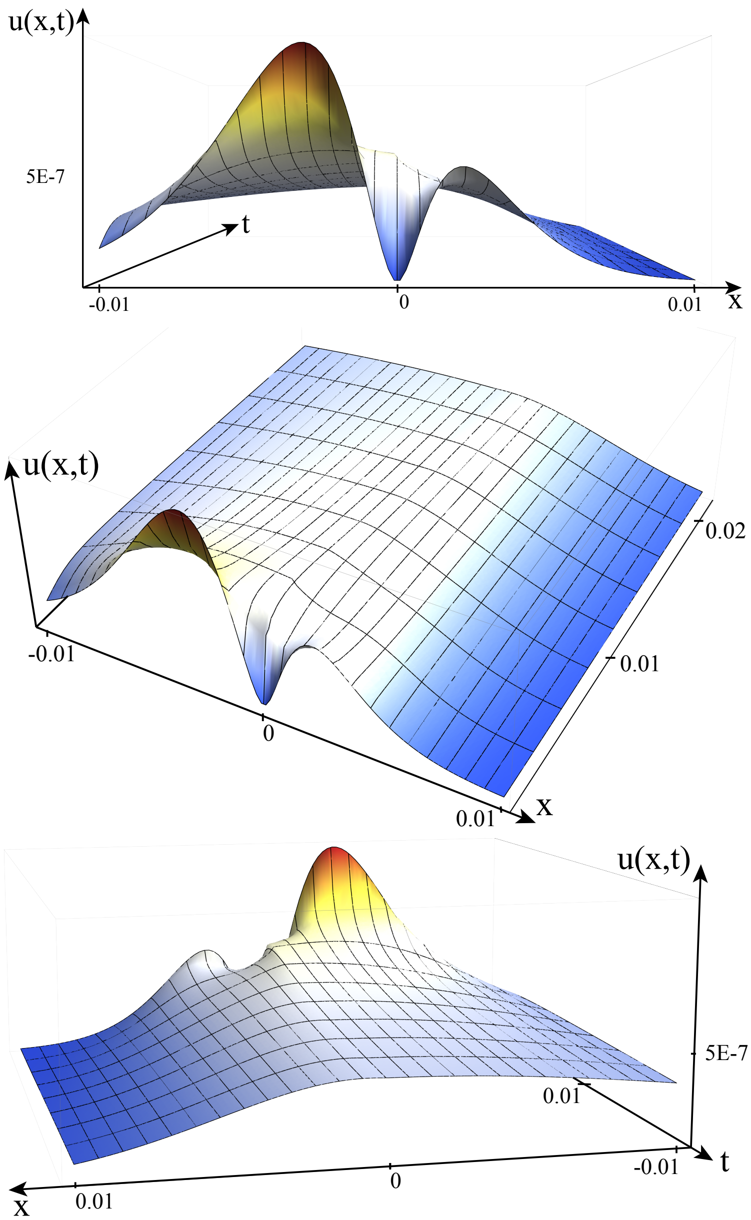

Figure 2.3: Results for the solution (2.17) and (2.19) with $u_0^{(1)}(x) = x^2 e^{(25)^2 x}$, $u_0^{(2)}(x) = x^2 e^{-(30)^2 x}$ and $\sigma_1 = .02$, $\sigma_2 = .06$, $\gamma^{(1)} = \gamma^{(2)} = 0$, $t \in [0, 0.02]$ using the hybrid analytical-numerical method of [20].

and the continuity conditions

$$u^{(1)}(x_1,t) = u^{(2)}(x_1,t), \qquad t > 0, \tag{2.24a}$$

$$\sigma_1^2 u_x^{(1)}(x_1,t) = \sigma_2^2 u_x^{(2)}(x_1,t), \qquad t > 0, \tag{2.24b}$$

as illustrated in Figure 2.4, where x_0 and x_2 are positive, $x_1 = 0$, and β_i, $1 \le i \le 4$ are constants. If $\beta_1 = \beta_3 = 0$ then Neumann boundary conditions are prescribed, whereas if $\beta_2 = \beta_4 = 0$ then Dirichlet conditions are given.

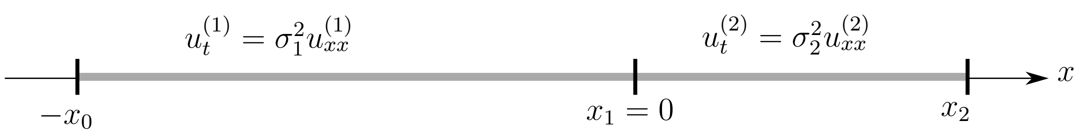

Figure 2.4: The heat equation for two finite domains.

As before we have the local relations

$$(e^{-ikx+\omega_1 t}u^{(1)}(x,t))_t = (\sigma_1^2 e^{-ikx+\omega_1 t}(u_x^{(1)}(x,t) + iku^{(1)}(x,t)))_x,$$
$$(e^{-ikx+\omega_2 t}u^{(2)}(x,t))_t = (\sigma_2^2 e^{-ikx+\omega_2 t}(u_x^{(2)}(x,t) + iku^{(2)}(x,t)))_x,$$

where $\omega_j = (\sigma_j k)^2$. We define the time transforms of the initial and boundary data and the spatial transforms of u for $k \in \mathbb{C}$ as follows:

$$\begin{aligned}
\hat{u}_0^{(1)}(k) &= \int_{-x_0}^{0} e^{-ikx} u_0^{(1)}(x)\,\mathrm{d}x, & \hat{u}^{(1)}(k,t) &= \int_{-x_0}^{0} e^{-ikx} u^{(1)}(x,t)\,\mathrm{d}x,\\
\hat{u}_0^{(2)}(k) &= \int_{0}^{x_2} e^{-ikx} u_0^{(2)}(x)\,\mathrm{d}x, & \hat{u}^{(2)}(k,t) &= \int_{0}^{x_2} e^{-ikx} u^{(2)}(x,t)\,\mathrm{d}x,\\
\tilde{f}_1(\omega,t) &= \int_0^t e^{\omega s} f_1(s)\,\mathrm{d}s, & \tilde{f}_2(\omega,t) &= \int_0^t e^{\omega s} f_2(s)\,\mathrm{d}s,\\
h_1^{(1)}(\omega,t) &= \int_0^t e^{\omega s} u_x^{(1)}(-x_0,s)\,\mathrm{d}s, & h_0^{(1)}(\omega,t) &= \int_0^t e^{\omega s} u^{(1)}(-x_0,s)\,\mathrm{d}s,\\
h_1^{(2)}(\omega,t) &= \int_0^t e^{\omega s} u_x^{(2)}(x_2,s)\,\mathrm{d}s, & h_0^{(2)}(\omega,t) &= \int_0^t e^{\omega s} u^{(2)}(x_2,s)\,\mathrm{d}s,\\
g_1(\omega,t) &= \int_0^t e^{\omega s} u_x^{(1)}(0,s)\,\mathrm{d}s = \frac{\sigma_2^2}{\sigma_1^2}\int_0^t e^{\omega s} u_x^{(2)}(0,s)\,\mathrm{d}s, & &\\
g_0(\omega,t) &= \int_0^t e^{\omega s} u^{(1)}(0,s)\,\mathrm{d}s = \int_0^t e^{\omega s} u^{(2)}(0,s)\,\mathrm{d}s. & &
\end{aligned}$$

Using Green's Theorem on the domains $[-x_0,0]\times[0,t]$ and $[0,x_2]\times[0,t]$ respectively (see Figure 2.5, we have the global relations

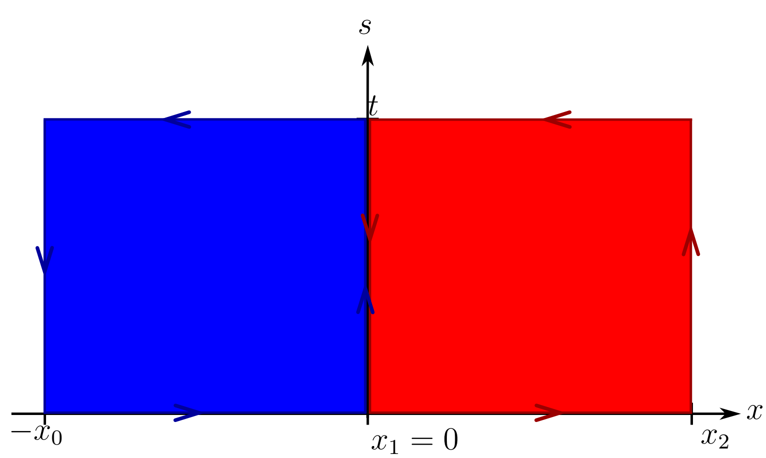

Figure 2.5: Domains for the application of Green's Theorem in the case of two finite rods.

$$
\begin{aligned}
e^{\omega_1 t}\hat{u}^{(1)}(k,t) =& \sigma_1^2\left(g_1(\omega_1,t)+ikg_0(\omega_1,t)\right)-e^{ikx_0}\sigma_1^2\left(h_1^{(1)}(\omega_1,t)+ikh_0^{(1)}(\omega_1,t)\right)\\
&+\hat{u}_0^{(1)}(k),
\end{aligned}
\tag{2.26a}
$$

$$
\begin{aligned}
e^{\omega_2 t}\hat{u}^{(2)}(k,t) =& e^{-ikx_2}\sigma_2^2\left(h_1^{(2)}(\omega_2,t)+ikh_0^{(2)}(\omega_2,t)\right)-\sigma_1^2 g_1(\omega_2,t)-ik\sigma_2^2 g_0(\omega_2,t)\\
&+\hat{u}_0^{(2)}(k).
\end{aligned}
\tag{2.26b}
$$

Both equations are valid for all $k\in\mathbb{C}$, in contrast to (2.8) and (2.10). Using the invariance of $\omega_1(k)=(\sigma_1 k)^2$ and $\omega_2(k)=(\sigma_2 k)^2$ under $k\to -k$ we obtain

$$
\begin{aligned}
e^{\omega_1 t}\hat{u}^{(1)}(-k,t) =& \sigma_1^2\left(g_1(\omega_1,t)-ikg_0(\omega_1,t)\right)-e^{-ikx_0}\sigma_1^2\left(h_1^{(1)}(\omega_1,t)-ikh_0^{(1)}(\omega_1,t)\right)\\
&+\hat{u}_0^{(1)}(-k),
\end{aligned}
\tag{2.27a}
$$

$$
\begin{aligned}
e^{\omega_2 t}\hat{u}^{(2)}(-k,t) =& e^{ikx_2}\sigma_2^2\left(h_1^{(2)}(\omega_2,t)-ikh_0^{(2)}(\omega_2,t)\right)-\sigma_1^2 g_1(\omega_2,t)+ik\sigma_2^2 g_0(\omega_2,t)\\
&+\hat{u}_0^{(2)}(-k).
\end{aligned}
\tag{2.27b}
$$

As in Section 2.1 we need to use all the symmetries of the set of dispersion relations. Using the symmetries $k\to k\frac{\sigma_1}{\sigma_2}$ and $k\to k\frac{\sigma_2}{\sigma_1}$ we have

$$
\begin{aligned}
e^{\omega_2 t}\hat{u}^{(1)}\left(k\frac{\sigma_2}{\sigma_1},t\right) =& -e^{ikx_0\frac{\sigma_2}{\sigma_1}}\left(\sigma_1^2 h_1^{(1)}(\omega_2,t)+ik\sigma_1\sigma_2 h_0^{(1)}(\omega_2,t)\right)\\
&+\sigma_1^2 g_1(\omega_2,t)+ik\sigma_1\sigma_2 g_0(\omega_2,t)+\hat{u}_0^{(1)}\left(k\frac{\sigma_2}{\sigma_1}\right),
\end{aligned}
\tag{2.28a}
$$

$$
\begin{aligned}
e^{\omega_1 t}\hat{u}^{(2)}\left(k\frac{\sigma_1}{\sigma_2},t\right) =& e^{-ikx_2\frac{\sigma_1}{\sigma_2}}\left(\sigma_2^2 h_1^{(2)}(\omega_1,t)+ik\sigma_1\sigma_2 h_0^{(2)}(\omega_1,t)\right)\\
&-\sigma_1^2 g_1(\omega_1,t)-ik\sigma_1\sigma_2 g_0(\omega_1,t)+\hat{u}_0^{(2)}(k),
\end{aligned}
\tag{2.28b}
$$

$$
\begin{aligned}
e^{\omega_2 t}\hat{u}^{(1)}\left(-k\frac{\sigma_2}{\sigma_1},t\right) =& -e^{-ikx_0\frac{\sigma_2}{\sigma_1}}\left(\sigma_1^2 h_1^{(1)}(\omega_2,t)-ik\sigma_1\sigma_2 h_0^{(1)}(\omega_2,t)\right)\\
&+\sigma_1^2 g_1(\omega_2,t)-ik\sigma_1\sigma_2 g_0(\omega_2,t)+\hat{u}_0^{(1)}\left(-k\frac{\sigma_2}{\sigma_1}\right),
\end{aligned}
\tag{2.28c}
$$

$$
\begin{aligned}
e^{\omega_1 t}\hat{u}^{(2)}\left(-k\frac{\sigma_1}{\sigma_2},t\right) =& e^{ikx_2\frac{\sigma_1}{\sigma_2}}\left(\sigma_2^2 h_1^{(2)}(\omega_1,t) - ik\sigma_1\sigma_2 h_0^{(2)}(\omega_1,t)\right) \\
&- \sigma_1^2 g_1(\omega_1,t) + ik\sigma_1\sigma_2 g_0(\omega_1,t) + \hat{u}_0^{(2)}\left(-k\frac{\sigma_1}{\sigma_2}\right).
\end{aligned}
\tag{2.28d}
$$

Inverting the Fourier transform in (2.26a) we have,

$$
\begin{aligned}
u^{(1)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\sigma_1^2(g_1(\omega_1,t) + ikg_0(\omega_1,t))\,\mathrm{d}k \\
&- \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x_0+x)-\omega_1 t}\sigma_1^2(h_1^{(1)}(\omega_1,t) + ikh_0^{(1)}(\omega_1,t))\,\mathrm{d}k \\
&+ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{u}_0^{(1)}(k)\,\mathrm{d}k.
\end{aligned}
$$

The integrand of the first integral is entire and decays as $k \to \infty$ for $k \in \mathbb{C}^- \setminus D^-$. The second integral has an integrand that is entire and decays as $k \to \infty$ for $k \in \mathbb{C}^+ \setminus D^+$. It is convenient to deform both contours away from $k = 0$ to avoid singularities in the integrands that become apparent in what follows. Initially, these singularities are removable, since the integrands are entire. Writing integrals of sums as sums of integrals, the singularities may cease to be removable. With the deformations away from $k = 0$, the apparent singularities are no cause for concern. In other words, we deform D^+ to D_R^+ and D^- to D_R^- as show in Figure 2.6 where

$$
D_R^\pm = \{k \in D^\pm : |k| > R\},
$$

and $R > 0$ is an arbitrary constant. An appropriate (sufficiently large) value of this constant may be chosen for any individual problem. Thus

$$
\begin{aligned}
u^{(1)}(x,t) =& \frac{-1}{2\pi}\int_{\partial D_R^-} e^{ikx-\omega_1 t}\sigma_1^2(g_1(\omega_1,t) + ikg_0(\omega_1,t))\,\mathrm{d}k \\
&- \frac{1}{2\pi}\int_{\partial D_R^+} e^{ik(x_0+x)-\omega_1 t}\sigma_1^2(h_1^{(1)}(\omega_1,t) + ikh_0^{(1)}(\omega_1,t))\,\mathrm{d}k \\
&+ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{u}_0^{(1)}(k)\,\mathrm{d}k.
\end{aligned}
\tag{2.29}
$$

To obtain the solution on the right we apply the inverse Fourier transform to (2.26b):

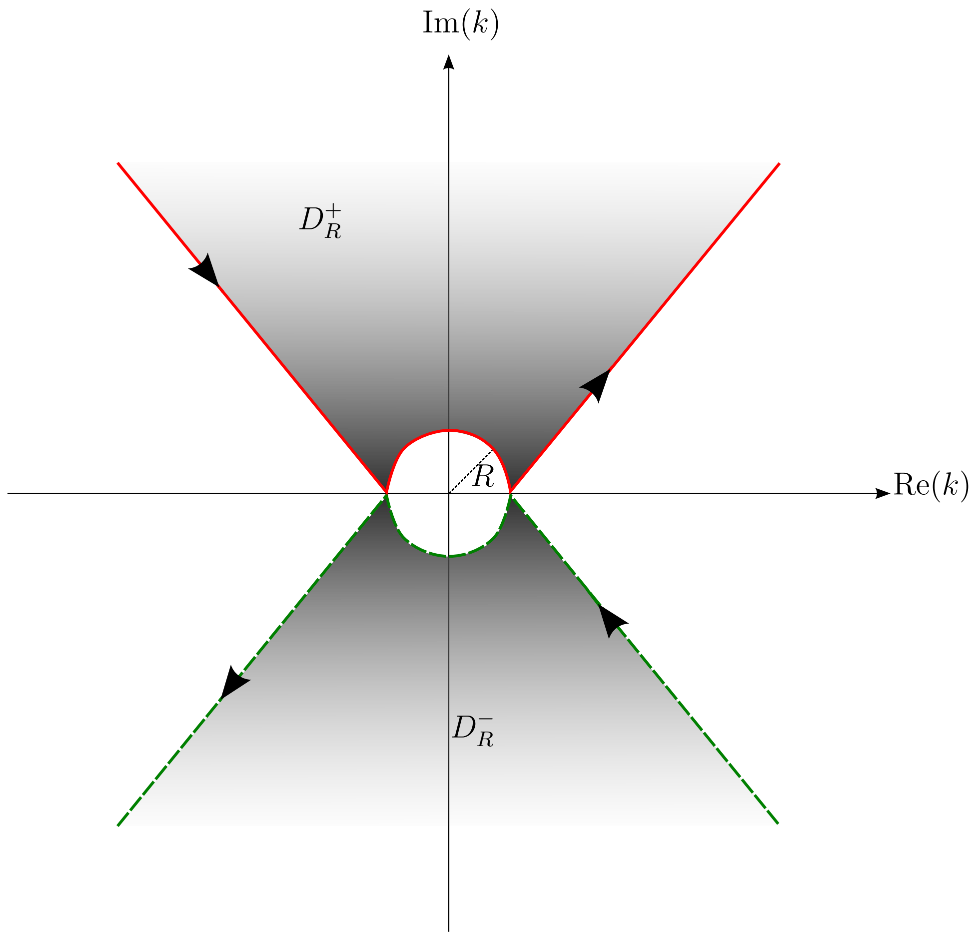

Figure 2.6: Deformation of the contours in Figure 1.2 for $|k| > R$.

$$
\begin{aligned}
u^{(2)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x_2)-\omega_2 t}\sigma_2^2(h_1^{(2)}(\omega_2,t)+ikh_0^{(2)}(\omega_2,t))\,\mathrm{d}k\\
&-\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}(ik\sigma_2^2 g_0(\omega_2,t)+\sigma_1^2 g_1(\omega_2,t))\,\mathrm{d}k\\
&+\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{u}_0^{(2)}(k)\,\mathrm{d}k.
\end{aligned}
$$

The integrand of the first integral is entire and decays as $k\to\infty$ for $k\in\mathbb{C}^-\setminus D^-$. The second integral has an integrand that is entire and decays as $k\to\infty$ for $k\in\mathbb{C}^+\setminus D^+$. We deform the contours as above to obtain

$$
\begin{aligned}
u^{(2)}(x,t) =& \frac{-1}{2\pi}\int_{\partial D_R^-} e^{ik(x-x_2)-\omega_2 t}\sigma_2^2(h_1^{(2)}(\omega_2,t)+ikh_0^{(2)}(\omega_2,t))\,\mathrm{d}k\\
&-\frac{1}{2\pi}\int_{\partial D_R^+} e^{ikx-\omega_2 t}(ik\sigma_2^2 g_0(\omega_2,t)+\sigma_1^2 g_1(\omega_2,t))\,\mathrm{d}k\\
&+\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{u}_0^{(2)}(k)\,\mathrm{d}k.
\end{aligned}
\tag{2.30}
$$

Taking the time transform of the boundary conditions and evaluating at the appropriate arguments results in

$$
\tilde{f}_1(\omega_1,t)=\beta_1 h_0^{(1)}(\omega_1,t)+\beta_2 h_1^{(1)}(\omega_1,t),
$$

and

$$
\tilde{f}_2(\omega_2,t)=\beta_3 h_0^{(2)}(\omega_2,t)+\beta_4 h_1^{(2)}(\omega_2,t).
$$

These two equations together with (2.26), (2.27) and (2.28) may be solved for the necessary unknowns $h_0^{(1)}(\omega_1,t)$, $h_0^{(2)}(\omega_2,t)$, $h_1^{(1)}(\omega_1,t)$, $h_1^{(2)}(\omega_2,t)$, $g_0(\omega_1,t)$, $g_1(\omega_1,t)$, $g_0(\omega_2,t)$, and $g_1(\omega_2,t)$. The resulting expressions are substituted in (2.29) and (2.30).

Although we could solve this problem in its full generality, we restrict ourselves to the case of Dirichlet boundary conditions ($\beta_2=\beta_4=0$, $\beta_1=\beta_3=1$), to simplify the already cumbersome formulae below. The system is not solvable if $\Delta_1(k)=0$ or $\Delta_2(k)=0$, where

$$
\begin{aligned}
\Delta_1(k) =& 2\pi \left(\sigma_1(e^{2ikx_0}+1)\left(e^{2ikx_2\frac{\sigma_1}{\sigma_2}}-1\right)+\sigma_2(e^{2ikx_0}-1)\left(e^{2ikx_2\frac{\sigma_1}{\sigma_2}}+1\right)\right) \\
&= 2i\pi\left(e^{2ikx_0}+1\right)\left(e^{2ikx_2\frac{\sigma_1}{\sigma_2}}+1\right)\left(\sigma_1\tan\left(x_2k\frac{\sigma_1}{\sigma_2}\right)+\sigma_2\tan(kx_0)\right),
\end{aligned}
$$

and

$$
\Delta_2(k) = \Delta_1\left(k\frac{\sigma_2}{\sigma_1}\right).
$$

It is easily seen that all values of k satisfying either of these equations (including $k=0$) are on the real line. Thus, on the contours we consider, the equations are solved without problem, resulting in the expressions below. As before, the right-hand sides of these expressions involve $\hat{u}^{(1)}(\cdot,t)$ and $\hat{u}^{(2)}(\cdot,t)$. All terms with such dependence are written out explicitly below. Terms that depend on known quantities only are contained in $\mathcal{K}^{(1)}$ and $\mathcal{K}^{(2)}$, the expressions for which are given later.

$$
\begin{aligned}
u^{(1)}(x,t) = \mathcal{K}^{(1)} &+ \int_{\partial D_R^-} \frac{e^{ikx}(\sigma_1+\sigma_2)+e^{ik(x+2x_2\frac{\sigma_1}{\sigma_2})}(\sigma_2-\sigma_1)}{\Delta_1(k)}\hat{u}^{(1)}(k,t)\,\mathrm{d}k \\
&- \int_{\partial D_R^-} \frac{e^{ik(x+2x_0)}(\sigma_1+\sigma_2)+e^{ik(x+2x_0+2x_2\frac{\sigma_1}{\sigma_2})}(\sigma_1-\sigma_2)}{\Delta_1(k)}\hat{u}^{(1)}(-k,t)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{2\sigma_1 e^{ik(x+2x_0+2x_2\frac{\sigma_1}{\sigma_2})}}{\Delta_1(k)}\hat{u}^{(2)}\left(k\frac{\sigma_1}{\sigma_2},t\right)\,\mathrm{d}k \\
&- \int_{\partial D_R^-} \frac{\sigma_1 e^{ik(x+2x_0)}}{\Delta_1(k)}\hat{u}^{(2)}\left(-k\frac{\sigma_1}{\sigma_2},t\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{e^{ik(x+2x_0)}(\sigma_2-\sigma_1)+e^{ik(x+2x_0+2x_2\frac{\sigma_1}{\sigma_2})}(\sigma_1+\sigma_2)}{\Delta_1(k)}\hat{u}^{(1)}(k,t)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{e^{ik(x+2x_0)}-(\sigma_2+\sigma_1)+e^{ik(x+2x_0+2x_2\frac{\sigma_1}{\sigma_2})}(\sigma_1-\sigma_2)}{\Delta_1(k)}\hat{u}^{(1)}(-k,t)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{2\sigma_1 e^{ik(x+2x_0+2x_2\frac{\sigma_1}{\sigma_2})}}{\Delta_1(k)}\hat{u}^{(2)}\left(k\frac{\sigma_1}{\sigma_2},t\right)\,\mathrm{d}k \\
&- \int_{\partial D_R^+} \frac{2\sigma_1 e^{ik(x+2x_0)}}{\Delta_1(k)}\hat{u}^{(2)}\left(-k\frac{\sigma_1}{\sigma_2},t\right)\,\mathrm{d}k,
\end{aligned}
\tag{2.31}
$$

and

$$\begin{aligned}
u^{(2)}(x,t) =&\mathcal{K}^{(2)} + \int_{\partial D_R^+} \frac{2\sigma_2 e^{ikx}}{\Delta_2(k)} \hat{u}^{(1)}\left(k\frac{\sigma_2}{\sigma_1}, t\right)\,\mathrm{d}k \\
&- \int_{\partial D_R^+} \frac{2\sigma_2 e^{ik(x+2x_0\frac{\sigma_2}{\sigma_1})}}{\Delta_2(k)} \hat{u}^{(1)}\left(-k\frac{\sigma_2}{\sigma_1}, t\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{e^{ikx+2x_2)}(\sigma_1-\sigma_2) + e^{ik(x+2x_2+2x_0\frac{\sigma_2}{\sigma_1})}(\sigma_1+\sigma_2)}{\Delta_2(k)} \hat{u}^{(2)}(k,t)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{e^{ikx}(\sigma_2-\sigma_1) - e^{ik(x+2x_0\frac{\sigma_2}{\sigma_1})}(\sigma_1+\sigma_2)}{\Delta_2(k)} \hat{u}^{(2)}(-k,t)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{2\sigma_2 e^{ikx}}{\Delta_2(k)} \hat{u}^{(1)}\left(k\frac{\sigma_2}{\sigma_1}, t\right)\,\mathrm{d}k \\
&- \int_{\partial D_R^-} \frac{\sigma_2 e^{ik(x+2x_0\frac{\sigma_2}{\sigma_1})}}{\Delta_2(k)} \hat{u}^{(1)}\left(-k\frac{\sigma_2}{\sigma_1}, t\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{e^{ikx)}(\sigma_1+\sigma_2) + e^{ik(x+2x_0\frac{\sigma_2}{\sigma_1})}(\sigma_1-\sigma_2)}{\Delta_2(k)} \hat{u}^{(2)}(k,t)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{e^{ikx}(\sigma_2-\sigma_1) - e^{ik(x+2x_0\frac{\sigma_2}{\sigma_1})}(\sigma_1+\sigma_2)}{\Delta_2(k)} \hat{u}^{(2)}(-k,t)\,\mathrm{d}k.
\end{aligned} \tag{2.32}$$

The integrands written explicitly in (2.31) and (2.32) decay in the regions around whose boundaries they are integrated. Thus, using Jordan's Lemma and Cauchy's Theorem, these integrals are shown to vanish. Thus the final solution is given by $\mathcal{K}^{(1)}$ and $\mathcal{K}^{(2)}$.

Proposition 2.2. *The solution of the heat transfer problem* (2.21)-(2.24) *is given by*

$$\begin{aligned}
u^{(1)}(x,t) =& \mathcal{K}^{(1)} = \int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{u}_0^{(1)}(k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{-4ik\sigma_1^2\sigma_2 e^{ik(x+2x_0+x_2\frac{\sigma_1}{\sigma_2})-\omega_1 t}}{\Delta_1(k)}\tilde{f}_2(\omega_1,t)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{2ik\sigma_1^2 e^{ik(x+x_0)-\omega_1 t}(\sigma_1+\sigma_2) - ik\sigma_1^2 e^{ik(x+x_0+2x_2\frac{\sigma_1}{\sigma_2})-\omega_1 t}(\sigma_1-\sigma_2)}{\Delta_1(k)}\tilde{f}_1(\omega_1,t)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{-e^{ikx-\omega_1 t}(\sigma_1+\sigma_2) + e^{ik(x+2x_2\frac{\sigma_1}{\sigma_2})-\omega_1 t}(\sigma_1-\sigma_2)}{\Delta_1(k)}\hat{u}_0^{(1)}(k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{e^{ik(x+2x_0)-\omega_1 t}(\sigma_1+\sigma_2) + e^{ik(x+2x_0+2x_2\frac{\sigma_1}{\sigma_2})-\omega_1 t}(\sigma_2-\sigma_1)}{\Delta_1(k)}\hat{u}_0^{(1)}(-k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{-2\sigma_1 e^{ik(x+2x_0+2x_2\frac{\sigma_1}{\sigma_2})-\omega_1 t}}{\Delta_1(k)}\hat{u}_0^{(2)}\left(k\frac{\sigma_1}{\sigma_2}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{\sigma_1 e^{ik(x+2x_0)-\omega_1 t}}{\Delta_1(k)}\hat{u}_0^{(2)}\left(-k\frac{\sigma_1}{\sigma_2}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{2ik\sigma_1^2 e^{ik(x+x_0)-\omega_1 t}(\sigma_1+\sigma_2) - ik\sigma_1^2 e^{ik(x+x_0+2x_2\frac{\sigma_1}{\sigma_2})-\omega_1 t}(\sigma_1-\sigma_2)}{\Delta_1(k)}\tilde{f}_1(\omega_1,t)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{-4ik\sigma_1^2\sigma_2 e^{ik(x+2x_0+x_2\frac{\sigma_1}{\sigma_2})-\omega_1 t}(1+\sigma_1\sigma_2)}{\Delta_1(k)}\tilde{f}_2(\omega_1,t)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{-e^{ik(x+2x_0)-\omega_1 t}(\sigma_2-\sigma_1) - e^{ik(x+2x_0+2x_2\frac{\sigma_1}{\sigma_2})-\omega_1 t}(\sigma_1+\sigma_2)}{\Delta_1(k)}\hat{u}_0^{(1)}(k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{e^{ik(x+2x_0)-\omega_1 t}(\sigma_1+\sigma_3) + e^{ik(x+2x_0+2x_2\frac{\sigma_1}{\sigma_3})-\omega_1 t}(\sigma_3-\sigma_1)}{\Delta_1(k)}\hat{u}_0^{(1)}(-k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{-2\sigma_1 e^{ik(x+2x_0+2x_2\frac{\sigma_1}{\sigma_2})-\omega_1 t}}{\Delta_1(k)}\hat{u}_0^{(2)}\left(k\frac{\sigma_1}{\sigma_2}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{2\sigma_1 e^{ik(x+2x_0)-\omega_1 t}}{\Delta_1(k)}\hat{u}_0^{(2)}\left(-k\frac{\sigma_1}{\sigma_2}\right)\,\mathrm{d}k,
\end{aligned} \tag{2.33}$$

for $-x_0 < x < 0$, *and, for* $0 < x < x_2$

$$
\begin{aligned}
u^{(2)}(x,t) =& \mathcal{K}^{(2)} = \int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{u}_0^{(2)}(k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{4ik\sigma_1\sigma_2^2 e^{ik(x+x_0\frac{\sigma_2}{\sigma_1})-\omega_2 t}}{\Delta_2(k)}\tilde{f}_1(\omega_w,t)\,\mathrm{d}k \\
&- \int_{\partial D_R^-} \frac{2ik\sigma_2^2 e^{ik(x+x_2)-\omega_2 t}(\sigma_1-\sigma_2) - ik\sigma_2^2 e^{ik(x+x_2+2x_0\frac{\sigma_2}{\sigma_1})-\omega_2 t}}{\Delta_2(k)}\tilde{f}_2(\omega_2,t)\,\mathrm{d}k \\
&- \int_{\partial D_R^-} \frac{2\sigma_2 e^{ikx-\omega_2 t}}{\Delta_2(k)}\hat{u}_0^{(1)}\left(k\frac{\sigma_2}{\sigma_1}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{\sigma_2 e^{ik(x+2x_0\frac{\sigma_2}{\sigma_1})-\omega_2 t}}{\Delta_2(k)}\hat{u}_0^{(1)}\left(-k\frac{\sigma_2}{\sigma_1}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{-e^{ik(x+2x_2)-\omega_2 t}(\sigma_1-\sigma_2) - e^{ik(x+2x_2+2x_0\frac{\sigma_2}{\sigma_1})-\omega_2 t}(\sigma_1+\sigma_2)}{\Delta_2(k)}\hat{u}_0^{(2)}(k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{e^{ikx-\omega_2 t}(\sigma_1-\sigma_2) + e^{ik(x+2x_0\frac{\sigma_2}{\sigma_1})-\omega_2 t}(\sigma_1+\sigma_2)}{2\Delta_2(k)}\hat{u}_0^{(2)}(-k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{4ik\sigma_1\sigma_2^2 e^{ik(x+x_0\frac{\sigma_2}{\sigma_1})-\omega_2 t}}{\Delta_2(k)}\tilde{f}_1(\omega_2,t)\,\mathrm{d}k \\
&- \int_{\partial D_R^+} \frac{2ik\sigma_2 e^{ik(x+x_2)-\omega_2 t}(\sigma_1-\sigma_2)+ik\sigma_2^2 e^{ik(x+x_2+2x_0\frac{\sigma_2}{\sigma_1})-\omega_2 t}(\sigma_1+\sigma_2)}{\Delta_2(k)}\tilde{f}_2(\omega_2,t)\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{-2\sigma_2 e^{ikx-\omega_2 t}}{\Delta_2(k)}\hat{u}_0^{(1)}\left(k\frac{\sigma_2}{\sigma_1}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{2\sigma_2 e^{ik(x+2x_0\frac{\sigma_2}{\sigma_1})-\omega_2 t}}{\Delta_2(k)}\hat{u}_0^{(1)}\left(-k\frac{\sigma_2}{\sigma_1}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{-e^{ikx-\omega_2 t}(\sigma_1+\sigma_2) + e^{ik(x+2x_0\frac{\sigma_2}{\sigma_1})-\omega_2 t}(\sigma_2-\sigma_1)}{\Delta_2(k)}\hat{u}_0^{(2)}(k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{e^{ikx-\omega_2 t}(\sigma_1-\sigma_2) + e^{ik(x+2x_0\frac{\sigma_2}{\sigma_1})-\omega_2 t}(\sigma_1+\sigma_2)}{\Delta_2(k)}\hat{u}_0^{(2)}(-k)\,\mathrm{d}k.
\end{aligned}
\tag{2.34}
$$

Remarks:

- The solution of the problem posed in (2.21)-(2.24) may be obtained using the classical method of separation of variables and superposition, see [33]. The solutions $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ are given by series of eigenfunctions with eigenvalues that satisfy a transcendental equation, closely related to the equation $\Delta_1(k) = 0$. This series solution may be obtained from Proposition 2.2 by deforming the contours along ∂D_R^- and ∂D_R^+ to the real line, including small semi-circles around each root of either $\Delta_1(k)$ or $\Delta_2(k)$, depending on whether $u^{(1)}(x,t)$ or $u^{(2)}(x,t)$ is being calculated. Indeed, this is allowed since all integrands decay in the wedges between these contours and the real line, and the zeros of $\Delta_1(k)$ and $\Delta_2(k)$ occur only on the real line, as stated above. Careful calculation of all different contributions, following the examples in [15, 25], shows that the contributions along the real line cancel, leaving only residue contributions from the small circles. Each residue contribution corresponds to a term in the classical series solution. It is not necessarily beneficial to leave the form of the solution in Proposition 2.2 for the series representation, as the latter depends on the roots of $\Delta_1(k)$ and $\Delta_2(k)$, which are not known explicitly. In contrast, the representation of Proposition 2.2 depends on known quantities only and may be readily computed, using a parameterization of the contours ∂D_R^- and ∂D_R^+.

- Similarly, the familiar piecewise linear steady-state solution of (2.21)-(2.24) with Dirichlet boundary conditions [33] can be observed from (2.33) and (2.34) by choosing initial conditions that decay appropriately and constant boundary conditions $f_1(t) = \gamma^{(1)}$ and $f_2(t) = \gamma^{(2)}$. It is convenient to choose zero initial conditions, since the initial conditions do not affect the steady state. As above, the contours are deformed so that they are along the real line with semi-circular paths around the zeros of $\Delta_1(k)$ and $\Delta_2(k)$, including $k = 0$. Since one of these deformations arises from D_R^+ while the other comes from D_R^-, the contributions along the real line cancel each other, while the semi-circles add to give full residue contributions from the poles associated with the zeros of $\Delta_1(k)$. All such residues vanish as $t \to \infty$, except at $k = 0$. It follows that the steady state behavior is determined by the residue at the origin. This results in

$$\begin{aligned}
u^{(1)}(x,t) = &\quad \frac{\sigma_2^2(\gamma^{(2)}-\gamma^{(1)})}{x_2\sigma_1^2+x_0\sigma_2^2}x+\frac{x_2\gamma^{(1)}\sigma_1^2+x_0\gamma^{(2)}\sigma_2^2}{x_2\sigma_1^2+x_0\sigma_2^2}, && -x_0<x<0,\\
u^{(2)}(x,t) = &\quad \frac{\sigma_1^2(\gamma^{(2)}-\gamma^{(1)})}{x_2\sigma_1^2+x_0\sigma_2^2}x+\frac{x_2\gamma^{(1)}\sigma_1^2+x_0\gamma^{(2)}\sigma_2^2}{x_2\sigma_1^2+x_0\sigma_2^2}, && 0<x<x_2,
\end{aligned}$$

which is piecewise linear and continuous.

- A more direct way to recover only the steady-state solution to (2.21) with $\lim_{t\to\infty} f_1(t) = \bar{f}_1$ and $\lim_{t\to\infty} f_2(t) = \bar{f}_2$ constant is to write the solution as the superposition of two parts: $u^{(1)}(x,t) = \bar{u}^{(1)}(x) + \check{u}^{(1)}(x,t)$ and $u^{(2)}(x,t) = \bar{u}^{(2)}(x) + \check{u}^{(2)}(x,t)$. The first parts $\bar{u}^{(1)}$ and $\bar{u}^{(2)}$ satisfy the boundary conditions as $t \to \infty$ and the stationary heat equation. In other words

$$\begin{aligned}
0 =&\sigma_1^2\bar{u}^{(1)}_{xx}(x), && -x_0<x<0,\\
0 =&\sigma_2^2\bar{u}^{(2)}_{xx}(x), && 0<x<x_2,\\
\bar{f}_1 =&\beta_1\bar{u}^{(1)}(-x_0)+\beta_2\bar{u}^{(1)}_x(-x_0),\\
\bar{f}_2 =&\beta_3\bar{u}^{(2)}(x_2)+\beta_4\bar{u}^{(2)}_x(x_2).
\end{aligned}$$

A piecewise linear ansatz with the imposition of the interface conditions results in linear algebra for the unknown coefficients, see [33]. With the steady state solution in hand, the second (time-dependent) parts $\check{u}^{(1)}$ and $\check{u}^{(2)}$ satisfy the initial conditions modified by the steady state solution and the boundary conditions minus their value as $t \to \infty$:

$$
\begin{aligned}
\check{u}_t^{(1)}(x,t) &= \sigma_1^2 \check{u}_{xx}^{(1)}(x,t), && x_0 < x < 0, && t > 0,\\
\check{u}_t^{(2)}(x,t) &= \sigma_2^2 \check{u}_{xx}^{(2)}(x,t), && 0 < x < x_2, && t > 0,\\
\check{u}^{(1)}(x,0) &= u^{(1)}(x,0) - \bar{u}^{(1)}(x), && -x_0 < x < 0, &&\\
\check{u}^{(2)}(x,0) &= u^{(2)}(x,0) - \bar{u}^{(2)}(x), && 0 < x < x_2, &&\\
f_1(t) - \bar{f}_1 &= \beta_1 \check{u}^{(1)}(-x_0,t) + \beta_2 \check{u}_x^{(1)}(-x_0,t), && && t > 0,\\
f_2(t) - \bar{f}_2 &= \beta_3 \check{u}^{(2)}(x_2,t) + \beta_4 \check{u}_x^{(2)}(x_2,t), && && t > 0,
\end{aligned}
$$

where, as usual, we impose continuity of temperature and heat flux at the interface $x = 0$. The dynamics of the solution is described by $\check{u}^{(1)}$ and $\check{u}^{(2)}$, both of which decay to zero as $t \to \infty$. Their explicit form is easily found using the method described in this section.

2.3 An infinite domain with three layers

In this section we consider the heat equation defined on two semi-infinite rods enclosing a single rod of length $2x_0$. That is, we seek three functions

$$
\begin{aligned}
&u^{(1)}(x,t), && x < -x_0, && t \geq 0,\\
&u^{(2)}(x,t), && -x_0 < x < x_0, && t \geq 0,\\
&u^{(3)}(x,t), && x > x_0, && t \geq 0,
\end{aligned}
$$

satisfying the equations

$$u_t^{(1)}(x,t) = \sigma_1^2 u_{xx}^{(1)}(x,t), \qquad x < -x_0, \qquad t > 0, \tag{2.35a}$$

$$u_t^{(2)}(x,t) = \sigma_2^2 u_{xx}^{(2)}(x,t), \qquad -x_0 < x < x_0, \qquad t > 0, \tag{2.35b}$$

$$u_t^{(3)}(x,t) = \sigma_3^2 u_{xx}^{(3)}(x,t), \qquad x > x_0, \qquad t > 0, \tag{2.35c}$$

the initial conditions

$$u_t^{(1)}(x,0) = u_0^{(1)}(x), \qquad x < -x_0, \tag{2.36a}$$

$$u_t^{(2)}(x,0) = u_0^{(2)}(x), \qquad -x_0 < x < x_0, \tag{2.36b}$$

$$u_t^{(3)}(x,0) = u_0^{(3)}(x), \qquad x > x_0, \tag{2.36c}$$

the asymptotic conditions

$$\lim_{x\to-\infty} u_t^{(1)}(x,t) = 0, \qquad t \geq 0, \tag{2.37a}$$

$$\lim_{x\to\infty} u_t^{(3)}(x,t) = 0, \qquad t \geq 0, \tag{2.37b}$$

and the continuity conditions

$$u^{(1)}(-x_0,t) = u^{(2)}(-x_0,t), \qquad t \geq 0, \tag{2.38a}$$

$$u^{(2)}(x_0,t) = u^{(3)}(x_0,t), \qquad t \geq 0, \tag{2.38b}$$

$$\sigma_1^2 u_x^{(1)}(-x_0,t) = \sigma_2^2 u_x^{(2)}(-x_0,t), \qquad t \geq 0, \tag{2.38c}$$

$$\sigma_2^2 u_x^{(2)}(x_0,t) = \sigma_3^2 u_x^{(3)}(x_0,t), \qquad t \geq 0, \tag{2.38d}$$

as in Figure 2.7.

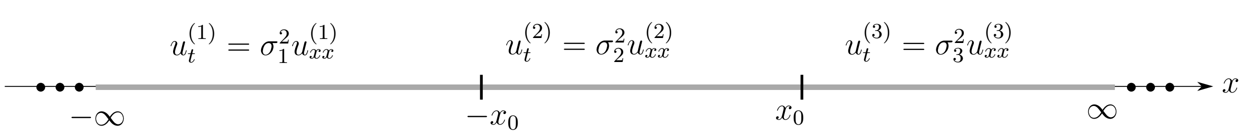

Figure 2.7: The heat equation for an infinite domain with three layers.

After defining the transforms

$$
\begin{aligned}
\hat{u}_0^{(1)}(k) &= \int_{-\infty}^{-x_0} e^{-ikx} u_0^{(1)}(x)\,\mathrm{d}x, & \hat{u}^{(1)}(k,t) &= \int_{-\infty}^{-x_0} e^{-ikx} u^{(1)}(x,t)\,\mathrm{d}x,\\
\hat{u}_0^{(2)}(k) &= \int_{-x_0}^{x_0} e^{-ikx} u_0^{(2)}(x)\,\mathrm{d}x, & \hat{u}^{(2)}(k) &= \int_{-x_0}^{x_0} e^{-ikx} u^{(2)}(x,t)\,\mathrm{d}x,\\
\hat{u}_0^{(3)}(k) &= \int_{x_0}^{\infty} e^{-ikx} u_0^{(3)}(x)\,\mathrm{d}x, & \hat{u}^{(3)}(k,t) &= \int_{x_0}^{\infty} e^{-ikx} u^{(3)}(x,t)\,\mathrm{d}x,\\
h_1(\omega,t) &= \int_0^t e^{\omega s} u_x^{(1)}(-x_0,s)\,\mathrm{d}s = \frac{\sigma_2^2}{\sigma_1^2}\int_0^t e^{\omega s} u_x^{(2)}(-x_0,s)\,\mathrm{d}s,\\
h_0(\omega,t) &= \int_0^t e^{\omega s} u^{(1)}(-x_0,s)\,\mathrm{d}s = \int_0^t e^{\omega s} u^{(2)}(-x_0,s)\,\mathrm{d}s,\\
g_1(\omega,t) &= \int_0^t e^{\omega s} u_x^{(2)}(x_0,s)\,\mathrm{d}s = \frac{\sigma_3^2}{\sigma_2^2}\int_0^t e^{\omega s} u_x^{(3)}(x_0,s)\,\mathrm{d}s,\\
g_0(\omega,t) &= \int_0^t e^{\omega s} u^{(2)}(x_0,s)\,\mathrm{d}s = \int_0^t e^{\omega s} u^{(3)}(x_0,s)\,\mathrm{d}s,
\end{aligned}
$$

we proceed as outlined in the preceding sections. The solution formulae are given in the following proposition:

Proposition 2.3. *The solution of the heat transfer problem* (2.35)-(2.38) *is*

$$
\begin{aligned}
u^{(1)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{u}_0^{(1)}(k)\,\mathrm{d}k\\
&- \sigma_1(\sigma_2+\sigma_3)\int_{\partial D_R^-} \frac{e^{ik(x+x_0+3x_0\frac{\sigma_1}{\sigma_2})-\omega_1 t}}{\Delta_1(k)}\hat{u}_0^{(2)}\left(\frac{k\sigma_1}{\sigma_2}\right)\mathrm{d}k\\
&- \int_{\partial D_R^-} \frac{e^{ik(x+2x_0)-\omega_1 t}}{2\Delta_1(k)}\Big((\sigma_1+\sigma_2)(\sigma_2-\sigma_3) + e^{4ikx_0\frac{\sigma_1}{\sigma_2}}(\sigma_1-\sigma_2)(\sigma_2+\sigma_3)\Big)\hat{u}_0^{(1)}(-k)\,\mathrm{d}k\\
&+ \sigma_1(\sigma_3-\sigma_2)\int_{\partial D_R^-} \frac{e^{ik(x+x_0+x_0\frac{\sigma_1}{\sigma_2})-\omega_1 t}}{\Delta_1(k)}\hat{u}_0^{(2)}\left(\frac{-k\sigma_1}{\sigma_2}\right)\mathrm{d}k\\
&- 2\sigma_1\sigma_2\int_{\partial D_R^-} \frac{e^{ik(x+x_0+x_0\frac{\sigma_1}{\sigma_3}+2x_0\frac{\sigma_1}{\sigma_2})-\omega_1 t}}{\Delta_1(k)}\hat{u}_0^{(3)}\left(\frac{k\sigma_1}{\sigma_3}\right)\mathrm{d}k,
\end{aligned}
$$

for $-\infty < x < -x_0$ *with* $\Delta_1(k) = \pi\Big((\sigma_1-\sigma_2)(\sigma_2-\sigma_3) + e^{4ix_0k\frac{\sigma_1}{\sigma_2}}(\sigma_1+\sigma_2)(\sigma_2+\sigma_3)\Big)$,

$$
\begin{aligned}
u^{(2)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{u}_0^{(2)}(k) \\
& - \sigma_2(\sigma_2-\sigma_3)\int_{\partial D_R^-} \frac{e^{ik(x+x_0+x_0\frac{\sigma_2}{\sigma_1})-\omega_2 t}}{\Delta_1(k\frac{\sigma_2}{\sigma_1})}\hat{u}_0^{(1)}\left(\frac{-k\sigma_2}{\sigma_1}\right)\mathrm{d}k \\
& + \frac{(\sigma_1-\sigma_2)(\sigma_2-\sigma_3)}{2}\int_{\partial D_R^-} \frac{e^{ikx-\omega_2 t}\hat{u}_0^{(2)}(k)}{\Delta_1(k\frac{\sigma_2}{\sigma_1})}\mathrm{d}k \\
& + \sigma_2(\sigma_2+\sigma_3)\int_{\partial D_R^+} \frac{e^{ik(x+x_0-x_0\frac{\sigma_2}{\sigma_1})-\omega_2 t}}{\Delta_3(k\frac{\sigma_2}{\sigma_3})}\hat{u}_0^{(1)}\left(\frac{k\sigma_2}{\sigma_1}\right)\mathrm{d}k \\
& - \sigma_2(\sigma_1+\sigma_2)\int_{\partial D_R^-} \frac{e^{ik(x+3x_0+x_0\frac{\sigma_2}{\sigma_3})-\omega_2 t}}{\Delta_1(k\frac{\sigma_2}{\sigma_1})}\hat{u}_0^{(3)}\left(\frac{k\sigma_2}{\sigma_3}\right)\mathrm{d}k \\
& - \frac{(\sigma_1+\sigma_2)(\sigma_2-\sigma_3)}{2}\int_{\partial D_R^-} \frac{e^{ik(x+2x_0)-\omega_2 t}\hat{u}_0^{(2)}(-k)}{\Delta_1(k\frac{\sigma_2}{\sigma_1})}\mathrm{d}k \\
& + \frac{(\sigma_2-\sigma_1)(\sigma_2+\sigma_3)}{2}\int_{\partial D_R^+} \frac{e^{ik(x+2x_0)-\omega_2 t}\hat{u}_0^{(2)}(k)}{\Delta_3(k\frac{\sigma_2}{\sigma_3})}\mathrm{d}k \\
& + \frac{(\sigma_2-\sigma_1)(\sigma_2-\sigma_3)}{2}\int_{\partial D_R^+} \frac{e^{ik(x+4x_0)-\omega_2 t}\hat{u}_0^{(2)}(-k)}{\Delta_3(k\frac{\sigma_2}{\sigma_3})}\mathrm{d}k \\
& + \sigma_2(\sigma_2-\sigma_1)\int_{\partial D_R^+} \frac{e^{ik(x+3x_0-x_0\frac{\sigma_2}{\sigma_3})-\omega_2 t}}{\Delta_3(k\frac{\sigma_2}{\sigma_3})}\hat{u}_0^{(3)}\left(\frac{-k\sigma_2}{\sigma_3}\right)\mathrm{d}k,
\end{aligned}
$$

for $-x_0 < x < x_0$ *with* $\Delta_3(k) = \pi\left((\sigma_1+\sigma_2)(\sigma_2+\sigma_3) + e^{4ix_0k\frac{\sigma_3}{\sigma_2}}(\sigma_1-\sigma_2)(\sigma_2-\sigma_3)\right)$. *Lastly, the expression for* $u^{(3)}(x,t)$*, valid for* $x > x_0$*, is identical to that for* $u^{(1)}(x,t)$ *with the replacements* $x_0 \leftrightarrow -x_0$*,* $(1) \leftrightarrow (2)$*, and* $\partial D_R^- \leftrightarrow -\partial D_R^+$*.*

2.4 A finite domain with three layers

In this section we consider the heat conduction problem in three rods of finite length. That is, we seek three functions

$$u^{(1)}(x,t), \quad -x_0 < x < 0, \quad t \geq 0,$$

$$u^{(2)}(x,t), \quad 0 < x < x_2, \quad t \geq 0,$$

$$u^{(3)}(x,t), \quad x_2 < x < x_3, \quad t \geq 0,$$

satisfying the equations

$$u_t^{(1)}(x,t) = \sigma_1^2 u_{xx}^{(1)}(x,t), \qquad -x_0 < x < 0, \; t > 0, \tag{2.39a}$$

$$u_t^{(2)}(x,t) = \sigma_2^2 u_{xx}^{(2)}(x,t), \qquad 0 < x < x_2, \; t > 0, \tag{2.39b}$$

$$u_t^{(3)}(x,t) = \sigma_3^2 u_{xx}^{(3)}(x,t), \qquad x_2 < x < x_3, \; t > 0, \tag{2.39c}$$

the initial conditions

$$u_t^{(1)}(x,0) = u_0^{(1)}(x), \qquad -x_0 < x < 0, \tag{2.40a}$$

$$u_t^{(2)}(x,0) = u_0^{(2)}(x), \qquad 0 < x < x_2, \tag{2.40b}$$

$$u_t^{(3)}(x,0) = u_0^{(3)}(x), \qquad x_2 < x < x_3, \tag{2.40c}$$

the boundary conditions

$$f_1(t) = \beta_1 u^{(1)}(-x_0,t) + \beta_2 u_x^{(1)}(-x_0,t), \qquad t > 0, \tag{2.41a}$$

$$f_3(t) = \beta_3 u^{(3)}(x_3,t) + \beta_4 u_x^{(3)}(x_3,t), \qquad t > 0, \tag{2.41b}$$

$$\tag{2.41c}$$

and the continuity conditions

$$u^{(1)}(0,t) = u^{(2)}(0,t), \qquad t \geq 0, \tag{2.42a}$$

$$u^{(2)}(x_2,t) = u^{(3)}(x_2,t), \qquad t \geq 0, \tag{2.42b}$$

$$\sigma_1^2 u_x^{(1)}(0,t) = \sigma_2^2 u_x^{(2)}(0,t), \qquad t \geq 0, \tag{2.42c}$$

$$\sigma_2^2 u_x^{(2)}(x_2,t) = \sigma_3^2 u_x^{(3)}(x_2,t), \qquad t \geq 0, \tag{2.42d}$$

as in Figure 2.8.

The solution process is as before, following the steps outlined in Section 2.3. For simplicity we assume Neumann boundary data ($\beta_1 = \beta_3 = 0$) and zero boundary conditions ($f_1(t) = f_3(t) = 0$). We define

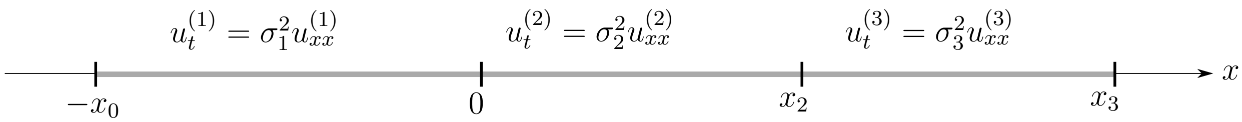

Figure 2.8: The heat equation for three finite domains.

$$
\begin{aligned}
\hat{u}_0^{(1)}(k) &= \int_{-x_0}^{0} e^{-ikx} u_0^{(1)}(x)\,\mathrm{d}x, & \hat{u}^{(1)}(k,t) &= \int_{-x_0}^{0} e^{-ikx} u^{(1)}(x,t)\,\mathrm{d}x,\\
\hat{u}_0^{(2)}(k) &= \int_{0}^{x_2} e^{-ikx} u_0^{(2)}(x)\,\mathrm{d}x, & \hat{u}^{(2)}(k) &= \int_{0}^{x_2} e^{-ikx} u^{(2)}(x,t)\,\mathrm{d}x,\\
\hat{u}_0^{(3)}(k) &= \int_{x_2}^{x_3} e^{-ikx} u_0^{(3)}(x)\,\mathrm{d}x, & \hat{u}^{(3)}(k,t) &= \int_{x_2}^{x_3} e^{-ikx} u^{(3)}(x,t)\,\mathrm{d}x,\\
g_1^{(1)}(\omega,t) &= \int_0^t e^{\omega s} u_x^{(1)}(-x_0,s)\,\mathrm{d}s, & g_0^{(1)}(\omega,t) &= \int_0^t e^{\omega s} u^{(1)}(-x_0,s)\,\mathrm{d}s,\\
h_1^{(1)}(\omega,t) &= \int_0^t e^{\omega s} u_x^{(1)}(0,s)\,\mathrm{d}s = \frac{\sigma_2^2}{\sigma_1^2}\int_0^t e^{\omega s} u_x^{(2)}(0,s)\,\mathrm{d}s, & &\\
h_0^{(1)}(\omega,t) &= \int_0^t e^{\omega s} u^{(1)}(0,s)\,\mathrm{d}s = \int_0^t e^{\omega s} u^{(2)}(0,s)\,\mathrm{d}s, & &\\
g_1^{(3)}(\omega,t) &= \int_0^t e^{\omega s} u_x^{(2)}(x_2,s)\,\mathrm{d}s = \frac{\sigma_3^2}{\sigma_2^2}\int_0^t e^{\omega s} u_x^{(3)}(x_2,s)\,\mathrm{d}s, & &\\
g_0^{(3)}(\omega,t) &= \int_0^t e^{\omega s} u^{(2)}(x_2,s)\,\mathrm{d}s = \int_0^t e^{\omega s} u^{(3)}(x_2,s)\,\mathrm{d}s, & &\\
h_1^{(3)}(\omega,t) &= \int_0^t e^{\omega s} u_x^{(3)}(x_3,s)\,\mathrm{d}s, & h_0^{(3)}(\omega,t) &= \int_0^t e^{\omega s} u^{(3)}(x_3,s)\,\mathrm{d}s.
\end{aligned}
$$

The solution is given by the following proposition.

Proposition 2.4. *The solution to the heat transfer problem* (2.39)-(2.42) *with* $\beta_1 = \beta_3 = 0$ *and* $f_1(t) = f_3(t) = 0$ *is*

$$
\begin{aligned}
u^{(1)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{u}_0^{(1)}(k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{e^{ikx-\omega_1 t}}{2\Delta_1(k)} \Big(e^{2ix_2k\frac{\sigma_1}{\sigma_3}}(\sigma_1-\sigma_2)(\sigma_2-\sigma_3) + e^{2ik\sigma_1(\frac{x_3}{\sigma_3}+\frac{x_2}{\sigma_2})}(\sigma_1+\sigma_2)(\sigma_2-\sigma_3) \\
&\quad + e^{2ix_3k\frac{\sigma_1}{\sigma_3}}(\sigma_1-\sigma_2)(\sigma_2+\sigma_3) + e^{2ix_2k\sigma_1(\frac{1}{\sigma_3}+\frac{1}{\sigma_2})}(\sigma_1+\sigma_2)(\sigma_2+\sigma_3)\Big)\,\hat{u}_0^{(1)}(k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{e^{ik(x+2x_0)-\omega_1 t}}{2\Delta_1(k)} \left(e^{2ix_2k\frac{\sigma_1}{\sigma_3}}(\sigma_1-\sigma_2)(\sigma_2-\sigma_3) + e^{2ik\sigma_1\left(\frac{x_3}{\sigma_3}+\frac{x_2}{\sigma_2}\right)}(\sigma_1+\sigma_2)(\sigma_2-\sigma_3)\right. \\
&\quad \left. + e^{2ix_3k\frac{\sigma_1}{\sigma_3}}(\sigma_1-\sigma_2)(\sigma_2+\sigma_3) + e^{2ix_2k\sigma_1\left(\frac{1}{\sigma_3}+\frac{1}{\sigma_2}\right)}(\sigma_1+\sigma_2)(\sigma_2+\sigma_3)\right)\hat{u}_0^{(1)}(-k)\,\mathrm{d}k \\
&- \int_{\partial D_R^-} \frac{e^{ik\left(x+2x_0+x_2\frac{\sigma_1}{\sigma_2}\right)-\omega_1 t}\sigma_1}{\Delta_1(k)} \left(e^{2ix_2k\frac{\sigma_1}{\sigma_3}}(\sigma_2-\sigma_3) + e^{2ix_3k\frac{\sigma_1}{\sigma_3}}(\sigma_2+\sigma_3)\right)\hat{u}_0^{(2)}\left(k\frac{\sigma_1}{\sigma_2}\right)\,\mathrm{d}k \\
&- \int_{\partial D_R^-} \frac{e^{ik\left(x+2x_0+x_2\frac{\sigma_1}{\sigma_2}\right)-\omega_1 t}\sigma_1}{\Delta_1(k)} \left(e^{2ix_3k\frac{\sigma_1}{\sigma_3}}(\sigma_2-\sigma_3) + e^{2ix_2k\frac{\sigma_1}{\sigma_3}}(\sigma_2+\sigma_3)\right)\hat{u}_0^{(2)}\left(-k\frac{\sigma_1}{\sigma_2}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{2\sigma_1\sigma_2}{\Delta_1(k)} e^{ik\left(x+2x_0+2x_3\frac{\sigma_1}{\sigma_3}+x_2\frac{\sigma_1}{\sigma_3}+x_2\frac{\sigma_1}{\sigma_2}\right)-\omega_1 t}\hat{u}_0^{(3)}\left(k\frac{\sigma_1}{\sigma_2}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{2\sigma_1\sigma_2}{\Delta_1(k)} e^{ik\left(x+2x_0+x_2\frac{\sigma_1}{\sigma_3}+x_2\frac{\sigma_1}{\sigma_2}\right)-\omega_1 t}\hat{u}_0^{(3)}\left(-k\frac{\sigma_1}{\sigma_2}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{e^{ik(x+2x_0)-\omega_1 t}}{2\Delta_1(k)} \left(e^{2ik\sigma_1\left(\frac{x_3}{\sigma_3}+\frac{x_2}{\sigma_2}\right)}(\sigma_1-\sigma_2)(\sigma_2-\sigma_3) + e^{2ix_2k\frac{\sigma_1}{\sigma_3}}(\sigma_1+\sigma_2)(\sigma_2-\sigma_3)\right. \\
&\quad \left. + e^{2ix_2k\sigma_1(\frac{1}{\sigma_3}+\frac{1}{\sigma_2})}(\sigma_1-\sigma_2)(\sigma_2+\sigma_3) + e^{2ix_3k\frac{\sigma_1}{\sigma_3}}(\sigma_1+\sigma_2)(\sigma_2+\sigma_3)\right)\hat{u}_0^{(1)}(k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{e^{ik(x+2x_0)-\omega_1 t}}{2\Delta_1(k)} \left(e^{2ix_2k\frac{\sigma_1}{\sigma_3}}(\sigma_1-\sigma_2)(\sigma_2-\sigma_3) + e^{2ik\sigma_1\left(\frac{x_3}{\sigma_3}+\frac{x_2}{\sigma_2}\right)}(\sigma_1+\sigma_2)(\sigma_2-\sigma_3)\right. \\
&\quad \left. + e^{2ix_3k\frac{\sigma_1}{\sigma_3}}(\sigma_1-\sigma_2)(\sigma_2+\sigma_3) + e^{2ix_2k\sigma_1(\frac{1}{\sigma_3}+\frac{1}{\sigma_2})}(\sigma_1+\sigma_2)(\sigma_2+\sigma_3)\right)\hat{u}_0^{(1)}(-k)\,\mathrm{d}k \\
&- \int_{\partial D_R^+} \frac{e^{ik\left(x+2x_0+x_2\frac{\sigma_1}{\sigma_2}\right)-\omega_1 t}\sigma_1}{\Delta_1(k)} \left(e^{2ix_2k\frac{\sigma_1}{\sigma_3}}(\sigma_2-\sigma_3) + e^{2ix_3k\frac{\sigma_1}{\sigma_3}}(\sigma_2+\sigma_3)\right)\hat{u}_0^{(2)}\left(k\frac{\sigma_1}{\sigma_2}\right)\,\mathrm{d}k \\
&- \int_{\partial D_R^+} \frac{e^{ik\left(x+2x_0+x_2\frac{\sigma_1}{\sigma_2}\right)-\omega_1 t}\sigma_1}{\Delta_1(k)} \left(e^{2ix_3k\frac{\sigma_1}{\sigma_3}}(\sigma_2-\sigma_3) + e^{2ix_2k\frac{\sigma_1}{\sigma_3}}(\sigma_2+\sigma_3)\right)\hat{u}_0^{(2)}\left(-k\frac{\sigma_1}{\sigma_2}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{2\sigma_1\sigma_2}{\Delta_1(k)} e^{ik\left(x+2x_0+2x_3\frac{\sigma_1}{\sigma_3}+x_2\sigma_1\left(\frac{1}{\sigma_3}+\frac{1}{\sigma_2}\right)\right)-\omega_1 t}\hat{u}_0^{(3)}\left(k\frac{\sigma_1}{\sigma_2}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{2\sigma_1\sigma_2}{\Delta_1(k)} e^{ik\left(x+2x_0+x_2\sigma_1\left(\frac{1}{\sigma_3}+\frac{1}{\sigma_2}\right)\right)-\omega_1 t}\hat{u}_0^{(3)}\left(-k\frac{\sigma_1}{\sigma_2}\right)\,\mathrm{d}k,
\end{aligned}
$$

for $-x_0 < x < 0$ *with*

$$\begin{aligned}\Delta_1(k) = \quad & \pi\Bigg(e^{2ix_2k\frac{\sigma_1}{\sigma_3}}(\sigma_1-\sigma_2)(\sigma_2-\sigma_3)+e^{2ik\left(x_3\frac{\sigma_1}{\sigma_3}+x_2\frac{\sigma_1}{\sigma_2}+x_0\right)}(\sigma_2-\sigma_1)(\sigma_2-\sigma_3)\\ & +e^{2ik\sigma_1\left(\frac{x_3}{\sigma_3}+\frac{x_2}{\sigma_2}\right)}(\sigma_1+\sigma_2)(\sigma_2-\sigma_3)+e^{2ik\left(x_2\frac{\sigma_1}{\sigma_3}+x_0\right)}(\sigma_1+\sigma_2)(\sigma_3-\sigma_2)\\ & +e^{2ix_3k\frac{\sigma_1}{\sigma_3}}(\sigma_1-\sigma_2)(\sigma_2+\sigma_3)+e^{2ik\left(x_0+x_2\frac{\sigma_1}{\sigma_3}+x_2\frac{\sigma_1}{\sigma_2}\right)}(\sigma_2-\sigma_1)(\sigma_2+\sigma_3)\\ & +e^{2ix_2k\left(\frac{\sigma_1}{\sigma_3}+\frac{\sigma_1}{\sigma_2}\right)}(\sigma_1+\sigma_2)(\sigma_2+\sigma_3)-e^{2ik\left(x_3\frac{\sigma_1}{\sigma_3}+x_0\right)}(\sigma_1+\sigma_2)(\sigma_2+\sigma_3)\Bigg).\end{aligned}$$

Next,

$$
\begin{aligned}
u^{(2)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{u}_0^{(2)}(k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{-e^{ik(x+x_2)-\omega_2 t}\sigma_2}{\Delta_2(k)} \left(e^{2ix_3k\frac{\sigma_2}{\sigma_3}}(\sigma_2-\sigma_3)+e^{2ix_2k\frac{\sigma_2}{\sigma_3}}(\sigma_2+\sigma_3)\right)\hat{u}_0^{(1)}\left(k\frac{\sigma_2}{\sigma_1}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{-e^{ik\left(x+x_2+2x_0\frac{\sigma_2}{\sigma_1}\right)-\omega_2 t}\sigma_2}{\Delta_2(k)} \left(e^{2ix_3k\frac{\sigma_2}{\sigma_3}}(\sigma_2-\sigma_3)+e^{2ix_2k\frac{\sigma_2}{\sigma_3}}(\sigma_2+\sigma_3)\right)\hat{u}_0^{(1)}\left(-k\frac{\sigma_2}{\sigma_1}\right)\,\mathrm{d}k \\
&- \int_{\partial D_R^-} \frac{e^{ikx-\omega_2 t}}{2\Delta_2(k)}\left(\sigma_2-\sigma_1+e^{2ix_0k\frac{\sigma_2}{\sigma_1}}(\sigma_1+\sigma_2)\right) \\
&\qquad * \left(e^{2ix_2k\frac{\sigma_2}{\sigma_3}}(\sigma_2-\sigma_3)+e^{2ix_3k\frac{\sigma_2}{\sigma_3}}(\sigma_2+\sigma_3)\right)\hat{u}_0^{(2)}(k)\,\mathrm{d}k \\
&- \int_{\partial D_R^-} \frac{e^{ikx-\omega_2 t}}{2\Delta_2(k)}\left(\sigma_2-\sigma_1+e^{2ix_0k\frac{\sigma_2}{\sigma_1}}(\sigma_1+\sigma_2)\right) \\
&\qquad * \left(e^{2ix_3k\frac{\sigma_2}{\sigma_3}}(\sigma_2-\sigma_3)+e^{2ix_2k\frac{\sigma_2}{\sigma_3}}(\sigma_2+\sigma_3)\right)\hat{u}_0^{(2)}(-k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{-e^{ik\left(x+\frac{\sigma_2}{\sigma_3}(x_2+2x_3)\right)-\omega_2 t}\sigma_2}{\Delta_2(k)}\left(\sigma_2-\sigma_1+e^{2iak\frac{\sigma_2}{\sigma_1}}(\sigma_1+\sigma_2)\right)\hat{u}_0^{(3)}(k\frac{\sigma_2}{\sigma_3})\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{-e^{ik\left(x+x_2\frac{\sigma_2}{\sigma_3}\right)-\omega_2 t}\sigma_2}{\Delta_2(k)}\left(\sigma_2-\sigma_1+e^{2iak\frac{\sigma_2}{\sigma_1}}(\sigma_1+\sigma_2)\right)\hat{u}_0^{(3)}\left(-k\frac{\sigma_2}{\sigma_3}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{-e^{ik(x+x_2)-\omega_2 t}\sigma_2}{\Delta_2(k)} \left(e^{2ix_3k\frac{\sigma_2}{\sigma_3}}(\sigma_2-\sigma_3)+e^{2ix_2k\frac{\sigma_2}{\sigma_3}}(\sigma_2+\sigma_3)\right)\hat{u}_0^{(1)}\left(k\frac{\sigma_2}{\sigma_1}\right)\,\mathrm{d}k \\
&- \int_{\partial D_R^+} \frac{e^{ik\left(x+x_2+2x_0\frac{\sigma_2}{\sigma_1}\right)-\omega_2 t}\sigma_2}{\Delta_2(k)} \left(e^{2ix_3k\frac{\sigma_2}{\sigma_3}}(\sigma_2-\sigma_3)+e^{2ix_2k\frac{\sigma_2}{\sigma_3}}(\sigma_2+\sigma_3)\right)\hat{u}_0^{(1)}\left(-k\frac{\sigma_2}{\sigma_1}\right)\,\mathrm{d}k \\
&- \int_{\partial D_R^+} \frac{e^{ik(x+2x_2)-\omega_2 t}}{2\Delta_2(k)}\left(\sigma_2+\sigma_1+e^{2ix_0k\frac{\sigma_2}{\sigma_1}}(\sigma_2-\sigma_1)\right) \\
&\qquad * \left(e^{2ix_3k\frac{\sigma_2}{\sigma_3}}(\sigma_2-\sigma_3)+e^{2ix_2k\frac{\sigma_2}{\sigma_3}}(\sigma_2+\sigma_3)\right)\hat{u}_0^{(2)}(k)\,\mathrm{d}k \\
&- \int_{\partial D_R^+} \frac{e^{ikx-\omega_2 t}}{2\Delta_2(k)}\left(\sigma_2-\sigma_1+e^{2ix_0k\frac{\sigma_2}{\sigma_1}}\,(\sigma_2+\sigma_1)\right) \\
&\qquad * \left(e^{2ix_3k\frac{\sigma_2}{\sigma_3}}(\sigma_2-\sigma_3)+e^{2ix_2k\frac{\sigma_2}{\sigma_3}}(\sigma_2+\sigma_3)\right)\hat{u}_0^{(2)}(-k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{-e^{ik\left(x+x_2\frac{\sigma_2}{\sigma_3}+2x_3\frac{\sigma_2}{\sigma_3}\right)-\omega_2 t}\sigma_2}{\Delta_2(k)}\left(\sigma_2-\sigma_1+e^{2ix_0k\frac{\sigma_2}{\sigma_1}}(\sigma_1+\sigma_2)\right)\hat{u}_0^{(3)}\left(k\frac{\sigma_2}{\sigma_3}\right)\,\mathrm{d}k+ \\
&+ \int_{\partial D_R^+} \frac{-e^{ik(x+x_2\frac{\sigma_2}{\sigma_3})-\omega_2 t}\sigma_2}{\Delta_2(k)}\left(\sigma_2-\sigma_1+e^{2ix_0k\frac{\sigma_2}{\sigma_1}}(\sigma_1+\sigma_2)\right)\hat{u}_0^{(3)}\left(-k\frac{\sigma_2}{\sigma_3}\right)\,\mathrm{d}k,
\end{aligned}
$$

for $0 < x < x_2$, *with*

$$\begin{aligned}
\Delta_2(k) = \quad & \pi \Bigg(e^{ik\left(x_0 \frac{\sigma_2}{\sigma_1} + x_2 + x_3 \frac{\sigma_2}{\sigma_3}\right)} (\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3) + e^{2x_2 ik \frac{\sigma_2}{\sigma_3}} (\sigma_2 - \sigma_1)(\sigma_2 - \sigma_3) \\
& + e^{2ik\sigma_2 \left(\frac{x_2}{\sigma_3} + \frac{x_0}{\sigma_1}\right)} (\sigma_1 + \sigma_2)(\sigma_2 - \sigma_3) + e^{2ik\left(x_3 \frac{\sigma_2}{\sigma_3} + x_2\right)} (\sigma_1 + \sigma_2(\sigma_3 - \sigma_2) \\
& + e^{2ik\left(x_0 \frac{\sigma_2}{\sigma_1} + x_2 \frac{\sigma_2}{\sigma_3} + x_2\right)} (\sigma_1 - \sigma_2)(\sigma_2 + \sigma_3) + e^{2ix_3 k \frac{\sigma_2}{\sigma_3}} (\sigma_2 - \sigma_1)(\sigma_2 + \sigma_3) \\
& - e^{2ix_2 k \left(\frac{\sigma_2}{\sigma_3} + 1\right)} (\sigma_1 + \sigma_2)(\sigma_2 + \sigma_3) + e^{2ik\sigma_2 \left(\frac{x_3}{\sigma_3} + \frac{x_0}{\sigma_1}\right)} (\sigma_1 + \sigma_2)(\sigma_2 + \sigma_3) \Bigg),
\end{aligned}$$

and

$$
\begin{aligned}
u^{(3)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_3 t}\hat{u}_0^{(3)}(k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{-2e^{ik\left(x+x_2+x_2\frac{\sigma_3}{\sigma_2}\right)-\omega_3 t}}{\Delta_3(k)}\sigma_2\sigma_3\hat{u}_0^{(1)}\left(k\frac{\sigma_2}{\sigma_1}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{-2e^{ik\left(x+x_2+x_2\frac{\sigma_3}{\sigma_2}+2x_0\frac{\sigma_3}{\sigma_1}\right)-\omega_3 t}}{\Delta_3(k)}\sigma_2\sigma_3\hat{u}_0^{(1)}\left(-k\frac{\sigma_2}{\sigma_1}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{e^{ik\left(x+x_2+x_2\frac{\sigma_3}{\sigma_2}\right)-\omega_3 t}}{\Delta_3(k)}\left(\sigma_1+\sigma_2+e^{2ix_0k\frac{\sigma_3}{\sigma_1}}(\sigma_2-\sigma_1)\right)\hat{u}_0^{(2)}\left(k\frac{\sigma_3}{\sigma_2}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{e^{ik(x+x_2)-\omega_3 t}}{\Delta_3(k)}\left(\sigma_2-\sigma_1+e^{2ix_0k\frac{\sigma_3}{\sigma_1}}(\sigma_2+\sigma_1)\right)\hat{u}_0^{(2)}\left(-k\frac{\sigma_3}{\sigma_2}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{e^{ik(x+2x_2)-\omega_3 t}}{2\Delta_3(k)}\left((\sigma_3-\sigma_2)\left(\sigma_1-\sigma_2-e^{2ix_0k\frac{\sigma_3}{\sigma_1}}(\sigma_1+\sigma_2)\right)\right. \\
&\qquad \left. -e^{2ix_2k\frac{\sigma_3}{\sigma_2}}(\sigma_2+\sigma_3)\left(\sigma_1+\sigma_2+e^{2ix_0k\frac{\sigma_3}{\sigma_1}}(\sigma_2-\sigma_1)\right)\right)\hat{u}_0^{(3)}(k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{e^{ikx-\omega_3 t}}{2\Delta_3(k)}\left((\sigma_3+\sigma_2)\left(\sigma_1-\sigma_2-e^{2ix_0k\frac{\sigma_3}{\sigma_1}}(\sigma_1+\sigma_2)\right)\right. \\
&\qquad \left. -e^{2ix_2k\frac{\sigma_3}{\sigma_2}}(\sigma_3-\sigma_2)\left(\sigma_1+\sigma_2+e^{2iak\frac{\sigma_3}{\sigma_1}}(\sigma_2-\sigma_1)\right)\right)\hat{u}_0^{(3)}(-k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{2e^{ik(x+x_2+x_2\frac{\sigma_3}{\sigma_2})-\omega_3 t}}{\Delta_3(k)}\sigma_2\sigma_3\hat{u}_0^{(1)}\left(k\frac{\sigma_2}{\sigma_1}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{2e^{ik(x+x_2+x_2\frac{\sigma_3}{\sigma_2}+2x_0\frac{\sigma_3}{\sigma_1})-\omega_3 t}}{\Delta_3(k)}\sigma_2\sigma_3\hat{u}_0^{(1)}\left(-k\frac{\sigma_2}{\sigma_1}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{-e^{ik\left(x+x_2+2x_2\frac{\sigma_3}{\sigma_2}\right)-\omega_3 t}\sigma_3}{\Delta_3(k)}\left(\sigma_1+\sigma_2+e^{2ix_0k\frac{\sigma_3}{\sigma_1}}(\sigma_2-\sigma_1)\right)\hat{u}_0^{(2)}\left(k\frac{\sigma_3}{\sigma_2}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{e^{ik(x+x_2)-\omega_3 t}\sigma_3}{\Delta_3(k)}\left(\sigma_2-\sigma_1+e^{2ix_0k\frac{\sigma_3}{\sigma_1}}(\sigma_2+\sigma_1)\right)\hat{u}_0^{(2)}\left(-k\frac{\sigma_3}{\sigma_2}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{e^{ik(x+2x_3)-\omega_3 t}}{2\Delta_3(k)}\left(e^{2ix_2k\frac{\sigma_3}{\sigma_2}}(\sigma_3-\sigma_2)\left(\sigma_1+\sigma_2+e^{2ix_0k\frac{\sigma_3}{\sigma_1}}\right)\right. \\
&\qquad \left. +(\sigma_2+\sigma_3)\left(\sigma_2-\sigma_1+e^{2ix_0k\frac{\sigma_3}{\sigma_1}}(\sigma_1+\sigma_2)\right)\right)\hat{u}_0^{(3)}(k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{e^{ikx-\omega_3 t}}{2\Delta_3(k)}\left(e^{2ix_2k\frac{\sigma_3}{\sigma_2}}(\sigma_3-\sigma_2)\left(\sigma_1+\sigma_2+e^{2ix_0k\frac{\sigma_3}{\sigma_1}}\right)\right. \\
&\qquad \left. (\sigma_2+\sigma_3)\left(\sigma_2-\sigma_1+e^{2ix_0k\frac{\sigma_3}{\sigma_1}}(\sigma_1+\sigma_2)\right)\right)\hat{u}_0^{(3)}(-k)\,\mathrm{d}k,
\end{aligned}
$$

for $x_2 < x < x_3$ with

$$\begin{aligned}\Delta_3(k) =\pi \Bigg(& e^{2ix_2k}(\sigma_1-\sigma_2)(\sigma_2-\sigma_3) + e^{2ik\left(x_0\frac{\sigma_3}{\sigma_1}+x_2\frac{\sigma_3}{\sigma_2}+x_3\right)}(\sigma_2-\sigma_1)(\sigma_2-\sigma_3)\\ & + e^{2ik\left(x_2\frac{\sigma_3}{\sigma_2}+x_3\right)}(\sigma_1+\sigma_2)(\sigma_2-\sigma_3) + e^{2ik\left(x_0\frac{\sigma_3}{\sigma_1}+x_2\right)}(\sigma_1+\sigma_2)(\sigma_3-\sigma_2)\\ & + e^{2ix_3k}(\sigma_1-\sigma_2)(\sigma_2+\sigma_3) + e^{2ik\left(\frac{x_0}{\sigma_1}+x_2+x_2\frac{\sigma_3}{\sigma_2}\right)}(\sigma_2-\sigma_1)(\sigma_2+\sigma_3)\\ & + e^{2ix_2k\left(1+\frac{\sigma_3}{\sigma_2}\right)}(\sigma_1+\sigma_2)(\sigma_2+\sigma_3) - e^{2ik\left(x_0\frac{\sigma_3}{\sigma_1}+x_3\right)}(\sigma_1+\sigma_2)(\sigma_2+\sigma_3)\Bigg).\end{aligned}$$

2.5 Periodic boundary conditions

We consider the problem of heat conduction in a ring consisting of two different materials as in Figure 2.9. We seek two functions:

$$u^{(1)}(x,t),\quad x_0<x<x_1,\quad t\geq 0,$$

$$u^{(2)}(x,t),\quad x_1<x<x_2,\quad t\geq 0,$$

satisfying the equations, initial, boundary, interface continuity conditions:

$$\begin{aligned}
&u^{(1)}_t = \sigma_1^2 u^{(1)}_{xx}, && u^{(1)}(x,0)=u^{(1)}_0(x), && x_0 <x<x_1, && t>0, && (2.43a)\\
&u^{(2)}_t = \sigma_2^2 u^{(2)}_{xx}, && u^{(2)}(x,0)=u^{(2)}_0(x), && x_1 <x<x_2, && t>0, && (2.43b)\\
&u^{(1)}(x_0,t)=u^{(2)}(x_2,t), && u^{(1)}(x_1,t)=u^{(2)}(x_1,t), && && t>0, && (2.43c)\\
&\sigma_1^2u^{(1)}_x(x_0,t)=\sigma_2^2u^{(2)}_x(x_2,t), && \sigma_1^2u^{(1)}_x(x_1,t)=\sigma_2^2u^{(2)}_x(x_1,t), && && t>0. && (2.43d)
\end{aligned}$$

As in previous sections we have the local relations

$$\begin{aligned}
&(e^{-ikx+\omega_1t}u^{(1)})_t=(\sigma_1^2e^{-ikx+\omega_1t}(u^{(1)}_x+iku^{(1)}))_x, && x_0<x<x_1,\\
&(e^{-ikx+\omega_2t}u^{(2)})_t=(\sigma_2^2e^{-ikx+\omega_2t}(u^{(2)}_x+iku^{(2)}))_x, && x_1<x<x_2,
\end{aligned}$$

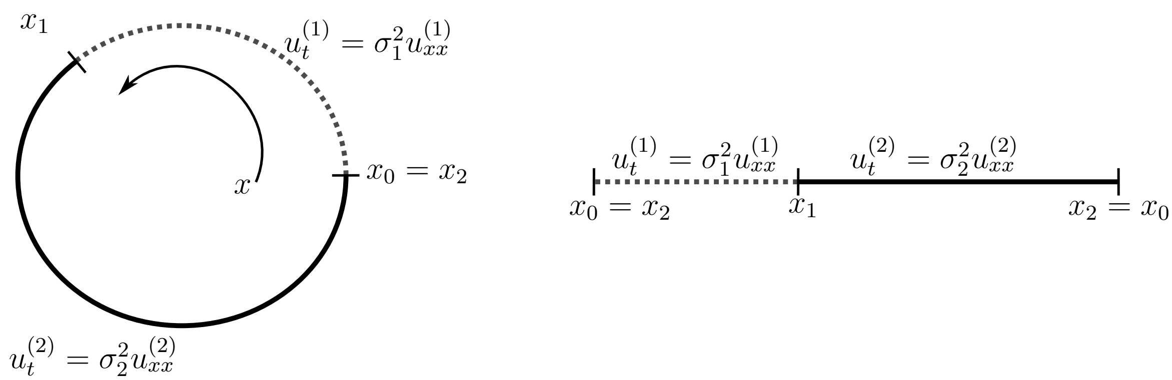

Figure 2.9: The heat equation with an interface posed on a ring.

where $\omega_j = (\sigma_j k)^2$. We define the time transforms of the initial and boundary data and the spatial transforms of u for $k \in \mathbb{C}$ as follows:

$$\begin{aligned}
\hat{u}_0^{(1)}(k) &= \int_{x_0}^{x_1} e^{-ikx} u_0^{(1)}(x)\,\mathrm{d}x, \\
\hat{u}^{(1)}(k,t) &= \int_{x_0}^{x_1} e^{-ikx} u^{(1)}(x,t)\,\mathrm{d}x, \\
\hat{u}_0^{(2)}(k) &= \int_{x_1}^{x_2} e^{-ikx} u_0^{(2)}(x)\,\mathrm{d}x, \\
\hat{u}^{(2)}(k,t) &= \int_{x_1}^{x_2} e^{-ikx} u^{(2)}(x,t)\,\mathrm{d}x, \\
g_0(\omega,t) &= \int_0^t e^{\omega s} u^{(1)}(x_1,s)\,\mathrm{d}s = \int_0^t e^{\omega s} u^{(2)}(x_1,s)\,\mathrm{d}s, \\
g_1(\omega,t) &= \int_0^t e^{\omega s} u_x^{(1)}(x_1,s)\,\mathrm{d}s = \frac{\sigma_2^2}{\sigma_1^2}\int_0^t e^{\omega s} u_x^{(2)}(x_1,s)\,\mathrm{d}s, \\
h_0(\omega,t) &= \int_0^t e^{\omega s} u^{(1)}(x_0,s)\,\mathrm{d}s = \int_0^t e^{\omega s} u^{(2)}(x_2,s)\,\mathrm{d}s, \\
h_1(\omega,t) &= \int_0^t e^{\omega s} u_x^{(1)}(x_0,s)\,\mathrm{d}s = \frac{\sigma_2^2}{\sigma_1^2}\int_0^t e^{\omega s} u_x^{(2)}(x_2,s)\,\mathrm{d}s.
\end{aligned}$$

Using Green's Theorem on the domains $[x_0, x_1] \times [0,t]$, and $[x_1, x_2] \times [0,t]$ respectively, we have the global relations

$$\begin{aligned} e^{\omega_1 t}\hat{u}^{(1)}(k,t) =&\sigma_1^2 e^{-ikx_1}\left(g_1(\omega_1,t)+ikg_0(\omega_1,t)\right) \\ &-\sigma_1^2 e^{-ikx_0}\left(h_1(\omega_1,t)+ikh_0(\omega_1,t)\right)+\hat{u}_0^{(1)}(k), \end{aligned} \tag{2.45a}$$

$$\begin{aligned} e^{\omega_2 t}\hat{u}^{(2)}(k,t) =&e^{-ikx_2}\left(\sigma_1^2 h_1(\omega_2,t)+ik\sigma_2^2 h_0(\omega_2,t)\right) \\ &-e^{-ikx_1}\left(\sigma_1^2 g_1(\omega_2,t)+ik\sigma_2^2 g_0(\omega_2,t)\right)+\hat{u}_0^{(2)}(k). \end{aligned} \tag{2.45b}$$

Both equations are valid for $k \in \mathbb{C}$ as is to be expected for the Fokas Method in bounded domains. Using the invariance of $\omega_1(k) = (\sigma_1 k)^2$ and $\omega_2(k) = (\sigma_2 k)^2$ under $k \to -k$ as well as the transformation $k \to \frac{\sigma_1}{\sigma_2}k$ and $k \to \frac{\sigma_2}{\sigma_1}k$ which transform $\omega_1(k) \leftrightarrow \omega_2(k)$ we obtain

$$\begin{aligned} e^{\omega_1 t}\hat{u}^{(1)}(-k,t) =&\sigma_1^2 e^{ikx_1}\left(g_1(\omega_1,t)-ikg_0(\omega_1,t)\right) \\ &-\sigma_1^2 e^{ikx_0}\left(h_1(\omega_1,t)-ikh_0(\omega_1,t)\right)+\hat{u}_0^{(1)}(-k), \end{aligned} \tag{2.46a}$$

$$\begin{aligned} e^{\omega_2 t}\hat{u}^{(2)}(k,t) =&e^{ikx_2}\left(\sigma_1^2 h_1(\omega_2,t)-ik\sigma_2^2 h_0(\omega_2,t)\right) \\ &-e^{ikx_1}\left(\sigma_1^2 g_1(\omega_2,t)-ik\sigma_2^2 g_0(\omega_2,t)\right)+\hat{u}_0^{(2)}(-k), \end{aligned} \tag{2.46b}$$

$$\begin{aligned} e^{\omega_2 t}\hat{u}^{(1)}\left(\frac{\sigma_2}{\sigma_1}k,t\right) =&e^{-ikx_1\frac{\sigma_2}{\sigma_1}}\left(\sigma_1^2 g_1(\omega_2,t)+ik\sigma_1\sigma_2 g_0(\omega_2,t)\right) \\ &-e^{-ikx_0\frac{\sigma_2}{\sigma_1}}\left(\sigma_1^2 h_1(\omega_2,t)+ik\sigma_1\sigma_2 h_0(\omega_2,t)\right)+\hat{u}_0^{(1)}\left(k\frac{\sigma_2}{\sigma_1}\right), \end{aligned} \tag{2.46c}$$

$$\begin{aligned} e^{\omega_1 t}\hat{u}^{(2)}\left(\frac{\sigma_1}{\sigma_2}k,t\right) =&e^{-ikx_2\frac{\sigma_1}{\sigma_2}}\left(\sigma_1^2 h_1(\omega_1,t)+ik\sigma_1\sigma_2 h_0(\omega_1,t)\right) \\ &-e^{-ikx_1\frac{\sigma_1}{\sigma_2}}\left(\sigma_1^2 g_1(\omega_1,t)+ik\sigma_1\sigma_2 g_0(\omega_1,t)\right)+\hat{u}_0^{(2)}\left(k\frac{\sigma_1}{\sigma_2}\right). \end{aligned} \tag{2.46d}$$

Inverting the Fourier transforms in (2.45a)

$$\begin{aligned} u^{(1)}(x,t) =&\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{u}_0^{(1)}(k)\,\mathrm{d}k \\ &+\frac{\sigma_1^2}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x_1)-\omega_1 t}(g_1(\omega_1,t)+ikg_0(\omega_1,t))\,\mathrm{d}k \\ &-\frac{\sigma_1^2}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x_0)-\omega_1 t}(h_1(\omega_1,t)+ikh_0(\omega_1,t))\,\mathrm{d}k. \end{aligned}$$

The integrand of the second integral is entire and decays as $k \to \infty$ for $k \in \mathbb{C}^- \setminus D^-$. The third integral has an integrand that is entire and decays as $k \to \infty$ for $k \in \mathbb{C}^+ \setminus D^+$. It is convenient to deform both contours away from $k = 0$ to avoid singularities in the integrands below as shown in Figure 2.6. Thus

$$\begin{aligned} u^{(1)}(x,t) = & \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-\omega_1 t} \hat{u}_0^{(1)}(k)\,\mathrm{d}k \\ & - \frac{\sigma_1^2}{2\pi} \int_{\partial D_R^-} e^{ik(x-x_1)-\omega_1 t}(g_1(\omega_1,t) + ikg_0(\omega_1,t))\,\mathrm{d}k \\ & - \frac{\sigma_1^2}{2\pi} \int_{\partial D_R^+} e^{ik(x-x_0)-\omega_1 t}(h_1(\omega_1,t) + ikh_0(\omega_1,t))\,\mathrm{d}k. \end{aligned} \tag{2.47}$$

To obtain the solution $u_2(x,t)$ for $x_1 < x < x_2$ we apply the inverse Fourier transform to (2.45b) and again deform where appropriate to find

$$\begin{aligned} u^{(2)}(x,t) = & \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-\omega_2 t} \hat{u}_0^{(2)}(k)\,\mathrm{d}k \\ & - \frac{1}{2\pi} \int_{\partial D_R^+} e^{ik(x-x_1)-\omega_2 t}(\sigma_1^2 g_1(\omega_2,t) + ik\sigma_2^2 g_0(\omega_2,t))\,\mathrm{d}k \\ & - \frac{1}{2\pi} \int_{\partial D_R^-} e^{ik(x-x_2)-\omega_2 t}(\sigma_1^2 h_1(\omega_1,t) + ik\sigma_2^2 h_0(\omega_2,t))\,\mathrm{d}k. \end{aligned}$$

All the global relations, using the symmetries of the set of dispersion relations, that is Equations (2.45) and (2.46), are used to solve for the unknown functions at the interface. Substituting these expressions into (2.47) we have equations for $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ which involve $\hat{u}^{(1)}(k,t)$ and $\hat{u}^{(2)}(k,t)$ evaluated at a variety of arguments but without the factor $e^{\omega_j t}$. Such integrands decay in the regions around whose boundaries they are integrated. Making extensive use of Jordan's Lemma and Cauchy's Theorem, these integrals are shown to vanish. Thus the final solution is given by

$$
\begin{aligned}
u^{(1)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{u}_0^{(1)}(k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{e^{ikx-\omega_1 t}}{2\Delta_1(k)}\left((\sigma_1+\sigma_2)^2 e^{ik(x_0-x_1)+\frac{ik\sigma_1}{\sigma_2}(x_1-x_2)} - 4\sigma_1\sigma_2\right. \\
&\qquad \left. -(\sigma_1-\sigma_2)^2 e^{ik(x_0-x_1)+\frac{ik\sigma_1}{\sigma_2}(x_2-x_1)}\right)\hat{u}_0^{(1)}(k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{(\sigma_1^2-\sigma_2^2)e^{ik(x-x_0-x_1)-\omega_1 t}}{2\Delta_1(k)}\left(e^{\frac{ik\sigma_1}{\sigma_2}(x_1-x_2)} - e^{\frac{ik\sigma_1}{\sigma_2}(x_2-x_1)}\right)\hat{u}_0^{(1)}(-k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{\sigma_1(\sigma_1+\sigma_2)e^{ikx-\omega_1 t}}{\Delta_1(k)}\left(e^{-ikx_1+\frac{ik\sigma_1}{\sigma_2}x_1} - e^{-ikx_0+\frac{ik\sigma_1}{\sigma_2}x_2}\right)\hat{u}_0^{(2)}\left(\frac{k\sigma_1}{\sigma_2}\right)\mathrm{d}k \\
&+ \int_{\partial D_R^-} \frac{\sigma_1(\sigma_1-\sigma_2)e^{ikx-\omega_1 t}}{\Delta_1(k)}\left(e^{-ikx_0-\frac{ik\sigma_1}{\sigma_2}x_2} - e^{-ikx_1-\frac{ik\sigma_1}{\sigma_2}x_1}\right)\hat{u}_0^{(2)}\left(-\frac{k\sigma_1}{\sigma_2}\right)\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{e^{ikx-\omega_1 t}}{2\Delta_1(k)}\left((\sigma_1-\sigma_2)^2 e^{ik(x_1-x_0)+\frac{ik\sigma_1}{\sigma_2}(x_2-x_1)} + 4\sigma_1\sigma_2\right. \\
&\qquad \left. -(\sigma_1+\sigma_2)^2 e^{ik(x_0-x_1)+\frac{ik\sigma_1}{\sigma_2}(x_2-x_1)}\right)\hat{u}_0^{(1)}(k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{(\sigma_1^2-\sigma_2^2)e^{ik(x-x_0-x_1)-\omega_1 t}}{2\Delta_1(k)}\left(e^{\frac{ik\sigma_1}{\sigma_2}(x_1-x_2)} - e^{\frac{ik\sigma_1}{\sigma_2}(x_2-x_1)}\right)\hat{u}_0^{(1)}(-k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{\sigma_1(\sigma_1+\sigma_2)e^{ikx-\omega_1 t}}{\Delta_1(k)}\left(e^{-ikx_1+\frac{ik\sigma_1}{\sigma_2}x_1} - e^{-ikx_0+\frac{ik\sigma_1}{\sigma_2}x_2}\right)\hat{u}_0^{(2)}\left(\frac{k\sigma_1}{\sigma_2}\right)\mathrm{d}k \\
&+ \int_{\partial D_R^+} \frac{\sigma_1(\sigma_2-\sigma_1)e^{ikx-\omega_1 t}}{\Delta_1(k)}\left(e^{-ikx_0-\frac{ik\sigma_1}{\sigma_2}x_2} - e^{-ikx_1-\frac{ik\sigma_1}{\sigma_2}x_1}\right)\hat{u}_0^{(2)}\left(-\frac{k\sigma_1}{\sigma_2}\right)\mathrm{d}k,
\end{aligned}
$$

for $x_0 < x < x_1$ where

$$
\begin{aligned}
\Delta_1(k) =& \pi\left(\sigma_1\left(e^{-ik\frac{\sigma_1}{\sigma_2}x_1} - e^{-ik\frac{\sigma_1}{\sigma_2}x_2}\right)\left(e^{-ikx_0}+e^{-ikx_1}\right) + \sigma_2\left(e^{-ik\frac{\sigma_1}{\sigma_2}x_1} + e^{-ik\frac{\sigma_1}{\sigma_2}x_2}\right)\left(e^{-ikx_0}-e^{-ikx_1}\right)\right) * \\
&\left(\sigma_1\left(e^{-ik\frac{\sigma_1}{\sigma_2}x_1} + e^{-ik\frac{\sigma_1}{\sigma_2}x_2}\right)\left(e^{-ikx_0}-e^{-ikx_1}\right) + \sigma_2\left(e^{-ik\frac{\sigma_1}{\sigma_2}x_1} - e^{-ik\frac{\sigma_1}{\sigma_2}x_2}\right)\left(e^{-ikx_0}+e^{-ikx_1}\right)\right).
\end{aligned}
$$

For $x_1 < x < x_2$, the solution $u^{(2)}(x,t)$ is found by switching the indices (1) and (2) on σ_1, σ_2, $u^{(1)}(\cdot)$ and $u^{(2)}(\cdot)$, replacing $\Delta_1(k)$ with $\Delta_2(k) = -\Delta_1(k\sigma_2/\sigma_1)$, and interchanging the integration paths $\int_{D_R^+}$ and $-\int_{D_R^-}$.

Note that $\Delta_1(k) = 0$ whenever $k = 0$, $\cot\left(\frac{k(x_0-x_1)}{2}\right)\tan\left(\frac{k\sigma_1(x_1-x_2)}{2\sigma_2}\right) = -\frac{\sigma_2}{\sigma_1}$, or $\cot\left(\frac{k\sigma_1(x_1-x_2)}{2\sigma_2}\right)\tan\left(\frac{k(x_0-x_1)}{2}\right) = -\frac{\sigma_2}{\sigma_1}$ are satisfied. Observe that $\Delta_j(k) = 0$ only for real values of k for $j = 1, 2$. Thus, through the deformation to D_R^+ and D_R^- we have avoided any singularities and, on the contours, the quantities needed are evaluated without problem.

Remarks:

- To solve (2.43) using separation of variables results in a solution defined implicitly in terms of the eigenvalues which solve $\Delta_1(\sigma_1 k) = 0$. Thus, the Fokas Method provides an alternative to this important classical problem. It is interesting to note that although the solution is periodic, it is not useful to assume a solution of the form $\sum_{j=-\infty}^{\infty} a_j(t)e^{ijx}$ because the equation defined on the whole domain $u_t = \sigma^2(x)u_{xx}$ with $\sigma(x)$ piecewise constant is no longer diagonal in Fourier space since $\sigma(x)$ has an infinite-term Fourier series.

- It is straightforward to generalize this work to the problem of n domains with periodic boundary conditions by combining what was done in [14] and [4, 47] for the heat equation with multiple domains with what we present here for periodic boundary conditions.

2.6 Burgers' Equation

Burgers' equation is a nonlinear PDE which models gas dynamics and traffic flow [38]. For a velocity $q(x,t)$ and viscosity coefficient σ^2 a general form of the viscous Burgers' equation in one spatial dimension is [5, 8, 9]

$$q_t = \sigma^2 q_{xx} + q q_x. \tag{2.48}$$

This equation can be linearized via the Cole-Hopf transformation

$$q(x,t) = -2\sigma^2 \frac{u(x,t)}{u_x(x,t)}.$$

The Burgers' equation is then reduced to the linear heat equation $u_t = \sigma^2 u_{xx}$ [38, Section 1.7] which can be easily solved.

If σ is piecewise constant then we can pose (2.48) as an interface problem as in Section 2.1. Using the Cole-Hopf transformation

$$q^{(j)}(x,t) = -2\sigma_j^2 \frac{u_x^{(j)}(x,t)}{u^{(j)}(x,t)}$$

for $j = 1, 2$ we find that $q(x,t)$ satisfies

$$q(x,t) = \begin{cases} q^{(1)}(x,t), & x < 0, t \geq 0, \\ q^{(2)}(x,t), & x > 0, t \geq 0, \end{cases}$$

which solve the Burgers' equations,

$$q_t^{(1)} = \sigma_1^2 q_{xx}^{(1)} - q^{(1)} q_x^{(1)}, \qquad x < 0,\ t \geq 0, \tag{2.49a}$$

$$q_t^{(2)} = \sigma_2^2 q_{xx}^{(2)} - q^{(2)} q_x^{(2)}, \qquad x > 0,\ t \geq 0, \tag{2.49b}$$

the initial conditions

$$q^{(1)}(x,0) = q_0^{(1)}(x) = -2\sigma_1^2 \frac{u_x^{(1)}(x,0)}{u_0^{(1)}(x)}, \qquad x < 0, \tag{2.50a}$$

$$q^{(2)}(x,0) = q_0^{(2)}(x) = -2\sigma_2^2 \frac{u_x^{(2)}(x,0)}{u_0^{(2)}(x)}, \qquad x > 0, \tag{2.50b}$$

the asymptotic conditions

$$\lim_{x \to -\infty} q^{(1)}(x,t) = 0, \qquad t \geq 0, \tag{2.51a}$$

$$\lim_{x \to \infty} q^{(2)}(x,t) = 0, \qquad t \geq 0, \tag{2.51b}$$

and the continuity interface conditions

$$q^{(1)}(0,t) = q^{(2)}(0,t), \qquad t > 0, \tag{2.52a}$$

$$q_x^{(1)}(0,t) + 2\sigma_1^2 u_{xx}^{(1)}(0,t) = q_x^{(2)}(0,t) + 2\sigma_2^2 u_{xx}^{(2)}(0,t), \qquad t > 0. \tag{2.52b}$$

We are not aware of any physically relevant problems which can be modeled by the Burgers' equation as presented in (2.49)-(2.52). However, the Cole-Hopf transformation and the methods presented in Section 2.1 give an analytical, closed-form solution for this nonlinear PDE with an interface.

Chapter 3

The heat equation on a graph

In Chapter 2 we applied the Fokas Method to a subclass of the problems described in this chapter in which each vertex has at most two edges: a configuration of metal rods with different diffusivities joined end-to-end along a line. The principal contribution of the work in this chapter is to show that the Fokas Method may also be applied to the more general configurations of connected rods described below. Examples of such configurations are shown in Figure 3.1. This work was done in collaboration with D.A. Smith [58].

The examples presented in this chapter and the paper [58] were selected because they can be formulated as linear systems which are relatively easy to solve by hand. It should be noted that this is not always the case. In general, for a graph with m_f finite edges and m_i infinite edges, it is always possible to express the generalized spectral map as a linear system of dimension $4m_f + 2m_i$ but it may not be practical to solve this system by hand.

Suppose V is a set of vertices and E is a set of directed edges between those vertices so that (V, E) is a finite connected directed graph. Associated with each edge $r \in E$ is the length $L_r \in (0, \infty]$. Suppose additionally if $L_r = \infty$, then the vertex at which r terminates has no other edges. For each edge $r \in E$, we define the open interval $\Omega_r = (0, L_r)$. We consider the graph to represent a configuration of narrow rods joined at the vertices in perfect thermal contact.

(a) *Eagle Catching Fish*, sculpted by Amy Goodman

(b) *virtual woman*, sculpted by Toby Short

Figure 3.1: Examples of connected rods in art.

3.1 Interface conditions for graphs

In order to derive the proper interface conditions we generalize what is outlined in [38] and repeated in Section 1.1 for problems with multiple rods. Assume each edge $r \in E$ is a rod made of some heat-conducting material with density $\rho_r(x)$ and unit cross-sectional area. We assume the surface of the rod is perfectly insulated so no heat is lost or gained through this surface. If we consider an infinitesimal section of length $\mathrm{d}x$ for $x \in \Omega_r$, then $\mathrm{d}C^{(r)}$, the heat content in the section, is proportional to the mass and the temperature, $q^{(r)}(x,t)$. That is

$$\mathrm{d}C^{(r)}(t) = c_r(x)\rho_r(x)q^{(r)}(x,t)\,\mathrm{d}x,$$

where $c_r(x)$ is the specific heat on r and $x \in \Omega_r$. Thus, the total heat content in the interval $x_1 \leq x \leq x_2$ where $x_1, x_2 \in \Omega_r$ is

$$C^{(r)}(t) = \int_{x_1}^{x_2} c_r(x)\rho_r(x)q^{(r)}(x,t)\,\mathrm{d}x.$$

Fourier's Law for heat conduction [30] states that the rate of heat flowing into a body is proportional to the area of that element and to the outward normal derivative of the temperature at that location. The constant of proportionality is the thermal conductivity, $k_r(x)$. In our example, the net inflow of heat through the boundaries x_1 and x_2 is

$$R(t) = k_r(x_2)q_x^{(r)}(x_2,t) - k_r(x_1)q_x^{(r)}(x_1,t), \tag{3.1}$$

where $x_1, x_2 \in \Omega_r$. Conservation of heat implies $\frac{\mathrm{d}}{\mathrm{d}t}C^{(r)}(t) = R(t)$ along each rod. That is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} c_r(x)\rho_r(x)q^{(r)}(x,t)\,\mathrm{d}x = k_r(x_2)q_x^{(r)}(x_2,t) - k_r(x_1)q_x^{(r)}(x_1,t), \tag{3.2}$$

for every r. This is a typical conservation law. On each rod we assume constant material properties. That is $c_r(x) = c_r$, $\rho_r(x) = \rho_r$, and $k_r(x) = k_r$ for $x \in \Omega_r$ and $t > 0$. Expressing $R(t)$ as the integral of a derivative and rewriting (3.2), we have

$$\int_{x_1}^{x_2} \left(c_r\rho_r q_t^{(r)}(x,t)\,\mathrm{d}x - \frac{\partial}{\partial x}\left(k_r q_x^{(r)}(x,t)\right)\right)\mathrm{d}x = 0,$$

for $x_1, x_2 \in \Omega_r$. Since this is true for any x_1 and x_2 in this range, it follows that the integrand must vanish. That is,

$$q_t^{(r)}(x,t) - \frac{k_r}{c_r\rho_r}q_{xx}^{(r)}(x,t) = 0.$$

Next, we scale x such that $\hat{x} = \frac{x}{c_r\rho_r}$ for $x \in \Omega_r$. Note that such a scaling also affects L_r and Ω_r and in what follows we assume all quantities are properly scaled. Dropping the $\hat{}$ and defining $\sigma_r^2 = k_r c_r \rho_r$ as the (scaled) thermal diffusivity in rod r we have

$$q_t^{(r)} = \sigma_r^2 q_{xx}^{(r)}, \qquad x \in \Omega_r,\ t > 0, \qquad r \in E, \tag{3.3a}$$

$$q^{(r)}(x,0) = q_0^{(r)}(x), \qquad x \in \overline{\Omega_r}, \qquad r \in E, \tag{3.3b}$$

where $q_0^{(r)} \in \mathcal{S}\left(\overline{\Omega_r}\right)$ is given initial data for each $r \in E$ with $\mathcal{S}\left(\overline{\Omega}\right)$ a Schwartz space of smooth, rapidly decaying functions restricted to the closure of the interval Ω.

We also assume that the rods at each vertex v are in perfect thermal contact [10]. The temperature at $x = 0$ of the rods emanating from v, $q^{(s)}(0,t)$, is the same as the temperature at $x = L_r$ of the rods terminating at v, $q^{(r)}(L_r,t)$. That is,

$$q^{(s)}(0,t) = q^{(r)}(L_r,t), \qquad t \geq 0, \tag{3.4}$$

where $q^{(s)}(x,t)$ emanates from the same vertex where $q^{(r)}(x,t)$ terminates. The relation (3.4) is true for all $r, s \in E$ that meet at a given vertex. If vertex v has p edges then (3.4) gives $p-1$ conditions.

In order to find a condition on the set of spatial derivatives of $q^{(r)}(x,t)$ we require an appropriately scaled Condition (3.2) valid for on a region centered at the vertex v. Consider a ball centered at a vertex v, with radius $\epsilon > 0$ sufficiently small so that all rods are straight lines within the ball and no other rods intersect the ball. Then, similar to Equation (3.1) we have

$$R(t) = \sum_{\substack{s \in E: \\ s \text{ emanates} \\ \text{from } v}} \sigma_s^2(\epsilon) q_x^{(s)}(\epsilon,t) - \sum_{\substack{r \in E: \\ r \text{ terminates} \\ \text{at } v}} \sigma_r^2(L_r - \epsilon) q_x^{(r)}(L_r - \epsilon, t).$$

By conservation of energy,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{\substack{s \in E: \\ s \text{ emanates} \\ \text{from } v}} c_s^2 \rho_s^2 \lim_{\epsilon \to 0} \int_0^\epsilon q^{(s)}(x,t)\,\mathrm{d}x + \sum_{\substack{r \in E: \\ r \text{ terminates} \\ \text{at } v}} c_r^2 \rho_r^2 \lim_{\epsilon \to 0} \int_{L_r - \epsilon}^{L_r} q^{(r)}(x,t)\,\mathrm{d}x \right) \\ = \lim_{\epsilon \to 0} \sum_{\substack{s \in E: \\ s \text{ emanates} \\ \text{from } v}} \sigma_s^2 q_x^{(s)}(\epsilon,t) - \lim_{\epsilon \to 0} \sum_{\substack{r \in E: \\ r \text{ terminates} \\ \text{at } v}} \sigma_r^2 q_x^{(r)}(L_r - \epsilon, t). \end{aligned} \tag{3.5}$$

The left-hand-side of (3.5) is zero by (3.4). This implies

$$\sum_{\substack{s \in E: \\ s \text{ emanates} \\ \text{from } v}} \sigma_s^2 q_x^{(s)}(0,t) - \sum_{\substack{r \in E: \\ r \text{ terminates} \\ \text{at } v}} \sigma_r^2 q_x^{(r)}(L_r, t) = 0. \tag{3.6}$$

Thus the appropriate sum of the heat flux is continuous across the interface.

For each finite endpoint vertex $v \in V$ which has only one finite-length edge $r_v \in E$ (and no infinite-length edges) we prescribe a general Robin boundary condition

$$\beta_0^v q^{(r_v)}(X,t) + \beta_1^v \partial_x q^{(r_v)}(X,t) = f_v(t), \qquad t \geq 0, \tag{3.7}$$

where f_v is known boundary datum and $\beta_0^v, \beta_1^v \in \mathbb{R}$ where $X = 0$ if r_v emanates from v, and $X = L_{r_v}$ if r_v terminates at v. If the graph has m_f finite edges and m_i infinite edges, then (3.4), (3.6), and (3.7) together prescribe a total of $2m_f + m_i$ boundary and interface conditions.

We insist that the initial data and interface conditions are compatible. That is, we require for each interface vertex v,

$$\sum_{\substack{s \in E:\\ s \text{ emanates}\\ \text{from } v}} \sigma_s^2 \partial_x q_0^s(0) - \sum_{\substack{r \in E:\\ r \text{ terminates}\\ \text{at } v}} \sigma_r^2 \partial_x q_0^{(r)}(L_r) = 0,$$

$$q_0^{(s)}(0) = q_0^{(r)}(L_r),$$

for $q^{(s)}(x,t)$ emanating from v and $q^{(r)}(x,t)$ terminating at v. We also require the initial and boundary conditions are compatible in the sense that for each finite endpoint vertex v with edge r_v,

$$\beta_0^v q_0^{r_v}(X) + \beta_1^v \partial_x q_0^{r_v}(X) = f_v(0).$$

3.2 Implicit integral representation of the solution

There is one global relation for each domain Ω_r. Each global relation links the boundary and interface values with the initial value for a particular $r \in E$. In contrast with Chapter 2, but following the innovation of [4, 47], we scale k in a way that will simplify later calculations.

For $x \in \Omega_r$, $t \geq 0$, and $k \in \mathbb{C}$, we have the local relations

$$\left(e^{-ixk/\sigma_r + k^2 t} q^{(r)}(x,t)\right)_t = \left(e^{-ixk/\sigma_r + k^2 t}\left(\sigma_r i k q^{(r)}(x,t) + \sigma_r^2 q_x^{(r)}(x,t)\right)\right)_x, \tag{3.8}$$

which are a one parameter family rewrite of (3.3a). We define

$$
\begin{aligned}
\hat{q}_0^{(r)}(k) &= \int_0^{L_r} e^{-ixk} q_0(x)\,\mathrm{d}x, && && k \in \mathbb{C},\\
\hat{q}^{(r)}(k,t) &= \int_0^{L_r} e^{-ixk} q^{(r)}(x,t)\,\mathrm{d}x, && t \geq 0, && k \in \mathbb{C},\\
g_0^{(r)}(\omega,t) &= \int_0^{t} e^{\omega s} q^{(r)}(0,s)\,\mathrm{d}s, && t \geq 0,\\
g_1^{(r)}(\omega,t) &= \int_0^{t} e^{\omega s} q_x^{(r)}(0,s)\,\mathrm{d}s, && t \geq 0,\\
h_0^{(r)}(\omega,t) &= \int_0^{t} e^{\omega s} q^{(r)}(L_r,s)\,\mathrm{d}s, && t \geq 0,\\
h_1^{(r)}(\omega,t) &= \int_0^{t} e^{\omega s} q_x^{(r)}(L_r,s)\,\mathrm{d}s, && t \geq 0.
\end{aligned}
$$

Applying Green's Theorem to $[0, L_r] \times [0, t]$ (where $L_r < \infty$) yields the global relation

$$
\begin{aligned}
\hat{q}_0^{(r)}\left(\frac{k}{\sigma_r}\right) - e^{k^2 t}\hat{q}^{(r)}\left(\frac{k}{\sigma_r}, t\right) = {}& \sigma_r ik g_0^{(r)}(k^2,t) + \sigma_r^2 g_1^{(r)}(k^2,t)\\
&- e^{-ikL_r/\sigma_r}\left(\sigma_r ik h_0^{(r)}(k^2,t) + \sigma_r^2 h_1^{(r)}(k^2,t)\right),
\end{aligned}
\tag{3.9a}
$$

for $k \in \mathbb{C}$, $t \geq 0$, and each $r \in E$. If $L_r = \infty$ the global relation is

$$
\hat{q}_0^{(r)}\left(\frac{k}{\sigma_r}\right) - e^{k^2 t}\hat{q}^{(r)}\left(\frac{k}{\sigma_r}, t\right) = \sigma_r ik g_0^{(r)}(k^2,t) + \sigma_r^2 g_1^{(r)}(k^2,t), \tag{3.9b}
$$

for $k \in \mathbb{C}^-$, $t \geq 0$, and each $r \in E$.

Inverting the Fourier transform in (3.9a) and (3.9b) respectively we have

$$
\begin{aligned}
q^{(r)}(x,t) = {}& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\sigma_r^2 k^2 t}\hat{q}_0^{(r)}(k)\,\mathrm{d}k\\
&- \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixk/\sigma_r - k^2 t}\left(ik g_0^{(r)}(k^2,t) + \sigma_r g_1^{(r)}(k^2,t)\right)\,\mathrm{d}k\\
&+ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik/\sigma_r(x-L_r) - k^2 t}\left(ik h_0^{(r)}(k^2,t) + \sigma_r h_1^{(r)}(k^2,t)\right)\,\mathrm{d}k,
\end{aligned}
\tag{3.10a}
$$

and

$$
\begin{aligned}
q^{(r)}(x,t) = {}& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\sigma_r^2 k^2 t}\hat{q}_0^{(r)}(k)\,\mathrm{d}k\\
&- \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixk/\sigma_r - k^2 t}\left(ik g_0^{(r)}(k^2,t) + \sigma_r g_1^{(r)}(k^2,t)\right)\,\mathrm{d}k.
\end{aligned}
\tag{3.10b}
$$

Define

$$D = \{k \in \mathbb{C} : \mathrm{Re}(k^2) < 0\}, \qquad D^{\pm} = \{k \in \mathbb{C} : D \cap \mathbb{C}^{\pm}\},$$

as shown in Figure 1.2. The integrand of the second integral in both (3.10a) and (3.10b) is entire and decays as $k \to \infty$ for $k \in \mathbb{C}^+ \setminus D^+$. Hence, by Jordan's Lemma we can replace the contour of integration of the second integral by $\int_{\partial D^+}$. It is convenient to deform both contours away from $k = 0$ to avoid singularities in the integrands that become apparent in what follows. Initially, these singularities are removable, since the integrands are entire. Writing integrals of sums as sums of integrals, the singularities may cease to be removable. In other words we deform $\partial D^{\pm}$ to $\partial D_R^{\pm}$, where

$$D_R^{\pm} = \{k \in D^{\pm} : |k| > R\},$$

and $R > 0$ is an arbitrary constant. An appropriate (sufficiently large) value of this constant may be chosen for any individual problem as in Figure 2.6. A similar deformation can be done for the third integral in (3.10a) from $\int_{-\infty}^{\infty} \cdot \, \mathrm{d}k$ to $-\int_{\partial D_R^-} \cdot \, \mathrm{d}k$.

Then, for each $r \in E$ such that $L_r < \infty$,

$$\begin{aligned} q^{(r)}(x,t) =& \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx - \sigma_r^2 k^2 t} \hat{q}_0^{(r)}(k) \, \mathrm{d}k \\ &- \frac{1}{2\pi} \int_{\partial D_R^+} e^{ixk/\sigma_r - k^2 t} \left(ikg_0^{(r)}(k^2,t) + \sigma_r g_1^{(r)}(k^2,t) \right) \mathrm{d}k \\ &- \frac{1}{2\pi} \int_{\partial D_R^-} e^{i(x-L_r)k/\sigma_r - k^2 t} \left(ikh_0^{(r)}(k^2,t) + \sigma_r h_1^{(r)}(k^2,t) \right) \mathrm{d}k. \end{aligned} \tag{3.11a}$$

Similarly, for each $r \in E$ such that $L_r = \infty$,

$$\begin{aligned} q^{(r)}(x,t) =& \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx - \sigma_r^2 k^2 t} \hat{q}_0^{(r)}(k) \, \mathrm{d}k \\ &- \frac{1}{2\pi} \int_{\partial D_R^+} e^{ixk/\sigma_r - k^2 t} \left(ikg_0^{(r)}(k^2,t) + \sigma_r g_1^{(r)}(k^2,t) \right) \mathrm{d}k. \end{aligned} \tag{3.11b}$$

3.3 m semi-infinite rods

In this section, we consider the case of $m \in \mathbb{N}$ semi-infinite rods of differing thermal diffusivity, joined at a single vertex. The graph is as shown in Figure 3.2. Of the $m+1$ vertices, m are each the terminus of a single edge, and all m edges emanate from the other vertex. Identify the set of edges E with the set $\{1, 2, \ldots, m\}$. The single interface vertex is denoted as v.

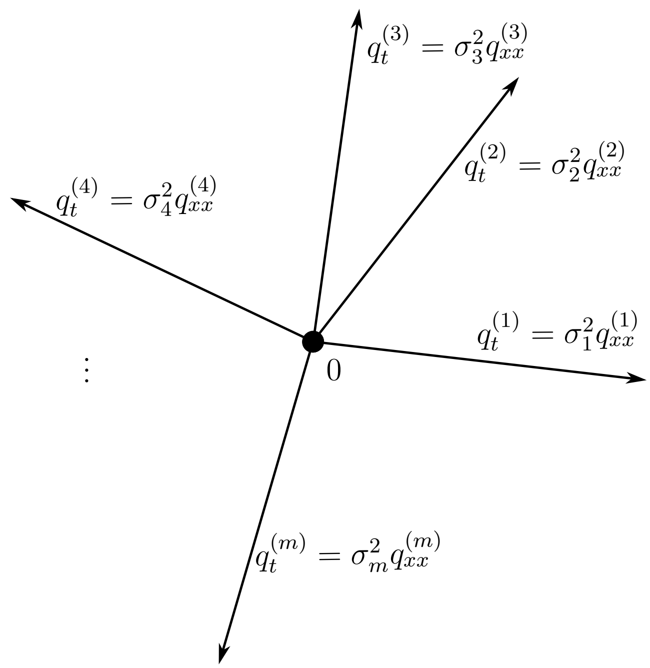

Figure 3.2: m semi-infinite rods.

Since $g_j^{(r)}(k^2, t)$ is invariant under the transformation $k \to -k$ we supplement the global relation (3.9b) with its evaluation at $-k$. Namely,

$$-\sigma_r ik g_0^{(r)}(k^2, t) + \sigma_r^2 g_1^{(r)}(k^2, t) = \hat{q}_0^{(r)}\left(\frac{-k}{\sigma_r}\right) - e^{k^2 t}\hat{q}^{(r)}\left(\frac{-k}{\sigma_r}, t\right), \tag{3.12}$$

for all $k \in \mathbb{C}^+$. Notice that this transformation is essential because our "solution" (3.11b) requires equations for $g_j^{(r)}(k^2, t)$ valid for $k \in D^+$, whereas the original global relation (3.9b) is valid only for $k \in \mathbb{C}^-$. Equations (3.12) represent a system of m linear equations in the $2m$ unknowns.

Appropriately transforming the interface conditions (3.4) and (3.6) for this problem we find

$$g_0(k^2,t) := g_0^{(s)}(k^2,t) = \int_0^t e^{k^2\tau} q^{(s)}(0,\tau)\,\mathrm{d}\tau, \quad 1 \le s \le m, \tag{3.13a}$$

$$\sum_{s=1}^{m} \sigma_s^2 g_1^{(s)}(k^2,t) = 0, \tag{3.13b}$$

for all $k \in \mathbb{C}$. Equation (3.13a) reduces the number of unknown functions from $2m$ to $m+1$. Equations (3.12), (3.13b) provide $m+1$ linear equations in these unknown functions. We represent this system as a matrix:

$$\mathcal{A}X = Y,$$

with

$$\begin{aligned}
X &= \left(g_0(k^2,t), \sigma_1^2 g_1^{(1)}(k^2,t), \dots, \sigma_m^2 g_1^{(m)}(k^2,t)\right)^\top, \\
Y &= \left(0, \hat{q}_0^{(1)}\left(\frac{-k}{\sigma_1}\right) - e^{k^2t}\hat{q}^{(1)}\left(\frac{-k}{\sigma_1},t\right), \dots, \hat{q}_0^{(m)}\left(\frac{-k}{\sigma_1}\right) - e^{k^2t}\hat{q}^{(m)}\left(\frac{-k}{\sigma_m},t\right)\right)^\top, \\
\mathcal{A} &= \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ -ik\sigma_1 & 1 & 0 & \cdots & 0 \\ -ik\sigma_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -ik\sigma_m & 0 & 0 & \cdots & 1 \end{pmatrix}.
\end{aligned}$$

For notational convenience, the entries of Y are denoted Y_r, where r is enumerated $0,\dots,m$. Similarly, the rows of the other matrices are counted from 0 to m.

Clearly, Y depends not only upon the initial data, but also upon the Fourier transforms of the solutions at time t which are not given data of the problem. Nevertheless, we proceed to solve the system as if Y is composed purely of known data and show afterwards that terms involving $\hat{q}^{(r)}(\cdot,t)$ make no contribution to the solution representation.

Subtracting the sum of rows $1,\dots,m$ from row 0 in matrix $\mathcal{A}$ it is immediate that

$$\det(\mathcal{A}) = ik\sum_{r=1}^{m}\sigma_r.$$

Let $\mathcal{A}_j$ be the matrix formed by replacing column j of $\mathcal{A}$ with the column vector Y. By Cramer's rule [12],

$$g_0(k^2,t) = \frac{\det(\mathcal{A}_0)}{\det(\mathcal{A})}, \qquad g_1^{(r)}(k^2,t) = \frac{\det(\mathcal{A}_r)}{\sigma_r^2 \det(\mathcal{A})}, \qquad 1 \le r \le m.$$

Trivially,

$$\det(\mathcal{A}_0) = -\sum_{p=1}^{m} Y_p.$$

To find $\det(\mathcal{A}_r)$ we subtract the j^{th} row for $j = 1, \ldots, r-1, r+1, \ldots, m$ from the first row. Then, for each $j = r$, $j = r-1$, $\ldots$, $j = 2$, we switch the j^{th} and $(j-1)^{\text{st}}$ rows and switch the j^{th} and $(j-1)^{\text{st}}$ columns. The resulting matrix is

$$\mathcal{B}_r = \begin{pmatrix} ik\sum\limits_{\substack{p=1\\p\neq r}}^{m}\sigma_p & -\sum\limits_{\substack{p=1\\p\neq r}}^{m} Y_p & 0 & \cdots & 0 \\ -ik\sigma_r & Y_r & 0 & \cdots & 0 \\ -ik\sigma_{r'} & Y_{r'} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -ik\sigma_{r''} & Y_{r''} & 0 & \cdots & 1 \end{pmatrix},$$

where the sequence $(r', \ldots, r'')$ of length $m-1$ is the sequence $(1, 2, \ldots, m)$ with the term r excluded. Clearly, $\det(\mathcal{B}_r) = \det(\mathcal{A}_r)$. Hence

$$\det(\mathcal{A}_r) = ik\left(Y_r\sum_{\substack{p=1\\p\neq r}}^{m}\sigma_p - \sigma_r\sum_{\substack{p=1\\p\neq r}}^{m} Y_p\right),$$

and it follows that

$$g_0(k^2,t) = \frac{-\sum_{p=1}^{m} Y_p}{ik\sum_{p=1}^{m}\sigma_p}, \tag{3.14}$$

$$g_1^{(r)}(k^2,t) = \frac{Y_r\sum_{p=1}^{m}\sigma_p - \sigma_r\sum_{p=1}^{m} Y_p}{\sigma_r^2\sum_{p=1}^{m}\sigma_p}, \qquad 1 \le r \le m. \tag{3.15}$$

Substituting Equations (3.14) and (3.15) into Equation (3.11b), we obtain expressions for each $q^{(r)}$ in terms of the Fourier transforms of every initial datum $\hat{q}_0^{(r)}(k^2,t)$ and of every solution $\hat{q}^{(r)}(\cdot,t)$:

$$
\begin{aligned}
q^{(r)}(x,t)=&\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixk-\sigma_r^2k^2t}\hat{q}_0^{(r)}(k)\,\mathrm{d}k\\
&+\int_{\partial D^+}\frac{e^{\frac{ixk}{\sigma_r}-k^2t}}{2\pi}\left(\frac{\sum_{p=1}^m \hat{q}_0^{(p)}\left(\frac{-k}{\sigma_p}\right)}{\sum_{p=1}^m \sigma_p}-\frac{\hat{q}_0^{(r)}\left(\frac{-k}{\sigma_r}\right)\sum_{p=1}^m\sigma_p-\sigma_r\sum_{p=1}^m \hat{q}_0^{(p)}\left(\frac{-k}{\sigma_p}\right)}{\sigma_r\sum_{p=1}^m\sigma_p}\right)\mathrm{d}k\\
&-\int_{\partial D^+}\frac{e^{\frac{ixk}{\sigma_r}}}{2\pi}\left(\frac{\sum_{p=1}^m \hat{q}^{(p)}\left(\frac{-k}{\sigma_p},t\right)}{\sum_{p=1}^m \sigma_p}-\frac{\hat{q}^{(r)}\left(\frac{-k}{\sigma_r},t\right)\sum_{p=1}^m\sigma_p-\sigma_r\sum_{p=1}^m \hat{q}_0^{(p)}\left(\frac{-k}{\sigma_p}\right)}{\sigma_r\sum_{p=1}^m\sigma_p}\right)\mathrm{d}k.
\end{aligned}
\tag{3.16}
$$

Note that the integrand of the third integral in (3.16) is analytic for all $k \in \mathbb{C}^+$ and decays for $k \to \infty$ for $k \in \mathbb{C}^+$. Thus, by Jordan's Lemma, the final integral of Equation (3.16) is 0. Hence, we obtain an explicit integral representation of the solution,

$$
\begin{aligned}
q^{(r)}(x,t) = &\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixk-\sigma_r^2k^2t}\hat{q}_0^{(r)}(k)\,\mathrm{d}k\\
&+\int_{\partial D^+}\frac{e^{ixk/\sigma_r-k^2t}}{2\pi}\left(\frac{2\sum_{p=1}^m \hat{q}_0^{(p)}\left(\frac{-k}{\sigma_p}\right)}{\sum_{p=1}^m\sigma_p}-\frac{\hat{q}_0^{(r)}\left(\frac{-k}{\sigma_r}\right)}{\sigma_r}\right)\,\mathrm{d}k,
\end{aligned}
\tag{3.17}
$$

in terms of only the initial data.

Notice that in this example we were able to set $R = 0$. This will not generally occur even for the semi-infinite rods when the network also contains finite rods (see, for example, [14, Proposition 3]). Indeed, $R = 0$ is possible here because $\det(\mathcal{A})$ is monomial which occurs for no configurations of rods except the one considered in this section.

3.4 m parallel finite rods

In this section we consider the case of $m \in \mathbb{N}$ parallel finite rods of differing thermal diffusivity extending between a pair of vertices. Note that we have chosen the rods to all be oriented in one direction. This is purely for notational convenience, the parameterization of the rod from 0 to L_r or from L_r to 0 is not important. The graph is as shown in Figure 3.3. One of the two vertices is the terminus of every edge, and every edge emanates from the other vertex. Again, it is notationally convenient to identify the set of edges E with the set $\{1, 2, \ldots, m\}$.

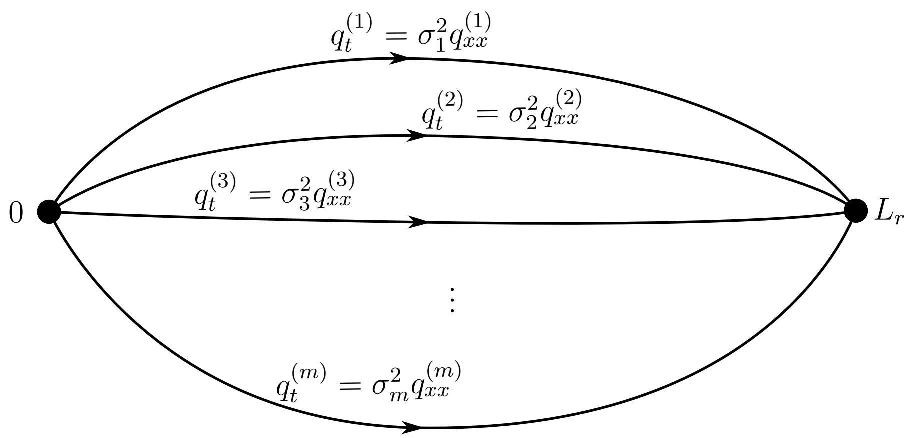

Figure 3.3: m parallel finite rods.

The continuity interface conditions allow us to define

$$g_0(k^2, t) := g_0^{(r)}(k^2, t), \qquad 1 \le r \le m, \tag{3.18a}$$

$$h_0(k^2, t) := h_0^{(r)}(k^2, t), \qquad 1 \le r \le m. \tag{3.18b}$$

Similarly, the continuity of flux interface conditions imply

$$\sum_{r=1}^{m} \sigma_r^2 g_1^{(r)}(k^2, t) = 0, \tag{3.19a}$$

$$\sum_{r=1}^{m} \sigma_r^2 h_1^{(r)}(k^2, t) = 0. \tag{3.19b}$$

Using the notation (3.18) the global relation (3.9a) reduces to

$$\begin{aligned} \hat{q}_0^{(r)}\left(\frac{k}{\sigma_r}\right) - e^{k^2 t}\hat{q}^{(r)}\left(\frac{k}{\sigma_r}, t\right) =& \sigma_r i k g_0(k^2, t) + \sigma_r^2 g_1^{(r)}(k^2, t) \\ & - e^{-ikL_r/\sigma_r}\left(\sigma_r i k h_0(k^2, t) + \sigma_r^2 h_1^{(r)}(k^2, t)\right). \end{aligned}$$

Applying the transformation $k \to -k$ which leaves the functions $g_j^{(r)}(k^2,t)$ and $h_j^{(r)}(k^2,t)$ invariant we obtain

$$\begin{aligned}\hat{q}_0^{(r)}\left(\frac{-k}{\sigma_r}\right) - e^{k^2t}\hat{q}^{(r)}\left(\frac{-k}{\sigma_r},t\right) =& -\sigma_r ikg_0(k^2,t) + \sigma_r^2 g_1^{(r)}(k^2,t) \\ & - e^{ikL_r/\sigma_r}\left(-\sigma_r ikh_0(k^2,t) + \sigma_r^2 h_1^{(r)}(k^2,t)\right).\end{aligned}$$

This provides a system of $2m$ equations valid for $k \in \mathbb{C}$ in $2m+2$ unknowns. System (3.19) provides two further equations in a subset of the same unknowns. We express this linear system as

$$\mathcal{A}X = Y,$$

where

$$X = \left(g_0(k^2,t), h_0(k^2,t), \sigma_1^2 g_1^{(1)}(k^2,t), \ldots, \sigma_m^2 g_1^{(m)}(k^2,t), \sigma_1^2 h_1^{(1)}(k^2,t), \ldots, \sigma_m^2 h_1^{(m)}(k^2,t)\right)^\top, \tag{3.20a}$$

$$Y = (0, Y_1(k), \ldots, Y_m(k), Y_1(-k), \ldots, Y_m(-k))^\top, \tag{3.20b}$$

$$Y_r(k) = \hat{q}_0^{(r)}\left(\frac{k}{\sigma_r}\right) - e^{k^2t}\hat{q}^{(r)}\left(\frac{k}{\sigma_r},t\right), \tag{3.20c}$$

and

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ ik\sigma_1 & -ik\sigma_1 e^{\frac{-ikL_1}{\sigma_1}} & 1 & 0 & \cdots & 0 & -e^{\frac{-ikL_1}{\sigma_1}} & 0 & \cdots & 0 \\ ik\sigma_2 & -ik\sigma_2 e^{\frac{-ikL_2}{\sigma_2}} & 0 & 1 & \cdots & 0 & 0 & -e^{\frac{-ikL_2}{\sigma_2}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ ik\sigma_m & -ik\sigma_m e^{\frac{-ikL_m}{\sigma_m}} & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & -e^{\frac{-ikL_m}{\sigma_m}} \\ -ik\sigma_1 & ik\sigma_1 e^{\frac{ikL_1}{\sigma_1}} & 1 & 0 & \cdots & 0 & -e^{\frac{ikL_1}{\sigma_1}} & 0 & \cdots & 0 \\ -ik\sigma_2 & ik\sigma_2 e^{\frac{ikL_2}{\sigma_2}} & 0 & 1 & \cdots & 0 & 0 & -e^{\frac{ikL_2}{\sigma_2}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -ik\sigma_m & ik\sigma_m e^{\frac{ikL_m}{\sigma_m}} & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & -e^{\frac{ikL_m}{\sigma_m}} \end{pmatrix}.$$

Solving the system via Cramer's rule, we obtain

$$g_0(k^2,t) = \frac{AC - BD}{k(A^2 - B^2)}, \qquad h_0(k^2,t) = \frac{BC - AD}{k(A^2 - B^2)},$$

$$g_1^{(r)}(k^2,t) = \frac{-\det\begin{pmatrix} B & A & C \\ A & B & D \\ \sigma_r \frac{1}{S_r(k)} & \sigma_r \frac{C^{(r)}(k)}{S_r(k)} & \frac{Y_r(k)e^{ikL_r/\sigma_r} - Y_r(-k)e^{-ikL_r/\sigma_r}}{e^{ikL_r/\sigma_r} - e^{-ikL_r/\sigma_r}} \end{pmatrix}}{\sigma_r^2(A^2 - B^2)}, \tag{3.21}$$

$$h_1^{(r)}(k^2,t) = \frac{\det\begin{pmatrix} B & A & D \\ A & B & C \\ \sigma_r \frac{1}{S_r(k)} & \sigma_r \frac{C^{(r)}(k)}{S_r(k)} & \frac{Y_r(k) - Y_r(-k)}{e^{ikL_r/\sigma_r} - e^{-ikL_r/\sigma_r}} \end{pmatrix}}{\sigma_r^2(A^2 - B^2)},$$

where

$$A = \sum_{p=1}^{m} \sigma_p \frac{1}{S_p(k)}, \qquad C = \sum_{p=1}^{m} \frac{Y_p(k) - Y_p(-k)}{e^{ikL_p/\sigma_p} - e^{-ikL_p/\sigma_p}},$$

$$B = \sum_{p=1}^{m} \sigma_p \frac{C_p(k)}{S_p(k)}, \qquad D = \sum_{p=1}^{m} \frac{Y_p(k)e^{ikL_p/\sigma_p} - Y_p(-k)e^{-ikL_p/\sigma_p}}{e^{ikL_p/\sigma_p} - e^{-ikL_p/\sigma_p}},$$

and

$$S_p(k) = \sin(kL_p/\sigma_p), \qquad C_p(k) = \cos(kL_p/\sigma_p).$$

Equation (3.21) for the t-transformed boundary and interface values depends on the Fourier transform of the solution $\hat{q}^{(r)}(\cdot,t)$ through Expression (3.20c) for Y_p. In order for Equation (3.11a) to represent an effective integral representation for each solution $q^{(r)}(x,t)$ we must remove this dependence.

The product of a meromorphic function with a holomorphic function must include as zeros every zero of the meromorphic function. As

$$\left(\prod_{p=1}^{m} S_p(k)\right)^2 (A^2 - B^2) \tag{3.22}$$

is an exponential polynomial (indeed an exponential sum), it is possible to obtain bounds on its zeros. By [42, Theorem 3], the zeros of (3.22) are confined within a finite-width horizontal strip. Hence, for sufficiently large $R > 0$, there are no zeros of $A^2 - B^2$ within $\overline{D_R^+} \cup \overline{D_R^-}$. Indeed, a simple calculation reveals

$$R = \frac{(m-1)\log 2 + \log(8 + m(m-1))}{\sqrt{2}\min_{1\le r\le m}\{\frac{L_r}{\sigma_r}\}}$$

is sufficient. A standard asymptotic analysis as in Chapter 2 shows that the terms involving $\hat{q}^{(p)}(\cdot, t)$ decay as $k \to \infty$ from within the appropriate domains $D_R^\pm$, and Jordan's Lemma establishes that these terms make no contribution to the integral representation.

We have established the effective integral representation for the solution given by equation (3.11a) with $g_0(k^2, t)$, $h_0(k^2, t)$, $g_1^{(r)}(k^2, t)$, and $h_1^{(r)}(k^2, t)$ given by equations (3.21) where $Y_r(k)$ in (3.20c) is replaced by

$$Y_r(k) = \hat{q}_0^{(r)}\left(\frac{k}{\sigma_r}\right), \qquad 1 \le r \le m.$$

3.5 m finite rods connected at a single point

In this section we consider the case of $m \in \mathbb{N}$ finite rods of differing thermal diffusivity joined at a single vertex. The graph is as shown in Figure 3.4. Of the $m+1$ vertices m are the terminus of a single edge and all m edges emanate from the other vertex. As before we identify the set of edges E with the set $\{1, 2, \ldots, m\}$. We enumerate the vertices so that each edge r emanates from vertex 0 and terminates at vertex r.

We use the condition of continuity across the interface to establish

$$g_0(k^2, t) := g_0^{(s)}(k^2, t) = \int_0^t e^{k^2\tau} q^{(s)}(0, \tau)\,\mathrm{d}\tau \tag{3.23}$$

for $1 \le s \le m$, $k \in \mathbb{C}$. Taking the time transform of the Robin boundary condition (3.7) we have

$$\tilde{f}_r(k, t) := \int_0^t e^{k^2 s} f_r(s)\,\mathrm{d}s = \beta_0^{(r)} h_0^{(r)}(k^2, t) + \beta_1^{(r)} h_1^{(r)}(k^2, t). \tag{3.24}$$

In order to reduce the size of the linear system that will eventually have to be solved, it is helpful to use the Robin condition to express either the 0^{th} order or the 1^{st} order

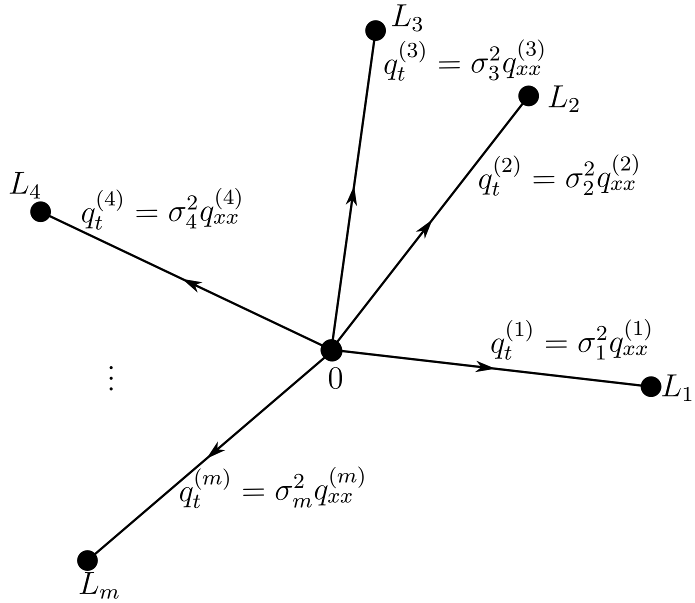

Figure 3.4: m finite rods joined at a single vertex.

boundary value at L_r in terms of the other. Suppose that precisely $1 \le m_N \le m$ of the boundary conditions are in fact Neumann conditions (that is the corresponding $\beta_0^{(r)} = 0$). Moreover suppose, without loss of generality, that these Neumann conditions are the boundary conditions at vertices $1, 2, \ldots, m_N$, that $\beta_1^{(r)} = 1$ for each $1 \le r \le m_N$, and that $\beta_0^{(r)} = 1$ for each $m_N + 1 \le r \le m$. We use the boundary conditions (3.24) and Equation (3.23) to rewrite the global relation (3.9a) as

$$\begin{aligned}\hat{q}_0^{(r)}\left(\frac{k}{\sigma_r}\right) - e^{k^2 t}\hat{q}^{(r)}\left(\frac{k}{\sigma_r}, t\right) &= \sigma_r ikg_0(k^2,t) + \sigma_r^2 g_1^{(r)}(k^2,t) - e^{-ikL_r/\sigma_r} ik\sigma_r h_0^{(r)}(k^2,t) \\ &\quad - e^{-ikL_r/\sigma_r}\sigma_r^2 \tilde{f}_r(k,t),\end{aligned} \tag{3.25a}$$

for $1 \le r \le m_N$, $k \in \mathbb{C}$ and

$$\begin{aligned}\hat{q}_0^{(r)}\left(\frac{k}{\sigma_r}\right) - e^{k^2 t}\hat{q}^{(r)}\left(\frac{k}{\sigma_r}, t\right) &= \sigma_r ikg_0(k^2,t) + \sigma_r^2 g_1^{(r)}(k^2,t) \\ &\quad - e^{-ikL_r/\sigma_r}\sigma_r\left(\left(\sigma_r - ik\beta_1^{(r)}\right) h_1^{(r)}(k^2,t) + ik\tilde{f}_r(k,t)\right),\end{aligned} \tag{3.25b}$$

for $m_N + 1 \le r \le m$ and $k \in \mathbb{C}$. As before, another set of m global relations is obtained by evaluating (3.25) for $k = -k$. Together with a t-transform of the interface condition (3.6) the $2m$ global relations form a system of linear equations in the $2m+1$ functions

$$\begin{aligned}
&g_0(k^2,t), && && t>0,\\
&g_1^{(r)}(k^2,t), && 1\le r\le m, && t>0,\\
&h_0^{(r)}(k^2,t), && 1\le r\le m_N, && t>0,\\
&h_1^{(r)}(k^2,t), && m_N+1\le r\le m, && t>0.
\end{aligned}$$

The system can be expressed as

$$\mathcal{A}X = Y, \tag{3.26a}$$

where

$$X = \left(g_0, \sigma_1^2 g_1^{(1)}, \ldots, \sigma_m^2 g_1^{(m)}, \sigma_1 h_0^{(1)}, \ldots, \sigma_{m_N} h_0^{(m_N)}, \sigma_{m_N+1}^2 h_1^{(m_N+1)}, \ldots, \sigma_m^2 h_1^{(m)}\right)^\top, \tag{3.26b}$$

$$Y = \left(0, Y_1(k), \ldots, Y_m(k), Y_1(-k), \ldots, Y_m(-k)\right)^\top, \tag{3.26c}$$

$$Y_r(k,t) = \begin{cases} \hat{q}_0^{(r)}\left(\frac{k}{\sigma_r}\right) - e^{k^2 t}\hat{q}^{(r)}\left(\frac{k}{\sigma_r}, t\right) + e^{\frac{-ikL_r}{\sigma_r}}\sigma_r^2 \tilde{f}_r(k,t), & 1\le r\le m_N,\\ \hat{q}_0^{(r)}\left(\frac{k}{\sigma_r}\right) - e^{k^2 t}\hat{q}^{(r)}\left(\frac{k}{\sigma_r}, t\right) + ike^{\frac{-ikL_r}{\sigma_r}}\sigma_r^2 \tilde{f}_r(k,t), & m_N+1\le r\le m, \end{cases} \tag{3.26d}$$

$$\mathcal{A} = \begin{pmatrix} 0 & 1_{1\times m} & 0_{1\times m} \\ \mathcal{A}_0(k) & I_{m\times m} & \mathcal{A}_1(-k) \\ \mathcal{A}_0(-k) & I_{m\times m} & \mathcal{A}_1(k) \end{pmatrix}, \tag{3.26e}$$

$$\mathcal{A}_0(k) = \left(ik\sigma_1, \ldots, ik\sigma_m\right)^\top, \tag{3.26f}$$

and

$$\mathcal{A}_1(k) =$$
$$\begin{pmatrix} ike^{\frac{ikL_1}{\sigma_1}} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & ike^{\frac{ikL_{m_N}}{\sigma_{m_N}}} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -\left(\frac{ik\beta_1^{m_N+1}}{\sigma_{m_N+1}}+1\right)e^{\frac{ikL_{m_N+1}}{\sigma_{m_N+1}}} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & -\left(\frac{ik\beta_1^m}{\sigma_m}+1\right)e^{\frac{ikL_m}{\sigma_m}}, \end{pmatrix}, \quad (3.26g)$$

where $1_{m\times n}$ is an $m\times n$ matrix with every entry equal to 1 and $I_{m\times n}$ is an $m\times n$ identity matrix. The arguments of each of the functions in X are (k^2,t). The boundary conditions (3.24) give the remaining spectral functions in terms of those appearing in the vector X.

Solving this system and substituting the solution into (3.11a) provides an integral representation of the solution in terms of the initial and boundary data, and the Fourier transform of the solution $\hat{q}^{(r)}(\cdot,t)$. It remains to show that it is possible to remove the dependence on the Fourier transform of the solution.

When the system (3.26) is solved using Cramer's rule, each entry X_j of X is expressed as the ratio of two determinants, each of which is an analytic function of k. The denominator is an exponential polynomial in which every exponent is purely imaginary. Hence the zeros of the denominator lie asymptotically within a horizontal logarithmic strip [42, Theorem 6]. Moreover, X_j is holomorphic except at the zeros of the denominator, so choosing R sufficiently large ensures that X_j is holomorphic on $\overline{D_R^+}\cup\overline{D_R^-}$.

As the growing entries of $\mathcal{A}$ lie in the same rows as the growing entries of Y involving $\hat{q}^{(r)}(\cdot,t)$, but each such entry in Y grows like $\mathcal{O}(k^{-1})$ multiplied by the corresponding entry in $\mathcal{A}$, each integrand involving $\hat{q}^{(r)}(\cdot,t)$ decays as $k\to\infty$ from within the relevant sector $\overline{D_R^\pm}$. Hence, by Jordan's Lemma, the terms involving $\hat{q}^{(r)}(\cdot,t)$ make no contribution to the solution representation.

Chapter 4

Interface problems for dispersive equations

In Chapter 2 the Fokas Method was used to solve the classical problem of the heat equation with interfaces. The same is done here for the linear Schrödinger (LS) equation with an interface. We restrict ourselves to the case of a continuous wave function with a jump in the derivative across the interface. Although the problem considered here is similar to the one presented in Chapter 2, the dispersive nature of the problem makes it more difficult to solve both classically and using the Fokas Method.

The LS equation is arguably the simplest dispersive equation, having the dispersion relation $\omega(k) = -ik^2$. It arises in its own right in quantum mechanics [54], and as the linearization of various nonlinear equations, most notably the nonlinear Schrödinger (NLS) equations $iq_t(x,t) = -q_{xx}(x,t) \pm |q(x,t)|^2 q(x,t)$. As such, it arises in a large variety of application areas, whenever the modulation of nonlinear wave trains is considered. Indeed, it has been derived in such diverse fields as waves in deep water [68], plasma physics [69], nonlinear fiber optics [34, 35], magneto-static spin waves [71], and many other settings.

The LS equation describes the behavior of solutions of the NLS equation in the small amplitude limit and understanding the linear dynamics is fundamental in understanding the dynamics of the more complicated nonlinear problem.

Recently, Cascaval and Hunter [11] have considered the time-dependent LS on simple networks. Their solution formulas are not explicit, as they contain implicit integral equations for the interface conditions. Their analysis is easily extended to more than two domains and also considers the NLS equation.

The LS equation in two semi-infinite domains with an interface is considered in Section 4.1. The method is adapted to the problem of two finite domains in Section 4.2. As before, the solution formulae given are easily computed numerically using techniques presented in [44, 62]. Throughout, our emphasis is on non-steady state solutions. The solutions presented here using the Fokas Method are explicit and depend only on known quantities. Although we present solution formulas only for the case of two domains (both finite or both infinite) it is straightforward to generalize this method to n domains.

4.1 Two semi-infinite domains

We seek two functions

$$q^{(1)}(x,t), \quad x<0, \quad t\geq 0,$$

and

$$q^{(2)}(x,t), \quad x>0, \quad t\geq 0,$$

satisfying the equations

$$iq_t^{(1)}(x,t) = \sigma_1 q_{xx}^{(1)}(x,t), \qquad x<0,\ t>0, \tag{4.1a}$$

$$iq_t^{(2)}(x,t) = \sigma_2 q_{xx}^{(2)}(x,t), \qquad x>0,\ t>0, \tag{4.1b}$$

the initial conditions

$$q^{(1)}(x,0) = q_0^{(1)}(x), \qquad x<0, \tag{4.2a}$$

$$q^{(2)}(x,0) = q_0^{(2)}(x), \qquad x>0, \tag{4.2b}$$

the asymptotic conditions

$$\lim_{x\to-\infty} q^{(1)}(x,t) = \gamma^{(1)}, \qquad t \geq 0, \tag{4.3a}$$

$$\lim_{x\to\infty} q^{(2)}(x,t) = \gamma^{(2)}, \qquad t \geq 0, \tag{4.3b}$$

and the continuity interface conditions

$$q^{(1)}(0,t) = q^{(2)}(0,t), \qquad t > 0, \tag{4.4a}$$

$$\rho_1 q_x^{(1)}(0,t) = \rho_2 q_x^{(2)}(0,t), \qquad t > 0, \tag{4.4b}$$

as shown in Figure 4.1 where $\gamma^{(1)}, \gamma^{(2)}, \sigma_1, \sigma_2, \rho_1$ and ρ_2 are t-independent nonzero constants. The sub- and super-indices 1 and 2 denote the left and right domain, respectively. In what follows we assume that σ_1 and σ_2 are both positive for convenience. First, we shift

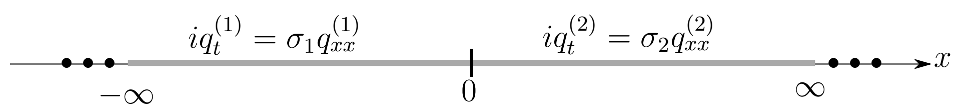

Figure 4.1: The LS equation for two semi-infinite domains.

the problem so that the asymptotic conditions are identically zero. We define $v^{(1)}(x,t) = q^{(1)}(x,t) - \gamma^{(1)}$ and $v^{(2)}(x,t) = q^{(2)}(x,t) - \gamma^{(2)}$ as the functions that satisfy

$$iv_t^{(1)}(x,t) = \sigma_1 v_{xx}^{(1)}(x,t), \qquad x < 0 \qquad t \geq 0, \tag{4.5a}$$

$$iv_t^{(2)}(x,t) = \sigma_2 v_{xx}^{(2)}(x,t), \qquad x > 0 \qquad t \geq 0, \tag{4.5b}$$

$$\lim_{x\to-\infty} v^{(1)}(x,t) = 0, \qquad t \geq 0, \tag{4.5c}$$

$$\lim_{x\to\infty} v^{(2)}(x,t) = 0, \qquad t \geq 0, \tag{4.5d}$$

$$v^{(1)}(x,0) = v_0^{(1)}(x), \qquad x < 0, \tag{4.5e}$$

$$v^{(2)}(x,0) = v_0^{(2)}(x), \qquad x > 0, \tag{4.5f}$$

$$v^{(1)}(0,t) + \gamma^{(1)} = v^{(2)}(0,t) + \gamma^{(2)}, \qquad t \geq 0, \tag{4.5g}$$

$$\rho_1 v_x^{(1)}(0,t) = \rho_2 v_x^{(2)}(0,t), \qquad t \geq 0. \tag{4.5h}$$

We begin with the local relations

$$
\begin{aligned}
\left(e^{-ikx+\omega_1 t}v^{(1)}(x,t)\right)_t &= \left(\sigma_1 e^{-ikx+\omega_1 t}(kv^{(1)}(x,t) - iv^{(1)}_x(x,t))\right)_x, && x<0,\\
\left(e^{-ikx+\omega_2 t}v^{(2)}(x,t)\right)_t &= \left(\sigma_2 e^{-ikx+\omega_2 t}(kv^{(2)}(x,t) - iv^{(2)}_x(x,t))\right)_x, && x>0.
\end{aligned}
$$

These are one-parameter relations obtained by rewriting (4.5a) and (4.5b). This also gives $\omega_1(k) = -i\sigma_1 k^2$ and $\omega_2(k) = -i\sigma_2 k^2$. The use of these functions instead of the dispersion relations proper is common when using the Fokas Method and we continue this use. Applying Green's Theorem in the strip $(-\infty,0)\times(0,t)$ (see Figure 2.5), we find the global relations

$$
\begin{aligned}
0 = &\int_{-\infty}^{0} e^{-ikx}v^{(1)}_0(x)\,\mathrm{d}x - \int_{-\infty}^{0} e^{-ikx+\omega_1 t}v^{(1)}(x,t)\,\mathrm{d}x\\
&+ \int_0^t \sigma_1 e^{\omega_1 s}\left(kv^{(1)}(0,s) - iv^{(1)}_x(0,s)\right)\,\mathrm{d}s,
\end{aligned}
\tag{4.6a}
$$

$$
\begin{aligned}
0 = &\int_{0}^{\infty} e^{-ikx}v^{(2)}_0(x)\,\mathrm{d}x - \int_{0}^{\infty} e^{-ikx+\omega_2 t}v^{(2)}(x,t)\,\mathrm{d}x\\
&- \int_0^t \sigma_2 e^{\omega_2 s}\left(kv^{(2)}(0,s) - iv^{(2)}_x(0,s)\right)\,\mathrm{d}s.
\end{aligned}
\tag{4.6b}
$$

The Fourier integrals in (4.6) require $k\in\mathbb{C}^+$ in (4.6a) and $k\in\mathbb{C}^-$ in (4.6b). For $k\in\mathbb{C}$, we define the following transforms:

$$\begin{aligned}
g_0(\omega,t) =& \int_0^t e^{\omega s} v^{(1)}(0,s)\,\mathrm{d}s = \int_0^t e^{\omega s}(v^{(2)}(0,s) + \gamma^{(2)} - \gamma^{(1)})\,\mathrm{d}s \\
=& \frac{(\gamma^{(2)} - \gamma^{(1)})(e^{\omega t} - 1)}{\omega} + \int_0^t e^{\omega s} v^{(2)}(0,s)\,\mathrm{d}s, \\
g_1(\omega,t) =& \int_0^t e^{\omega s} v_x^{(1)}(0,s)\,\mathrm{d}s = \frac{\rho_2}{\rho_1}\int_0^t e^{\omega s} v_x^{(2)}(0,s)\,\mathrm{d}s, \\
\hat{v}^{(1)}(k,t) =& \int_{-\infty}^0 e^{-ikx} v^{(1)}(x,t)\,\mathrm{d}x, \\
\hat{v}_0^{(1)}(k) =& \int_{-\infty}^0 e^{-ikx} v_0^{(1)}(x)\,\mathrm{d}x, \\
\hat{v}^{(2)}(k,t) =& \int_0^{\infty} e^{-ikx} v^{(2)}(x,t)\,\mathrm{d}x, \\
\hat{v}_0^{(2)}(k) =& \int_0^{\infty} e^{-ikx} v_0^{(2)}(x)\,\mathrm{d}x.
\end{aligned}$$

Using these definitions, the global relations (4.6) are rewritten as

$$0 = \hat{v}_0^{(1)}(k) - e^{\omega_1 t}\hat{v}^{(1)}(k,t) + k\sigma_1 g_0(\omega_1,t) - i\sigma_1 g_1(\omega_1,t), \tag{4.7a}$$

$$\begin{aligned}
0 =& \hat{v}_0^{(2)}(k) - e^{\omega_2 t}\hat{v}^{(2)}(k,t) - k\sigma_2 g_0(\omega_2,t) \\
&+ \frac{i(\gamma^{(2)} - \gamma^{(1)})}{k}(e^{\omega_2 t} - 1) + \frac{i\sigma_2\rho_1}{\rho_2} g_1(\omega_2,t),
\end{aligned} \tag{4.7b}$$

where $k \in \mathbb{C}^+$ for (4.7a) and $k \in \mathbb{C}^-$ for (4.7b). Since the functions $\omega_2(k)$ and $\omega_1(k)$ are invariant under $k \to -k$ we can supplement (4.7) with their evaluation at $-k$, namely

$$0 = \hat{v}_0^{(1)}(-k) - e^{\omega_1 t}\hat{v}^{(1)}(-k,t) - k\sigma_1 g_0(\omega_1,t) - i\sigma_1 g_1(\omega_1,t), \tag{4.8a}$$

$$\begin{aligned}
0 =& \hat{v}_0^{(2)}(-k) - e^{\omega_2 t}\hat{v}^{(2)}(-k,t) + k\sigma_2 g_0(\omega_2,t) - \frac{i(\gamma^{(2)} - \gamma^{(1)})}{k}(e^{\omega_2 t} - 1) \\
&+ \frac{i\sigma_2\rho_1}{\rho_2} g_1(\omega_2,t),
\end{aligned} \tag{4.8b}$$

where $k \in \mathbb{C}^-$ for (4.8a) and $k \in \mathbb{C}^+$ for (4.8b).

Inverting the Fourier transform in (4.7a) we have

$$\begin{aligned}
v^{(1)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx - \omega_1 t}\hat{v}_0^{(1)}(k)\,\mathrm{d}k \\
&+ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx - \omega_1 t}\sigma_1(k g_0(\omega_1,t) - i g_1(\omega_1,t))\,\mathrm{d}k,
\end{aligned}$$

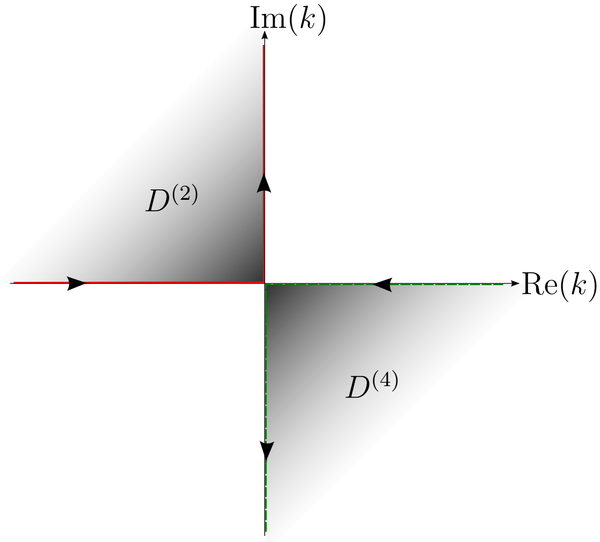

Figure 4.2: The domains $D^{(2)}$ and $D^{(4)}$ for the LS equation.

for $x < 0$ and $t > 0$. Let $D = \{k \in \mathbb{C} : \mathrm{Re}(-ik^2) < 0\} = D^{(2)} \cup D^{(4)}$. The region D is shown in Figure 4.2. The integrand of the second integral is entire and decays as $k \to \infty$ for $k \in \mathbb{C}^- \setminus D^{(4)}$. Using the analyticity of the integrand and applying Jordan's Lemma we can replace the contour of integration of the second integral by $-\int_{\partial D^{(4)}}$:

$$\begin{aligned} v^{(1)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{v}_0^{(1)}(k)\,\mathrm{d}k \\ &- \frac{1}{2\pi}\int_{\partial D^{(4)}} e^{ikx-\omega_1 t}\sigma_1(kg_0(\omega_1,t) - ig_1(\omega_1,t))\,\mathrm{d}k. \end{aligned} \tag{4.9}$$

Similarly, inverting the Fourier transform in (4.7b) we have

$$\begin{aligned} v^{(2)}(x,t) =& \frac{(\gamma^{(2)}-\gamma^{(1)})}{2}\phi(\sigma_2,x,t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{v}_0^{(2)}(k)\,\mathrm{d}k \\ &+ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\sigma_2\left(-kg_0(\omega_2,t) + \frac{i\rho_1}{\rho_2}g_1(\omega_2,t)\right)\mathrm{d}k, \end{aligned}$$

for $x > 0$, $t > 0$, and

$$\phi(\sigma,x,t) = \begin{cases} 0, & t = 0, \\ -\operatorname{sgn}(x) + \frac{1}{\sqrt{\pi\sigma t}}e^{i\pi/4}\int_0^x e^{-iy^2/(4\sigma t)}\,\mathrm{d}y, & t > 0. \end{cases}$$

The integrand of the third integral is entire and decays as $k \to \infty$ for $k \in \mathbb{C}^+ \setminus D^{(2)}$. Using the analyticity of the integrand and applying Jordan's Lemma we can replace the contour of integration $\int_{-\infty}^{\infty} \cdot \, \mathrm{d}k$ by $\int_{\partial D^{(2)}} \cdot \, \mathrm{d}k$:

$$\begin{aligned} v^{(2)}(x,t) =& \frac{\gamma^{(2)} - \gamma^{(1)}}{2}\phi(\sigma_2, x, t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{v}_0^{(2)}(k)\,\mathrm{d}k \\ &+ \frac{1}{2\pi}\int_{\partial D^{(2)}} e^{ikx-\omega_2 t}\sigma_2\left(-kg_0(\omega_2,t) + \frac{i\rho_1}{\rho_2}g_1(\omega_2,t)\right)\mathrm{d}k. \end{aligned} \tag{4.10}$$

The Expressions (4.9) and (4.10) for $v^{(1)}(x,t)$ and $v^{(2)}(x,t)$ depend on the unknown functions $g_0(\cdot,t)$ and $g_1(\cdot,t)$, evaluated at different arguments. These functions need to be expressed in terms of known quantities. To obtain a system of two equations for the two unknown functions we use the four global relations. We use (4.7b) and (4.8a) to solve for $g_0(\omega_1,t)$, and $g_1(\omega_1,t)$. This requires use of all the symmetries of the set of $\{\omega_1(k), \omega_2(k)\}$. Namely, the transformation $k \to \sqrt{\sigma_1/\sigma_2}k$ in (4.7b). Substituting these into (4.9) we have

$$\begin{aligned} v^{(1)}(x,t) =& \frac{\rho_2\sigma_1(\gamma^{(2)} - \gamma^{(1)})}{\rho_2\sigma_1 + \rho_1\sqrt{\sigma_1\sigma_2}}\phi(\sigma_1,x,t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{v}_0^{(1)}(k)\,\mathrm{d}k \\ &+ \frac{\rho_2\sigma_1 - \rho_1\sqrt{\sigma_1\sigma_2}}{2\pi(\rho_2\sigma_1 + \rho_1\sqrt{\sigma_1\sigma_2})}\int_{\partial D^{(4)}} e^{ikx-\omega_1 t}\hat{v}_0^{(1)}(-k)\,\mathrm{d}k \\ &- \frac{\rho_2\sigma_1}{\pi(\sigma_2\rho_1 + \rho_2\sqrt{\sigma_1\sigma_2})}\int_{\partial D^{(4)}} e^{ikx-\omega_1 t}\hat{v}_0^{(2)}\left(k\sqrt{\frac{\sigma_1}{\sigma_2}}\right)\mathrm{d}k \\ &- \frac{\rho_2\sigma_1 - \rho_1\sqrt{\sigma_1\sigma_2}}{2\pi(\rho_2\sigma_1 + \rho_1\sqrt{\sigma_1\sigma_2})}\int_{\partial D^{(4)}} e^{ikx}\hat{v}^{(1)}(-k,t)\,\mathrm{d}k \\ &+ \frac{\rho_2\sigma_1}{\pi(\sigma_2\rho_1 + \rho_2\sqrt{\sigma_1\sigma_2})}\int_{\partial D^{(4)}} e^{ikx}\hat{v}^{(2)}\left(k\sqrt{\frac{\sigma_1}{\sigma_2}}, t\right)\mathrm{d}k, \end{aligned} \tag{4.11}$$

for $x < 0$, $t > 0$. The first four terms depend only on known functions. The integrand of the second-to-last term is analytic for all $k \in \mathbb{C}^-$. Further, $\hat{v}^{(1)}(-k,t)$ decays for $k \to \infty$ when $k \in \mathbb{C}^-$. Thus, by Jordan's Lemma, the integral of $\exp(ikx)\hat{v}^{(1)}(-k,t)$ along a closed, bounded curve in $\mathbb{C}^-$ vanishes. In particular we consider the closed curve $\mathcal{L}^{(4)} = \mathcal{L}_{D^{(4)}} \cup \mathcal{L}_C^{(4)}$ where $\mathcal{L}_{D^{(4)}} = \partial D^{(4)} \cap \{k : |k| < C\}$ and $\mathcal{L}_C^{(4)} = \{k \in D^{(4)} : |k| = C\}$, see Figure 4.3.

Since the integral along $\mathcal{L}_C^-$ vanishes as $C \to \infty$, the fourth integral on the right-hand side of (4.11) must vanish since the contour $\mathcal{L}_{D^{(4)}}$ becomes $\partial D^{(4)}$ as $R \to \infty$. The uniform decay

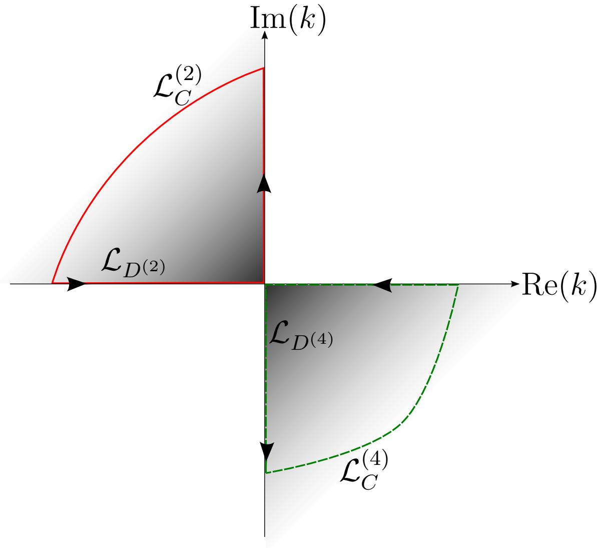

Figure 4.3: The contour $\mathcal{L}^{(4)}$ is shown in green as a dashed line. An application of Cauchy's Integral Theorem using this contour allows elimination of the contribution of $\hat{v}^{(1)}(-k,t)$ from the integral (4.11). Similarly, the contour $\mathcal{L}^{(2)}$ is shown in red and application of Cauchy's Integral Theorem using this contour allows elimination of the contribution of $\hat{v}^{(2)}(-k,t)$ from (4.12).

of $\hat{v}^{(1)}(-k,t)$ for large k is exactly the condition required for the integral to vanish using Jordan's Lemma. For the final integral in (4.11) we use that $\hat{v}^{(2)}(k\sqrt{\sigma_1/\sigma_2},t)$ is analytic and bounded for $k \in \mathbb{C}^-$. Using the same argument as above, the fifth integral in (4.11) vanishes and we have an explicit representation for $v^{(1)}(x,t)$ in terms of initial conditions:

$$\begin{aligned} v^{(1)}(x,t) =& \frac{\rho_2\sigma_1(\gamma^{(2)}-\gamma^{(1)})}{\rho_2\sigma_1+\rho_1\sqrt{\sigma_1\sigma_2}}\phi(\sigma_1,x,t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{v}_0^{(1)}(k)\,\mathrm{d}k \\ &+ \frac{\rho_2\sigma_1-\rho_1\sqrt{\sigma_1\sigma_2}}{2\pi(\rho_2\sigma_1+\rho_1\sqrt{\sigma_1\sigma_2})}\int_{\partial D^{(4)}} e^{ikx-\omega_1 t}\hat{v}_0^{(1)}(-k)\,\mathrm{d}k \\ &- \frac{\rho_2\sigma_1}{\pi(\sigma_2\rho_1+\rho_2\sqrt{\sigma_1\sigma_2})}\int_{\partial D^{(4)}} e^{ikx-\omega_1 t}\hat{v}_0^{(2)}\left(k\sqrt{\frac{\sigma_1}{\sigma_2}}\right)\mathrm{d}k, \end{aligned}$$

To find an explicit expression for $v^{(2)}(x,t)$ we need to find expressions for $g_0(\omega_2,t)$ and

$g_1(\omega_2, t)$ valid for $k \in \mathbb{C}^+ \setminus D^{(2)}$. To this end we solve (4.7a) with $k \to k\sqrt{\sigma_2/\sigma_1}$ and (4.8b) for $g_0(\omega_2, t)$ and $g_1(\omega_2, t)$. Substituting into equation (4.10), we obtain

$$\begin{aligned} v^{(2)}(x,t) =& \frac{\rho_1\sqrt{\sigma_1\sigma_2}(\gamma^{(2)}-\gamma^{(1)})}{\rho_2\sigma_1+\rho_1\sqrt{\sigma_2\sigma_1}}\phi(\sigma_2,x,t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{v}_0^{(2)}(k)\,\mathrm{d}k \\ &+ \frac{\rho_1\sigma_2}{\pi\left(\rho_2\sigma_1+\rho_1\sqrt{\sigma_2\sigma_1}\right)}\int_{\partial D^{(2)}} e^{ikx-\omega_2 t}\hat{v}_0^{(1)}\left(k\sqrt{\frac{\sigma_2}{\sigma_1}}\right)\mathrm{d}k \\ &+ \frac{\rho_2\sigma_1-\rho_1\sqrt{\sigma_1\sigma_2}}{2\pi\left(\rho_2\sigma_1+\rho_1\sqrt{\sigma_2\sigma_1}\right)}\int_{\partial D^{(2)}} e^{ikx-\omega_2 t}\hat{v}_0^{(2)}(-k)\,\mathrm{d}k \\ &- \frac{\rho_1\sigma_2}{\pi\left(\rho_2\sigma_1+\rho_1\sqrt{\sigma_2\sigma_1}\right)}\int_{\partial D^{(2)}} e^{ikx}\hat{v}^{(1)}\left(k\sqrt{\frac{\sigma_2}{\sigma_1}},t\right)\mathrm{d}k \\ &- \frac{\rho_2\sigma_1-\rho_1\sqrt{\sigma_1\sigma_2}}{2\pi\left(\rho_2\sigma_1+\rho_1\sqrt{\sigma_2\sigma_1}\right)}\int_{\partial D^{(2)}} e^{ikx}\hat{v}^{(2)}(-k,t)\,\mathrm{d}k, \end{aligned} \tag{4.12}$$

for $x > 0$, $t > 0$. As before, the first four integrals are known. To compute the fifth and sixth integrals we proceed as we did for $v^{(1)}(x,t)$ and eliminate integrals that decay in the regions over which they are integrated. The final solution is

$$\begin{aligned} v^{(2)}(x,t) =& \frac{\rho_1\sqrt{\sigma_1\sigma_2}(\gamma^{(2)}-\gamma^{(1)})}{\rho_2\sigma_1+\rho_1\sqrt{\sigma_2\sigma_1}}\phi(\sigma_2,x,t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{v}_0^{(2)}(k)\,\mathrm{d}k \\ &+ \frac{\rho_1\sigma_2}{\pi\left(\rho_2\sigma_1+\rho_1\sqrt{\sigma_2\sigma_1}\right)}\int_{\partial D^{(2)}} e^{ikx-\omega_2 t}\hat{v}_0^{(1)}\left(k\sqrt{\frac{\sigma_2}{\sigma_1}}\right)\mathrm{d}k \\ &+ \frac{\rho_2\sigma_1-\rho_1\sqrt{\sigma_1\sigma_2}}{2\pi\left(\rho_2\sigma_1+\rho_1\sqrt{\sigma_2\sigma_1}\right)}\int_{\partial D^{(2)}} e^{ikx-\omega_2 t}\hat{v}_0^{(2)}(-k)\,\mathrm{d}k. \end{aligned}$$

Returning to the original variables we have the following proposition which determines $q^{(2)}(x,t)$ and $q^{(1)}(x,t)$ fully explicitly in terms of the given initial conditions and the prescribed boundary conditions as $|x| \to \infty$.

Proposition 4.1. *The solution of the linear Schrödinger problem* (4.1)-(4.4) *is given by*

$$\begin{aligned} q^{(1)}(x,t) =& \gamma^{(1)} + \frac{\rho_2\sigma_1(\gamma^{(2)}-\gamma^{(1)})}{\rho_2\sigma_1+\rho_1\sqrt{\sigma_1\sigma_2}}\phi(\sigma_1,x,t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{v}_0^{(1)}(k)\,\mathrm{d}k \\ &+ \frac{\rho_2\sigma_1-\rho_1\sqrt{\sigma_1\sigma_2}}{2\pi(\rho_2\sigma_1+\rho_1\sqrt{\sigma_1\sigma_2})}\int_{\partial D^{(4)}} e^{ikx-\omega_1 t}\hat{v}_0^{(1)}(-k)\,\mathrm{d}k \\ &- \frac{\rho_2\sigma_1}{\pi(\rho_1\sigma_2+\rho_2\sqrt{\sigma_1\sigma_2})}\int_{\partial D^{(4)}} e^{ikx-\omega_1 t}\hat{v}_0^{(2)}\left(k\sqrt{\frac{\sigma_1}{\sigma_2}}\right)\mathrm{d}k, \end{aligned} \tag{4.13}$$

for $x < 0$ and for $x > 0$,

$$
\begin{aligned}
q^{(2)}(x,t) = &\gamma^{(2)} + \frac{\rho_1\sqrt{\sigma_1\sigma_2}(\gamma^{(2)}-\gamma^{(1)})}{\rho_2\sigma_1+\rho_1\sqrt{\sigma_2\sigma_1}}\phi(\sigma_2,x,t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{v}_0^{(2)}(k)\,\mathrm{d}k \\
&+ \frac{\rho_1\sigma_2}{\pi\left(\rho_2\sigma_1+\rho_1\sqrt{\sigma_2\sigma_1}\right)}\int_{\partial D^{(2)}} e^{ikx-\omega_2 t}\hat{v}_0^{(1)}\left(k\sqrt{\frac{\sigma_2}{\sigma_1}}\right)\mathrm{d}k \\
&+ \frac{\rho_2\sigma_1-\rho_1\sqrt{\sigma_1\sigma_2}}{2\pi\left(\rho_2\sigma_1+\rho_1\sqrt{\sigma_2\sigma_1}\right)}\int_{\partial D^{(2)}} e^{ikx-\omega_2 t}\hat{v}_0^{(2)}(-k)\,\mathrm{d}k.
\end{aligned}
\tag{4.14}
$$

Remarks:

- The use of the discrete symmetries of the functions $\omega_1(k)$ and $\omega_2(k)$ is an important aspect of the Fokas Method. When solving the LS equation in a single medium, the only discrete symmetry required is $k \to -k$, which was used here to obtain (4.8a). Due to the two media, there are two dispersion relations in the present problem: $\omega_1(k) = -i\sigma_1 k^2$ and $\omega_2(k) = -i\sigma_2 k^2$. The collection of both functions $\{\omega_1(k), \omega_2(k)\}$ retains the discrete symmetry $k \to -k$, but admits an additional one, namely: $k \to k\sqrt{\sigma_2/\sigma_1}$ which transforms the two functions to each other. All nontrivial discrete symmetries of $\{\omega_1(k), \omega_2(k)\}$ are needed to derive the final solution representation.

- In Equations (4.13) and (4.14) it is possible to deform the integration paths back to the real line. This deformation hints that a classical solution in terms of Fourier-like integral transforms should be possible. However, a priori it is not clear how to obtain the appropriate transforms for general initial conditions and boundary conditions. In effect, as in [25], the Fokas Method can be seen as a method to construct the appropriate transform to solve the problem.

- It is interesting to note that when $\sigma_1 = \sigma_2$, $\rho_1 = \rho_2$ and $\gamma^{(1)} = \gamma^{(2)} = 0$, the solution formulae in their proper x-domain of definition reduce to the solution of the whole line problem. Also, if $\gamma^{(1)} = 0 = \gamma^{(2)}$, Cascaval and Hunter [11] find a solution to the LS equation with an interface by imposing the solution for the LS problem on the half-line given in [25] and viewing the interface problem as a forced problem on the real line where the forcing is occurring at the interface. This leads to a solution of the interface problem which requires the numerical solution of an integral equation.

- The leading-order behavior in time for (4.1) with initial conditions which decay sufficiently fast to the boundary values (4.3) at $\pm\infty$ is easily obtained by using integration by parts and the method of stationary phase [6]. In the limit as $t \to \infty$ for x/t constant,

$$
\begin{aligned}
q^{(1)}(x,t) \sim{} & \frac{\rho_2\gamma^{(2)}\sqrt{\sigma_1}+\rho_1\gamma^{(1)}\sqrt{\sigma_2}}{\rho_2\sqrt{\sigma_1}+\rho_1\sqrt{\sigma_2}} + \frac{e^{\frac{i\pi}{4}-\frac{ix^2}{4\sigma_1 t}}}{\sqrt{4\sigma_1 t}}\left(\frac{\rho_2\sigma_1(\gamma^{(2)}-\gamma^{(1)})x}{\rho_2\sigma_1+\rho_1\sqrt{\sigma_1\sigma_2}}\right.\\
& +\frac{\hat{v}_0^{(1)}\left(-\frac{x}{2\sigma_1 t}\right)}{\sqrt{\pi}} - \frac{(\rho_2\sigma_1-\rho_1\sqrt{\sigma_1\sigma_2})\hat{v}_0^{(1)}\left(\frac{x}{2\sigma_1 t}\right)}{(\rho_2\sigma_1+\rho_1\sqrt{\sigma_1\sigma_2})\sqrt{\pi}}\\
& \left.+\frac{\rho_2\sigma_1\hat{v}_0^{(2)}\left(-\frac{x}{2t\sqrt{\sigma_1\sigma_2}}\right)}{(\rho_1\sigma_2+\rho_2\sqrt{\sigma_1\sigma_2})\sqrt{\pi}}\right),
\end{aligned}
\tag{4.15}
$$

for $x<0$ and, for $x>0$,

$$
\begin{aligned}
q^{(2)}(x,t) \sim{} & \frac{\rho_2\gamma^{(2)}\sqrt{\sigma_1}+\rho_1\gamma^{(1)}\sqrt{\sigma_2}}{\rho_2\sqrt{\sigma_1}+\rho_1\sqrt{\sigma_2}} + \frac{e^{\frac{i\pi}{4}-\frac{ix^2}{4\sigma_2 t}}}{\sqrt{4\sigma_2 t}}\left(\frac{\rho_1\sqrt{\sigma_1\sigma_2}(\gamma^{(2)}-\gamma^{(1)})x}{\rho_2\sigma_1+\rho_1\sqrt{\sigma_2\sigma_1}}\right.\\
& +\frac{\hat{v}_0^{(2)}\left(-\frac{x}{2\sigma_2 t}\right)}{\sqrt{\pi}} + \frac{\rho_1\sigma_2\hat{v}_0^{(1)}\left(-\frac{x}{2t\sqrt{\sigma_2\sigma_1}}\right)}{\left(\rho_2\sigma_1+\rho_1\sqrt{\sigma_2\sigma_1}\right)\sqrt{\pi}}\\
& \left.+\frac{(\rho_2\sigma_1-\rho_1\sqrt{\sigma_1\sigma_2})\hat{v}_0^{(2)}\left(\frac{x}{2\sigma_2 t}\right)}{\left(\rho_2\sigma_1+\rho_1\sqrt{\sigma_2\sigma_1}\right)\sqrt{\pi}}\right).
\end{aligned}
\tag{4.16}
$$

The constant factor in (4.15) and (4.16) is the weighted average of the boundary conditions at infinity with weights given by $\rho_2\sqrt{\sigma_1}$ and $\rho_1\sqrt{\sigma_2}$. The oscillations are contained in the terms $\exp(-ix^2/(4\sigma_1 t))$ and $\exp(-ix^2/(4\sigma_2 t))$. In Figure 4.4 the envelope of the real (imaginary) part of the solution is plotted in gray (black) as a dot-dashed line. The real part of the solution (plotted as a solid line in blue) is centered around the weighted average (plotted in black as a dotted line) and the imaginary part of the

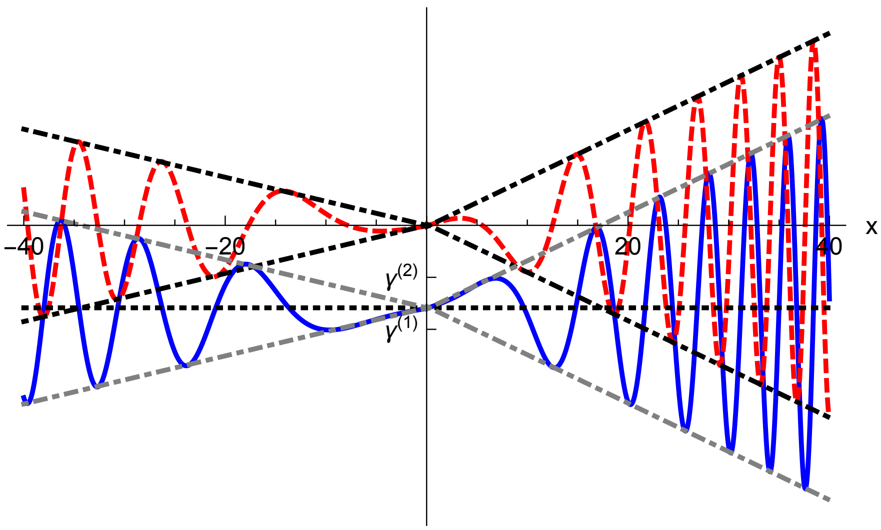

Figure 4.4: The leading order behavior of $q(x,t)$ as given in (4.15) and (4.16) with $t = 10, \gamma^{(1)} = -20, \gamma^{(2)} = -10, \rho_1 = 2, \rho_2 = 1, \sigma_1 = 2$, and $\sigma_2 = 1$ and initial conditions $q_0^{(1)}(x) = \frac{\rho_1\gamma^{(1)}+\rho_2\gamma^{(2)}}{\rho_1+\rho_2} + \rho_2\frac{\gamma^{(2)}-\gamma^{(1)}}{\rho_1+\rho_2}\tanh(x)$ and $q_0^{(2)}(x) = \frac{\rho_1\gamma^{(1)}+\rho_2\gamma^{(2)}}{\rho_1+\rho_2} + \rho_1\frac{\gamma^{(2)}-\gamma^{(1)}}{\rho_1+\rho_2}\tanh(x)$.

solution (plotted as a dashed line in red) is centered around zero. In using the method of stationary phase one must look in directions of constant x/t. Using integration by parts there is no such restriction. When x is large the term from integration by parts is dominant and so in Figure 4.4 there is no need to fix x/t.

- In quantum mechanics one considers only the finite energy case, that is, $\gamma^{(1)} = 0 = \gamma^{(2)}$. In this case, asymptotics requires the use of the method of stationary phase only. Thus, in Figure 4.5 we consider solutions for $x/t = \pm 1$. The real and imaginary parts of the solution are centered around 0. In Figure 4.5 the real part of the solution is plotted as a solid line in blue and the imaginary part of the solution is plotted as a dashed line in red. The envelope of the real (imaginary) part of the solution is plotted in gray (black) as a dot-dashed line.

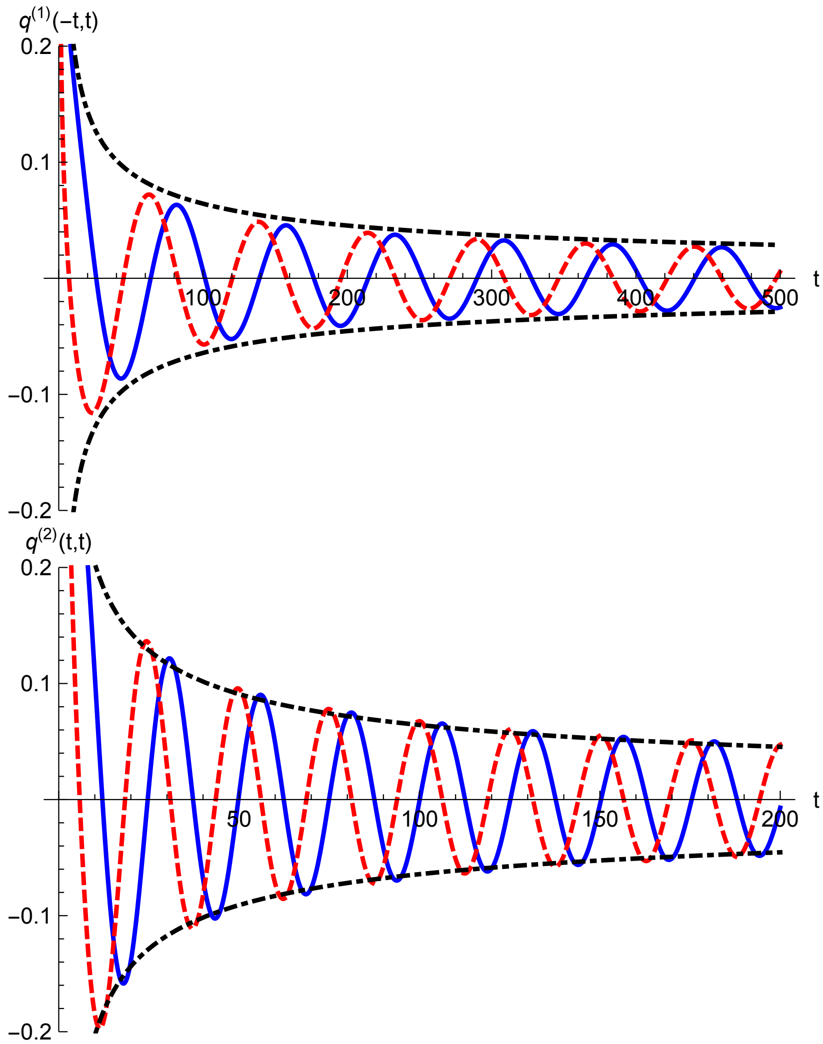

Figure 4.5: The leading order behavior of $q^{(1)}(-t,t)$ and $q^{(2)}(t,t)$ as given in (4.15) and (4.16) respectively with $\gamma^{(1)} = 0, \gamma^{(2)} = 0, \rho_1 = 4, \rho_2 = 1, \sigma_1 = 3,$ and $\sigma_2 = 1$ with $q_0^{(1)}(x) = (1+\rho_2 x)e^{-x^2}$ and $q_0^{(2)}(x) = (1+\rho_1 x)e^{-x^2}$.

4.2 Two finite domains

We wish to find two functions:

$$q^{(1)}(x,t), \quad -x_0 < x < x_1, \;\; t \geq 0,$$

and

$$q^{(2)}(x,t), \quad x_1 < x < x_2, \quad t \geq 0,$$

satisfying the equations

$$iq_t^{(1)}(x,t) = \sigma_1 q_{xx}^{(1)}(x,t) \qquad -x_0 < x < x_1, \; t > 0, \tag{4.17a}$$

$$iq_t^{(2)}(x,t) = \sigma_2 q_{xx}^{(2)}(x,t), \qquad x_1 < x < x_2, t > 0, \tag{4.17b}$$

the initial conditions

$$q^{(1)}(x,0) = q_0^{(1)}(x), \qquad -x_0 < x < x_1, \tag{4.18a}$$

$$q^{(2)}(x,0) = q_0^{(2)}(x), \qquad x_1 < x < x_2, \tag{4.18b}$$

the Robin boundary conditions

$$\beta_1 q^{(1)}(-x_0,t) + \beta_2 q_x^{(1)}(-x_0,t) = f_1(t), \qquad t > 0, \tag{4.19a}$$

$$\beta_3 q^{(2)}(x_2,t) + \beta_4 q_x^{(2)}(x_2,t) = f_2(t), \qquad t > 0, \tag{4.19b}$$

and the continuity interface conditions

$$q^{(1)}(0,t) = q^{(2)}(0,t), \qquad t > 0, \tag{4.20a}$$

$$\rho_1 q_x^{(1)}(0,t) = \rho_2 q_x^{(2)}(0,t), \qquad t > 0, \tag{4.20b}$$

as illustrated in Figure 4.6 where x_0 and x_2 are positive, $x_1 = 0$, and $\sigma_1, \sigma_2, \rho_1, \rho_2$ and β_j, $1 \leq j \leq 4$ are nonzero constants. As before we assume σ_1 and σ_2 are positive for convenience. If $\beta_1 = \beta_3 = 0$ then Neumann boundary conditions are prescribed, whereas if $\beta_2 = \beta_4 = 0$ then Dirichlet conditions are given.

Figure 4.6: The LS equation for two finite domains.

As before we begin with the local relations

$$\left(e^{-ikx+\omega_1 t} q^{(1)}(x,t)\right)_t = \left(\sigma_1 e^{-ikx+\omega_1 t}(kq^{(1)}(x,t) - iq_x^{(1)}(x,t))\right)_x,$$
$$\left(e^{-ikx+\omega_2 t} q^{(1)}(x,t)\right)_t = \left(\sigma_2 e^{-ikx+\omega_2 t}(kq^{(2)}(x,t) - iq_x^{(2)}(x,t))\right)_x.$$

For $k \in \mathbb{C}$ we define the following transforms:

$$\tilde{f}_1(\omega,t) = \int_0^t e^{\omega s} f_1(s)\,\mathrm{d}s, \qquad \tilde{f}_2(\omega,t) = \int_0^t e^{\omega s} f_2(s)\,\mathrm{d}s,$$
$$h_1^{(1)}(\omega,t) = \int_0^t e^{\omega s} q_x^{(1)}(-x_0,s)\,\mathrm{d}s, \qquad h_0^{(1)}(\omega,t) = \int_0^t e^{\omega s} q^{(1)}(-x_0,s)\,\mathrm{d}s,$$
$$h_1^{(2)}(\omega,t) = \int_0^t e^{\omega s} q_x^{(2)}(x_2,s)\,\mathrm{d}s, \qquad h_0^{(2)}(\omega,t) = \int_0^t e^{\omega s} q^{(2)}(x_2,s)\,\mathrm{d}s,$$
$$\hat{q}^{(1)}(k,t) = \int_{-x_0}^0 e^{-ikx} q^{(1)}(x,t)\,\mathrm{d}x, \qquad \hat{q}_0^{(1)}(k) = \int_{-x_0}^0 e^{-ikx} q_0^{(1)}(x)\,\mathrm{d}x,$$
$$\hat{q}^{(2)}(k,t) = \int_0^{x_2} e^{-ikx} q^{(2)}(x,t)\,\mathrm{d}x, \qquad \hat{q}_0^{(2)}(k) = \int_0^{x_2} e^{-ikx} q_0^{(2)}(x)\,\mathrm{d}x,$$
$$g_0(\omega,t) = \int_0^t e^{\omega s} q^{(1)}(0,s)\,\mathrm{d}s = \int_0^t e^{\omega s} q^{(2)}(0,s)\,\mathrm{d}s,$$
$$g_1(\omega,t) = \int_0^t e^{\omega s} q_x^{(1)}(0,s)\,\mathrm{d}s = \frac{\rho_2}{\rho_1}\int_0^t e^{\omega s} q_x^{(2)}(0,s)\,\mathrm{d}s.$$

Applying Green's Theorem in the domains $[-x_0,0]\times[0,t]$ and $[0,x_2]\times[0,t]$ respectively, we find the global relations

$$\begin{aligned} e^{\omega_1 t}\hat{q}^{(1)}(k,t) =& \hat{q}_0^{(1)}(k) + k\sigma_1 g_0(\omega_1,t) - i\sigma_1 g_1(\omega_1,t) \\ & - \sigma_1 e^{ikx_0}(kh_0^{(1)}(\omega_1,t) - ih_1^{(1)}(\omega_1,t)), \end{aligned} \tag{4.21a}$$

$$
\begin{aligned}
e^{\omega_2 t}\hat{q}^{(2)}(k,t) =& \hat{q}_0^{(2)}(k) - k\sigma_2 g_0(\omega_2,t) + \frac{i\sigma_2\rho_1}{\rho_2} g_1(\omega_2,t) \\
& + \sigma_2 e^{-ikx_2}(kh_0^{(2)}(\omega_2,t) - ih_1^{(2)}(\omega_2,t)),
\end{aligned} \tag{4.21b}
$$

which are valid for all $k \in \mathbb{C}$ in contrast to (4.6). Using the invariance of $\omega_1(k)$ and $\omega_2(k)$ under $k \to -k$ we supplement (4.21) with their evaluation at $-k$, namely

$$
\begin{aligned}
e^{\omega_1 t}\hat{q}^{(1)}(-k,t) =& \hat{q}_0^{(1)}(-k) - k\sigma_1 g_0(\omega_1,t) - i\sigma_1 g_1(\omega_1,t) \\
& - \sigma_1 e^{-ikx_0}(-kh_0^{(1)}(\omega_1,t) - ih_1^{(1)}(\omega_1,t)),
\end{aligned} \tag{4.22a}
$$

$$
\begin{aligned}
e^{\omega_2 t}\hat{q}^{(2)}(-k,t) =& \hat{q}_0^{(2)}(-k) + k\sigma_2 g_0(\omega_2,t) + \frac{i\sigma_2\rho_1}{\rho_2} g_1(\omega_2,t) \\
& + \sigma_2 e^{ikx_2}(-kh_0^{(2)}(\omega_2,t) - ih_1^{(2)}(\omega_2,t)).
\end{aligned} \tag{4.22b}
$$

Inverting the Fourier transform in (4.21a) we have

$$
\begin{aligned}
q^{(1)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{q}_0^{(1)}(k)\,\mathrm{d}k \\
& + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\sigma_1(kg_0(\omega_1,t) - ig_1(\omega_1,t))\,\mathrm{d}k \\
& + \frac{1}{2\pi}\int_{-\infty}^{\infty} \sigma_1 e^{ik(x+x_0)-\omega_1 t}(kh_0^{(1)}(\omega_1,t) - ih_1^{(1)}(\omega_1,t))\,\mathrm{d}k,
\end{aligned}
$$

for $-x_0 < x < 0$ and $t > 0$. The integrand of the second integral is entire and decays as $k \to \infty$ for $k \in \mathbb{C}^- \setminus D^{(4)}$. The last integral is entire and decays as $k \to \infty$ for $k \in \mathbb{C}^+ \setminus D^{(2)}$. It is convenient to deform both contours away from the real axis to avoid singularities in the integrands that become apparent in what follows. Initially these singularities are removable since the integrands are entire. Writing integrals of sums as sums of integrals, these singularities may cease to be removable. With the deformation away from the real axis the singularities are no cause for concern. In other words, we deform $D^{(2)}$ to $D_0^{(2)}$ and $D^{(4)}$ to $D_0^{(4)}$ as show in Figure 4.7 where the deformed contours approach the real axis asymptotically. Thus,

$$\begin{aligned} q^{(1)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{q}_0^{(1)}(k)\,\mathrm{d}k \\ &- \frac{1}{2\pi}\int_{\partial D_0^{(4)}} e^{ikx-\omega_1 t}\sigma_1(kg_0(\omega_1,t) - ig_1(\omega_1,t))\,\mathrm{d}k \\ &+ \frac{1}{2\pi}\int_{\partial D_0^{(2)}} \sigma_1 e^{ik(x+x_0)-\omega_1 t}(kh_0^{(1)}(\omega_1,t) - ih_1^{(1)}(\omega_1,t))\,\mathrm{d}k. \end{aligned}$$

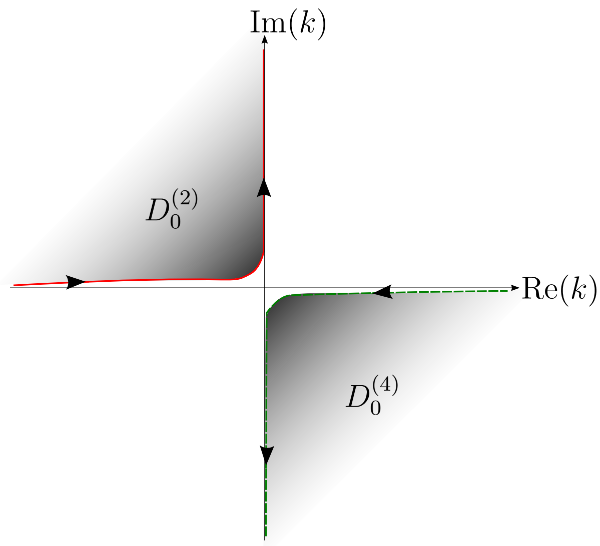

Figure 4.7: Deformation of the contours in Figure 4.2 away from the real axis.

Similarly, inverting the Fourier transform in (4.21b) we have

$$\begin{aligned} q^{(2)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{q}_0^{(2)}(k)\,\mathrm{d}k \\ &+ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\sigma_2\left(-kg_0(\omega_2,t) + \frac{i\rho_1}{\rho_2}g_1(\omega_2,t)\right)\,\mathrm{d}k \\ &+ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x_2)-\omega_2 t}\sigma_2(kh_0^{(2)}(\omega_2,t) - ih_1^{(2)}(\omega_2,t))\,\mathrm{d}k, \end{aligned}$$

for $0 < x < x_2$ and $t > 0$. The integrand of the second integral is entire and decays as $k \to \infty$ for $k \in \mathbb{C}^+ \setminus D^{(2)}$. The integrand of the third integral is entire and decays as $k \to \infty$

for $k \in \mathbb{C}^- \setminus D^{(4)}$. We deform as above to find

$$\begin{aligned} q^{(2)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{q}_0^{(2)}(k)\,\mathrm{d}k \\ &+ \frac{1}{2\pi}\int_{\partial D_0^{(2)}} e^{ikx-\omega_2 t}\sigma_2\left(-kg_0(\omega_2,t) + \frac{i\rho_1}{\rho_2}g_1(\omega_2,t)\right)\mathrm{d}k \\ &- \frac{1}{2\pi}\int_{\partial D_0^{(4)}} e^{ik(x-x_2)-\omega_2 t}\sigma_2(kh_0^{(2)}(\omega_2,t) - ih_1^{(2)}(\omega_2,t))\,\mathrm{d}k. \end{aligned}$$

Taking the time transform of the boundary conditions results in

$$\beta_1 h_0^{(1)}(\omega,t) + \beta_2 h_1^{(1)}(\omega,t) = \tilde{f}_1(\omega,t), \tag{4.23a}$$

and

$$\beta_3 h_0^{(2)}(\omega,t) + \beta_4 h_1^{(2)}(\omega,t) = \tilde{f}_2(\omega,t). \tag{4.23b}$$

To obtain a system of six equations for the six unknown functions $g_0(\omega,t)$, $g_1(\omega,t)$, $h_0^{(1)}(\omega,t)$, $h_1^{(1)}(\omega,t)$, $h_0^{(2)}(\omega,t)$, and $h_1^{(2)}(\omega,t)$ we use the global relations evaluated at k, (4.21), and $-k$, (4.22), and the time transform of the boundary conditions (4.23).

Although we could solve this problem in its full generality, we restrict to the case of Dirichlet boundary conditions ($\beta_2 = \beta_4 = 0$, $\beta_1 = \beta_3 = 1$) to simplify the already cumbersome formulae below. The system is not solvable for $h_1^{(1)}(\omega,t)$ and $h_1^{(2)}(\omega,t)$ if $\Delta_1(k) = 0$, where

$$\begin{aligned} \Delta_1(k) =& 4i\pi\sigma_2 e^{ik\left(x_0 + x_2\sqrt{\frac{\sigma_1}{\sigma_2}}\right)}\left(\rho_1\cos(x_0 k)\sin\left(x_2 k\sqrt{\frac{\sigma_1}{\sigma_2}}\right)\right. \\ &\left. +\rho_2\sqrt{\frac{\sigma_1}{\sigma_2}}\sin(x_0 k)\cos\left(x_2 k\sqrt{\frac{\sigma_1}{\sigma_2}}\right)\right). \end{aligned} \tag{4.24}$$

It is easily seen that all values of k satisfying this relation (including $k = 0$) are on the real line. Thus on the contours, the equations are solved without problem, resulting in the expressions below. As in the previous section, the right-hand sides of these expressions involve $\hat{q}^{(1)}(\cdot,t)$ and $\hat{q}^{(2)}(\cdot,t)$, evaluated at a variety of arguments. All terms with such dependence are written out explicitly below. Terms that depend on known quantities only are contained in $\mathcal{K}^{(1)}(x,t)$ and $\mathcal{K}^{(2)}(x,t)$, the expressions for which are given in Proposition 4.2.

$$\begin{aligned}
q^{(1)}(x,t) =&\mathcal{K}^{(1)}(x,t) + \int_{\partial D_0^{(4)}} \frac{\rho_2\sigma_1}{\Delta_1(k)} e^{ik\left(2x_0+2x_2\sqrt{\frac{\sigma_1}{\sigma_2}}+x\right)} \hat{q}^{(2)}\left(k\sqrt{\frac{\sigma_1}{\sigma_2}},t\right)\,\mathrm{d}k\\
&-\int_{\partial D_0^{(4)}} \frac{\beta_3\sigma_2 e^{ikx}}{2\Delta_1(k)}\left(\rho_1\left(e^{\frac{2ix_2k\sqrt{\sigma_1}}{\sqrt{\sigma_2}}}-1\right)-\rho_2\sqrt{\frac{\sigma_1}{\sigma_2}}\left(e^{\frac{2ix_2k\sqrt{\sigma_1}}{\sqrt{\sigma_2}}}+1\right)\right)\hat{q}^{(1)}(k,t)\,\mathrm{d}k\\
&+\int_{\partial D_0^{(4)}} \frac{\sigma_2 e^{ik(2x_0+x)}}{2\Delta_1(k)}\left(\rho_1\left(e^{\frac{2ix_2k\sqrt{\sigma_1}}{\sqrt{\sigma_2}}}-1\right)-\rho_2\sqrt{\frac{\sigma_1}{\sigma_2}}\left(e^{\frac{2ix_2k\sqrt{\sigma_1}}{\sqrt{\sigma_2}}}+1\right)\right)\hat{q}^{(1)}(-k,t)\,\mathrm{d}k\\
&-\int_{\partial D_0^{(4)}} \frac{\rho_2\sigma_1 e^{ik(2x_0+x)}}{\Delta_1(k)}\hat{q}^{(2)}\left(-k\sqrt{\frac{\sigma_1}{\sigma_2}},t\right)\,\mathrm{d}k\\
&-\int_{\partial D_0^{(2)}} \frac{\sigma_2 e^{ik(2x_0+x)}}{2\Delta_1(k)}\left(\rho_1\left(e^{\frac{2ix_2k\sqrt{\sigma_1}}{\sqrt{\sigma_2}}}-1\right)+\rho_2\sqrt{\frac{\sigma_1}{\sigma_2}}\rho_2\left(e^{\frac{2ix_2k\sqrt{\sigma_1}}{\sqrt{\sigma_2}}}+1\right)\right)\hat{q}^{(1)}(k,t)\,\mathrm{d}k\\
&-\int_{\partial D_0^{(2)}} \frac{\sigma_2 e^{ik(2x_0+x)}}{2\Delta_1(k)}\left(\rho_1\left(e^{\frac{2ix_2k\sqrt{\sigma_1}}{\sqrt{\sigma_2}}}-1\right)-\rho_2\sqrt{\frac{\sigma_1}{\sigma_2}}\rho_2\left(e^{\frac{2ix_2k\sqrt{\sigma_1}}{\sqrt{\sigma_2}}}+1\right)\right)\hat{q}^{(1)}(-k,t)\,\mathrm{d}k\\
&-\int_{\partial D_0^{(2)}} \frac{\rho_2\sigma_1 e^{ik\left(2x_0+2x_2\sqrt{\frac{\sigma_1}{\sigma_2}}+x\right)}}{\Delta_1(k)}\hat{q}^{(2)}\left(k\sqrt{\frac{\sigma_1}{\sigma_2}},t\right)\,\mathrm{d}k\\
&+\int_{\partial D_0^{(2)}} \frac{\rho_2\sigma_1 e^{ik(2x_0+x)}}{\Delta_1(k)}\hat{q}^{(2)}\left(-k\sqrt{\frac{\sigma_1}{\sigma_2}},t\right)\,\mathrm{d}k,
\end{aligned} \tag{4.25}$$

for $-x_0 < x < 0$, $t > 0$ and

$$
\begin{aligned}
q^{(2)}(x,t) =& \mathcal{K}^{(2)}(x,t) + \int_{\partial D_0^{(2)}} \frac{\rho_1 \sigma_2 e^{ikx}}{\Delta_2(k)} \hat{q}^{(1)}\left(k\sqrt{\frac{\sigma_2}{\sigma_1}}, t\right) \mathrm{d}k \\
& - \int_{\partial D_0^{(2)}} \frac{\rho_1 \sigma_2 e^{ik\left(2x_0\sqrt{\frac{\sigma_2}{\sigma_1}}+x\right)}}{\Delta_2(k)} \hat{q}^{(1)}\left(-k\sqrt{\frac{\sigma_2}{\sigma_1}}, t\right) \mathrm{d}k \\
& + \int_{\partial D_0^{(2)}} \frac{\sigma_1 e^{ik(2x_2+x)}}{2\Delta_2(k)} \left(\rho_2\left(e^{\frac{2ix_0k\sqrt{\sigma_2}}{\sqrt{\sigma_1}}} - 1\right) + \rho_2\sqrt{\frac{\sigma_2}{\sigma_1}}\left(e^{\frac{2ix_0k\sqrt{\sigma_2}}{\sqrt{\sigma_1}}} + 1\right)\right) \hat{q}^{(2)}(k,t)\,\mathrm{d}k \\
& - \int_{\partial D_0^{(2)}} \frac{\sigma_1 e^{ikx}}{2\Delta_2(k)} \left(\rho_2\left(e^{\frac{2ix_0k\sqrt{\sigma_2}}{\sqrt{\sigma_1}}} - 1\right) + \rho_1\sqrt{\frac{\sigma_2}{\sigma_1}}\left(e^{\frac{2ix_0k\sqrt{\sigma_2}}{\sqrt{\sigma_1}}} + 1\right)\right) \hat{q}^{(2)}(-k,t)\,\mathrm{d}k \\
& - \int_{\partial D_0^{(4)}} \frac{\rho_1 \sigma_2 e^{ikx}}{\Delta_2(k)} \hat{q}^{(1)}\left(k\sqrt{\frac{\sigma_2}{\sigma_1}}, t\right) \mathrm{d}k \\
& + \int_{\partial D_0^{(4)}} \frac{\rho_1 \sigma_2 e^{ik\left(2x_0\sqrt{\frac{\sigma_2}{\sigma_1}}+x\right)}}{\Delta_2(k)} \hat{q}^{(1)}\left(-k\sqrt{\frac{\sigma_2}{\sigma_1}}, t\right) \mathrm{d}k \\
& + \int_{\partial D_0^{(4)}} \frac{\sigma_1 e^{ikx}}{2\Delta_2(k)} \left(\rho_2\left(e^{\frac{2ix_0k\sqrt{\sigma_2}}{\sqrt{\sigma_1}}} - 1\right) - \rho_1\sqrt{\frac{\sigma_2}{\sigma_1}}\left(e^{\frac{2ix_0k\sqrt{\sigma_2}}{\sqrt{\sigma_1}}} + 1\right)\right) \hat{q}^{(2)}(k,t)\,\mathrm{d}k \\
& + \int_{\partial D_0^{(4)}} \frac{\sigma_1 e^{ikx}}{2\Delta_2(k)} \left(\rho_2\left(e^{\frac{2ix_0k\sqrt{\sigma_2}}{\sqrt{\sigma_1}}} - 1\right) + \rho_1\sqrt{\frac{\sigma_2}{\sigma_1}}\left(e^{\frac{2ix_0k\sqrt{\sigma_2}}{\sqrt{\sigma_1}}} + 1\right)\right) \hat{q}^{(2)}(-k,t)\,\mathrm{d}k,
\end{aligned}
\tag{4.26}
$$

for $0 < x < x_2$, $t > 0$, where

$$
\Delta_2(k) = \sqrt{\frac{\sigma_2}{\sigma_1}} \Delta_1\left(k\sqrt{\frac{\sigma_2}{\sigma_1}}\right).
$$

The integrands written explicitly in (4.25) and (4.26) decay in the regions around whose boundaries they are integrated. Thus, using Jordan's Lemma and Cauchy's Theorem these integrals are shown to vanish. The final solution is given by $\mathcal{K}^{(1)}(x,t)$ and $\mathcal{K}^{(2)}(x,t)$.

Proposition 4.2. *The solution of the linear Schrödinger interface problem* (4.17)-(4.20) *is*

$$
\begin{aligned}
q^{(1)}(x,t) =& \mathcal{K}^{(1)}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx+\omega_1 t}\hat{q}_0^{(1)}(k)\,\mathrm{d}k + \int_{\partial D_0^{(4)}} \frac{\rho_2\sigma_1 e^{ik(2x_0+x)+\omega_1 t}}{\Delta_1(k)}\hat{q}_0^{(1)}\left(-k\sqrt{\frac{\sigma_1}{\sigma_2}}\right)\mathrm{d}k \\
&+ \int_{\partial D_0^{(4)}} \frac{\sigma_2 e^{ikx+\omega_1 t}}{2\Delta_1(k)}\left(\rho_1\left(e^{\frac{2ix_2k\sqrt{\sigma_1}}{\sqrt{\sigma_2}}}-1\right)-\rho_2\sqrt{\frac{\sigma_1}{\sigma_2}}\left(e^{\frac{2ix_2k\sqrt{\sigma_1}}{\sqrt{\sigma_2}}}+1\right)\right)\hat{q}_0^{(1)}(k)\,\mathrm{d}k \\
&- \int_{\partial D_0^{(4)}} \frac{\sigma_2 e^{ik(2x_0+x)+\omega_1 t}}{2\Delta_1(k)}\left(\rho_1\left(e^{\frac{2ix_2k\sqrt{\sigma_1}}{\sqrt{\sigma_2}}}-1\right)-\rho_2\sqrt{\frac{\sigma_1}{\sigma_2}}\left(e^{\frac{2ix_2k\sqrt{\sigma_1}}{\sqrt{\sigma_2}}}+1\right)\right)\hat{q}_0^{(1)}(-k)\,\mathrm{d}k \\
&- \int_{\partial D_0^{(4)}} \frac{\rho_2\sigma_1 e^{ik\left(2x_0+2x_2\sqrt{\frac{\sigma_1}{\sigma_2}}+x\right)+\omega_1 t}}{\Delta_1(k)}\hat{q}_0^{(2)}\left(k\sqrt{\frac{\sigma_1}{\sigma_2}}\right)\mathrm{d}k \\
&- \int_{\partial D_0^{(4)}} \frac{k\sigma_1\sigma_2 e^{ik(x_0+x)+\omega_1 t}}{\Delta_1(k)}\left(\rho_1\left(e^{\frac{2ix_2k\sqrt{\sigma_1}}{\sqrt{\sigma_2}}}-1\right)-\rho_2\sqrt{\frac{\sigma_1}{\sigma_2}}\left(e^{\frac{2ix_2k\sqrt{\sigma_1}}{\sqrt{\sigma_2}}}+1\right)\right)\tilde{f}_1(\omega_1,t)\,\mathrm{d}k \\
&- \int_{\partial D_0^{(4)}} \frac{2k\rho_2\sigma_1\sqrt{\sigma_1\sigma_2}e^{ik\left(2x_0+x_2\sqrt{\frac{\sigma_1}{\sigma_2}}+x\right)+\omega_1 t}}{\Delta_1(k)}\tilde{f}_2(\omega_2,t)\,\mathrm{d}k \\
&+ \int_{\partial D_0^{(2)}} \frac{\sigma_2 e^{ik(2x_0+x)+\omega_1 t}}{2\Delta_1(k)}\left(\rho_1\left(e^{\frac{2ix_2k\sqrt{\sigma_1}}{\sqrt{\sigma_2}}}-1\right)+\rho_2\sqrt{\frac{\sigma_1}{\sigma_2}}\left(e^{\frac{2ix_2k\sqrt{\sigma_1}}{\sqrt{\sigma_2}}}+1\right)\right)\hat{q}_0^{(1)}(k)\,\mathrm{d}k \\
&+ \int_{\partial D_0^{(2)}} \frac{\sigma_2 e^{ik(2x_0+x)+\omega_1 t}}{2\Delta_1(k)}\left(\rho_1\left(e^{\frac{2ix_2k\sqrt{\sigma_1}}{\sqrt{\sigma_2}}}-1\right)-\rho_2\sqrt{\frac{\sigma_1}{\sigma_2}}\left(e^{\frac{2ix_2k\sqrt{\sigma_1}}{\sqrt{\sigma_2}}}+1\right)\right)\hat{q}_0^{(1)}(-k)\,\mathrm{d}k \\
&+ \int_{\partial D_0^{(2)}} \frac{\rho_2\sigma_1 e^{ik\left(2x_0+2x_2\sqrt{\frac{\sigma_1}{\sigma_2}}+x\right)+\omega_1 t}}{\Delta_1(k)}\hat{q}_0^{(2)}\left(k\sqrt{\frac{\sigma_1}{\sigma_2}}\right)\mathrm{d}k \\
&- \int_{\partial D_0^{(2)}} \frac{\rho_2\sigma_1 e^{ik(2x_0+x)+\omega_1 t}}{\Delta_1(k)}\hat{q}_0^{(1)}\left(-k\sqrt{\frac{\sigma_1}{\sigma_2}}\right)\mathrm{d}k \\
&+ \int_{\partial D_0^{(2)}} \frac{k\sigma_1\sigma_2 e^{ik(x_0+x)+\omega_1 t}}{\Delta_1(k)}\left(\rho_1\left(e^{\frac{2ix_2k\sqrt{\sigma_1}}{\sqrt{\sigma_2}}}-1\right)-\rho_2\sqrt{\frac{\sigma_1}{\sigma_2}}\left(e^{\frac{2ix_2k\sqrt{\sigma_1}}{\sqrt{\sigma_2}}}+1\right)\right)\tilde{f}_1(\omega_1,t)\,\mathrm{d}k \\
&+ \int_{\partial D_0^{(2)}} \frac{2k\rho_2\sigma_1\sqrt{\sigma_1\sigma_2 e^{ik\left(2x_0+x_2\sqrt{\frac{\sigma_1}{\sigma_2}}+x\right)+\omega_1 t}}}{\Delta_1(k)}\tilde{f}_2(\omega_2,t)\,\mathrm{d}k,
\end{aligned}
$$

$$
\begin{aligned}
q^{(2)}(x,t) =\mathcal{K}^{(2)} = &\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx+\omega_2 t}\hat{q}_0^{(2)}(k)\,\mathrm{d}k - \int_{\partial D_0^{(2)}} \frac{\rho_1\sigma_2 e^{ikx+\omega_2 t}}{\Delta_2(k)}\hat{q}_0^{(1)}\left(k\sqrt{\frac{\sigma_2}{\sigma_1}}\right)\mathrm{d}k \\
&+ \int_{\partial D_0^{(2)}} \frac{\rho_1\sigma_2 e^{ik(2x_0\sqrt{\frac{\sigma_2}{\sigma_1}}+x)+\omega_2 t}}{\Delta_2(k)}\hat{q}_0^{(1)}\left(-k\sqrt{\frac{\sigma_2}{\sigma_1}}\right)\mathrm{d}k \\
&- \int_{\partial D_0^{(2)}} \frac{\sigma_1 e^{ik(2x_2+x)+\omega_2 t}}{2\Delta_2(k)}\left(\rho_2\left(e^{\frac{2ix_0k\sqrt{\sigma_2}}{\sqrt{\sigma_1}}}-1\right)+\rho_1\sqrt{\frac{\sigma_2}{\sigma_1}}\left(e^{\frac{2ix_0k\sqrt{\sigma_2}}{\sqrt{\sigma_1}}}+1\right)\right)\hat{q}_0^{(2)}(k)\,\mathrm{d}k \\
&+ \int_{\partial D_0^{(2)}} \frac{\sigma_1 e^{ikx+\omega_2 t}}{2\Delta_2(k)}\left(\rho_2\left(e^{\frac{2ix_0k\sqrt{\sigma_2}}{\sqrt{\sigma_1}}}-1\right)+\rho_1\sqrt{\frac{\sigma_2}{\sigma_1}}\left(e^{\frac{2ix_0k\sqrt{\sigma_2}}{\sqrt{\sigma_1}}}+1\right)\right)\hat{q}_0^{(2)}(-k)\,\mathrm{d}k \\
&+ \int_{\partial D_0^{(2)}} \frac{2k\rho_1\sigma_2\sqrt{\sigma_1\sigma_2}e^{ik(x_0\sqrt{\frac{\sigma_2}{\sigma_1}}+x)+\omega_2 t}}{\Delta_2(k)}\tilde{f}_1(\omega_2,t)\,\mathrm{d}k \\
&- \int_{\partial D_0^{(2)}} \frac{k\sigma_1\sigma_2 e^{ik(x_2+x)+\omega_2 t}}{\Delta_2(k)}\left(\rho_2\left(e^{\frac{2ix_0k\sqrt{\sigma_2}}{\sqrt{\sigma_1}}}-1\right)+\rho_1\sqrt{\frac{\sigma_2}{\sigma_1}}\left(e^{\frac{2ix_0k\sqrt{\sigma_2}}{\sqrt{\sigma_1}}}+1\right)\right)\tilde{f}_2(\omega_2,t)\,\mathrm{d}k \\
&+ \int_{\partial D_0^{(4)}} \frac{\rho_1\sigma_2 e^{ikx+\omega_2 t}}{\Delta_2(k)}\hat{q}_0^{(1)}\left(k\sqrt{\frac{\sigma_2}{\sigma_1}}\right)\mathrm{d}k \\
&- \int_{\partial D_0^{(4)}} \frac{\rho_1\sigma_2 e^{ik(2x_0\sqrt{\frac{\sigma_2}{\sigma_1}}+x)+\omega_2 t}}{\Delta_2(k)}\hat{q}_0^{(1)}\left(-k\sqrt{\frac{\sigma_2}{\sigma_1}}\right)\mathrm{d}k \\
&- \int_{\partial D_0^{(4)}} \frac{\sigma_1 e^{ikx+\omega_2 t}}{2\Delta_2(k)}\left(\rho_2\left(e^{\frac{2ix_0k\sqrt{\sigma_2}}{\sqrt{\sigma_1}}}-1\right)-\rho_1\sqrt{\frac{\sigma_2}{\sigma_1}}\left(e^{\frac{2ix_0k\sqrt{\sigma_2}}{\sqrt{\sigma_1}}}+1\right)\right)\hat{q}_0^{(2)}(k)\,\mathrm{d}k \\
&- \int_{\partial D_0^{(4)}} \frac{\sigma_1 e^{ikx+\omega_2 t}}{2\Delta_2(k)}\left(\rho_2\left(e^{\frac{2ix_0k\sqrt{\sigma_2}}{\sqrt{\sigma_1}}}-1\right)+\rho_1\sqrt{\frac{\sigma_2}{\sigma_1}}\left(e^{\frac{2ix_0k\sqrt{\sigma_2}}{\sqrt{\sigma_1}}}+1\right)\right)\hat{q}_0^{(2)}(-k)\,\mathrm{d}k \\
&- \int_{\partial D_0^{(4)}} \frac{2k\rho_1\sigma_2\sqrt{\sigma_1\sigma_2}e^{ik(x_0\sqrt{\frac{\sigma_2}{\sigma_1}}+x)+\omega_2 t}}{\Delta_2(k)}\tilde{f}_1(\omega_2,t)\,\mathrm{d}k \\
&+ \int_{\partial D_0^{(4)}} \frac{k\sigma_1\sigma_2 e^{ik(x+x_2)+\omega_2 t}}{\Delta_2(k)}\left(\rho_2\left(e^{\frac{2ix_0k\sqrt{\sigma_2}}{\sqrt{\sigma_1}}}-1\right)+\rho_1\sqrt{\frac{\sigma_2}{\sigma_1}}\left(e^{\frac{2ix_0k\sqrt{\sigma_2}}{\sqrt{\sigma_1}}}+1\right)\right)\tilde{f}_2(\omega_2,t)\,\mathrm{d}k,
\end{aligned}
$$

for $-x_0 < x < 0$ *and* $0 < x < x_2$ *respectively.*

Remarks:

- The solution of the problem posed in (4.17)-(4.20) may be obtained using the classical method of separation of variables and superposition as was done for the heat equation in [33]. The solutions $q^{(1)}(x,t)$ and $q^{(2)}(x,t)$ are given by a series of eigenfunctions with eigenvalues that satisfy a transcendental equation. The classical series solution may be obtained from the solution in Proposition 4.2 by deforming the contours along $\partial D_0^{(4)}$ and $\partial D_0^{(2)}$ to the real line, including small semi-circles around each root of either $\Delta_1(k)$ or $\Delta_2(k)$, depending on whether $q^{(1)}(x,t)$ or $q^{(2)}(x,t)$ is being calculated. Indeed, careful calculation of all different contributions, following the examples in [15, 25, 64], is allowed since all integrands decay in the wedges between these contours and the real line, and the zeros of $\Delta_1(k)$ and $\Delta_2(k)$ occur only on the real line as stated above. It is not necessarily beneficial to leave the form of the solution in Proposition 4.2 for the series representation, as the latter depends on the roots of $\Delta_1(k)$ and $\Delta_2(k)$, which are not known explicitly. In contrast, the representation of Proposition 4.2 depends on known quantities only and may be readily computed using one's favorite parameterization of the contours $\partial D_0^{(4)}$ and $\partial D_0^{(2)}$.

- In the case of the heat equation on the finite interval with an interface there are also an infinite number of poles on the real-k axis. The major difference here is that the boundary of D coincides with the real-k axis whereas in the heat equation the only intersection between the real axis and D is at $k = 0$.

- As stated earlier, this method applies to general boundary conditions although we chose to present the details only for the Dirichlet case. When genuine Robin boundary conditions are used (the case in which all of the coefficients β_1, β_2, β_3, and β_4 are nonzero) the analogous denominator to (4.24) may have zeros on the interior of $D_0^{(2)}$ and $D_0^{(4)}$ depending on the relative signs of the coefficients. Thus, special care is needed to eliminate unknown boundary values in these cases. This can be worked out in a straightforward way, as is done for problems without interfaces in [15].

- Similar to Section 4.1, long time asymptotics are easily computed using the method of stationary phase [6]. The asymptotic behavior is centered around zero for x/t constant and the shape of the envelope is determined by the integrands of the solution given in

Proposition 4.2 as in (4.15) and (4.16).

Chapter 5

The time-dependent Schrödinger equation with piecewise constant potentials

The N-particle time-dependent (linear) Schrödinger equation is given by

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\sum_{n=1}^{N}\frac{p_n^2}{2m_n} + V(x_1,\dots,x_N,t)\right)\psi. \tag{5.1}$$

Here $\hbar$ is the Planck constant, x_j denotes the 3-dimensional coordinate vector of the j^{th} particle with mass m_j, p_j denotes the momentum operator $i\hbar\nabla_{x_j}$ for the j^{th} particle, and $V(x_1,\dots,x_N,t)$ is the N-particle potential. One can argue that (5.1) is the most important PDE in all of mathematical physics. Standard textbooks such as [19, 41, 48] rightfully emphasize the solution of (5.1) in simplified settings, so as to build up the intuition using exact solutions and their properties. Favorite textbook scenarios consider the one-particle case $N = 1$ in one (1) spatial dimension with time-independent potential $V(x)$. The linear Schrödinger (LS) equation reduces to

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\psi_{xx} + V(x)\psi, \tag{5.2}$$

where m is the particle mass. Since $V(x)$ is time independent, separation of variables $\psi(x,t) = \phi(x)T(t)$ leads to

$$T(t) = T_0 e^{-iEt/\hbar}, \quad -\frac{\hbar^2}{2m}\phi'' + V(x)\phi = E\phi, \tag{5.3}$$

where the energy E is a (real) separation constant. The second equation above is the one-dimensional one-particle time-independent Schrödinger equation. Even at this point, the problem is solvable in closed form in few cases, such as the free particle ($V = 0$) and the harmonic oscillator ($V = kx^2/2$, k constant) [19, 41, 48].

The study of Schrödinger equations with piecewise constant potentials is important for a number of reasons. First, to some extent (see below), analytical solutions are available, allowing the development of more physical intuition using scenarios such as the particle in a box, and the piecewise constant potential barrier [48]. Piecewise constant potentials also provide the simplest example of a periodic potential, using the Kronig-Penney model [48]. Second, multiple-scale perturbation theory [6, 37, 39] shows that the approximation of a complicated potential using a few constant levels results in accurate leading-order behavior, provided the levels are adequately chosen. This is also evident from the Rayleigh-Ritz characterization of the eigenvalues of (5.3) [41, 48], which depends only on weighted averages of the potential. As such, the understanding of (5.2) with piecewise constant potential is of central importance to the study of quantum mechanics. From a physical point of view, the qualitative features of a potential can often be approximated well using a potential which is pieced together from a number of constant parts [31, 48]. For instance, although the forces acting between a proton and a neutron are not accurately known on theoretical grounds, it is known that they are short-range forces, *i.e.*, they extend a short distance, then drop to zero quickly. These forces are well modeled using a piecewise constant potential [48].

Nonetheless, the solutions that are found in the piecewise constant setting are often restricted to single-mode solutions of (5.3), explaining the phenomena of tunneling and trapping [19, 41, 48]. Solutions of the initial-value problem for (5.2) are not readily available. The presence of both discrete and continuous spectrum exacerbates the use of straightforward linear superposition. Extensive discussions of this are found in [45, 46], but even there the required superposition result is not immediately found. The goal of this chapter is to solve the initial-value problem (IVP) for (5.2) using the Fokas Method, combined with more recent ideas generalizing the Fokas Method to allow for the explicit solution of interface problems as in earlier chapters. In what follows we present *explicit, closed-form* solutions of the IVP for (5.2) with square integrable initial data.

We apply the same techniques as in previous chapters to the IVP consisting of (5.2) with $\psi(x,0)=\psi_0(x)\in L_2(\mathbb{R})$. We regard this problem as an interface problem with interfaces located at the discontinuities of the potential $V(x)$. The wave function $\psi(x,t)$ and its derivative $\psi_x(x,t)$ are assumed to be continuous across the interfaces. The first condition is a requirement following from the probabilistic interpretation of the wave function, while the second condition follows from integrating the equation across an interface and allowing the length of the integration interval to limit to zero [48]. For simplicity, the independent variables occurring in (5.2) are rescaled so that, in effect, we may equate $m=1$, $\hbar=1$. Thus in what follows, we consider

$$i\frac{\partial\psi}{\partial t}=-\psi_{xx}+V(x)\psi, \tag{5.4}$$

where $V(x)$ is a piecewise constant potential.

5.1 A step potential

We wish to solve the classical IVP

$$i\psi_t=-\psi_{xx}+\alpha(x)\psi, \qquad -\infty<x<\infty, \tag{5.5a}$$

$$\psi(x,0)=\psi_0(x), \qquad -\infty<x<\infty, \tag{5.5b}$$

where

$$\alpha(x) = \begin{cases} \alpha_1, & x < 0, \\ \alpha_2, & x > 0, \end{cases} \tag{5.6}$$

and $\lim_{x\to\pm\infty}\psi(x,t) = 0$. We treat this as an interface problem solved by

$$\psi(x,t) = \begin{cases} \psi^{(1)}(x,t), & x < 0, \\ \psi^{(2)}(x,t), & x > 0, \end{cases} \tag{5.7}$$

where $\psi^{(1)}(x,t)$ and $\psi^{(2)}(x,t)$ solve

$$i\psi_t^{(1)} = -\psi_{xx}^{(1)} + \alpha_1\psi^{(1)}, \qquad x < 0, \tag{5.8a}$$

$$i\psi_t^{(2)} = -\psi_{xx}^{(2)} + \alpha_2\psi^{(2)}, \qquad x > 0, \tag{5.8b}$$

with initial conditions

$$\psi^{(1)}(x,0) = \psi_0^{(1)}(x), \qquad x < 0, \tag{5.9a}$$

$$\psi^{(2)}(x,0) = \psi_0^{(2)}(x), \qquad x > 0, \tag{5.9b}$$

and the interface continuity conditions

$$\psi^{(1)}(0,t) = \psi^{(2)}(0,t), \qquad t > 0, \tag{5.10a}$$

$$\psi_x^{(1)}(0,t) = \psi_x^{(2)}(0,t), \qquad t > 0, \tag{5.10b}$$

as in Figure 5.1.

We follow the standard steps in the application of the Fokas Method and begin with the local relations:

$$(e^{-ikx+\omega_1 t}\psi^{(1)})_t = (e^{-ikx+\omega_1 t}(i\psi_x^{(1)} - k\psi^{(1)}))_x, \qquad x < 0, \tag{5.11a}$$

$$(e^{-ikx+\omega_2 t}\psi^{(2)})_t = (e^{-ikx+\omega_2 t}(i\psi_x^{(2)} - k\psi^{(2)}))_x, \qquad x > 0, \tag{5.11b}$$

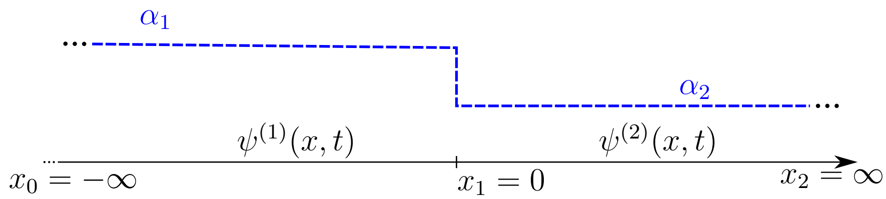

Figure 5.1: A cartoon of the potential $\alpha(x)$ in the case of one interface.

where $\omega_j(k) = i(\alpha_j + k^2)$ for $j = 1, 2$. Note that, as is common in the Fokas Method, the ω_j differ from the standard convention for dispersion relations by a factor of i. Thus for dispersive problems ω_j is purely imaginary. Integrating over the strips $(-\infty, 0) \times (0, t)$ and $(0, \infty) \times (0, t)$ respectively (see Figure 2.2), and applying Green's Theorem, we have the global relations

$$\int_{-\infty}^{0} e^{-ikx+\omega_1 t}\psi^{(1)}(x,t)\,\mathrm{d}x = \int_{-\infty}^{0} e^{-ikx}\psi_0^{(1)}(x)\,\mathrm{d}x + \int_0^t e^{\omega_1 s}(i\psi_x^{(1)}(0,s) - k\psi^{(1)}(0,s))\,\mathrm{d}s, \quad (5.12a)$$

$$\int_{0}^{\infty} e^{-ikx+\omega_2 t}\psi^{(2)}(x,t)\,\mathrm{d}x = \int_{0}^{\infty} e^{-ikx}\psi_0^{(2)}(x)\,\mathrm{d}x - \int_0^t e^{\omega_2 s}(i\psi_x^{(2)}(0,s) - k\psi^{(2)}(0,s))\,\mathrm{d}s. \quad (5.12b)$$

We define the following transforms:

$$\begin{aligned}
\hat{\psi}^{(1)}(k,t) &= \int_{-\infty}^{0} e^{-ikx}\psi^{(1)}(x,t)\,\mathrm{d}x, && x<0, \quad t>0,\ \mathrm{Im}(k)>0,\\
\hat{\psi}_0^{(1)}(k) &= \int_{-\infty}^{0} e^{-ikx}\psi_0^{(1)}(x)\,\mathrm{d}x, && x<0, \quad \mathrm{Im}(k)>0,\\
\hat{\psi}^{(2)}(k,t) &= \int_{0}^{\infty} e^{-ikx}\psi^{(2)}(x,t)\,\mathrm{d}x, && x>0, \quad t>0,\ \mathrm{Im}(k)<0,\\
\hat{\psi}_0^{(2)}(k) &= \int_{0}^{\infty} e^{-ikx}\psi_0^{(2)}(x)\,\mathrm{d}x, && x>0, \quad \mathrm{Im}(k)<0,\\
g_0(\omega,t) &= \int_0^t e^{\omega s}\psi^{(1)}(0,s)\,\mathrm{d}s = \int_0^t e^{\omega s}\psi^{(2)}(0,s)\,\mathrm{d}s, && t>0,\ \omega\in\mathbb{C},\\
g_1(\omega,t) &= \int_0^t e^{\omega s}\psi_x^{(1)}(0,s)\,\mathrm{d}s = \int_0^t e^{\omega s}\psi_x^{(2)}(0,s)\,\mathrm{d}s, && t>0,\ \omega\in\mathbb{C},
\end{aligned}$$

where in the last two definitions we used the continuity conditions (5.10). With these definitions the global relations become

$$e^{\omega_1 t}\hat{\psi}^{(1)}(k,t) = \hat{\psi}_0^{(1)}(k) + ig_1(\omega_1,t) - kg_0(\omega_1,t), \qquad k \in \mathbb{C}^+, \tag{5.13a}$$

$$e^{\omega_2 t}\hat{\psi}^{(2)}(k,t) = \hat{\psi}_0^{(2)}(k) - ig_1(\omega_2,t) + kg_0(\omega_2,t), \qquad k \in \mathbb{C}^-. \tag{5.13b}$$

We wish to transform the global relations so that $g_0(\cdot,t)$ and $g_1(\cdot,t)$ depend on a common argument, $-ik^2$ as in [4, 47]. To this end, let

$$\nu^{(j)}(k) = ik(1+\alpha_j/k^2)^{1/2}.$$

In this chapter we define $z^{1/2}$ as follows. Given $z \in \mathbb{C}$, let $z = re^{i\theta}$ where $\theta = \theta_p + 2\pi n$ for $n \in \mathbb{Z}$ and $-\pi < \theta_p \leq \pi$. Then, $z^{1/2} = \sqrt{r}e^{i\theta_p/2}e^{in\pi}$. Thus, for a given value of z, $z^{1/2}$ takes two possible values corresponding to even or odd n. These two values are denoted $\sqrt{z} = \sqrt{r}e^{i\theta_p/2}$ and $-\sqrt{z} = -\sqrt{r}e^{i\theta_p/2}$. Thus, we have

$$\nu^{(j)}(k) = ik\sqrt{1+\frac{\alpha_j}{k^2}}, \quad \nu^{(j)}(-k) = -\nu^{(j)}(k),$$

which make up a two-sheeted expression with branch points at $\pm i\sqrt{\alpha_j}$ leading to branch cuts in the complex k plane along $[-i\sqrt{\alpha_1}, i\sqrt{\alpha_1}]$ and $[-i\sqrt{\alpha_2}, i\sqrt{\alpha_2}]$. These cuts are on the real or imaginary axis, depending on the signs of α_1 and α_2. Using the transformations $k \to \nu^{(j)}(\pm k)$, $j = 1, 2$, we have the transformed global relations

$$e^{-ik^2 t}\hat{\psi}^{(1)}\left(\nu^{(1)}(k),t\right) = \hat{\psi}_0^{(1)}\left(\nu^{(1)}(k)\right) + ig_1(-ik^2,t) - \nu^{(1)}(k)g_0(-ik^2,t), \tag{5.14a}$$

$$e^{-ik^2 t}\hat{\psi}^{(1)}\left(\nu^{(1)}(-k),t\right) = \hat{\psi}_0^{(1)}\left(\nu^{(1)}(-k)\right) + ig_1(-ik^2,t) - \nu^{(1)}(-k)g_0(-ik^2,t), \tag{5.14b}$$

$$e^{-ik^2 t}\hat{\psi}^{(2)}\left(\nu^{(2)}(k),t\right) = \hat{\psi}_0^{(2)}\left(\nu^{(2)}(k)\right) - ig_1(-ik^2,t) + \nu^{(2)}(k)g_0(-ik^2,t), \tag{5.14c}$$

$$e^{-ik^2 t}\hat{\psi}^{(2)}\left(\nu^{(2)}(-k),t\right) = \hat{\psi}_0^{(2)}\left(\nu^{(2)}(-k)\right) - ig_1(-ik^2,t) + \nu^{(2)}(-k)g_0(-ik^2,t), \tag{5.14d}$$

where $\mathrm{Re}(k) > 0$ in (5.14a) and (5.14d) and $\mathrm{Re}(k) < 0$ in (5.14b) and (5.14c). To determine the regions of validity of (5.14), one should note that $\mathrm{sgn}(\mathrm{Re}(-i\nu^{(j)}(\pm k))) = \mathrm{sgn}(\mathrm{Im}(\nu^{(j)}(\pm k))) = \pm\,\mathrm{sgn}(\mathrm{Re}(k))$.

Inverting the Fourier transform in (5.13) we have the solution formulae

$$\psi^{(1)}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{\psi}_0^{(1)}(k)\,\mathrm{d}k + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\left(ig_1(\omega_1,t) - kg_0(\omega_1,t)\right)\,\mathrm{d}k, \tag{5.15a}$$
$$\psi^{(2)}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{\psi}_0^{(2)}(k)\,\mathrm{d}k - \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\left(ig_1(\omega_2,t) - kg_0(\omega_2,t)\right)\,\mathrm{d}k, \tag{5.15b}$$

for $x < 0$ and $x > 0$ respectively. Examining the integrands in the formulae above we see we can deform farther into the complex plane as follows:

$$\psi^{(1)}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{\psi}_0^{(1)}(k)\,\mathrm{d}k - \frac{1}{2\pi}\int_{\partial D_R^{(3)}} e^{ikx-\omega_1 t}\left(ig_1(\omega_1,t) - kg_0(\omega_1,t)\right)\,\mathrm{d}k, \tag{5.16a}$$
$$\psi^{(2)}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{\psi}_0^{(2)}(k)\,\mathrm{d}k - \frac{1}{2\pi}\int_{\partial D_R^{(1)}} e^{ikx-\omega_2 t}\left(ig_1(\omega_2,t) - kg_0(\omega_2,t)\right)\,\mathrm{d}k, \tag{5.16b}$$

where

$$D_R^{(j)} = \{k \in D^{(j)} : |k| > R\}, \tag{5.17}$$

with $D^{(j)}$ the j^{th} quadrant of the complex plane. The regions $D_R^{(j)}$ for $j = 1, 2, 3, 4$ are as shown in Figure 5.2 where $\Lambda = \max_l\{|\alpha_l|\}$ and $R > \sqrt{2\Lambda}$ is a sufficiently large constant. The reason for integrating around $D_R^{(j)}$ rather than $D^{(j)}$ in (5.16) is to avoid singularities in what follows.

Next we let $k = \nu^{(j)}(\kappa)$ when integrating around $D_R^{(1)}$ and $k = \nu^{(j)}(-\kappa)$ when integrating around $D_R^{(3)}$ so that $g_0(\cdot,t)$ and $g_1(\cdot,t)$ have a common argument and all integrals with unknown terms are integrated around $D_R^{(4)}$. That is, (5.16) becomes

$$\begin{aligned}\psi^{(1)}(x,t) = &\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{\psi}_0^{(1)}(k)\,\mathrm{d}k \\ &- \frac{1}{2\pi}\int_{\partial D_R^{(4)}} e^{i\nu^{(1)}(-\kappa)x+i\kappa^2 t}\left(\frac{i\kappa}{\nu^{(1)}(\kappa)}g_1(-i\kappa^2,t) + \kappa g_0(-i\kappa^2,t)\right)\,\mathrm{d}\kappa,\end{aligned} \tag{5.18a}$$

$$\begin{aligned}\psi^{(2)}(x,t) = &\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{\psi}_0^{(2)}(k)\,\mathrm{d}k \\ &+ \frac{1}{2\pi}\int_{\partial D_R^{(4)}} e^{i\nu^{(2)}(\kappa)x+i\kappa^2 t}\left(\frac{i\kappa}{\nu^{(2)}(\kappa)}g_1(-i\kappa^2,t) - \kappa g_0(-i\kappa^2,t)\right)\,\mathrm{d}\kappa.\end{aligned} \tag{5.18b}$$

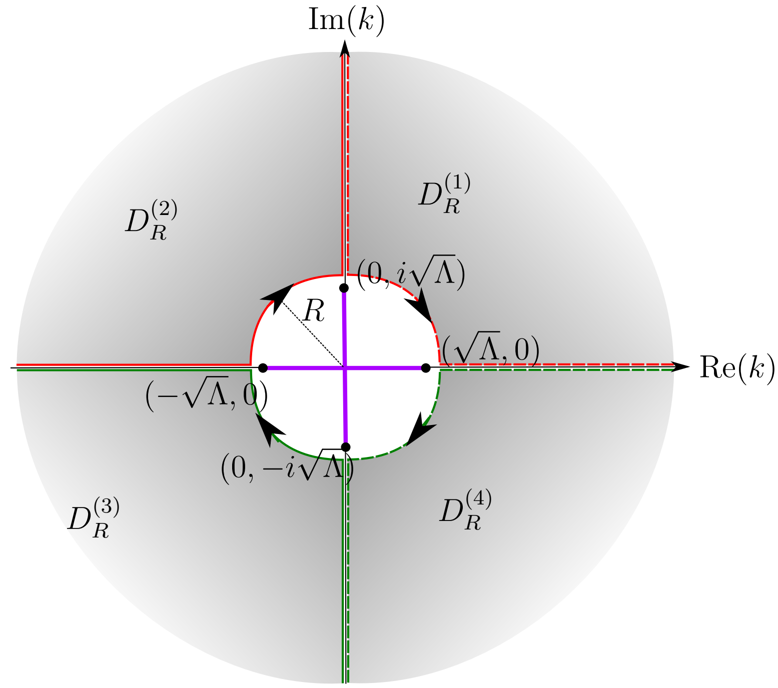

Figure 5.2: The regions $D_R^{(j)}$, $j = 1, 2, 3, 4$.

Using the transformed global relations (5.14a) and (5.14d) valid in $D^{(4)}$ one solves for $g_0(-i\kappa^2, t)$ and $g_1(-i\kappa^2, t)$. Noticing that $\nu^{(j)}(-\kappa) = -\nu^{(j)}(\kappa)$ we denote $\nu^{(j)}(\kappa)$ by $\nu^{(j)}$. In the remainder of this section the argument of all $\nu^{(j)}$ is κ. Substituting these into (5.18) one finds

$$\begin{aligned}
\psi^{(1)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{\psi}_0^{(1)}(k)\,\mathrm{d}k - \int_{\partial D_R^{(4)}} \frac{\kappa(\nu^{(1)}-\nu^{(2)})}{2\pi\nu^{(1)}(\nu^{(1)}+\nu^{(2)})} e^{-i\nu^{(1)}x+i\kappa^2 t}\hat{\psi}_0^{(1)}\left(\nu^{(1)}\right)\,\mathrm{d}\kappa \\
&- \int_{\partial D_R^{(4)}} \frac{\kappa}{\pi(\nu^{(1)}+\nu^{(2)})} e^{-i\nu^{(1)}x+i\kappa^2 t}\hat{\psi}_0^{(2)}\left(-\nu^{(2)}\right)\,\mathrm{d}\kappa \\
&+ \int_{\partial D_R^{(4)}} \frac{\kappa(\nu^{(1)}-\nu^{(2)})}{2\pi\nu^{(1)}(\nu^{(1)}+\nu^{(2)})}\left(\nu^{(2)}-\nu^{(1)}\right) e^{-i\nu^{(1)}x}\hat{\psi}^{(1)}\left(\nu^{(1)},t\right)\,\mathrm{d}\kappa \\
&- \int_{\partial D_R^{(4)}} \frac{\kappa\nu^{(1)}}{\pi(\nu^{(1)}+\nu^{(2)})} e^{-i\nu^{(1)}x}\hat{\psi}^{(2)}\left(-\nu^{(2)},t\right)\,\mathrm{d}\kappa,
\end{aligned} \tag{5.19}$$

for $x < 0$. Similarly,

$$\begin{aligned}\psi^{(2)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{\psi}_0^{(2)}(k)\,\mathrm{d}k - \int_{\partial D_R^{(4)}} \frac{\kappa}{\pi\left(\nu^{(1)}+\nu^{(2)}\right)}\nu^{(2)} e^{i\nu^{(2)}x+i\kappa^2 t}\hat{\psi}_0^{(1)}\left(\nu^{(1)}\right)\,\mathrm{d}\kappa \\ &+ \int_{\partial D_R^{(4)}} \frac{\kappa\left(\nu^{(1)}-\nu^{(2)}\right)}{2\pi\nu^{(2)}\left(\nu^{(1)}+\nu^{(2)}\right)} e^{i\nu^{(2)}x+i\kappa^2 t}\hat{\psi}_0^{(2)}\left(-\nu^{(2)}\right)\,\mathrm{d}\kappa \\ &+ \int_{\partial D_R^{(4)}} \frac{\kappa}{\pi\left(\nu^{(1)}+\nu^{(2)}\right)} e^{i\nu^{(2)}x}\hat{\psi}^{(1)}\left(\nu^{(1)},t\right)\,\mathrm{d}\kappa \\ &- \int_{\partial D_R^{(4)}} \frac{\kappa\left(\nu^{(1)}-\nu^{(2)}\right)}{2\pi\nu^{(2)}\left(\nu^{(1)}+\nu^{(2)}\right)} e^{i\nu^{(2)}x}\hat{\psi}^{(2)}\left(-\nu^{(2)},t\right)\,\mathrm{d}\kappa,\end{aligned} \tag{5.20}$$

for $x > 0$. The first three terms in the Expressions (5.19) and (5.20) depend only on known functions. The last two terms in (5.19) and (5.20) are analytic for $\mathrm{Re}(\kappa) > 0$. Note that $\hat{\psi}^{(1)}\left(\nu^{(1)},t\right)$ and $\hat{\psi}^{(2)}\left(-\nu^{(2)},t\right)$ decay exponentially fast for $|\kappa| \to \infty$ when $\mathrm{Re}(\kappa) > 0$. Thus, by Jordan's Lemma, the integrals of $\exp(-i\nu^{(1)}x)\hat{\psi}^{(1)}\left(\nu^{(1)},t\right)$ and $\exp(-i\nu^{(1)}x)\hat{\psi}^{(2)}\left(-\nu^{(2)},t\right)$ along a closed, bounded curve in the right-half of the complex κ plane vanish for $x < 0$. In particular we consider the closed curve $\mathcal{L}^{(4)} = \mathcal{L}_{D^{(4)}} \cup \mathcal{L}_C^{(4)}$ where $\mathcal{L}_{D^{(4)}} = \partial D_R^{(4)} \cap \{\kappa : |\kappa| < C\}$ and $\mathcal{L}_C^{(4)} = \{\kappa \in D_R^{(4)} : |\kappa| = C\}$, see Figure 5.3.

Since the integral along $\mathcal{L}_C$ vanishes for large C, the fourth and fifth integrals on the right-hand side of (5.19) must vanish since the contour $\mathcal{L}_{D^{(4)}}$ becomes $\partial D^{(4)}$ as $C \to \infty$. For the final two integrals in Equation (5.20) we use the fact that for $x > 0$ the integrals of $\exp(i\nu^{(2)}x)\hat{\psi}^{(1)}\left(\nu^{(1)},t\right)$ and $\exp(i\nu^{(2)}x)\hat{\psi}^{(2)}\left(-\nu^{(2)},t\right)$ along a closed, bounded curve in the right-half of the complex κ plane vanish. Thus, we have an explicit representation for $\psi^{(1)}(x,t)$ in terms of only initial conditions:

$$\begin{aligned}\psi^{(1)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{\psi}_0^{(1)}(k)\,\mathrm{d}k \\ &- \int_{\partial D_R^{(4)}} \frac{\kappa\left(\nu^{(1)}-\nu^{(2)}\right)}{2\pi\nu^{(1)}\left(\nu^{(1)}+\nu^{(2)}\right)} e^{-i\nu^{(1)}x+i\kappa^2 t}\hat{\psi}_0^{(1)}\left(\nu^{(1)}\right)\,\mathrm{d}\kappa \\ &- \int_{\partial D_R^{(4)}} \frac{\kappa}{\pi\left(\nu^{(1)}+\nu^{(2)}\right)} e^{-i\nu^{(1)}x+i\kappa^2 t}\hat{\psi}_0^{(2)}\left(-\nu^{(2)}\right)\,\mathrm{d}\kappa,\end{aligned} \tag{5.21}$$

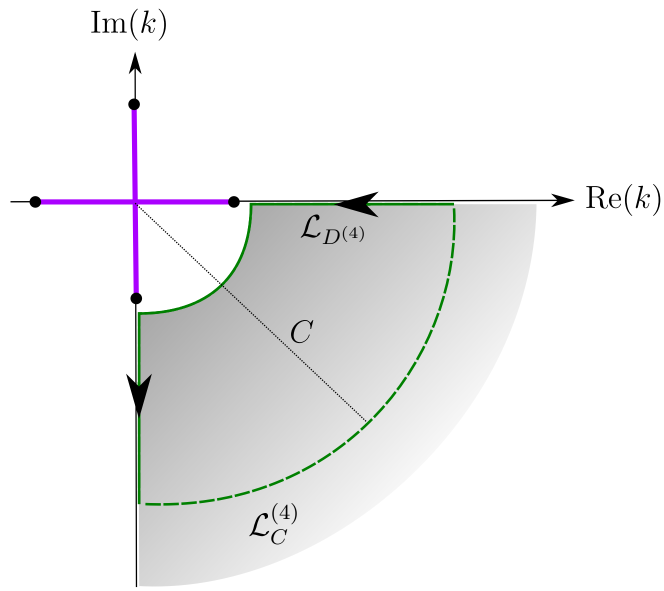

Figure 5.3: The contour $\mathcal{L}_{D^{(4)}}$ is shown as a green solid line and the contour $\mathcal{L}_C$ is shown as a green dashed line. An application of Cauchy's Integral Theorem using this contour shows that the contribution of $\hat{\psi}^{(1)}(\nu^{(1)}, t)$ and $\hat{\psi}^{(2)}(-\nu^{(2)}, t)$ vanishes from the integral expressions (5.19) and (5.20).

for $x < 0$, and

$$\begin{aligned}\psi^{(2)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{\psi}_0^{(2)}(k)\,\mathrm{d}k \\ &- \int_{\partial D_R^{(4)}} \frac{\kappa\nu^{(2)}}{\pi\left(\nu^{(1)}+\nu^{(2)}\right)} e^{i\nu^{(2)}x+i\kappa^2 t}\hat{\psi}_0^{(1)}\left(\nu^{(1)}\right)\,\mathrm{d}\kappa \\ &+ \int_{\partial D_R^{(4)}} \frac{\kappa\left(\nu^{(1)}-\nu^{(2)}\right)}{2\pi\nu^{(2)}\left(\nu^{(1)}+\nu^{(2)}\right)} e^{i\nu^{(2)}x+i\kappa^2 t}\hat{\psi}_0^{(2)}\left(-\nu^{(2)}\right)\,\mathrm{d}\kappa,\end{aligned} \tag{5.22}$$

for $x > 0$. Note that the denominators in (5.21) and (5.22) are zero at the branch points $\kappa = \pm i\sqrt{\alpha_j}$. However, these points are avoided by integrating over the boundary of $D_R^{(4)}$.

The expressions (5.21) and (5.22) provide fully explicit solutions for the IVP (5.5). They are written in a form containing more familiar exponents by letting $\kappa = ik\sqrt{1+\alpha_1/k^2}$ in the second and third integrals of (5.21) and $\kappa = -ik\sqrt{1+\alpha_2/k^2}$ in the second and third integrals of (5.22). Then

$$\begin{aligned}\psi^{(1)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{\psi}_0^{(1)}(k)\,\mathrm{d}k \\ &- \int_{\partial D_R^{(3)}} \frac{1-\sqrt{1+\frac{\alpha_1-\alpha_2}{k^2}}}{2\pi\left(1+\sqrt{1+\frac{\alpha_1-\alpha_2}{k^2}}\right)} e^{ikx-\omega_1 t}\hat{\psi}_0^{(1)}\left(-k\right)\,\mathrm{d}k \\ &- \int_{\partial D_R^{(3)}} \frac{1}{\pi\left(1+\sqrt{1+\frac{\alpha_1-\alpha_2}{k^2}}\right)} e^{ikx-\omega_1 t}\hat{\psi}_0^{(2)}\left(k\sqrt{1+\frac{\alpha_1-\alpha_2}{k^2}}\right)\,\mathrm{d}k,\end{aligned} \tag{5.23a}$$

for $x<0$, and

$$\begin{aligned}\psi^{(2)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{\psi}_0^{(2)}(k)\,\mathrm{d}k \\ &+ \int_{\partial D_R^{(1)}} \frac{1}{\pi\left(1+\sqrt{1-\frac{\alpha_1-\alpha_2}{k^2}}\right)} e^{ikx-\omega_2 t}\hat{\psi}_0^{(2)}\left(k\sqrt{1-\frac{\alpha_1-\alpha_2}{k^2}}\right)\,\mathrm{d}k \\ &+ \int_{\partial D_R^{(1)}} \frac{1-\sqrt{1-\frac{\alpha_1-\alpha_2}{k^2}}}{2\pi\left(1+\sqrt{1-\frac{\alpha_1-\alpha_2}{k^2}}\right)} e^{ikx-\omega_2 t}\hat{\psi}_0^{(1)}\left(-k\right)\,\mathrm{d}k,\end{aligned} \tag{5.23b}$$

for $x>0$. It appears that our solution depends on an extra parameter R. However, observe that $\int_{\partial D_R^{(3)}}\cdot\,\mathrm{d}k = \int_{\partial D_{\tilde{R}}^{(3)}}\cdot\,\mathrm{d}k + \oint_{\mathcal{R}}$ where $\partial D_R^{(3)}$, $\partial D_{\tilde{R}}^{(3)}$ and $\mathcal{R}$ are as in Figure 5.4. Since the integrands in (5.23a) are analytic in $D_R^{(3)}$ (and therefore $\mathcal{R}$), $\oint_{\mathcal{R}}\cdot\,\mathrm{d}k = 0$. Hence, $\int_{\partial D_R}\cdot\,\mathrm{d}k = \int_{\partial D_{\tilde{R}}}\cdot\,\mathrm{d}k$ for any $R>\Lambda$, and our solution is independent of the value of R chosen. The same argument is true for (5.23b).

It is useful to deform the contours in (5.23) back to the real line in order to do asymptotic analysis. We examine the branch cut introduced in (5.23) of the form $\sqrt{1+\frac{a}{k^2}}$. In (5.23) $a=\alpha_2-\alpha_1$ but it may be different in later sections. If $a>0$ the branch points are at $\pm i\sqrt{a}$. We fix the branch cut to be on the finite imaginary axis running from $-i\sqrt{a}$ to $-i\sqrt{a}$ by defining the local polar coordinates

$$k-i\sqrt{a} = r_1 e^{i\theta_1},$$

$$k+i\sqrt{a} = r_2 e^{i\theta_2},$$

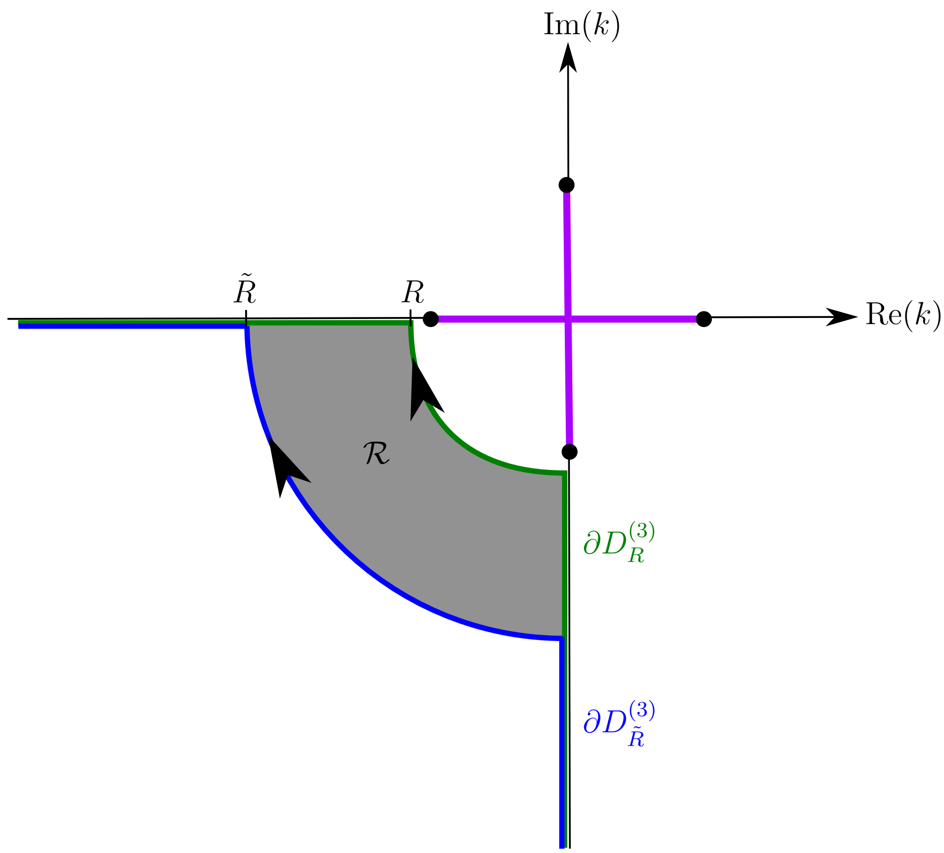

Figure 5.4: The contours $\partial D_R^{(3)}$ and $\partial D_{\tilde{R}}^{(3)}$ and the region $\mathcal{R}$.

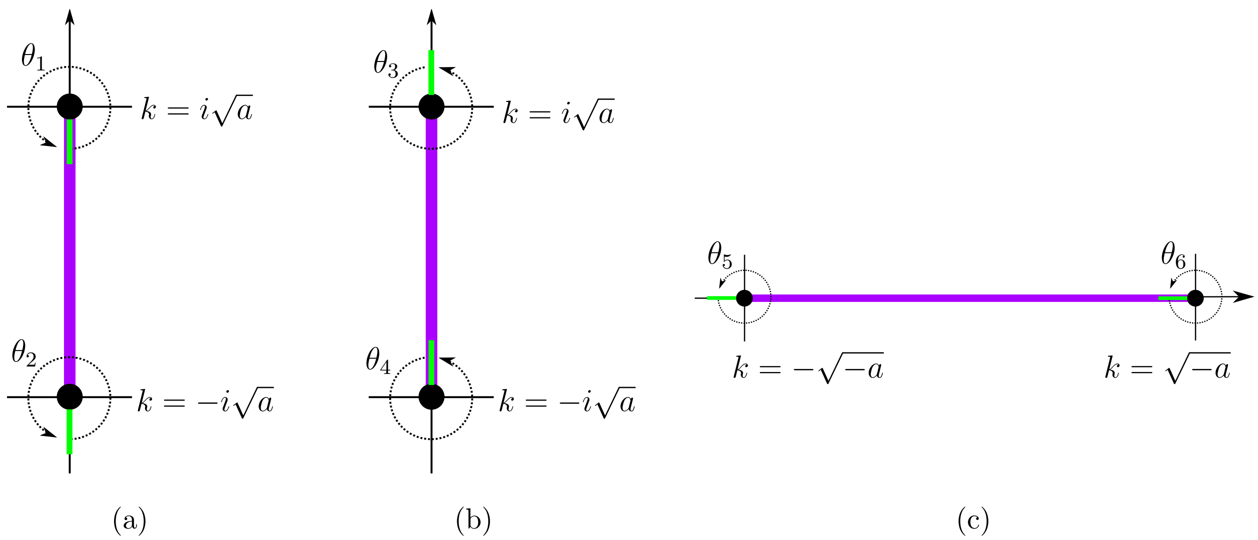

Figure 5.5: Branch cuts for $\sqrt{1+\frac{a}{k^2}}$ and the local parameterizations around the branch points. In (a), $a > 0$ and the local parameterization around the branch points $\pm i\sqrt{a}$ with $-\pi/2 < \theta_1, \theta_2 \leq 3\pi/2$. In (b), $a > 0$ and the local parameterization around the branch points $\pm i\sqrt{a}$ with $-3\pi/2 < \theta_3, \theta_4 \leq \pi/2$. In (c) $a < 0$ and the local parameterization around the branch points $\pm\sqrt{-a}$ with $-\pi < \theta_5, \theta_6 \leq \pi$.

where $-\pi/2 < \theta_1, \theta_2 \leq 3\pi/2$ as in Figure 5.5a or $-3\pi/2 < \theta_3, \theta_4 \leq \pi/2$ as in Figure 5.5b. Similarly, if $a < 0$, $\pm\sqrt{-a}$ are the branch points. We fix the branch cut to be on the finite real axis running from $-\sqrt{-a}$ to $-\sqrt{-a}$ by defining the local polar coordinates

$$k + \sqrt{-a} = r_3 e^{i\theta_5},$$
$$k - \sqrt{-a} = r_4 e^{i\theta_6},$$

where $-\pi < \theta_5, \theta_4 = 6 \leq \pi$ as in Figure 5.5c.

If the branch cut is on the imaginary axis then deforming $\partial D_R^{(1)}$ to the real axis and using the local parameterization $-\pi/2 < \theta_1, \theta_2 \leq 3\pi/2$ as in Figure 5.5a, one finds

$$\begin{aligned}\int_{\partial D_R^{(1)}} f(k)\,\mathrm{d}k =& \fint_{-\infty}^{\infty} f(k)\,\mathrm{d}k + i\int_0^{\sqrt{a}} f(re^{3\pi i/2}+i\sqrt{a})\,\mathrm{d}r \\ &- \lim_{\epsilon\to 0} i\epsilon \int_{-\pi/2}^{3\pi/2} f(\epsilon e^{i\theta}+i\sqrt{a})e^{i\theta}\,\mathrm{d}\theta - i\int_0^{\sqrt{a}} f(re^{-\pi i/2}+i\sqrt{a})\,\mathrm{d}r,\end{aligned} \tag{5.24}$$

as in the dashed red line in Figure 5.6 where $\fint_{-\infty}^{\infty}\cdot\,\mathrm{d}k$ is a Cauchy Principal Value integral. Deforming $\partial D_R^{(3)}$ to the real axis when the branch cut is on the imaginary axis requires the local parameterization with $-3\pi/2 < \theta_3, \theta_4 \leq \pi/2$ as in 5.5b. Then

$$\begin{aligned}\int_{\partial D_R^{(3)}} f(k)\,\mathrm{d}k =& -\fint_{-\infty}^{\infty} f(k)\,\mathrm{d}k + i\int_{-\sqrt{a}}^{0} f(re^{\pi i/2}-i\sqrt{a})\,\mathrm{d}r \\ &- \lim_{\epsilon\to 0} i\epsilon \int_{-3\pi/2}^{\pi/2} f(\epsilon e^{i\theta}-i\sqrt{a})e^{i\theta}\,\mathrm{d}\theta - i\int_{-\sqrt{a}}^{0} f(re^{-3\pi i/2}-i\sqrt{a})\,\mathrm{d}r,\end{aligned} \tag{5.25}$$

as in the solid green line in Figure 5.6. If the branch cut is on the real axis then deforming $\partial D_R^{(1)}$ and $\partial D_R^{(3)}$ to the real axis one finds

$$\int_{\partial D_R^{(1)}} f(k)\,\mathrm{d}k = \fint_{-\infty}^{\infty} f(k)\,\mathrm{d}k, \tag{5.26}$$

and

$$\int_{\partial D_R^{(3)}} f(k)\,\mathrm{d}k = -\fint_{-\infty}^{\infty} f(k)\,\mathrm{d}k, \tag{5.27}$$

as in Figure 5.7.

In what follows we consider $\alpha_2 > \alpha_1$. Then, $\partial D_R^{(3)}$ in (5.23a) can be deformed as in (5.27) and $\partial D_R^{(1)}$ in (5.23b) can be deformed as in (5.24).

$$\psi^{(1)}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{\psi}_0^{(1)}(k)\,\mathrm{d}k + \frac{1}{2\pi}\fint_{-\infty}^{\infty} a^{(1)}(k)e^{ikx-\omega_1 t}\,\mathrm{d}k, \tag{5.28}$$

for $x < 0$, where

$$a^{(1)}(k) = \frac{1}{1+\sqrt{1+\frac{\alpha_1-\alpha_2}{k^2}}}\left((1-\sqrt{1+\frac{\alpha_1-\alpha_2}{k^2}})\hat{\psi}_0^{(1)}(-k) + 2\hat{\psi}_0^{(2)}\left(k\sqrt{1+\frac{\alpha_1-\alpha_2}{k^2}}\right)\right),$$

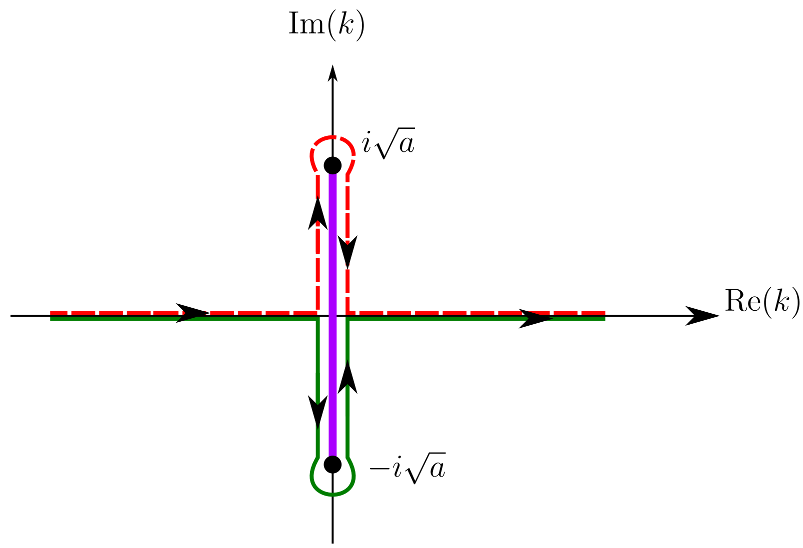

Figure 5.6: The deformations of $\partial D_R^{(1)}$ (as a red dashed line) and $\partial D_R^{(3)}$ (as a green solid line) to the real line when the branch cut is on the imaginary axis.

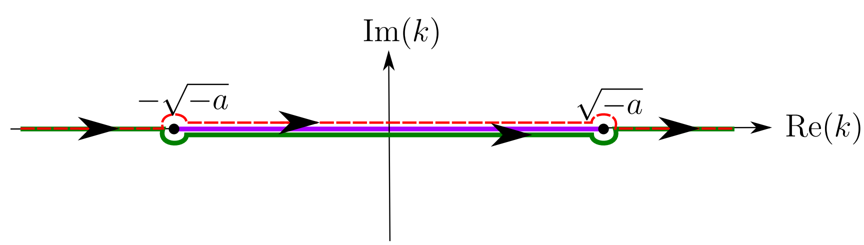

Figure 5.7: The deformations of $\partial D_R^{(1)}$ (as a red dashed line) and $\partial D_R^{(3)}$ (as a solid green line) to the real line for the case when the branch cut is on the real axis.

and

$$\begin{aligned}\psi^{(2)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{\psi}_0^{(2)}(k)\,\mathrm{d}k + \frac{1}{2\pi}\fint_{-\infty}^{\infty} a^{(2)}(k)e^{ikx-\omega_2 t}\,\mathrm{d}k \\ &+ i\int_0^{\sqrt{\alpha_2-\alpha_1}} \Big(a^{(2)}(re^{3\pi i/2}+i\sqrt{\alpha_2-\alpha_1}) - a^{(2)}(re^{-\pi i/2}+i\sqrt{\alpha_2-\alpha_1})\Big) \\ &\qquad * e^{(r-\sqrt{\alpha_2-\alpha_1})x+it(r^2-\alpha_1-2r\sqrt{\alpha_2-\alpha_1})}\,\mathrm{d}r,\end{aligned} \tag{5.29}$$

for $x > 0$, where

$$a^{(2)}(k) = \frac{1}{1+\sqrt{1-\frac{\alpha_1-\alpha_2}{k^2}}}\left((1-\sqrt{1-\frac{\alpha_1-\alpha_2}{k^2}})\hat{\psi}_0^{(1)}(-k) + 2\hat{\psi}_0^{(2)}\left(k\sqrt{1-\frac{\alpha_1-\alpha_2}{k^2}}\right)\right).$$

At this point the large-time leading-order behavior of (5.5) with initial conditions which decay sufficiently fast at $\pm\infty$ is easily obtained using the Method of Stationary Phase [6]. Notice that the third integral of (5.29) is decaying for x large and positive. Thus, it does not contribute using the Method of Stationary Phase. We choose $x/t = \gamma_1 < 0$ for $x < 0$ and $x/t = \gamma_2 > 0$ for $x > 0$. We obtain

$$\psi^{(1)} \sim \frac{e^{i\left(\frac{\gamma_1^2}{4}-\alpha_1\right)t-\frac{i\pi}{4}}}{2\sqrt{\pi t}}\left(\hat{\psi}_0^{(1)}\left(\frac{\gamma_1}{2}\right) + \frac{\left(1-\sqrt{1+\frac{4(\alpha_1-\alpha_2)}{\gamma_1^2}}\right)\hat{\psi}_0^{(1)}\left(\frac{-\gamma_1}{2}\right)+2\hat{\psi}_0^{(2)}\left(\frac{\gamma_1}{2}\sqrt{1+\frac{4(\alpha_1-\alpha_2)}{\gamma_1^2}}\right)}{1+\sqrt{1+\frac{4(\alpha_1-\alpha_2)}{\gamma_1^2}}}\right),$$

and

$$\psi^{(2)} \sim \frac{e^{i\left(\frac{\gamma_2^2}{4}-\alpha_2\right)t-\frac{i\pi}{4}}}{2\sqrt{\pi t}}\left(\hat{\psi}_0^{(2)}\left(\frac{\gamma_2}{2}\right) + \frac{\left(1-\sqrt{1-\frac{4(\alpha_1-\alpha_2)}{\gamma_2^2}}\right)\hat{\psi}_0^{(1)}\left(\frac{-\gamma_2}{2}\right)+2\hat{\psi}_0^{(2)}\left(\frac{\gamma_2}{2}\sqrt{1-\frac{4(\alpha_1-\alpha_2)}{\gamma_2^2}}\right)}{1+\sqrt{1-\frac{4(\alpha_1-\alpha_2)}{\gamma_2^2}}}\right).$$

The oscillations that are expected as a consequence of dispersion are contained in $\exp(it(\gamma_j^2/4-\alpha_j))$. In Figures 5.8 and 5.9 the envelopes of the solutions are plotted in black as a dot-dashed line. The real part of the solution (plotted as a solid line in blue) and the imaginary part of the solution (plotted as a dashed line in red) are centered around the t-axis. Using the Method of Stationary Phase one must look in directions of constant x/t.

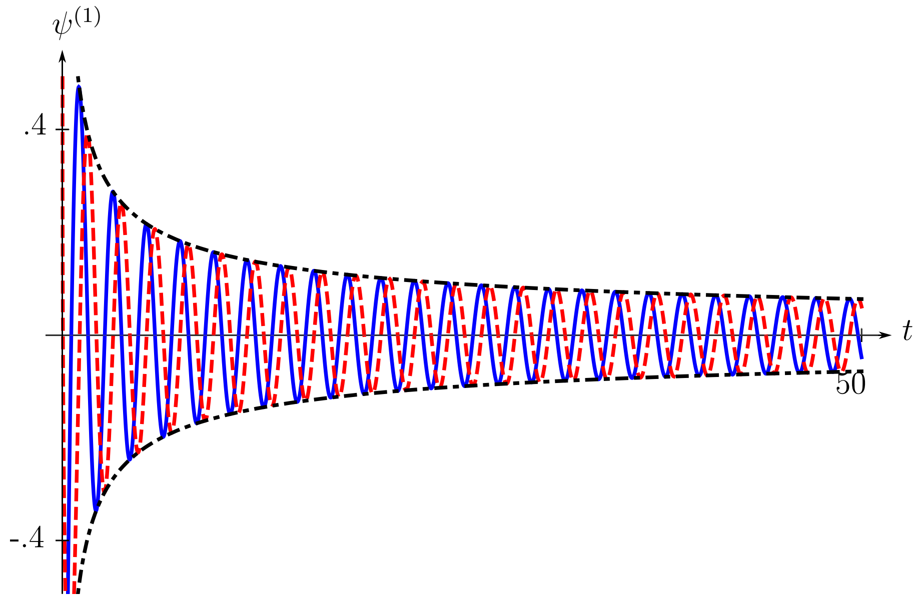

Figure 5.8: The real (red dashed) and imaginary (blue solid) parts of the leading order behavior as $t \to \infty$ of $\psi^{(1)}$ along rays of $x/t = -4$ with $\psi_0(x) = e^{-x^2}$, $\alpha_1 = 1$, and $\alpha_2 = 2$.

In Figure 5.8 we consider solutions for $x/t = -4$ and in Figure 5.9 we have solutions with $x/t = 2$. In both figures $\alpha_1 = 1, \alpha_2 = 2$ and $\psi_0(x) = e^{-x^2}$.

The Method of Stationary Phase is not useful for considering the nature of solutions near the barrier at $x = 0$, since requiring t to be large implies that x is large if x/t is to be constant. In order to evaluate the solution formulae numerically near the interface one could use techniques presented in [44, 62, 63]. It may also be possible to use asymptotic techniques similar to those in [7].

Notice that when $\alpha_1 = \alpha_2 = 0$ the problem reduces to the IVP for the linear Schrödinger equation on the whole line. It is easily seen that the solutions (5.21) and (5.22) reduce to the solution of the problem found using Fourier transforms split into the appropriate domains for the free particle problem.

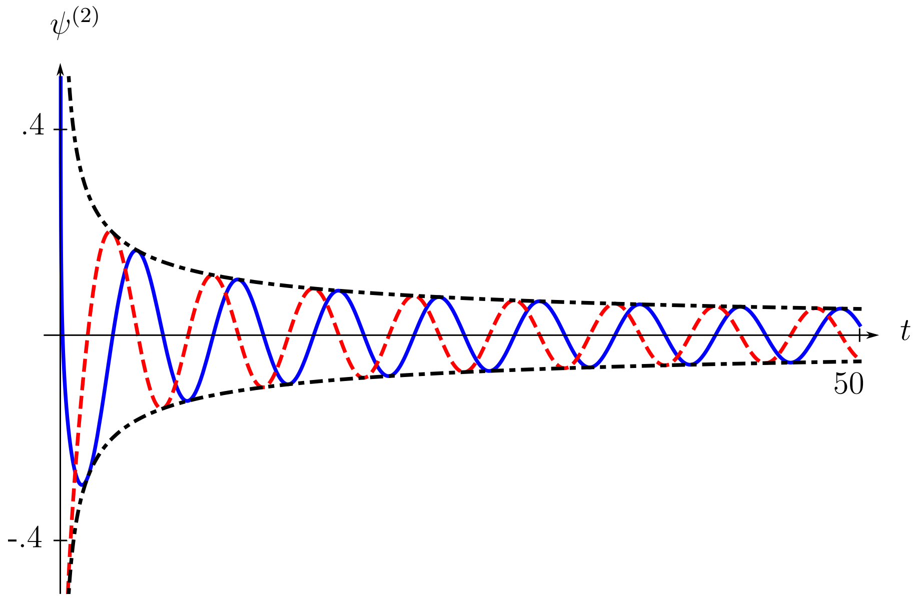

Figure 5.9: The real (red dashed) and imaginary (blue solid) parts of the leading order behavior as $t \to \infty$ of $\psi^{(2)}$ along rays of $x/t = 2$ with $\psi_0(x) = e^{-x^2}$, $\alpha_1 = 1$, and $\alpha_2 = 2$.

5.2 n potential jumps

We wish to solve the classical problem

$$i\psi_t = -\psi_{xx} + \alpha(x)\psi, \qquad -\infty < x < \infty, \tag{5.30a}$$

$$\psi(x,0) = \psi_0(x), \qquad -\infty < x < \infty, \tag{5.30b}$$

with

$$\alpha(x) = \begin{cases} \alpha_1, & x < x_1, \\ \alpha_2, & x_1 < x < x_2 \\ \vdots & \\ \alpha_n, & x_{n-1} < x < x_n, \\ \alpha_{n+1}, & x > x_n, \end{cases}$$

and $\lim_{|x|\to\infty}\psi(x,t)=0$. We repeat the same steps as in the previous section, but now for an arbitrary number n of constant levels of the potential $\alpha(x)$. As a consequence, the formulae obtained are significantly more involved, but no less explicit. The experience gained from the previous section provides the insight necessary to proceed with the general case presented here.

We treat the problem (5.30) as an interface problem solved by

$$\psi(x,t)=\begin{cases}\psi^{(1)}(x,t), & x<x_1,\\ \psi^{(2)}(x,t), & x_1<x<x_2,\\ \vdots & \\ \psi^{(n)}(x,t), & x_{n-1}<x<x_n,\\ \psi^{(n+1)}(x,t), & x>x_n,\end{cases} \tag{5.31}$$

which solve the $n+1$ IVPs

$$i\psi_t^{(j)}=-\psi_{xx}^{(j)}+\alpha_j\psi^{(j)}, \tag{5.32a}$$

$$\psi^{(j)}(x,0)=\psi_0^{(j)}(x), \tag{5.32b}$$

for $x_{j-1}<x<x_j$, with $x_0=-\infty$ and $x_{n+1}=\infty$, $j=1,\dots,n+1$. The solutions of the IVPs (5.32) are coupled by the interface conditions

$$\begin{aligned}\psi^{(j)}(x_j,t)&=\psi^{(j+1)}(x_j,t), && t>0,\\ \psi_x^{(j)}(x_j,t)&=\psi_x^{(j+1)}(x_j,t), && t>0,\end{aligned}$$

for $1\le j\le n$ as in Figure 5.10.

We begin with the n local relations

$$(e^{-ikx+\omega_j t}\psi^{(j)})_t=(e^{-ikx+\omega_j t}(i\psi_x^{(j)}-k\psi^{(j)}))_x, \qquad x_{j-1}<x<x_j, \tag{5.33}$$

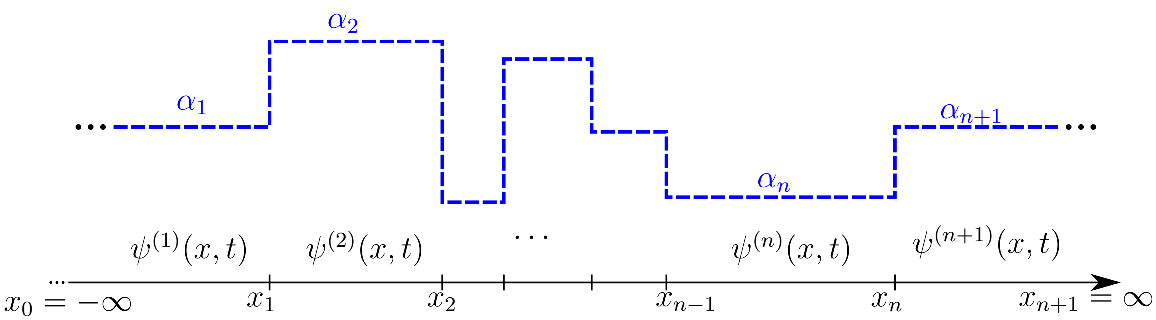

Figure 5.10: A cartoon of the potential $\alpha(x)$ in the case of n interfaces.

where $\omega_j(k) = i(\alpha_j + k^2)$ for $1 \le j \le n+1$ and $x_0 = -\infty$ and $x_{n+1} = \infty$. Applying Green's Theorem and integrating over the (possibly unbounded) strips $(x_{j-1}, x_j) \times (0,t)$ for $1 \le j \le n+1$, we have the n global relations

$$\begin{aligned}\int_{x_{j-1}}^{x_j} e^{-ikx+\omega_j t}\psi^{(j)}(x,t)\,\mathrm{d}x = \int_{x_{j-1}}^{x_j} e^{-ikx}\psi_0^{(j)}(x)\,\mathrm{d}x + \int_0^t e^{-ikx_j+\omega_j s}(i\psi_x^{(j)}(x_j,s) - k\psi^{(j)}(x_j,s))\,\mathrm{d}s \\ - \int_0^t e^{-ikx_{j-1}+\omega_j s}(i\psi_x^{(j)}(x_{j-1},s) - k\psi^{(j)}(x_{j-1},s))\,\mathrm{d}s.\end{aligned}$$

As before, we define the following transforms, for $j = 1, \ldots, n+1$:

$$\begin{aligned}
\hat{\psi}^{(j)}(k,t) &= \int_{x_{j-1}}^{x_j} e^{-ikx}\psi^{(j)}(x,t)\,\mathrm{d}x, && x_{j-1} < x < x_j, \quad t > 0,\\
\hat{\psi}_0^{(j)}(k) &= \int_{x_{j-1}}^{x_j} e^{-ikx}\psi_0^{(j)}(x)\,\mathrm{d}x, && x_{j-1} < x < x_j,\\
g_0^{(j)}(\omega,t) &= \int_0^t e^{\omega s}\psi^{(j)}(x_j,s)\,\mathrm{d}s = \int_0^t e^{\omega s}\psi^{(j+1)}(x_j,s)\,\mathrm{d}s, && t > 0\\
g_1^{(j)}(\omega,t) &= \int_0^t e^{\omega s}\psi_x^{(j)}(x_j,s)\,\mathrm{d}s = \int_0^t e^{\omega s}\psi_x^{(j+1)}(x_j,s)\,\mathrm{d}s, && t > 0.
\end{aligned}$$

For convenience we assume the x_j are shifted such that $x_1 = 0$ and $x_j > 0$ for all $j \ge 2$. All but four of these integrals are proper integrals, and they are defined for $k \in \mathbb{C}$. The only ones that are not valid in all of $\mathbb{C}$ are $\hat{\psi}^{(1)}(k,t)$, $\hat{\psi}_0^{(1)}(k)$ (valid for $\mathrm{Im}(k) \geq 0$) and $\hat{\psi}^{(n+1)}(k,t)$, $\hat{\psi}_0^{(n+1)}(k)$ (valid for $\mathrm{Im}(k) \leq 0$).

With these definitions the global relations become

$$
\begin{aligned}
e^{\omega_1 t}\hat{\psi}^{(1)}(k,t) =& \hat{\psi}_0^{(1)}(k) + ig_1^{(1)}(\omega_1,t) - kg_0^{(1)}(\omega_1,t), && \mathrm{Im}(k) \geq 0, \quad (5.34a)\\
e^{\omega_j t}\hat{\psi}^{(j)}(k,t) =& \hat{\psi}_0^{(j)}(k) + e^{-ikx_j}(ig_1^{(j)}(\omega_j,t) - kg_0^{(j)}(\omega_j,t))\\
&- e^{-ikx_{j-1}}(ig_1^{(j-1)}(\omega_j,t) - kg_0^{(j-1)}(\omega_j,t)), && k \in \mathbb{C}, \quad (5.34b)\\
e^{\omega_{n+1} t}\hat{\psi}^{(n+1)}(k,t) =& \hat{\psi}_0^{(n+1)}(k) - e^{-ikx_n}(ig_1^{(n)}(\omega_{n+1},t) - kg_0^{(n)}(\omega_{n+1},t)), && \mathrm{Im}(k) \leq 0, \quad (5.34c)
\end{aligned}
$$

where $2 \leq j \leq n$. As in the previous section we transform the global relations so that $g_0^{(j)}(\cdot,t)$ and $g_1^{(j)}(\cdot,t)$ depend on a common argument. Let

$$
\nu^{(j)}(k) = ik\sqrt{1 + \frac{\alpha_j}{k^2}}, \qquad j = 1,\ldots,n+1.
$$

Using the transformations $k = \pm\nu^{(j)}(\kappa)$, we have the transformed global relations

$$
e^{-i\kappa^2 t}\hat{\psi}^{(1)}(\nu^{(1)}(\kappa),t) = \hat{\psi}_0^{(1)}(\nu^{(1)}(\kappa)) + ig_1^{(1)} - \nu^{(1)}(\kappa)g_0^{(1)}, \tag{5.35a}
$$

$$
e^{-i\kappa^2 t}\hat{\psi}^{(1)}(-\nu^{(1)}(\kappa),t) = \hat{\psi}_0^{(1)}(-\nu^{(1)}(\kappa)) + ig_1^{(1)} + \nu^{(1)}(\kappa)g_0^{(1)}, \tag{5.35b}
$$

$$
\begin{aligned}
e^{-i\kappa^2 t}\hat{\psi}^{(j)}(\nu^{(j)}(\kappa),t) =& \hat{\psi}_0^{(j)}(\nu^{(j)}(\kappa)) + e^{-i\nu^{(j)}(\kappa)x_j}(ig_1^{(j)} - \nu^{(j)}(\kappa)g_0^{(j)})\\
&- e^{-i\nu^{(j)}(\kappa)x_{j-1}}(ig_1^{(j-1)} - \nu^{(j)}(\kappa)g_0^{(j-1)}),
\end{aligned} \tag{5.35c}
$$

$$
\begin{aligned}
e^{-i\kappa^2 t}\hat{\psi}^{(j)}(-\nu^{(j)}(\kappa),t) =& \hat{\psi}_0^{(j)}(-\nu^{(j)}(\kappa)) + e^{-i\nu^{(j)}(\kappa)x_j}(ig_1^{(j)} + \nu^{(j)}(\kappa)g_0^{(j)})\\
&- e^{i\nu^{(j)}(\kappa)x_{j-1}}(ig_1^{(j-1)} + \nu^{(j)}(\kappa)g_0^{(j-1)}),
\end{aligned} \tag{5.35d}
$$

$$
e^{-i\kappa^2 t}\hat{\psi}^{(n+1)}(\nu^{(n+1)}(\kappa),t) = \hat{\psi}_0^{(n+1)}(\nu^{(n+1)}(\kappa)) - e^{-i\nu^{(n+1)}(\kappa)x_n}(ig_1^{(n)} - \nu^{(n+1)}(\kappa)g_0^{(n)}), \tag{5.35e}
$$

$$
e^{-i\kappa^2 t}\hat{\psi}^{(n+1)}(-\nu^{(n+1)}(\kappa),t) = \hat{\psi}_0^{(n+1)}(-\nu^{(n+1)}(\kappa)) - e^{i\nu^{(n+1)}(\kappa)x_n}(ig_1^{(n)} + \nu^{(n+1)}(\kappa)g_0^{(n)}), \tag{5.35f}
$$

where $2 \leq j \leq n$ and $g_0^{(j)} = g_0^{(j)}(-i\kappa^2,t)$, $g_1^{(j)} = g_1^{(j)}(-i\kappa^2,t)$, for $1 \leq j \leq n$. In order for (5.35) to be well defined $\mathrm{Re}(\kappa) \geq 0$ for (5.35a) and (5.35f). Similarly, $\mathrm{Re}(\kappa) \leq 0$ in (5.35b) and (5.35e). Equations (5.35c) and (5.35d) are valid for all $\kappa \in \mathbb{C}$.

Inverting the Fourier transform in (5.34) we have the solution formulae

$$
\begin{aligned}
\psi^{(1)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{\psi}_0^{(1)}(k)\,\mathrm{d}k + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\left(ig_1^{(1)}(\omega_1,t) - kg_0^{(1)}(\omega_1,t)\right)\,\mathrm{d}k,\\
\psi^{(j)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_j t}\hat{\psi}_0^{(j)}(k)\,\mathrm{d}k + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x_j)-\omega_j t}\left(ig_1^{(j)}(\omega_j,t) - kg_0^{(j)}(\omega_j,t)\right)\mathrm{d}k\\
& - \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x_{j-1})-\omega_j t}\left(ig_1^{(j-1)}(\omega_j,t) - kg_0^{(j-1)}(\omega_j,t)\right)\,\mathrm{d}k,\\
\psi^{(n+1)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_{n+1} t}\hat{\psi}_0^{(n+1)}(k)\,\mathrm{d}k\\
& - \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x_n)-\omega_{n+1} t}\left(ig_1^{(n)}(\omega_{n+1},t) - kg_0^{(n)}(\omega_{n+1},t)\right)\,\mathrm{d}k,
\end{aligned}
$$

for $2 \le j \le n$, and $x_{j-1} < x < x_j$. As usual in the Fokas Method we deform these integrals into the complex plane. Using Cauchy's Theorem and Jordan's Lemma we have

$$
\begin{aligned}
\psi^{(1)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{\psi}_0^{(1)}(k)\,\mathrm{d}k - \frac{1}{2\pi}\int_{\partial D_R^{(3)}} e^{ikx-\omega_1 t}\left(ig_1^{(1)}(\omega_1,t) - kg_0^{(1)}(\omega_1,t)\right)\,\mathrm{d}k,\\
\psi^{(j)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_j t}\hat{\psi}_0^{(j)}(k)\,\mathrm{d}k - \frac{1}{2\pi}\int_{\partial D_R^{(3)}} e^{ik(x-x_j)-\omega_j t}\left(ig_1^{(j)}(\omega_j,t) - kg_0^{(j)}(\omega_j,t)\right)\mathrm{d}k\\
& - \frac{1}{2\pi}\int_{\partial D_R^{(1)}} e^{ik(x-x_{j-1})-\omega_j t}\left(ig_1^{(j-1)}(\omega_j,t) - kg_0^{(j-1)}(\omega_j,t)\right)\,\mathrm{d}k,\\
\psi^{(n+1)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_{n+1} t}\hat{\psi}_0^{(n+1)}(k)\,\mathrm{d}k\\
& - \frac{1}{2\pi}\int_{\partial D_R^{(1)}} e^{ik(x-x_n)-\omega_{n+1} t}\left(ig_1^{(n)}(\omega_{n+1},t) - kg_0^{(n)}(\omega_{n+1},t)\right)\,\mathrm{d}k,
\end{aligned}
$$

where $D_R^{(j)}$ is as in (5.17) and Figure 5.2. Again, we wish to transform the integrals involving $g_0^{(j)}(\cdot,t)$ and $g_1^{(j)}(\cdot,t)$ in each of the solution formulae above so these terms depend on $-ik^2$. As before, we deform to $D_R^{(j)}$ (with $\Lambda = \max_j |\alpha_j|$, $R > \sqrt{2\Lambda}$). Choosing $k = \nu^{(j)}(\kappa)$ on $\partial D_R^{(1)}$ and $k = -\nu^{(j)}(\kappa)$ on $\partial D_R^{(3)}$ we have

$$
\begin{aligned}
\psi^{(1)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{\psi}_0^{(1)}(k)\,\mathrm{d}k\\
& - \frac{1}{2\pi}\int_{\partial D_R^{(4)}} e^{-i\nu^{(1)}(\kappa)x+i\kappa^2 t}\left(\frac{i\kappa}{\nu^{(1)}(\kappa)}g_1^{(1)} + \kappa g_0^{(1)}\right)\,\mathrm{d}\kappa,
\end{aligned}
\tag{5.36a}
$$

$$\begin{aligned}\psi^{(j)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_j t}\hat{\psi}_0^{(j)}(k)\,\mathrm{d}k \\ &- \frac{1}{2\pi}\int_{\partial D_R^{(4)}} e^{-i\nu^{(j)}(\kappa)(x-x_j)+i\kappa^2 t}\left(\frac{i\kappa}{\nu^{(j)}(\kappa)}g_1^{(j)} + \kappa g_0^{(j)}\right)\mathrm{d}\kappa \\ &+ \frac{1}{2\pi}\int_{\partial D_R^{(4)}} e^{i\nu^{(j)}(\kappa)(x-x_{j-1})+i\kappa^2 t}\left(\frac{i\kappa}{\nu^{(j)}(\kappa)}g_1^{(j-1)} - \kappa g_0^{(j-1)}\right)\mathrm{d}\kappa,\end{aligned} \tag{5.36b}$$

$$\begin{aligned}\psi^{(n+1)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_{n+1} t}\hat{\psi}_0^{(n+1)}(k)\,\mathrm{d}k \\ &+ \frac{1}{2\pi}\int_{\partial D_R^{(4)}} e^{i\nu^{(n+1)}(\kappa)(x-x_n)+i\kappa^2 t}\left(\frac{i\kappa}{\nu^{(n+1)}(\kappa)}g_1^{(n)} - \kappa g_0^{(n)}\right)\mathrm{d}\kappa,\end{aligned} \tag{5.36c}$$

where $g_0^{(j)} \equiv g_0^{(j)}(-i\kappa^2, t)$ and $g_1^{(j)} \equiv g_1^{(j)}(-i\kappa^2, t)$.

Using the $2n$ transformed global relations valid in $D_R^{(4)}$ (5.35a), (5.35c), (5.35d), and (5.35f) one solves for $g_0^{(j)}$ and $g_1^{(j)}$. This amounts to solving the $2n \times 2n$ matrix problem

$$\mathcal{A}(\kappa)X(-i\kappa^2, t) = Y(\kappa) + \mathcal{Y}(\kappa, t)$$

where

$$X(-i\kappa^2,t) = \left(g_0^{(1)}, g_0^{(2)}, \ldots, g_0^{(n)}, ig_1^{(1)}, ig_1^{(2)}, \ldots, ig_1^{(n)}\right)^\top, \tag{5.37a}$$

$$Y(\kappa) = \left(\hat{\psi}_0^{(1)}(\nu^{(1)}), \ldots, \hat{\psi}_0^{(n)}(\nu^{(n)}), \hat{\psi}_0^{(2)}(-\nu^{(2)}), \ldots, \hat{\psi}_0^{(n+1)}(-\nu^{(n+1)})\right)^\top, \tag{5.37b}$$

$$\mathcal{Y}(\kappa,t) = \left(\hat{\psi}^{(1)}(\nu^{(1)},t), \ldots, \hat{\psi}^{(n)}(\nu^{(n)},t), \hat{\psi}^{(2)}(-\nu^{(2)},t), \ldots, \hat{\psi}^{(n+1)}(-\nu^{(n+1)},t)\right)^\top, \tag{5.37c}$$

and

$$\mathcal{A}(\kappa) =$$
$$\left(\begin{array}{ccc:ccc}
-\nu^{(1)}e^{-i\nu^{(1)}x_1} & & & e^{-i\nu^{(1)}x_1} & & \\
\nu^{(2)}e^{-i\nu^{(2)}x_1} & -\nu^{(2)}e^{-i\nu^{(2)}x_2} & & -e^{-i\nu^{(2)}x_1} & e^{-i\nu^{(2)}x_2} & \\
\ddots & \ddots & & \ddots & \ddots & \\
 & \nu^{(n)}e^{-i\nu^{(n)}x_{n-1}} & -\nu^{(n)}e^{-i\nu^{(n)}x_n} & & -e^{-i\nu^{(n)}x_{n-1}} & e^{-i\nu^{(n)}x_n} \\
\hdashline
-\nu^{(2)}e^{i\nu^{(2)}x_1} & \nu^{(2)}e^{i\nu^{(2)}x_2} & & -e^{i\nu^{(2)}x_1} & e^{i\nu^{(2)}x_2} & \\
\ddots & \ddots & & \ddots & \ddots & \\
 & -\nu^{(n)}e^{i\nu^{(n)}x_{n-1}} & \nu^{(n)}e^{i\nu^{(n)}x_n} & & -e^{i\nu^{(n)}x_{n-1}} & e^{i\nu^{(n)}x_n} \\
 & & -\nu^{(n+1)}e^{i\nu^{(n+1)}x_n} & & & -e^{i\nu^{(n+1)}x_n}
\end{array}\right), \tag{5.37d}$$

where all $\nu^{(j)}$ are evaluated at κ. The matrix $\mathcal{A}(\kappa)$ is made up of four $n \times n$ blocks as indicated by the dashed lines. The two blocks in the upper half of $\mathcal{A}(\kappa)$ are empty except for entries on the main and -1 diagonals. The lower two blocks of $\mathcal{A}(\kappa)$ are zero except on the main and $+1$ diagonals.

Every term in the linear equation $\mathcal{A}(\kappa)X(-i\kappa^2, t) = Y(\kappa)$ is known. By substituting the solutions of this equation into (5.36), we have solved the LS equation with a piecewise constant potential in terms of only known functions. It remains to show that the contribution to the solution from the linear equation $\mathcal{A}(\kappa)X(-i\kappa^2, t) = \mathcal{Y}(\kappa, t)$ is 0 when substituted into (5.36).

To this end consider $\mathcal{A}(\kappa)X(-i\kappa^2, t) = \mathcal{Y}(\kappa, t)$. We solve this system using Cramer's

Rule [12]. We factor $\mathcal{A}(\kappa)=\mathcal{A}^L(\kappa)\mathcal{A}^M(\kappa)$ where

$$\mathcal{A}^L(\kappa)=\left(\begin{array}{ccc|ccc} e^{-i\nu^{(1)}x_1} & & & & & \\ & \ddots & & & & \\ & & e^{-i\nu^{(n)}x_n} & & & \\ \hline & & & e^{i\nu^{(1)}x_1} & & \\ & & & & \ddots & \\ & & & & & e^{i\nu^{(n)}x_n} \end{array}\right),$$

Let $\mathcal{A}_j(\kappa,t)$ be the matrix $\mathcal{A}(\kappa)$ with the j^{th} column replaced by $\mathcal{Y}(\kappa,t)$. Similar to $\mathcal{A}(\kappa)$, this matrix can be factored as $\mathcal{A}_j(\kappa,t)=\mathcal{A}^L(\kappa)\mathcal{A}_j^M(\kappa,t)\mathcal{A}_j^R(\kappa,t)$ where $\mathcal{A}_j^R(\kappa,t)$ is the $2n\times 2n$ identity matrix with the (j,j) entry replaced by $e^{-i\kappa^2 t}$. Hence, $\det(\mathcal{A}_j(\kappa,t))=e^{-i\kappa^2 t}\det(\mathcal{A}_j^M(\kappa,t))$.

The terms we are trying to eliminate contribute to the solution (5.36) in the form

$$\int_{\partial D_R^{(4)}} e^{-i\nu^{(j)}(x-x_j)+i\kappa^2 t}\left(\frac{i\kappa}{\nu^{(j)}}g_1^{(j)}+\kappa g_0^{(j)}\right)\mathrm{d}\kappa, \tag{5.38a}$$

and

$$\int_{\partial D_R^{(4)}} e^{i\nu^{(j)}(x-x_{j-1})+i\kappa^2 t}\left(\frac{i\kappa}{\nu^{(j)}}g_1^{(j-1)}-\kappa g_0^{(j-1)}\right)\mathrm{d}\kappa, \tag{5.38b}$$

for $1\le j\le n$, $2\le j\le n+1$, respectively with $x_{j-1}<x<x_j$. Using Cramer's Rule these become

$$\int_{\partial D_R^{(4)}} e^{-i\nu^{(j)}(x-x_j)}\left(\frac{\kappa}{\nu^{(j)}}\frac{\det(\mathcal{A}_{j+n}^M(\kappa,t))}{\det(\mathcal{A}^M(\kappa))}+\kappa\frac{\det(\mathcal{A}_j^M(\kappa))}{\det(\mathcal{A}^M(\kappa))}\right)\mathrm{d}\kappa, \tag{5.39a}$$

and

$$\int_{\partial D_R^{(4)}} e^{i\nu^{(j)}(x-x_{j-1})}\left(\frac{\kappa}{\nu^{(j)}}\frac{\det(\mathcal{A}_{j-1+n}^M(\kappa,t))}{\det(\mathcal{A}^M(\kappa))}-\kappa\frac{\det(\mathcal{A}_{j-1}^M(\kappa,t))}{\det(\mathcal{A}^M(\kappa))}\right)\mathrm{d}\kappa, \tag{5.39b}$$

respectively. Here we have used the factorizations of $\det(\mathcal{A}(\kappa))$ and $\det(\mathcal{A}_j(\kappa,t))$. As is usual in the Fokas Method we use the large κ asymptotics to show the terms (5.39) are 0.

Observe the elements of $\mathcal{A}^M(\kappa)$ are either 0, $\mathcal{O}(\kappa)$ or decaying exponentially fast for $\kappa \in D_R^{(4)}$. Hence,

$$\det(\mathcal{A}^M(\kappa)) = c(\kappa) = \mathcal{O}(\kappa^n),$$

for large κ in $D_R^{(4)}$. Expanding the determinant of $\mathcal{A}_j^M(\kappa, t)$ along the j^{th} column we see that

$$\begin{aligned} e^{-i\nu^{(j)}(x-x_j)} \frac{\det(\mathcal{A}_j^M(\kappa,t))}{\det(\mathcal{A}^M(\kappa))} &= e^{-i\nu^{(j)}(x-x_j)} \kappa \frac{\det(\mathcal{A}_j^M(\kappa,t))}{c(\kappa)} \\ &= e^{-i\nu^{(j)}(x-x_j)} \sum_{\ell=1}^{n} c_\ell(\kappa) \left(e^{ix_\ell \nu^{(\ell)}} \hat{\psi}^{(\ell)}\left(\nu^{(\ell)}, t\right) + e^{-ix_\ell \nu^{(\ell)}} \hat{\psi}^{(\ell+1)}\left(-\nu^{(\ell+1)}, t\right) \right), \end{aligned}$$

where $c_\ell(\kappa) = \mathcal{O}(\kappa^{-1})$ and $x_{j-1} < x < x_j$. The terms $e^{ix_\ell \nu^{(\ell)}} \hat{\psi}^{(\ell)}\left(\nu^{(\ell)}, t\right)$ and $e^{-ix_\ell \nu^{(\ell)}} \hat{\psi}^{(\ell+1)}\left(-\nu^{(\ell+1)}, t\right)$ decay exponentially for $k \in D_R^{(4)}$. The integrands in (5.39) are analytic for $\text{Re}(\kappa) > 0$. Similar to the argument on page 106, since the integral along $\mathcal{L}_C^{(4)}$ vanishes for large C, the integrals (5.39) must vanish since the contour $\mathcal{L}_{D^{(4)}}$ becomes $\partial D^{(4)}$ as $C \to \infty$. The uniform decay of the ratios of the determinants for large κ is exactly the condition required for the integral to vanish using Jordan's Lemma. Hence, the solution to (5.30) is (5.36) where $g_0^{(j)}(-i\kappa^2, t)$ and $g_1^{(j)}(-i\kappa^2, t)$ for $1 \le j \le n+1$ are found by solving

$$\mathcal{A}(\kappa) X(-i\kappa^2, t) = Y(\kappa), \tag{5.40}$$

where $\mathcal{A}(\kappa), X(-i\kappa^2, t)$, and $Y(\kappa)$ are given in Equations (5.37d), (5.37a), and (5.37b) respectively. As in the previous section, deforming to the real line is possible using (5.24)-(5.27). However, one must be careful to also avoid any poles present in (5.36).

5.3 Potential well and barrier

As an example of the general method given in Section 5.2, in this section we solve the classical problem of the finite potential well or barrier:

$$i\psi_t = -\psi_{xx} + \alpha(x)\psi, \tag{5.41}$$

for $-\infty < x < \infty$ and

$$\alpha(x) = \begin{cases} 0, & x < x_1, \\ \alpha, & x_1 < x < x_2, \\ 0, & x > x_2, \end{cases}$$

with the initial condition $\psi(x,0) = \psi_0(x)$ and $\lim_{|x|\to\infty} \psi(x,t) = 0$.

The problem of a finite potential well or barrier is a standard textbook problem in quantum mechanics. In such texts this problem is usually solved using separation of variables, *i.e.*, assuming $\psi(x,t) = X(x)T(t)$. The x problem, $X'' + (\xi^2 - \alpha(x))X = 0$ is solved in the three different regions. Separation of variables is only allowed if the initial wave function $\psi(x,0)$ can be expanded in terms of solutions of the time-independent Schrödinger equation [48]. Solving the time-independent Schrödinger equation is equivalent to studying the forward scattering problem with the specified potential. The "scattering matrix" (see [2, Equation 1.3.3] or [17, p. 104]) is

$$\begin{pmatrix} a(\xi) & b(\xi) \\ \bar{b}(\xi) & -\bar{a}(\xi) \end{pmatrix}.$$

The zeros of $a(\xi)$ are the discrete eigenvalues for the problem. With some work we find

$$a(\xi) = e^{-i\xi x_2}\left(\cosh(x_2\sqrt{\alpha-\xi^2}) + \frac{i(2\xi^2-\alpha)}{2\xi\sqrt{\alpha-\xi^2}}\sinh(x_2\sqrt{\alpha-\xi^2})\right).$$

This problem is examined in many excellent texts including [2, 3, 16, 17].

The potential well or barrier problem is the standard example to introduce students to the concept of quantum tunneling which is a phenomenon where a particle "tunnels" over a barrier that it cannot overcome in the classical mechanics setting [53]. The closed form solutions we present at the end of this section all depend on the initial conditions from each of the three regions and quantum tunneling is clearly present. Finding the closed form solutions is as easy as letting $n = 2$, $\alpha_1 = \alpha_3 = 0$ and $\alpha_2 = \alpha$ in (5.36) as in Figure 5.11. Again we denote $g_0^{(j)} = g_0^{(j)}(-i\kappa^2, t)$, $g_1^{(j)} = g_1^{(j)}(-i\kappa^2, t)$, for $j = 1, 2$ and $\nu^{(j)} = \nu^{(j)}(\kappa)$. Solving (5.40)

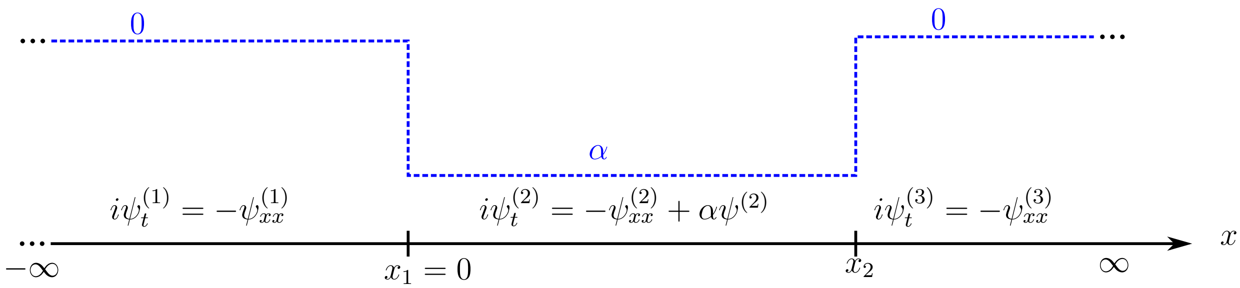

Figure 5.11: A cartoon of the potential $\alpha(x)$ for a potential well or barrier.

in the case of $n = 2$ we have solutions for $g_0^{(1)}, g_1^{(1)}, g_0^{(2)}, g_1^{(2)}$ valid in $D_R^{(4)}$. Let

$$\begin{aligned}\Delta(\kappa) =& 2\pi\left(i\kappa(1+e^{ix_2\nu^{(2)}}) + \nu^{(2)}(1-e^{ix_2\nu^{(2)}})\right)\left(i\kappa(1-e^{ix_2\nu^{(2)}}) + \nu^{(2)}(1+e^{ix_2\nu^{(2)}})\right)\\ =& 4i\pi e^{ix_2\nu^{(2)}}\left((\alpha+2\kappa^2)\sin(x_2\nu^{(2)}) + 2\kappa\nu^{(2)}\cos(x_2\nu^{(2)})\right)\end{aligned}$$

The solutions (5.36) with the appropriate values of $g_0^{(j)}$ and $g_1^{(j)}$ are

$$\begin{aligned}\psi^{(1)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{\psi}_0^{(1)}(k)\,\mathrm{d}k + \int_{\partial D_R^{(4)}} \frac{i\alpha(1-e^{2ix_2\nu^{(2)}})}{\Delta(\kappa)} e^{\kappa x + i\kappa^2 t}\hat{\psi}_0^{(1)}(i\kappa)\,\mathrm{d}\kappa\\ &+ \int_{\partial D_R^{(4)}} \frac{2i\kappa(\kappa - i\nu^{(2)})}{\Delta(\kappa)} e^{\kappa x + i\kappa^2 t}\hat{\psi}_0^{(2)}(-\nu^{(2)})\,\mathrm{d}\kappa\\ &+ \int_{\partial D_R^{(4)}} \frac{2i\kappa(\kappa + i\nu^{(2)})}{\Delta(\kappa)} e^{\kappa(x+2ix_2\nu^{(2)}) + i\kappa^2 t}\hat{\psi}_0^{(2)}(\nu^{(2)})\,\mathrm{d}\kappa\\ &- \int_{\partial D_R^{(4)}} \frac{4\kappa\nu^{(2)}}{\Delta(\kappa)} e^{\kappa(x+x_2) + ix_2\nu^{(2)} + i\kappa^2 t}\hat{\psi}_0^{(3)}(-i\kappa)\,\mathrm{d}\kappa,\end{aligned} \tag{5.42}$$

for $x < 0$,

$$\begin{aligned}
\psi^{(2)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{\psi}_0^{(2)}(k)\,\mathrm{d}k + \int_{\partial D_R^{(4)}} \frac{2i\kappa(\kappa+i\nu^{(2)})}{\Delta(\kappa)} e^{i\nu^{(2)}(2x_2-x)+i\kappa^2 t}\hat{\psi}_0^{(1)}(i\kappa)\,\mathrm{d}\kappa \\
&- \int_{\partial D_R^{(4)}} \frac{\kappa(\kappa+i\nu^{(2)})^2}{\nu^{(2)}\Delta(\kappa)} e^{i\nu^{(2)}(2x_2-x)+i\kappa^2 t}\hat{\psi}_0^{(2)}(-\nu^{(2)})\,\mathrm{d}\kappa \\
&+ \int_{\partial D_R^{(4)}} \frac{\alpha\kappa}{\nu^{(2)}\Delta(\kappa)} e^{i\nu^{(2)}(2x_2-x)+i\kappa^2 t}\hat{\psi}_0^{(2)}(\nu^{(2)})\,\mathrm{d}\kappa \\
&- \int_{\partial D_R^{(4)}} \frac{2i\kappa(\kappa-i\nu^{(2)})}{\Delta(\kappa)} e^{i\nu^{(2)}(x_2-x)+\kappa x_2+i\kappa^2 t}\hat{\psi}_0^{(3)}(-i\kappa)\,\mathrm{d}\kappa \\
&- \int_{\partial D_R^{(4)}} \frac{2i\kappa(\kappa-i\nu^{(2)})}{\Delta(\kappa)} e^{i\nu^{(2)}x+i\kappa^2 t}\hat{\psi}_0^{(1)}(i\kappa)\,\mathrm{d}\kappa \\
&+ \int_{\partial D_R^{(4)}} \frac{\alpha\kappa}{\nu^{(2)}\Delta(\kappa)} e^{i\nu^{(2)}x+i\kappa^2 t}\hat{\psi}_0^{(2)}(-\nu^{(2)})\,\mathrm{d}\kappa \\
&+ \int_{\partial D_R^{(4)}} \frac{\kappa(\kappa+i\nu^{(2)})^2}{\nu^{(2)}\Delta(\kappa)} e^{i\nu^{(2)}(2x_2+x)+i\kappa^2 t}\hat{\psi}_0^{(2)}(\nu^{(2)}(\kappa))\,\mathrm{d}\kappa \\
&+ \int_{\partial D_R^{(4)}} \frac{2i\kappa(\kappa+i\nu^{(2)})}{\Delta(\kappa)} e^{i\nu^{(2)}(x_2+x)+\kappa x_2+i\kappa^2 t}\hat{\psi}_0^{(3)}(-i\kappa)\,\mathrm{d}\kappa,
\end{aligned} \tag{5.43}$$

for $0 < x < x_2$, and

$$\begin{aligned}
\psi^{(3)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_3 t}\hat{\psi}_0^{(3)}(k)\,\mathrm{d}k - \int_{\partial D_R^{(4)}} \frac{4\kappa\nu^{(2)}}{\Delta(\kappa)} e^{\kappa(x_2-x)+ix_2\nu^{(2)}+i\kappa^2 t}\hat{\psi}_0^{(1)}(i\kappa)\,\mathrm{d}\kappa \\
&+ \int_{\partial D_R^{(4)}} \frac{2\kappa(\kappa+i\nu^{(2)})}{\Delta(\kappa)} e^{\kappa(x_2-x)+ix_2\nu^{(2)}+i\kappa^2 t}\hat{\psi}_0^{(2)}(-\nu^{(2)})\,\mathrm{d}\kappa \\
&- \int_{\partial D_R^{(4)}} \frac{2i\kappa(\kappa-i\nu^{(2)})}{\Delta(\kappa)} e^{\kappa(x_2-x)+ix_2\nu^{(2)}+i\kappa^2 t}\hat{\psi}_0^{(2)}(\nu^{(2)}(\kappa))\,\mathrm{d}\kappa \\
&+ \int_{\partial D_R^{(4)}} \frac{i\alpha\left(1-e^{2i\nu^{(2)}x_2}\right)}{\Delta(\kappa)} e^{\kappa(2x_2-x)+i\kappa^2 t}\hat{\psi}_0^{(3)}(-i\kappa)\,\mathrm{d}\kappa
\end{aligned} \tag{5.44}$$

when $x > x_2$.

Using the change of variables $\kappa = ik$ in (5.42), $\kappa = -ik$ in (5.44), $\kappa = ik\sqrt{1+\frac{\alpha}{k^2}}$ in the second, third, fourth, and fifth integrals of (5.43), and $\kappa = -ik\sqrt{1+\frac{\alpha}{k^2}}$ in the last four integrals of (5.43) we find

$$\begin{aligned}
\psi^{(1)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{\psi}_0^{(1)}(k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^{(3)}} \frac{\alpha\left(e^{-2ikx_2\sqrt{1-\frac{\alpha}{k^2}}}-1\right)}{\Delta(ik)} e^{ikx-\omega_1 t}\hat{\psi}_0^{(1)}(-k)\,\mathrm{d}k \\
&- \int_{\partial D_R^{(3)}} \frac{2k^2\left(1+\sqrt{1-\frac{\alpha}{k^2}}\right)}{\Delta(ik)} e^{ikx-\omega_1 t}\hat{\psi}_0^{(2)}\left(k\sqrt{1-\frac{\alpha}{k^2}}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^{(3)}} \frac{2k^2\left(1-\sqrt{1-\frac{\alpha}{k^2}}\right)}{\Delta(ik)} e^{ik\left(x-2x_2\sqrt{1-\frac{\alpha}{k^2}}\right)-\omega_1 t}\hat{\psi}_0^{(2)}\left(-k\sqrt{1-\frac{\alpha}{k^2}}\right)\,\mathrm{d}k \\
&- \int_{\partial D_R^{(3)}} \frac{4k^2\sqrt{1-\frac{\alpha}{k^2}}}{\Delta(ik)} e^{ik\left(x+x_2-x_2\sqrt{1-\frac{\alpha}{k^2}}\right)-\omega_1 t}\hat{\psi}_0^{(3)}(k)\,\mathrm{d}k,
\end{aligned} \tag{5.45}$$

for $x<0$,

$$\begin{aligned}
\psi^{(2)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{\psi}_0^{(2)}(k)\,\mathrm{d}k \\
&- \int_{\partial D_R^{(3)}} \frac{2k^2(1-\sqrt{1+\frac{\alpha}{k^2}})}{\Delta\left(ik\sqrt{1+\frac{\alpha}{k^2}}\right)} e^{ik(x-2x_2)-\omega_2 t}\hat{\psi}_0^{(1)}\left(-k\sqrt{1-\frac{\alpha}{k^2}}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^{(3)}} \frac{(\alpha+2k^2-2k^2\sqrt{1+\frac{\alpha}{k^2}})}{\Delta\left(ik\sqrt{1+\frac{\alpha}{k^2}}\right)} e^{ik(x-2x_2)-\omega_2 t}\hat{\psi}_0^{(2)}(k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^{(3)}} \frac{\alpha}{\Delta\left(ik\sqrt{1+\frac{\alpha}{k^2}}\right)} e^{ik(x-2x_2)-\omega_2 t}\hat{\psi}_0^{(2)}(-k)\,\mathrm{d}k \\
&- \int_{\partial D_R^{(3)}} \frac{2k^2(1+\sqrt{1+\frac{\alpha}{k^2}})}{\Delta\left(ik\sqrt{1+\frac{\alpha}{k^2}}\right)} e^{ik(x-x_2+x_2\sqrt{1+\frac{\alpha}{k^2}})-\omega_2 t}\hat{\psi}_0^{(3)}\left(k\sqrt{1-\frac{\alpha}{k^2}}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^{(1)}} \frac{2k^2(1+\sqrt{1+\frac{\alpha}{k^2}})}{\Delta\left(-ik\sqrt{1+\frac{\alpha}{k^2}}\right)} e^{ikx-\omega_2 t}\hat{\psi}_0^{(1)}\left(k\sqrt{1-\frac{\alpha}{k^2}}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^{(1)}} \frac{(\alpha+2k^2-2k^2\sqrt{1+\frac{\alpha}{k^2}})}{\Delta\left(-ik\sqrt{1+\frac{\alpha}{k^2}}\right)} e^{ik(x+2x_2)-\omega_2 t}\hat{\psi}_0^{(2)}(k)\,\mathrm{d}k \\
&- \int_{\partial D_R^{(1)}} \frac{\alpha}{\Delta\left(-ik\sqrt{1+\frac{\alpha}{k^2}}\right)} e^{ikx-\omega_2 t}\hat{\psi}_0^{(2)}(-k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^{(1)}} \frac{2k^2(1-\sqrt{1+\frac{\alpha}{k^2}})}{\Delta\left(-ik\sqrt{1+\frac{\alpha}{k^2}}\right)} e^{ik(x+x_2-x_2\sqrt{1+\frac{\alpha}{k^2}})-\omega_2 t}\hat{\psi}_0^{(3)}\left(-k\sqrt{1-\frac{\alpha}{k^2}}\right)\,\mathrm{d}k,
\end{aligned} \tag{5.46}$$

for $0 < x < x_2$, and

$$\begin{aligned}\psi^{(3)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_3 t}\hat{\psi}_0^{(3)}(k)\,\mathrm{d}k\\
&+\int_{\partial D_R^{(1)}} \frac{4k^2\sqrt{1-\frac{\alpha}{k^2}}}{\Delta(-ik)} e^{ik(x-x_2+x_2\sqrt{1-\frac{\alpha}{k^2}})-\omega_3 t}\hat{\psi}_0^{(1)}(k)\,\mathrm{d}k\\
&+\int_{\partial D_R^{(1)}} \frac{2k^2(1+\sqrt{1-\frac{\alpha}{k^2}})}{\Delta(-ik)} e^{ik(x-x_2)+ix_2\sqrt{1-\frac{\alpha}{k^2}}-\omega_3 t}\hat{\psi}_0^{(2)}\left(k\sqrt{1-\frac{\alpha}{k^2}}\right)\mathrm{d}k\\
&-\int_{\partial D_R^{(1)}} \frac{2k^2(1-\sqrt{1-\frac{\alpha}{k^2}})}{\Delta(-ik)} e^{ik(x-x_2)+ix_2\sqrt{1-\frac{\alpha}{k^2}}-\omega_3 t}\hat{\psi}_0^{(2)}\left(-k\sqrt{1-\frac{\alpha}{k^2}}\right)\mathrm{d}k\\
&+\int_{\partial D_R^{(1)}} \frac{\alpha(1-e^{2ikx_2\sqrt{1-\frac{\alpha}{k^2}}})}{\Delta(-ik)} e^{ik(x-2x_2)-\omega_3 t}\hat{\psi}_0^{(3)}(-k)\,\mathrm{d}k,\end{aligned} \tag{5.47}$$

when $x > x_2$.

Remarks:

- If one lets $\alpha = 0$ in (5.45)-(5.47) then the Fourier transform solution to the free Schrödinger equation on the whole line is recovered.

- In order to numerically or asymptotically evaluate these expressions one could use techniques presented in [7, 44, 62, 63]. The detailed asymptotic and numerical evaluation is currently under investigation.

- As stated at the beginning of this section, (5.41) is solved in standard quantum mechanics texts using separation of variables and the study of the forward scattering problem with the specified potential. The zeros of $a(\xi)$, the $(1,1)$ component of the scattering matrix, are the discrete eigenvalues for the problem. The zeros of $a(\xi)$ cannot be found explicitly but it is clear that the zeros of $a(\xi)$ for ξ purely imaginary correspond to the zeros of the denominators of (5.45)-(5.47) with $i\xi^2 = \omega_j(k)$.

Chapter 6

Linear Korteweg-de Vries equation with an interface

All previous chapters in this dissertation have dealt exclusively with problems that are second order in the spatial variable. This chapter is the first investigation into higher-order problems. The process presented in this chapter makes clear how to resolve new issues that arise when moving to a higher-order problem.

The nondimensionalized Korteweg-de Vries (KdV) equation

$$q_t + 6qq_x + q_{xxx} = 0,$$

is one of the most studied nonlinear PDEs [36, 49, 50, 70]. It arises in the study of long waves in shallow water, ion-acoustic waves in plasmas, and in general, describes the slow evolution of long waves in dispersive media [2]. In what follows we study the linearized KdV equation (LKdV) in a composite medium,

$$q_t = \sigma(x) q_{xxx}, \qquad -\infty < x < \infty, \tag{6.1}$$

where $\sigma(x)$, a real-valued function, is piecewise constant. This equation describes the behavior of solutions of the KdV equation in the small-amplitude limit and understanding its dynamics is fundamental in understanding the dynamics of the more complicated nonlinear problem.

In what follows an explicit solution method is given resulting in closed-form expressions. We provide a sufficient criterion on the interface conditions for such solutions to be obtainable via the Fokas Method. Although we do not prove uniqueness of the solution, we note some examples of interface conditions that do yield uniqueness. The numerical evaluation of the solution is not considered but should be possible via the methods presented in [7, 44, 62, 63]. As we do not have a physical application on hand, this paper addresses the mathematical question of the number and type of interface conditions required to ensure that (6.1) is well posed.

6.1 Background

Determining the number of boundary conditions necessary for a well-posed problem is a nontrivial issue, especially for BVPs with higher than second-order derivatives. Consider LKdV posed on the half line

$$q_t = \sigma^3 q_{xx}, \qquad x > 0, \quad t > 0, \tag{6.2}$$

where the form of the coefficient σ^3 is chosen for convenience. If $\sigma < 0$ then one boundary condition is needed, whereas if $\sigma > 0$, two boundary conditions must be prescribed in order for the problem to be well posed [15, 25]. This difference in seemingly very similar BVPs is understood at an intuitive level by considering the phase velocity $c(k) = -i\omega(k)/k$ where $\omega(k) = -i\sigma^3 k^3$ [38]. Thus, the phase velocity is $c(k) = -\sigma^3 k^2$. If $\sigma < 0$ the phase velocity is negative and information travels toward the boundary as in Figure 6.1a. If $\sigma > 0$, the phase velocity is positive and information travels away from the boundary as in Figure 6.1b. Therefore, it seems reasonable that one must prescribe more boundary information. Note that if we were solving (6.2) for $x < 0$ these results would be switched. This will become relevant in what follows for the interface problem on the whole line.

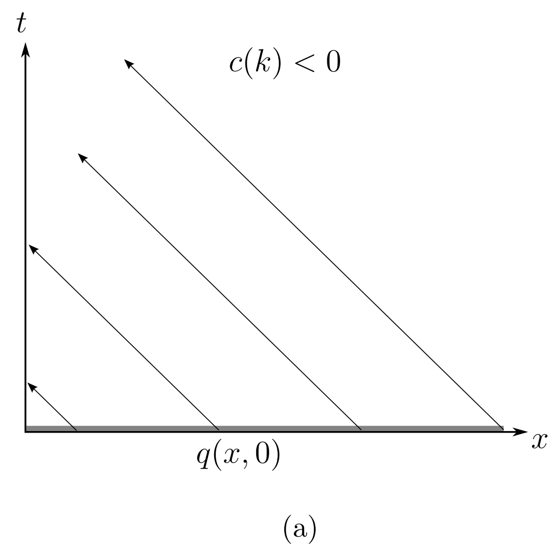

Figure 6.1: (a) When $\sigma < 0$ information from the initial condition propagates toward the boundary $x = 0$ and one boundary condition needs to be prescribed. (b) When $\sigma > 0$ information from the initial condition propagates away from the boundary and two boundary conditions need to be prescribed.

Remark: Note that the argument above is heuristic and one which we understand with hindsight. The problem we are concerned does not have propagation but rather, dispersion. Thus, although Figure 6.1 may hint at the Method of Characteristics, our argument should not be given the same credence.

One of the strengths of the Fokas Method for solving linear PDEs is the straightforward way it determines how many and what type of boundary conditions result in a well-posed problem [15, 25, 27]. Previous papers by us and others [4, 14, 47, 55, 56, 57, 58], as well as the previous chapters in this thesis, have shown that the Fokas Method is useful for finding explicit general solutions to interface problems. In the cases currently in the literature, only second-order problems are considered and the number of conditions required at each interface is clearly two. The example of LKdV on the half-line suggests that the number of interface conditions needed in the case of LKdV with an interface depends on the sign of σ. This is the case indeed. In Propositions 6.1–6.3 we describe exactly the number and type of conditions necessary.

6.2 Notation and set-up

We investigate (6.1) where $\sigma(x)$ is the piecewise constant real-valued function

$$\sigma(x) = \begin{cases} \sigma_1^3, & x < 0, \\ \sigma_2^3, & x > 0, \end{cases} \tag{6.3}$$

with the initial condition $q(x,0) = q_0(x)$, and appropriate conditions at the interface $x = 0$. The choice of the power 3 in the definition of $\sigma(x)$ is purely for convenience. We assume throughout this work that the solution decays rapidly to zero as $|x| \to \infty$. If nonzero conditions at $|x| = \infty$ are desired this can be treated easily in a manner similar to that for the heat equation in [14] and for the linear Schrödinger equation in [56]. We pose (6.1) as the following interface problem:

$$q_t^{(1)} = \sigma_1^3 q_{xxx}^{(1)}, \qquad x < 0, \qquad 0 < t \leq T, \tag{6.4a}$$

$$q_t^{(2)} = \sigma_2^3 q_{xxx}^{(2)}, \qquad x > 0, \qquad 0 < t \leq T, \tag{6.4b}$$

subject to the initial conditions

$$q^{(1)}(x,0) = q_0^{(1)}(x), \qquad x < 0, \tag{6.5a}$$

$$q^{(2)}(x,0) = q_0^{(2)}(x), \qquad x > 0, \tag{6.5b}$$

with $q^{(1)}(\cdot,t) \in S(-\infty,0)$ and $q^{(2)}(\cdot,t) \in S(0,\infty)$ where $S(X)$ is the Schwartz space of restrictions to X of rapidly decaying functions. Likewise, we assume rapid decay of the initial conditions, $q_0^{(1)}(\cdot) \in S(-\infty,0)$ and $q_0^{(2)}(\cdot) \in S(0,\infty)$.

Some number of interface conditions at $x = 0$ needs to be prescribed. The number and type of such conditions are given in Propositions 6.1–6.3. We make a distinction in this manuscript between "boundary problems" and "interface problems." Boundary problems are those in which the conditions given at the interface ($x = 0$) allow one to solve either (6.4a) or (6.4b) as a half-line BVP without knowing the solution on the other domain. For example, if one can solve a BVP for $q^{(1)}(x,t)$ then one can use that solution to provide any necessary conditions at $x = 0$ to solve the second BVP for $q^{(2)}(x,t)$. Conditions for a well-posed BVP are given in [25, 65]. Since these cases have been examined, we restrict to those interface conditions which do not decouple such that either (6.4a) or (6.4b) can be solved as a BVP.

It is of note that by making use of the PDE, interface conditions can always be written in terms of a (possibly) non-homogenous linear function of

$$\left.\frac{\partial^n}{\partial x^n} q^{(1)}(x,t)\right|_{x=0},$$

and

$$\left.\frac{\partial^n}{\partial x^n} q^{(2)}(x,t)\right|_{x=0},$$

for $n = 0, 1, 2$ and all t. For example, one might require $q^{(1)}_{xxx}(0,t) = q^{(2)}_{xxx}(0,t)$ as an interface condition. This can be imposed by applying the equation and integrating in t to give

$$\frac{1}{\sigma_1^3} q^{(1)}(0,t) - \frac{1}{\sigma_2^3} q^{(2)}(0,t) = \frac{1}{\sigma_1^3} q_0^{(1)}(0) - \frac{1}{\sigma_2^3} q_0^{(2)}(0), \tag{6.6}$$

for all t, which is clearly of the form we require with $f(t) = \frac{1}{\sigma_1^3} q_0^{(1)}(0) - \frac{1}{\sigma_2^3} q_0^{(2)}(0)$. Using a similar process for any conditions on derivatives greater than second order as well as elementary linear algebra one can always express the interface conditions in the reduced forms given in Propositions 6.1–6.3 after possibly letting $x \to -x$.

Remark: If an interface condition specifies a linear combination of $\partial_x^n q^{(1)}(0,t)$ (n=0,1,2) only or $\partial_x^n q^{(2)}(0,t)$ (n=0,1,2) only, then we say it is a *boundary condition*. Note that the interface conditions

$$q^{(1)}(0,t) = 0, \qquad \text{and} \qquad q^{(1)}(0,t) - q^{(2)}(0,t) = 0,$$

are equivalent to the interface conditions

$$q^{(1)}(0,t) = 0, \qquad \text{and} \qquad q^{(2)}(0,t) = 0,$$

so it is only meaningful to discuss the maximum number of boundary conditions for any equivalent expression of a given system of interface conditions. Henceforth any mention of a number of boundary conditions should be interpreted as such a maximum number of boundary conditions.

A problem with one boundary condition may or may not decouple into a pair of BVP. Even if such a decoupling is possible, it may or may not be possible to solve the BVPs sequentially. For example, the problem with $\sigma_1, \sigma_2 > 0$, boundary condition $q^{(1)}(0,t) = 0$, and interface conditions $q_x^{(1)}(0,t) = q_x^{(2)}(0,t)$ and $q_{xx}^{(1)}(0,t) = q_{xx}^{(2)}(0,t)$ decouples into a solvable BVP for $q^{(1)}$ and a subsequent solvable BVP for $q^{(2)}$. However, the problem with $\sigma_1, \sigma_2 < 0$, and the same boundary and interface conditions does not decouple.

6.3 Application of the Fokas Method

We follow the standard steps in the application of the Fokas Method. Assuming existence of a solution, we begin with the local relations:

$$(e^{-ikx+\omega_j t}q^{(j)})_t = (e^{-ikx+\omega_j t}\sigma_j^3(q_{xx}^{(j)} + ikq_x^{(j)} - k^2 q^{(j)}))_x,$$

with $\omega_j = \omega_j(k) = i\sigma_j^3 k^3$ for $j = 1, 2$. Applying Green's Theorem and integrating over the strips $(-\infty, 0) \times (0, t)$ and $(0, \infty) \times (0, t)$ respectively (see Figure 2.2) we have the global relations

$$\begin{aligned}
\int_{-\infty}^{0} e^{-ikx+\omega_1 t} q^{(1)}(x,t)\,\mathrm{d}x &= \int_{-\infty}^{0} e^{-ikx} q_0^{(1)}(x)\,\mathrm{d}x \\
&\quad + \int_0^t e^{\omega_1 s}\sigma_1^3 \left(q_{xx}^{(1)}(0,s) + ikq_x^{(1)}(0,s) - k^2 q^{(1)}(0,s)\right)\,\mathrm{d}s,
\end{aligned}$$

$$\begin{aligned}
\int_{0}^{\infty} e^{-ikx+\omega_2 t} q^{(2)}(x,t)\,\mathrm{d}x &= \int_{0}^{\infty} e^{-ikx} q_0^{(2)}(x)\,\mathrm{d}x \\
&\quad - \int_0^t e^{\omega_2 s}\sigma_2^3 \left(q_{xx}^{(2)}(0,s) + ikq_x^{(2)}(0,s) - k^2 q^{(2)}(0,s)\right)\,\mathrm{d}s.
\end{aligned}$$

We define the following:

$$\begin{aligned}
\hat{q}^{(1)}(k,t) &= \int_{-\infty}^{0} e^{-ikx} q^{(1)}(x,t)\,\mathrm{d}x, && \mathrm{Im}(k) \geq 0, && 0 < t < T,\\
\hat{q}_0^{(1)}(k) &= \int_{-\infty}^{0} e^{-ikx} q_0^{(1)}(x)\,\mathrm{d}x, && \mathrm{Im}(k) \geq 0, && \\
\hat{q}^{(2)}(k,t) &= \int_{0}^{\infty} e^{-ikx} q^{(2)}(x,t)\,\mathrm{d}x, && \mathrm{Im}(k) \leq 0, && 0 < t < T,\\
\hat{q}_0^{(2)}(k) &= \int_{0}^{\infty} e^{-ikx} q_0^{(2)}(x)\,\mathrm{d}x, && \mathrm{Im}(k) \leq 0, && \\
g_n(\omega,t) &= \int_0^t e^{\omega s}\frac{\partial^n}{\partial x^n} q^{(1)}(0,s)\,\mathrm{d}s, && n = 0,1,2, && 0 < t < T,\\
h_n(\omega,t) &= \int_0^t e^{\omega s}\frac{\partial^n}{\partial x^n} q^{(2)}(0,s)\,\mathrm{d}s, && n = 0,1,2, && 0 < t < T,
\end{aligned}$$

The global relations become

$$e^{\omega_1 t}\hat{q}^{(1)}(k,t) = \hat{q}_0^{(1)}(k) + \sigma_1^3\left(g_2(\omega_1,t) + ikg_1(\omega_1,t) - k^2 g_0(\omega_1,t)\right), \qquad \mathrm{Im}(k) \geq 0, \tag{6.7a}$$

$$e^{\omega_2 t}\hat{q}^{(2)}(k,t) = \hat{q}_0^{(2)}(k) - \sigma_2^3\left(h_2(\omega_2,t) + ikh_1(\omega_2,t) - k^2 h_0(\omega_2,t)\right), \qquad \mathrm{Im}(k) \leq 0. \tag{6.7b}$$

We wish to transform the global relations so that $g_n(\cdot,t)$ and $h_n(\cdot,t)$ for $n = 0,1,2$ depend on a common argument, ik^3 as in [4, 47]. Noting ik^3 is invariant under the transformations $k \to \alpha k$ and $k \to \alpha^2 k$ where $\alpha = e^{2i\pi/3}$ and evaluating at $t = T$ we have the following six global relations:

$$e^{ik^3T}\hat{q}^{(1)}\left(\frac{\alpha^j k}{\sigma_1}, T\right) = \hat{q}_0^{(1)}\left(\frac{\alpha^j k}{\sigma_1}\right) + \left(\sigma_1^3 g_2(ik^3,T) + i\alpha^j k\sigma_1^2 g_1(ik^3,T) - \alpha^{2j}k^2\sigma_1 g_0(ik^3,T)\right),$$
$$\sigma_1\,\mathrm{Im}(\alpha^j k) \geq 0, \tag{6.8a}$$

$$e^{ik^3T}\hat{q}^{(2)}\left(\frac{\alpha^j k}{\sigma_2}, T\right) = \hat{q}_0^{(2)}\left(\frac{\alpha^j k}{\sigma_2}\right) - \left(\sigma_2^3 h_2(ik^3,T) + i\alpha^j k\sigma_2^2 h_1(ik^3,T) - \alpha^{2j}k^2\sigma_2 h_0(ik^3,T)\right),$$
$$\sigma_2\,\mathrm{Im}(\alpha^j k) \leq 0, \tag{6.8b}$$

for $j = 0,1,2$.

Inverting the Fourier transform in (6.7) we have the solution formulas

$$\begin{aligned} q^{(1)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{q}_0^{(1)}(k)\,\mathrm{d}k \\ &+ \frac{\sigma_1^3}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\left(g_2(\omega_1,t) + ikg_1(\omega_1,t) - k^2 g_0(\omega_1,t)\right)\mathrm{d}k, \end{aligned} \tag{6.9a}$$

$$\begin{aligned} q^{(2)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{q}_0^{(2)}(k)\,\mathrm{d}k \\ &- \frac{\sigma_2^3}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\left(h_2(\omega_2,t) + ikh_1(\omega_2,t) - k^2 h_0(\omega_2,t)\right)\mathrm{d}k, \end{aligned} \tag{6.9b}$$

for $0 < t < T$ and $x < 0$ and $x > 0$ respectively. Next, we transform the second integral in each of the previous equations so that $g_n(\cdot,t)$ and $h_n(\cdot,t)$ depend on ik^3 for $n=0,1,2$.

$$\begin{aligned} q^{(1)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{q}_0^{(1)}(k)\,\mathrm{d}k \\ &+ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\frac{k}{\sigma_1}x-ik^3t}\left(\sigma_1^2 g_2(ik^3,t)+ik\sigma_1 g_1(ik^3,t)-k^2 g_0(ik^3,t)\right)\mathrm{d}k, \end{aligned} \tag{6.10a}$$

$$\begin{aligned} q^{(2)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{q}_0^{(2)}(k)\,\mathrm{d}k \\ &- \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\frac{k}{\sigma_2}x-ik^3t}\left(\sigma_2^2 h_2(ik^3,t)+ik\sigma_2 h_1(ik^3,t)-k^2 h_0(ik^3,t)\right)\mathrm{d}k, \end{aligned} \tag{6.10b}$$

Let $D=\{k\in\mathbb{C}:\operatorname{Re}(ik^3)<0\}=D^{(1)}\cup D^{(3)}\cup D^{(5)}$ as in Figure 6.2. The parenthetical numbers in the superscript of D represent an enumeration of the sectors of the complex plane, in contrast to the parenthetical numbers in the superscript of q (and Γ, below), which represent the two half-line domains $(-\infty,0)$ and $(0,\infty)$. Let $D_R=\{k\in\mathbb{C}:k\in D\cap|k|>R\}=D_R^{(1)}+D_R^{(3)}+D_R^{(5)}$ where $R>0$ is a positive constant as shown in Figure 6.3. Let $\Gamma^{(j)}$ be the contour that is the boundary of the region $\{k\in D_R:(-1)^j\sigma_j\operatorname{Im}(k)>0\}$, oriented so that $D_R^{(1)}$ and $D_R^{(3)}$ lie to the left, and $D_R^{(5)}$ lies to the right of any $\Gamma^{(j)}$ to which they are adjacent. Note that whether $\Gamma^{(j)}$ is the boundary of $D_R^{(1)}\cup D_R^{(3)}$ or $D_R^{(5)}$ depends not only on j but also upon the sign of σ_j. The integrand of the second integral in (6.10a) is analytic and decays as $k\to\infty$ from within the set bounded between $\mathbb{R}$ and $\Gamma^{(1)}$, and the integrand of the second integral in (6.10b) is analytic and decays as $k\to\infty$ from within the set bounded between $\mathbb{R}$ and $\Gamma^{(2)}$. Hence, by Jordan's Lemma and Cauchy's Theorem, the contours of integration can be deformed from $\mathbb{R}$ to $\Gamma^{(j)}$.

$$\begin{aligned} q^{(1)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{q}_0^{(1)}(k)\,\mathrm{d}k \\ &+ \frac{1}{2\pi}\int_{\Gamma^{(1)}} e^{i\frac{k}{\sigma_1}x-ik^3t}\left(\sigma_1^2 g_2(ik^3,t)+ik\sigma_1 g_1(ik^3,t)-k^2 g_0(ik^3,t)\right)\mathrm{d}k, \end{aligned} \tag{6.11a}$$

$$\begin{aligned} q^{(2)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{q}_0^{(2)}(k)\,\mathrm{d}k \\ &- \frac{1}{2\pi}\int_{\Gamma^{(2)}} e^{i\frac{k}{\sigma_2}x-ik^3t}\left(\sigma_2^2 h_2(ik^3,t)+ik\sigma_2 h_1(ik^3,t)-k^2 h_0(ik^3,t)\right)\mathrm{d}k. \end{aligned} \tag{6.11b}$$

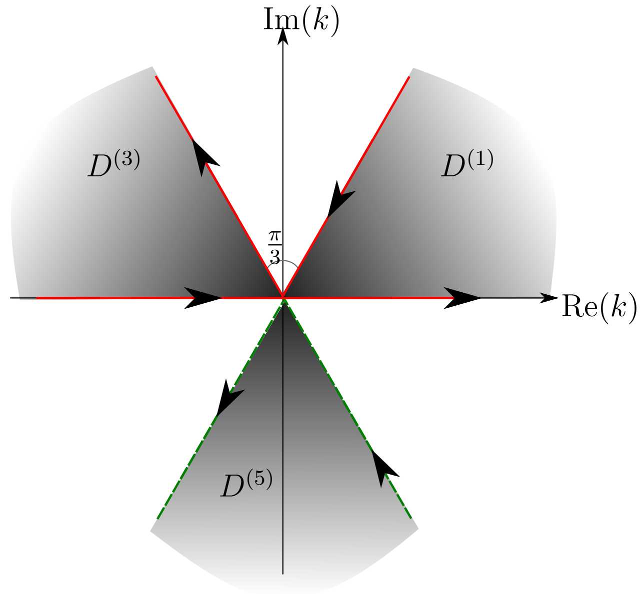

Figure 6.2: The evenly distributed regions $D^{(1)}$, $D^{(3)}$, $D^{(5)}$ where $\mathrm{Re}(ik^3) < 0$.

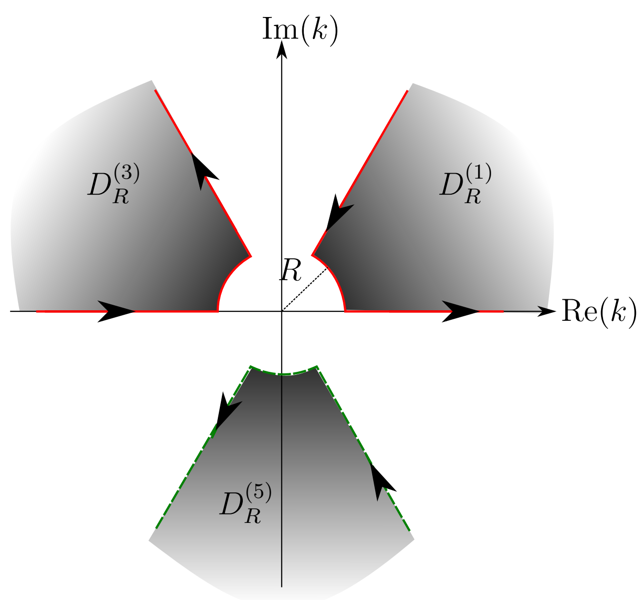

Figure 6.3: The regions $D_R^{(1)}$, $D_R^{(3)}$, $D_R^{(5)}$ where $\mathrm{Re}(ik^3) < 0$ and $|k| > R$.

We replace t by T in the arguments of g_j and h_j by noting that this is equivalent to replacing the integral $\int_0^t e^{ik^3} \frac{\partial^n}{\partial x^n} q^{(j)}(0,s)\,\mathrm{d}s$ with $\int_0^T e^{ik^3} \frac{\partial^n}{\partial x^n} q^{(j)}(0,s)\,\mathrm{d}s - \int_t^T e^{ik^3} \frac{\partial^n}{\partial x^n} q^{(j)}(0,s)\,\mathrm{d}s$. Using analyticity properties of the integrand and Jordan's Lemma, the contribution from the second integral is zero and thus we replace $g_j(ik^3,t)$ and $h_j(ik^3,t)$ with $g_j(ik^3,T)$ and $h_j(ik^3,T)$:

$$\begin{aligned} q^{(1)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{q}_0^{(1)}(k)\,\mathrm{d}k \\ &+ \frac{1}{2\pi}\int_{\Gamma^{(1)}} e^{i\frac{k}{\sigma_1}x-ik^3t}\left(\sigma_1^2 g_2(ik^3,T) + ik\sigma_1 g_1(ik^3,T) - k^2 g_0(ik^3,T)\right)\,\mathrm{d}k, \end{aligned} \tag{6.12a}$$

$$\begin{aligned} q^{(2)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{q}_0^{(2)}(k)\,\mathrm{d}k \\ &- \frac{1}{2\pi}\int_{\Gamma^{(2)}} e^{i\frac{k}{\sigma_2}x-ik^3t}\left(\sigma_2^2 h_2(ik^3,T) + ik\sigma_2 h_1(ik^3,T) - k^2 h_0(ik^3,T)\right)\,\mathrm{d}k. \end{aligned} \tag{6.12b}$$

Since (6.11) and (6.12) are equivalent we will switch between them whenever convenient in what follows.

In Section 6.4, we show how it is possible to obtain expressions for all six spectral functions g_j, h_j in the relevant domains by solving a linear system. Indeed, for any $r \in \{1,2,3\}$, if $k \in \overline{D_R^{(r)}}$ (the closure of $D_R^{(r)}$), then a certain number, say m, of the global relation equations (6.7) are valid for k. We must supplement these equations with $6-m$ interface conditions to obtain a solvable system. Given the coefficients of g_j, h_j in (6.7), it is clear that the determinant of the linear system must be a polynomial in k. The criteria of Propositions 6.1–6.3 identify the cases in which this determinant is not identically 0, that is the system is full rank. For such a full rank system, it is always possible to choose $R > 0$ sufficiently large that $\overline{D_R}$ contains no zeros of the determinant, which is essential in the proof of Proposition 6.4. We denote this linear system by

$$\mathcal{A}X = Y + \mathcal{Y}, \tag{6.13}$$

where

$$X = \left(g_0(ik^3,T), g_1(ik^3,T), g_2(ik^3,T), h_0(ik^3,T), h_1(ik^3,T), h_2(ik^3,T)\right)^\top. \tag{6.14}$$

The right-hand side of (6.13) is split into the sum of Y, which includes expressions that are known explicitly (*i.e.*, $q_0^{(j)}(\cdot)$, $j = 1, 2$ and non-homogenous terms from the interface conditions) and $\mathcal{Y}$ which includes unknown expressions (*i.e.*, $\hat{q}^{(j)}(\cdot, t)$, $j = 1, 2$).

6.4 Results

We show below in Proposition 6.4 that (6.11) is a solution of problem (6.1). It remains to show that this solution is unique in order to establish well posedness. In an attempt to show uniqueness, we assume there exist two solutions to (6.1). Let $u(x, t)$ be their difference. Then $u(x, t)$ satisfies (6.1) with homogenous initial and interface conditions. A standard energy argument shows

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}\left(\frac{\partial^n}{\partial x^n}u(x,t)\right)^2\mathrm{d}x = \lim_{\epsilon\to 0^-}\sigma_1\left(2\frac{\partial^n}{\partial x^n}u(\epsilon,t)\frac{\partial^{n+2}}{\partial x^{n+2}}u(\epsilon,t) - \left(\frac{\partial^{n+1}}{\partial x^{n+1}}u(\epsilon,t)\right)^2\right) \\ - \lim_{\epsilon\to 0^+}\sigma_2\left(2\frac{\partial^n}{\partial x^n}u(\epsilon,t)\frac{\partial^{n+2}}{\partial x^{n+2}}u(\epsilon,t) - \left(\frac{\partial^{n+1}}{\partial x^{n+1}}u(\epsilon,t)\right)^2\right)\end{aligned} \tag{6.15}$$

for any nonnegative integer n. If the interface conditions given are such that the right-hand side of (6.15) is always negative then, because $u(x, 0) = 0$ and the left-hand side of (6.15) is always non-negative, we have $u(x, t) \equiv 0$. Thus, one suitable choice of interface conditions is those that satisfy this relationship. For various signs of σ_1, σ_2, which we consider here, it is not clear how to establish that the solution (6.11) is unique in general.

In each of the following propositions, we assume that the interface conditions are not such that the problem reduces to a pair of BVP. It is a matter of trivial linear algebra to determine whether any particular problem has this property, and its well-posedness and solution are then known [25, 65].

In the case $\sigma_1 > 0$ and $\sigma_2 < 0$, the phase velocity for $x < 0$ is positive and the phase velocity for $x > 0$ is negative. Thus, information from the initial conditions propagates toward the interface as in Figure 6.4. In this case we expect the minimal number of interface conditions to be necessary for a well-posed problem.

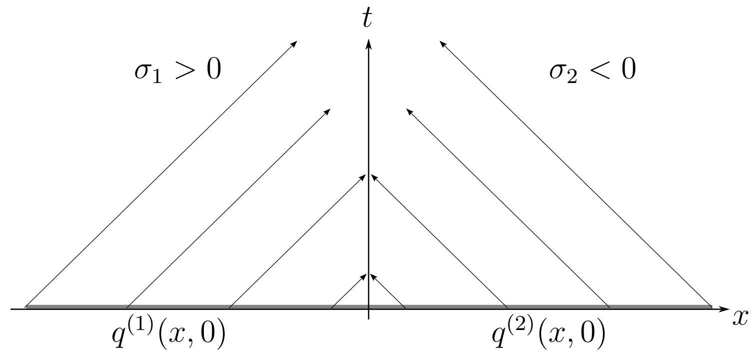

Figure 6.4: Information from the initial conditions $q^{(1)}(x,0)$ and $q^{(2)}(x,0)$ propagates toward the interface.

Proposition 6.1. *Assume $\sigma_1 > 0$ and $\sigma_2 < 0$. Equation (6.13) is solvable for X if and only if two interface conditions are given. These conditions must be of the form*

$$\beta_{11}q^{(1)}(0,t) + \beta_{12}q_x^{(1)}(0,t) + \beta_{13}q_{xx}^{(1)}(0,t) + \beta_{14}q^{(2)}(0,t) + \beta_{15}q_x^{(2)}(0,t) + \beta_{16}q_{xx}^{(2)}(0,t) = f_1(t), \tag{6.16a}$$

$$\beta_{21}q^{(1)}(0,t) + \beta_{22}q_x^{(1)}(0,t) + \beta_{23}q_{xx}^{(1)}(0,t) + \beta_{24}q^{(2)}(0,t) + \beta_{25}q_x^{(2)}(0,t) + \beta_{26}q_{xx}^{(2)}(0,t) = f_2(t). \tag{6.16b}$$

The solution to (6.13) is full rank, that is, solvable for X, whenever at least one of the following holds

1. $\beta_{14}\beta_{21} \neq \beta_{11}\beta_{24}$,

2. $\sigma_1(\beta_{15}\beta_{21} - \beta_{11}\beta_{25}) \neq \sigma_2(\beta_{12}\beta_{24} - \beta_{14}\beta_{22}$,

3. $\sigma_1^2(\beta_{16}\beta_{21} - \beta_{11}\beta_{26}) + \sigma_1\sigma_2(\beta_{15}\beta_{22} - \beta_{12}\beta_{25}) + \sigma_2^2(\beta_{14}\beta_{24} - \beta_{13}\beta_{24}) \neq 0$,

4. $\sigma_1(\beta_{16}\beta_{22} - \beta_{12}\beta_{26}) \neq \sigma_2(\beta_{13}\beta_{25} - \beta_{15}\beta_{23})$,

5. $\beta_{16}\beta_{23} \neq \beta_{13}\beta_{26}$.

Proof of Proposition 6.1. In the case $\sigma_1 > 0$ and $\sigma_2 < 0$, the second integrals of both (6.10a) and (6.10b) can be deformed from $\int_{-\infty}^{\infty} \cdot \,\mathrm{d}k$ to $-\int_{\partial D_R^{(5)}} \cdot \,\mathrm{d}k$. We rewrite the global relations (6.7) as

$$e^{ik^3t}\hat{q}^{(1)}\left(\frac{\alpha k}{\sigma_1}, T\right) - \hat{q}_0^{(1)}\left(\frac{\alpha k}{\sigma_1}\right) = \sigma_1^3 g_2(ik^3, T) + i\alpha k\sigma_1^2 g_1(ik^3, T) - (\alpha k)^2\sigma_1 g_0(ik^3, T), \quad (6.17a)$$

$$\begin{aligned} e^{ik^3t}\hat{q}^{(2)}\left(\frac{\alpha k}{\sigma_2}, T\right) - \hat{q}_0^{(2)}\left(\frac{\alpha k}{\sigma_2}\right) =& -\sigma_2^3 h_2(ik^3, T) - i\alpha k\sigma_2^2 h_1(ik^3, T) \\ &+ (\alpha k)^2\sigma_2 h_0(ik^3, T), \end{aligned} \quad (6.17b)$$

$$\begin{aligned} e^{ik^3t}\hat{q}^{(1)}\left(\frac{\alpha^2 k}{\sigma_1}, T\right) - \hat{q}_0^{(1)}\left(\frac{\alpha^2 k}{\sigma_1}\right) =& \sigma_1^3 g_2(ik^3, T) + i\alpha^2 k\sigma_1^2 g_1(ik^3, T) \\ &- (\alpha^2 k)^2\sigma_1 g_0(ik^3, T), \end{aligned} \quad (6.17c)$$

$$\begin{aligned} e^{ik^3t}\hat{q}^{(2)}\left(\frac{\alpha^2 k}{\sigma_2}, T\right) - \hat{q}_0^{(2)}\left(\frac{\alpha^2 k}{\sigma_2}\right) =& -\sigma_2^3 h_2(ik^3, T) - ik\alpha^2\sigma_2^2 h_1(ik^3, T) \\ &+ (\alpha^2 k)^2\sigma_2 h_0(ik^3, T), \end{aligned} \quad (6.17d)$$

which are all valid for $k \in \overline{D^{(5)}}$. Evaluating (6.16) for $t = s$, multiplying by e^{ik^3s}, and integrating from 0 to t one obtains

$$h_j(ik^3, T) + \sum_{\ell=0}^{2} \beta_{j+1,\ell+1} g_\ell(ik^3, T) = \tilde{f}_{j+1}(ik^3, T), \quad j \in \{0, 1, 2\},$$

where

$$\tilde{f}_j(\omega, T) = \int_0^T e^{\omega s} f_j(s)\,\mathrm{d}s, \quad j \in \{0, 1, 2\},$$

which is valid for $k \in \overline{D^{(r)}}$ (the closure of $D^{(r)}$).

In order to solve the full 6×6 system it is clear we must impose two "interface conditions," since the global relations (6.17) provide exactly four of the necessary six equations. If there is one boundary condition relating $q^{(1)}$ and $q^{(2)}$ and their spatial derivatives then one can solve the problem on the left (right) and use the solution and remaining interface conditions to solve the problem on the right (left). The half-line problem is well posed [25, 65] and its solution will not be considered here. Hence, "interface conditions" of the type (6.16) are all we need to consider.

The above argument only fails if $\det(\mathcal{A}) \equiv 0$ since all singularities are outside $D_R^{(5)}$. Examining $\det(\mathcal{A}) = 0$ one obtains a polynomial in k. Since we need this to hold for all k, we consider the coefficients of each power of k. Requiring at least one coefficient to be nonzero gives the conditions stated in (6.1). $\square$

In the case $\sigma_1 > 0$ and $\sigma_2 > 0$, the phase velocity for $x < 0$ and $x > 0$ is positive. Thus, information from $q_0^{(1)}(x)$ propagates toward the interface but information from $q_0^{(2)}(x)$ propagates away from the interface as in Figure 6.5. Hence, we expect that more interface conditions are necessary for a well-posed problem than in the case when $\sigma_1 > 0$ and $\sigma_2 < 0$ as in Proposition 6.1. Notice that the case of $\sigma_1 < 0$ and $\sigma_2 < 0$ could be considered in this case by letting $x \to -x$. Hence, we consider only the case where $\sigma_1 > 0$ and $\sigma_2 > 0$.

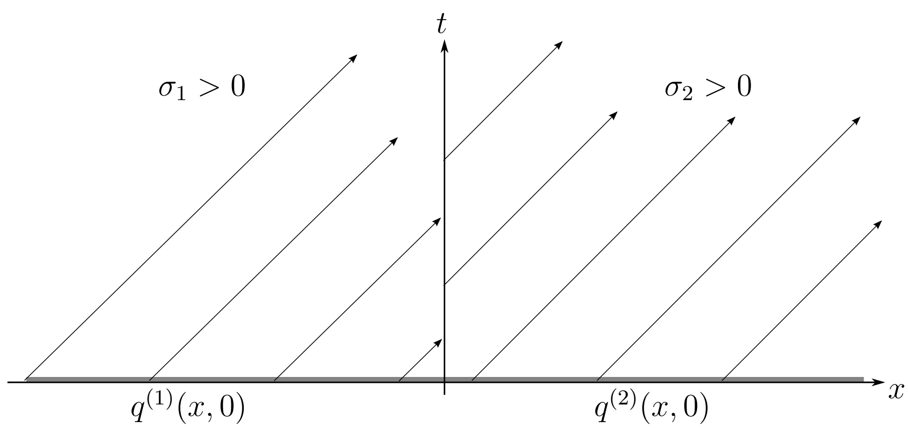

Figure 6.5: Information from the initial condition $q^{(1)}(x,0)$ propagates toward the interface while information from $q^{(2)}(x,0)$ propagates away from the interface.

Proposition 6.2. *Assume $\sigma_1 > 0$ and $\sigma_2 > 0$. The square matrix $\mathcal{A}$ in* (6.13) *is solvable for X if and only if three interface conditions of the following form are given.*

$$\beta_{11}q^{(1)}(0,t) + \beta_{12}q_x^{(1)}(0,t) + \beta_{13}q_{xx}^{(1)}(0,t) + q^{(2)}(0,t) = f_1(t), \tag{6.18a}$$

$$\beta_{21}q^{(1)}(0,t) + \beta_{22}q_x^{(1)}(0,t) + \beta_{23}q_{xx}^{(1)}(0,t) + q_x^{(2)}(0,t) = f_2(t), \tag{6.18b}$$

$$\beta_{31}q^{(1)}(0,t) + \beta_{32}q_x^{(1)}(0,t) + \beta_{33}q_{xx}^{(1)}(0,t) + q_{xx}^{(2)}(0,t) = f_3(t). \tag{6.18c}$$

The solution to (6.13) *is solvable for X whenever one or more of the following is satisfied:*

1. $\beta_{31} \neq 0$,
2. $\sigma_1\beta_{21} + \sigma_2\beta_{32} \neq 0$,
3. $\sigma_1^2\beta_{11} + \sigma_1\sigma_2\beta_{22} + \sigma_2^2\beta_{33} \neq 0$,
4. $\sigma_1\beta_{12} + \sigma_2\beta_{23} \neq 0$,
5. $\beta_{13} \neq 0$.

Remark: It may be possible to rewrite the interface conditions so that one is a boundary condition for $q^{(2)}$ and still have an interface problem. However, a single boundary condition for $q^{(1)}$ or a pair of boundary conditions for $q^{(2)}$ implies that the problem separates into a pair of BVPs.

Proof of Proposition 6.2. In the case $\sigma_1 > 0$ and $\sigma_2 > 0$, the second integrals of (6.10a) can be deformed from $\int_{-\infty}^{\infty} \cdot \, \mathrm{d}k$ to $-\int_{\partial D_R^{(5)}} \cdot \, \mathrm{d}k$. The second integral of (6.10b) can be deformed from $\int_{-\infty}^{\infty} \cdot \, \mathrm{d}k$ to $\int_{\partial D_R^{(1)}} \cdot \, \mathrm{d}k + \int_{\partial D_R^{(3)}} \cdot \, \mathrm{d}k$. We rewrite the global relations for each $r \in \{1, 3, 5\}$ as

$$\begin{aligned} e^{ik^3t}\hat{q}^{(1)}\left(\frac{\alpha^r k}{\sigma_1}, T\right) - \hat{q}_0^{(1)}\left(\frac{\alpha^r k}{\sigma_1}\right) =& \sigma_1^3 g_2(ik^3, T) + ik\alpha^r\sigma_1^2 g_1(ik^3, T) \\ & - (k\alpha^r)^2\sigma_1 g_0(ik^3, T), \end{aligned} \tag{6.19a}$$

$$
\begin{aligned}
e^{ik^3t}\hat{q}^{(1)}\left(\frac{\alpha^{r+2}k}{\sigma_1},T\right)-\hat{q}_0^{(1)}\left(\frac{\alpha^{r+2}k}{\sigma_1}\right)=&\sigma_1^3g_2(ik^3,T)+ik\alpha^{r+2}\sigma_1^2g_1(ik^3,T)\\
&-(k\alpha^{r+2})^2\sigma_1g_0(ik^3,T),
\end{aligned}
\tag{6.19b}
$$

$$
\begin{aligned}
e^{ik^3t}\hat{q}^{(2)}\left(\frac{\alpha^{r+1}k}{\sigma_2},T\right)-\hat{q}_0^{(2)}\left(\frac{\alpha^{r+1}k}{\sigma_2}\right)=&-\sigma_2^3h_2(ik^3,T)-ik\alpha^{r+1}\sigma_2^2h_1(ik^3,T)\\
&+(k\alpha^{r+1})^2\sigma_2h_0(ik^3,T),
\end{aligned}
\tag{6.19c}
$$

which are all valid for $k\in\overline{D^{(r)}}$. Evaluating (6.18) for $t=s$, multiplying by e^{ik^3s}, and integrating from 0 to t one obtains

$$
h_j(ik^3,T)+\sum_{\ell=0}^{2}\beta_{j+1,\ell+1}g_\ell(ik^3,T)=\tilde{f}_{j+1}(ik^3,T),\quad j\in\{0,1,2\},
$$

where

$$
\tilde{f}_j(\omega,T)=\int_0^T e^{\omega s}f_j(s)\,\mathrm{d}s,\quad j\in\{0,1,2\},
$$

which is valid for $k\in\overline{D^{(r)}}$.

In order to solve the full 6×6 system it is clear we must impose three “interface conditions”, since (6.19) provides just three equations. We must now examine the cases where one or more of these conditions decouples into a boundary condition on either $q^{(1)}$ or $q^{(2)}$. If there is one boundary condition relating $q^{(1)}$ and its spatial derivatives, then one can solve the problem on the left and use the solution and remaining interface conditions to solve the problem on the right. Solving the half-line problem is well posed [25, 65] and is not considered here. Hence, “interface conditions” of the type (6.18) are all we need to consider.

Examining $\det(\mathcal{A})=0$ in this case, one obtains a polynomial in k. Since we need this to hold for all k, we consider the coefficients of each power of k. Since we want conditions on $\det(\mathcal{A})\neq 0$ we need at least one of the coefficients to be nonzero. This gives the conditions stated in (6.2).

$\square$

In the case $\sigma_1 < 0$ and $\sigma_2 > 0$, the phase velocity for $x < 0$ is negative and the phase velocity for $x > 0$ is positive. Thus, information from the initial conditions propagates away from the interface as in Figure 6.6. Hence, we expect that more interface conditions are necessary for a well-posed problem than in the previous case when three interface conditions were necessary as in Proposition 6.2.

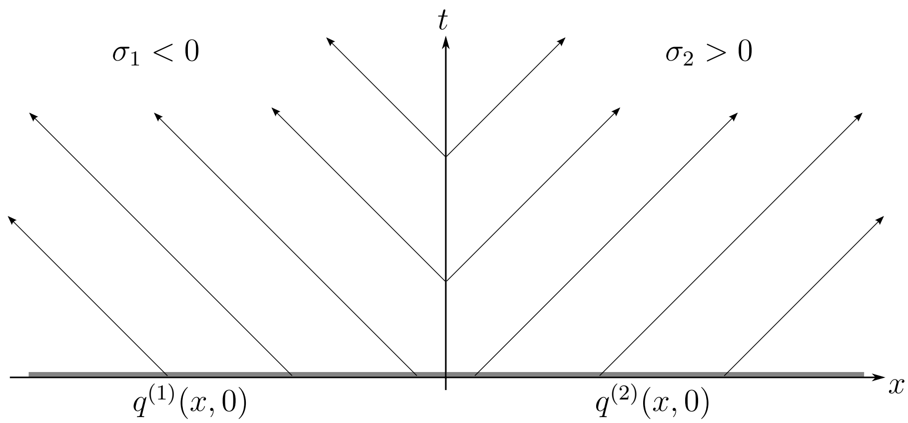

Figure 6.6: Information from the initial conditions $q^{(1)}(x, 0)$ and $q^{(2)}(x, 0)$ propagates away from the interface.

Proposition 6.3. *Assume $\sigma_1 < 0$ and $\sigma_2 > 0$. Equation* (6.13) *is full rank if and only if four interface conditions are given. These conditions must be of the form*

$$q^{(1)}(0,t) + \beta_{14}q^{(2)}(0,t) + \beta_{15}q_x^{(2)}(0,t) + \beta_{16}q_{xx}^{(2)}(0,t) = f_1(t), \tag{6.20a}$$

$$q_x^{(1)}(0,t) + \beta_{24}q^{(2)}(0,t) + \beta_{25}q_x^{(2)}(0,t) + \beta_{26}q_{xx}^{(2)}(0,t) = f_2(t), \tag{6.20b}$$

$$q_{xx}^{(1)}(0,t) + \beta_{34}q^{(2)}(0,t) + \beta_{35}q_x^{(2)}(0,t) + \beta_{36}q_{xx}^{(2)}(0,t) = f_3(t), \tag{6.20c}$$

$$\beta_{44}q^{(2)}(0,t) + \beta_{45}q_x^{(2)}(0,t) + \beta_{46}q_{xx}^{(2)}(0,t) = f_4(t). \tag{6.20d}$$

The solution to (6.13) *is full rank whenever one or more of the following is satisfied:*

1. $\beta_{35}\beta_{44} \neq \beta_{34}\beta_{45}$,

2. $\sigma_1(\beta_{34}\beta_{46} - \beta_{36}\beta_{44}) \neq \sigma_2(\beta_{24}\beta_{45} - \beta_{25}\beta_{44})$,

3. $\sigma_1^2(\beta_{35}\beta_{46} - \beta_{36}\beta_{45}) + \sigma_1\sigma_2(\beta_{26}\beta_{44} - \beta_{24}\beta_{46}) + \sigma_2^2(\beta_{14}\beta_{45} - \beta_{15}\beta_{44}) \neq 0$,

4. $\sigma_1(\beta_{26}\beta_{45} - \beta_{25}\beta_{46}) \neq \sigma_2(\beta_{16}\beta_{44} - \beta_{14}\beta_{46})$,

5. $\beta_{16}\beta_{45} \neq \beta_{15}\beta_{46}$.

Remark: As four interface conditions are required, it must be possible to write (at least) two as boundary conditions. If there are two boundary conditions for either $q^{(1)}$ or $q^{(2)}$, then the problem separates into a pair of BVP, so we only consider the case where there is precisely one boundary condition for each of $q^{(1)}$ and $q^{(2)}$. However, for the purposes of stating the result, it is more convenient to write the conditions in the form (6.20).

Proof of Proposition 6.3. In the case $\sigma_1 < 0$ and $\sigma_2 > 0$, the second integrals of both (6.10a) and (6.10b) can be deformed from $\int_{-\infty}^{\infty} \cdot \,\mathrm{d}k$ to $\int_{\partial D_R^{(1)}} \cdot \,\mathrm{d}k + \int_{\partial D_R^{(3)}} \cdot \,\mathrm{d}k$. The appropriate global relations can be rewritten for $r \in \{1,3\}$ as

$$\begin{aligned} e^{ik^3t}\hat{q}^{(1)}\left(\frac{\alpha^{r+1}k}{\sigma_1}, T\right) - \hat{q}_0^{(1)}\left(\frac{\alpha^{r+1}k}{\sigma_1}\right) &= \sigma_1^3 g_2(ik^3, T) + ik\alpha^{r+1}\sigma_1^2 g_1(ik^3, T) \\ &\quad - (k\alpha^{r+1})^2\sigma_1 g_0(ik^3, T), \end{aligned} \tag{6.21a}$$

$$\begin{aligned} e^{ik^3t}\hat{q}^{(2)}\left(\frac{\alpha^{r+1}k}{\sigma_2}, T\right) - \hat{q}_0^{(2)}\left(\frac{\alpha^{r+1}k}{\sigma_2}\right) &= -\sigma_2^3 h_2(ik^3, T) - ik\alpha^{r+1}\sigma_2^2 h_1(ik^3, T) \\ &\quad + (k\alpha^{r+1})^2\sigma_2 h_0(ik^3, T), \end{aligned} \tag{6.21b}$$

which are all valid for $k \in \overline{D^{(r)}}$. Evaluating (6.20) for $t = s$, multiplying by e^{ik^3s}, and integrating from 0 to t one obtains

$$\begin{aligned}
g_0(ik^3,T) + \beta_{15}h_1(ik^3,T) + \beta_{16}h_2(ik^3,T) &= \tilde{f}_1(ik^3,T),\\
g_1(ik^3,T) + \beta_{25}h_1(ik^3,T) + \beta_{26}h_2(ik^3,T) &= \tilde{f}_2(ik^3,T),\\
g_2(ik^3,T) + \beta_{35}h_1(ik^3,T) + \beta_{36}h_2(ik^3,T) &= \tilde{f}_3(ik^3,T),\\
h_0(ik^3,T) + \beta_{45}h_1(ik^3,T) + \beta_{46}h_2(ik^3,T) &= \tilde{f}_4(ik^3,T),
\end{aligned}$$

where

$$\tilde{f}_j(\omega,T) = \int_0^T e^{\omega s} f_j(s)\,\mathrm{d}s, \quad j \in \{1,2,3,4\},$$

which is valid for $k \in \overline{D^{(r)}}$.

In order to solve the full 6×6 system (6.13) we must impose four "interface conditions," since (6.21) gives only two equations. We need to examine the cases where one or more of these conditions decouples into a boundary condition on either $q^{(1)}$ or $q^{(2)}$. Using elementary linear algebra it is clear that at least one of these conditions must be a boundary condition. If there are two boundary conditions relating $q^{(1)}$, $q^{(2)}$ and their spatial derivatives then one can solve the problem on the left (right) and use the solution and remaining interface conditions to solve the problem on the right (left). Solving the half-line problem is well posed [25, 65] and will not be considered here. Hence, "interface conditions" of the type (6.20) are all we need to consider.

Examining $\det(\mathcal{A}) = 0$ one obtains a polynomial in k. We need this condition to hold for all k and we consider the coefficients of each power of k. Since we want conditions on $\det(\mathcal{A}) \neq 0$ we need at least one of the coefficients to be nonzero. This gives the conditions stated in (6.3). $\square$

Proposition 6.4. *Assume $\mathcal{A}$ in* (6.13) *is full rank. A solution to* (6.1) *is given by* (6.12) *where $g_j(ik^3,T)$ and $h_j(ik^3,T)$ for $j = 0,1,2$ are the solution to the linear system $\mathcal{A}X = Y$ where $\mathcal{A}$, X, and Y are given in* (6.14) *and the surrounding paragraph.*

Proof of Proposition 6.4. Consider $\mathcal{A}_j$, which is the matrix $\mathcal{A}$ with the j^{th} column replaced by $\mathcal{Y}$. This matrix can be factored as $\mathcal{A}_j = \mathcal{A}_j^R\mathcal{A}_j^M$ where $\mathcal{A}_j^R$ is the six by six identity matrix with the (j,j) element replaced by e^{ik^3T}. Hence, $\det(\mathcal{A}_j) = e^{ik^3T}\det(\mathcal{A}_j^M)$. We solve $\mathcal{A}X = \mathcal{Y}$ using Cramer's Rule [12]. If we show that the contribution to the solution from $\mathcal{Y}$ is zero, then we have proved the proposition. The terms we are concerned with from (6.12) are

$$\frac{1}{2\pi}\int_{\Gamma^{(1)}} e^{i\frac{k}{\sigma_1}x-ik^3t}\left(\sigma_1^2 g_2(ik^3,T)+ik\sigma_1 g_1(ik^3,T)-k^2 g_0(ik^3,T)\right)\,\mathrm{d}k.$$

and

$$\frac{1}{2\pi}\int_{\Gamma^{(2)}} e^{i\frac{k}{\sigma_2}x-ik^3t}\left(\sigma_2^2 h_2(ik^3,T)+ik\sigma_2 h_1(ik^3,T)-k^2 h_0(ik^3,T)\right)\,\mathrm{d}k.$$

Using Cramer's Rule and our factorization these become

$$\frac{1}{2\pi}\int_{\Gamma^{(1)}} e^{i\frac{k}{\sigma_1}x+ik^3(T-t)}\left(\sigma_1^2\frac{\det(\mathcal{A}_{(3)}^M)}{\det(\mathcal{A})}+ik\sigma_1\frac{\det(\mathcal{A}_{(2)}^M)}{\det(\mathcal{A})}-k^2\frac{\det(\mathcal{A}_{(1)}^M)}{\det(\mathcal{A})}\right)\,\mathrm{d}k. \tag{6.22a}$$

$$\frac{1}{2\pi}\int_{\Gamma^{(2)}} e^{i\frac{k}{\sigma_2}x+ik^3(T-t)}\left(\sigma_2^2\frac{\det(\mathcal{A}_{(6)}^M)}{\det(\mathcal{A})}+ik\sigma_2\frac{\det(\mathcal{A}_{(5)}^M)}{\det(\mathcal{A})}-k^2\frac{\det(\mathcal{A}_{(4)}^M)}{\det(\mathcal{A})}\right)\,\mathrm{d}k. \tag{6.22b}$$

We would like to show these integrand terms are analytic and decay for large k inside the domains around which they are integrated. Note that $\det(\mathcal{A}) \neq 0$ since (6.13) is full rank.

Case 1. $\sigma_1 < 0, \sigma_2 > 0$**:** For $\sigma_1 < 0$ and $\sigma_2 > 0$, $\Gamma^{(1)}, \Gamma^{(2)} = \partial D_R^{(1)} + \partial D_R^{(3)}$. Using the form of $\mathcal{Y}$ in this case each piece of the integrand in (6.22a) is of the form

$$\begin{aligned}
\frac{1}{2\pi}\int_{\Gamma^{(1)}} e^{i\frac{kx}{\sigma_1}+ik^3(T-t)}\frac{\det(\mathcal{A}_j^M)}{\det(\mathcal{A})} &= \frac{1}{2\pi}\int_{\Gamma^{(1)}} e^{i\frac{kx}{\sigma_1}}\left(c_1(k)\hat{q}^{(1)}\left(\frac{\alpha^{r+1}k}{\sigma_1},T\right)+c_2(k)\hat{q}^{(2)}\left(\frac{\alpha^{r+1}k}{\sigma_2},T\right)\right)\\
&= \frac{1}{2\pi}\int_{\Gamma^{(1)}} e^{i\frac{kx}{\sigma_1}+ik^3(T-t)}\left(\int_{-\infty}^{0} c_1(k)e^{-i\frac{\alpha^{r+1}ky}{\sigma_1}}q^{(1)}(y,T)\,\mathrm{d}y\right)\mathrm{d}k\\
&\quad+\frac{1}{2\pi}\int_{\Gamma^{(1)}} e^{i\frac{kx}{\sigma_1}+ik^3(T-t)}\left(\int_{0}^{\infty} c_2(k)e^{-i\frac{\alpha^{r+1}ky}{\sigma_2}}q^{(2)}(y,T)\,\mathrm{d}y\right)\mathrm{d}k,
\end{aligned}$$

where $c_1(k)$ and $c_2(k)$ involve the constants $\beta_{j,\ell}$, are $\mathcal{O}(k^{-1})$ as $k \to \infty$ from within $\Gamma^{(1)}$ and are analytic for all $k \in D_R^{(r)}$. For $k \to \infty$ with $k \in D^{(r)}$ the expression inside the parenthesis is decaying exponentially fast. Thus, by Jordan's Lemma, these integrals along a closed, bounded curve in the complex k plane vanish for $x < 0$. In particular we consider the closed curves $\mathcal{L}^{(1)} = \mathcal{L}_{D^{(1)}} \cup \mathcal{L}_{C^{(1)}}$ and $\mathcal{L}^{(3)} = \mathcal{L}_{D^{(3)}} \cup \mathcal{L}_{C^{(3)}}$ where $\mathcal{L}_{D^{(j)}} = \partial D_R^{(j)} \cap \{k : |k| < C\}$ and $\mathcal{L}_{C^{(j)}} = \{k \in D_R^{(j)} : |k| = C\}$, see Figure 6.7.

Since the integrals along $\mathcal{L}_{C^{(1)}}$ and $\mathcal{L}_{C^{(3)}}$ vanish for large C, the integrals must vanish since the contour $\mathcal{L}_{D^{(1)}}$ becomes $\partial D_R^{(1)}$ as $C \to \infty$. The same argument holds for $\mathcal{L}^{(3)}$ and $\partial D_R^{(3)}$. The uniform decay of the expressions in parentheses for large k is exactly the condition required for the integral to vanish using Jordan's Lemma. Hence Equation (6.22a) is zero. A similar argument holds for (6.22b).

Case 2. $\sigma_1 > 0, \sigma_2 > 0$**:** For $\sigma_1 > 0$ and $\sigma_2 > 0$, $\Gamma^{(1)} = \partial D_R^{(5)}$ and $\Gamma^{(2)} = \partial D_R^{(1)} + \partial D_R^{(3)}$. Using the form of $\mathcal{Y}$ in this case each piece of the integrand in (6.22a) is of the form

$$\begin{aligned}\frac{1}{2\pi}\int_{\Gamma^{(1)}} e^{i\frac{kx}{\sigma_1}+ik^3(T-t)}\frac{\det(\mathcal{A}_j^M)}{\det(\mathcal{A})}\,\mathrm{d}k = &\frac{1}{2\pi}\int_{\Gamma^{(1)}} e^{i\frac{kx}{\sigma_1}+ik^3(T-t)}\left(c_1(k)\hat{q}^{(1)}\left(\frac{\alpha^2 k}{\sigma_1},T\right)+c_2(k)\hat{q}^{(1)}\left(\frac{\alpha k}{\sigma_1},T\right)\right)\mathrm{d}k\\ &+\frac{1}{2\pi}\int_{\Gamma^{(1)}} e^{i\frac{kx}{\sigma_1}+ik^3(T-t)}c_3(k)\hat{q}^{(2)}\left(\frac{k}{\sigma_2},T\right)\,\mathrm{d}k\\ =&\frac{1}{2\pi}\int_{\Gamma^{(1)}} e^{i\frac{kx}{\sigma_1}+ik^3(T-t)}\left(c_1(k)\int_{-\infty}^0 e^{-i\frac{\alpha^2 ky}{\sigma_1}}q^{(1)}(y,T)\,\mathrm{d}y\right)\mathrm{d}k\\ &+\frac{1}{2\pi}\int_{\Gamma^{(1)}} e^{i\frac{kx}{\sigma_1}+ik^3(T-t)}\left(c_2(k)\int_{-\infty}^0 e^{-i\frac{\alpha ky}{\sigma_1}}q^{(1)}(y,T)\,\mathrm{d}y\right)\mathrm{d}k\\ &+\frac{1}{2\pi}\int_{\Gamma^{(1)}} e^{i\frac{kx}{\sigma_1}+ik^3(T-t)}\left(c_3(k)\int_0^{\infty} e^{-i\frac{ky}{\sigma_2}}q^{(2)}(y,T)\,\mathrm{d}y\right)\mathrm{d}k,\end{aligned}$$

where $c_1(k)$, $c_2(k)$, and $c_3(k)$ are $\mathcal{O}(k^{-1})$, analytic in $D^{(r)}$, and involve the constants $\beta_{j,\ell}$. For $k \to \infty$ with $k \in D^{(r)}$ the full expression inside the parentheses is decaying exponentially fast. Thus, we can apply Jordan's Lemma and Cauchy's Theorem as in Case 1, using curves shown in Figure 6.7. Hence, Equation (6.22a) is zero. Again, a similar argument holds for (6.22b).

Case 3. $\sigma_1 > 0, \sigma_2 < 0$**:** For $\sigma_1 > 0$ and $\sigma_2 < 0$, $\Gamma^{(1)}, \Gamma^{(2)} = \partial D_R^{(5)}$. Using the form of $\mathcal{Y}$ in this case each piece of the integrand in (6.22a) is of the form

$$
\begin{aligned}
\frac{1}{2\pi}\int_{\Gamma^{(1)}} e^{i\frac{kx}{\sigma_1}+ik^3(T-t)}\frac{\det(\mathcal{A}_j^M)}{\det(\mathcal{A})}\,\mathrm{d}k =& \frac{1}{2\pi}\int_{\Gamma^{(1)}} e^{i\frac{kx}{\sigma_1}+ik^3(T-t)}\left(c_1(k)\hat{q}^{(1)}\!\left(\frac{\alpha k}{\sigma_1},T\right)+c_2(k)\hat{q}^{(2)}\!\left(\frac{\alpha k}{\sigma_2},T\right)\right)\mathrm{d}k \\
&+\frac{1}{2\pi}\int_{\Gamma^{(1)}} e^{i\frac{kx}{\sigma_1}+ik^3(T-t)}\left(c_3(k)\hat{q}^{(1)}\left(\frac{\alpha^2 k}{\sigma_1},T\right)+c_4(k)\hat{q}^{(2)}\left(\frac{\alpha^2 k}{\sigma_2},T\right)\right)\mathrm{d}k \\
=& \frac{1}{2\pi}\int_{\Gamma^{(1)}} e^{i\frac{kx}{\sigma_1}+ik^3(T-t)}\left(c_1(k)\int_{-\infty}^{0} e^{-i\frac{\alpha k y}{\sigma_1}} q^{(1)}(y,T)\,\mathrm{d}y\right)\mathrm{d}k \\
&+\frac{1}{2\pi}\int_{\Gamma^{(1)}} e^{i\frac{kx}{\sigma_1}+ik^3(T-t)}\left(c_2(k)\int_{0}^{\infty} e^{-i\frac{\alpha k y}{\sigma_2}} q^{(2)}(y,T)\,\mathrm{d}y\right)\mathrm{d}k \\
&+\frac{1}{2\pi}\int_{\Gamma^{(1)}} e^{i\frac{kx}{\sigma_1}+ik^3(T-t)}\left(c_3(k)\int_{-\infty}^{0} e^{-i\frac{\alpha^2 k y}{\sigma_1}} q^{(1)}(y,T)\,\mathrm{d}y\right)\mathrm{d}k \\
&+\frac{1}{2\pi}\int_{\Gamma^{(1)}} e^{i\frac{kx}{\sigma_1}+ik^3(T-t)}\left(c_4(k)\int_{0}^{\infty} e^{-i\frac{\alpha^2 k y}{\sigma_2}} q^{(2)}(y,T)\,\mathrm{d}y\right)\mathrm{d}k,
\end{aligned}
$$

where $c_1(k), c_2(k), c_3(k)$ and $c_4(k)$ are $\mathcal{O}(k^{-1})$, involve the constants $\beta_{j,\ell}$, and are analytic for $k \in D^{(r)}$. For $k \to \infty$ with $k \in D^{(5)}$ the expressions inside the parentheses are decaying exponentially fast. As before we apply Jordan's Lemma to the appropriate curves in Figure 6.7 and use Cauchy's Theorem. Thus, Equation (6.22a) is zero. A similar argument holds for (6.22b).

Taking derivatives in x and t it is clear that the expression (6.11) satisfies the PDE. When $t = 0$ the second integrals in (6.11a) and (6.11b) evaluate to zero and the expressions clearly give the initial condition $q(x, 0) = q_0(x)$. □

Remark: We have not shown that (6.11) satisfies the interface conditions directly. Showing this is more involved than showing it satisfies the initial conditions. The process requires solving $\mathcal{A}X = Y$ for X and evaluating the explicit expression at the interface ($x = 0$) and using the properties of Fourier transforms.

6.5 Examples

In this section we give solutions to (6.1) for different signs of σ_1 and σ_2 with "canonical interface conditions." That is, we prescribe that the function and its first N spatial derivatives are continuous across the interface where $1 \leq N \leq 3$ depends on the signs of σ_1 and σ_2.

Example 1. $\sigma_1 < 0, \sigma_2 > 0$**:** This example requires four interface conditions. We impose that the function, as well as its first, second, and third derivatives are continuous across the boundary.

$$
\begin{aligned}
q^{(1)}(0,t) &= q^{(2)}(0,t),\\
q_x^{(1)}(0,t) &= q_x^{(2)}(0,t),\\
q_{xx}^{(1)}(0,t) &= q_{xx}^{(2)}(0,t),\\
q_{xxx}^{(1)}(0,t) &= q_{xxx}^{(2)}(0,t),
\end{aligned}
$$

The first three conditions can be imposed directly. The condition on the third spatial derivative can be imposed by applying the equation and integrating in t to give (6.6). Applying the t transform we have

$$\frac{1}{\sigma_1^3} g_0(ik^3, T) - \frac{1}{\sigma_2^3} h_0(ik^3, T) = \frac{e^{ik^3 t} - 1}{ik^3}\left(\frac{1}{\sigma_1^3} q_0^{(1)}(0) - \frac{1}{\sigma_2^3} q_0^{(2)}(0)\right).$$

Using elementary row operations, we have, in the notation of Proposition 6.3, $f_1(T) = f_2(T) = f_3(T) = 0$, $f_4(T) = \frac{e^{ik^3 T}-1}{ik^3}\left(\frac{1}{\sigma_1^3} q_0^{(1)}(0) - \frac{1}{\sigma_2^3} q_0^{(2)}(0)\right)$, $\beta_{25} = \beta_{36} = -1$, and the remaining $\beta_{j,\ell} = 0$. Using these interface conditions and solving (6.13), Equation (6.11) becomes

$$
\begin{aligned}
q^{(1)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{q}_0^{(1)}(k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^{(1)}} \frac{\alpha^2\sigma_1-\sigma_2}{2\pi\alpha^2\sigma_1(\sigma_1-\sigma_2)} e^{\frac{ikx}{\sigma_1}-ik^3t}\hat{q}_0^{(1)}\left(\frac{\alpha^2 k}{\sigma_1}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^{(1)}} \frac{i(\alpha-1)}{2\pi\alpha^2 k\sigma_1\sigma_2(\sigma_1-\sigma_2)}(e^{-ik^3t}-1)e^{\frac{ikx}{\sigma_1}} q_0^{(1)}(0)\,\mathrm{d}k \\
&+ \int_{\partial D_R^{(1)}} \frac{\sigma_1^2(\alpha^2-1)}{2\pi\alpha^2\sigma_2^2(\sigma_1-\sigma_2)} e^{\frac{ikx}{\sigma_1}-ik^3t}\hat{q}_0^{(2)}\left(\frac{\alpha^2 k}{\sigma_2}\right)\,\mathrm{d}k \\
&- \int_{\partial D_R^{(1)}} \frac{i\sigma_1^2(\alpha-1)}{2\pi\alpha^2 k\sigma_2^4(\sigma_1-\sigma_2)}(e^{-ik^3t}-1)e^{\frac{ikx}{\sigma_1}} q_0^{(2)}(0)\,\mathrm{d}k \\
&+ \int_{\partial D_R^{(3)}} \frac{\alpha\sigma_1-\sigma_2}{2\pi k\alpha\sigma_1(\sigma_1-\sigma_2)} e^{\frac{ikx}{\sigma_1}-ik^3t}\hat{q}_0^{(1)}\left(\frac{\alpha k}{\sigma_1}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^{(3)}} \frac{i(\alpha^2-1)}{2\pi k(\sigma_1^3-\sigma_2^3)}(e^{-ik^3t}-1)e^{\frac{ikx}{\sigma_1}} q_0^{(1)}(0)\,\mathrm{d}k \\
&+ \int_{\partial D_R^{(3)}} \frac{\sigma_1^2(\alpha-1)}{2\pi\alpha\sigma_2^2(\sigma_1-\sigma_2)} e^{\frac{ikx}{\sigma_1}-ik^3t}\hat{q}_0^{(2)}\left(\frac{\alpha k}{\sigma_2}\right)\,\mathrm{d}k \\
&- \int_{\partial D_R^{(3)}} \frac{i\sigma_1^2(\alpha^2-1)}{2\pi\sigma_2^4\alpha k(\sigma_1-\sigma_2)}(e^{-ik^3t}-1)e^{\frac{ikx}{\sigma_1}} q_0^{(2)}(0)\,\mathrm{d}k,
\end{aligned}
$$

$$\begin{aligned}
q^{(2)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{q}_0^{(2)}(k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^{(1)}} \frac{\sigma_2^2(\alpha^2-1)}{2\pi\alpha^2\sigma_1^2(\sigma_1-\sigma_2)} e^{\frac{ikx}{\sigma_2}-ik^3t}\hat{q}_0^{(1)}\left(\frac{\alpha^2 k}{\sigma_1}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^{(1)}} \frac{i(\alpha-1)(\sigma_2-\alpha\sigma_1+\alpha\sigma_2)}{2\pi\alpha^2 k\sigma_1^3(\sigma_1-\sigma_2)}(e^{-ik^3t}-1)e^{\frac{ikx}{\sigma_2}}q_0^{(1)}(0)\,\mathrm{d}k \\
&+ \int_{\partial D_R^{(1)}} \frac{\alpha^2\sigma_2-\sigma_1}{2\pi\alpha^2\sigma_2(\sigma_1-\sigma_2)} e^{\frac{ikx}{\sigma_2}-ik^3t}\hat{q}_0^{(2)}\left(\frac{\alpha^2 k}{\sigma_2}\right)\,\mathrm{d}k \\
&- \int_{\partial D_R^{(1)}} \frac{i(\alpha-1)(\alpha\sigma_1-\alpha\sigma_2-\sigma_2)}{2\pi\alpha^2 k\sigma_2^3(\sigma_1-\sigma_2)}(e^{-ik^3t}-1)e^{\frac{ikx}{\sigma_2}}q_0^{(2)}(0)\,\mathrm{d}k \\
&+ \int_{\partial D_R^{(3)}} \frac{\sigma_2^2(\alpha-1)}{2\pi\alpha\sigma_1^2(\sigma_1-\sigma_2)} e^{\frac{ikx}{\sigma_2}-ik^3t}\hat{q}_0^{(1)}\left(\frac{\alpha k}{\sigma_1}\right)\,\mathrm{d}k \\
&- \int_{\partial D_R^{(3)}} \frac{i(\alpha-1)(\sigma_2+\alpha\sigma_1)}{2\pi\alpha\sigma_1^3 k(\sigma_1-\sigma_2)}(e^{-ik^3t}-1)e^{\frac{ikx}{\sigma_2}}q_0^{(1)}(0)\,\mathrm{d}k \\
&+ \int_{\partial D_R^{(3)}} \frac{\alpha\sigma_2-\sigma_1}{2\pi\alpha\sigma_2(\sigma_1-\sigma_2)} e^{\frac{ikx}{\sigma_2}-ik^3t}\hat{q}_0^{(2)}\left(\frac{\alpha k}{\sigma_2}\right)\,\mathrm{d}k \\
&- \int_{\partial D_R^{(3)}} \frac{i(\alpha-1)(\sigma_2+\alpha\sigma_1)}{2\pi\alpha k\sigma_2^3(\sigma_1-\sigma_2)}(e^{-ik^3t}-1)e^{\frac{ikx}{\sigma_2}}q_0^{(2)}(0)\,\mathrm{d}k.
\end{aligned}$$

Example 2. $\sigma_1 > 0, \sigma_2 > 0$**:** This example requires three interface conditions. We impose that the function, as well as its first and second derivative are continuous across the boundary. That is,

$$\begin{aligned}
q^{(1)}(0,t) &= q^{(2)}(0,t), \\
q_x^{(1)}(0,t) &= q_x^{(2)}(0,t), \\
q_{xx}^{(1)}(0,t) &= q_{xx}^{(2)}(0,t).
\end{aligned} \tag{6.23}$$

In the notation of Proposition 6.2 $f_1(T) = f_2(T) = f_3(T) = 0$, $\beta_{11} = \beta_{22} = \beta_{33} = -1$, and the remaining $\beta_{j,\ell} = 0$. Using the interface conditions (6.23) and solving (6.13), Equation (6.11) becomes

$$
\begin{aligned}
q^{(1)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{q}_0^{(1)}(k)\,\mathrm{d}k \\
&+ \int_{\partial D_R^{(5)}} \frac{(\sigma_1-\sigma_2)(\sigma_1+\alpha\sigma_1+\alpha\sigma_2)}{2\pi\alpha\sigma_1(\sigma_1-\alpha\sigma_2)(\sigma_1+\sigma_2+\alpha\sigma_2)} e^{\frac{ikx}{\sigma_1}-ik^3t}\hat{q}_0^{(1)}\left(\frac{\alpha k}{\sigma_1}\right)\mathrm{d}k, \\
&+ \int_{\partial D_R^{(5)}} \frac{\sigma_2-\sigma_1}{2\pi\alpha\sigma_1(\sigma_1+\sigma_2+\alpha\sigma_2)} e^{\frac{ikx}{\sigma_1}-ik^3t}\hat{q}_0^{(1)}\left(\frac{\alpha^2 k}{\sigma_1}\right)\mathrm{d}k, \\
&- \int_{\partial D_R^{(5)}} \frac{3\sigma_1^3}{2\pi\sigma_2(\sigma_1-\alpha\sigma_2)(\sigma_1+\sigma_2+\alpha\sigma_2)} e^{\frac{ikx}{\sigma_1}-ik^3t}\hat{q}_0^{(2)}\left(\frac{k}{\sigma_2}\right)\mathrm{d}k,
\end{aligned}
$$

$$
\begin{aligned}
q^{(2)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{q}_0^{(2)}(k)\,\mathrm{d}k \\
&- \int_{\partial D_R^{(1)}} \frac{\sigma_2(\sigma_1^2+\sigma_1\sigma_2-\sigma_2^2)}{2\pi\sigma_1^2(\sigma_1^2+\alpha(1+\alpha)\sigma_1\sigma_2-\sigma_2^2)} e^{\frac{ikx}{\sigma_2}-ik^3t}\hat{q}_0^{(1)}\left(\frac{k}{\sigma_1}\right)\mathrm{d}k \\
&+ \int_{\partial D_R^{(1)}} \frac{\sigma_2(\sigma_2(\sigma_1+\sigma_2)+\alpha(\sigma_1^2+\sigma_2^2))}{2\pi\alpha\sigma_1^2(\sigma_1^2+\alpha(1+\alpha)\sigma_1\sigma_2-\sigma_2^2)} e^{\frac{ikx}{\sigma_2}-ik^3t}\hat{q}_0^{(1)}\left(\frac{\alpha k}{\sigma_1}\right)\mathrm{d}k \\
&- \int_{\partial D_R^{(1)}} \frac{\sigma_1^2+(1+\alpha)\sigma_1\sigma_2-\alpha\sigma_2^2}{2\pi\alpha\sigma_2(\sigma_1^2+\alpha(1+\alpha)\sigma_1\sigma_2-\sigma_2^2)} e^{\frac{ikx}{\sigma_2}-ik^3t}\hat{q}_0^{(2)}\left(\frac{\alpha^2 k}{\sigma_2}\right)\mathrm{d}k \\
&+ \int_{\partial D_R^{(3)}} \frac{\sigma_2(\sigma_1^2+\sigma_1\sigma_2-\sigma_2^2)}{2\pi\sigma_1^2(\alpha\sigma_1^2(1+\alpha)+\sigma_2(\sigma_1+\sigma_2))} e^{\frac{ikx}{\sigma_2}-ik^3t}\hat{q}_0^{(1)}\left(\frac{k}{\sigma_1}\right)\mathrm{d}k \\
&+ \int_{\partial D_R^{(3)}} \frac{\sigma_2(\alpha\sigma_1(\sigma_2-\sigma_1)+\sigma_2(\sigma_1+\sigma_2))}{2\pi\alpha\sigma_1^2(\alpha\sigma_1^2(1+\alpha)+\sigma_2(\sigma_1+\sigma_2))} e^{\frac{ikx}{\sigma_2}-ik^3t}\hat{q}_0^{(1)}\left(\frac{\alpha^2 k}{\sigma_1}\right)\mathrm{d}k \\
&- \int_{\partial D_R^{(3)}} \frac{(1+\alpha)\sigma_1^2+\sigma_1\sigma_2+\alpha\sigma_2^2}{2\pi\alpha\sigma_2(\alpha\sigma_1^2(1+\alpha)+\sigma_2(\sigma_1+\sigma_2))} e^{\frac{ikx}{\sigma_2}-ik^3t}\hat{q}_0^{(2)}\left(\frac{\alpha k}{\sigma_2}\right)\mathrm{d}k.
\end{aligned}
$$

Example 3. $\sigma_1 > 0, \sigma_2 < 0$**:** This example requires two interface conditions. We impose that the function and its first derivative are continuous across the boundary. That is,

$$
\begin{aligned}
q^{(1)}(0,t) &= q^{(2)}(0,t), \\
q_x^{(1)}(0,t) &= q_x^{(2)}(0,t).
\end{aligned} \tag{6.24}
$$

In the notation of Proposition 6.1 $f_1(T) = f_2(T) = 0$, $\beta_{15} = \beta_{25} = -1$, and the remaining $\beta_{j,\ell} = 0$. Using the interface conditions (6.24) and solving (6.13), Equation (6.11) becomes

$$
\begin{aligned}
q^{(1)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{q}_0^{(1)}(k)\,\mathrm{d}k + \int_{\partial D_R^{(5)}} \frac{\sigma_1+\alpha\sigma_1-\sigma_2}{2\pi\alpha\sigma_1(\sigma_1+\sigma_2)} e^{\frac{ikx}{\sigma_1}-ik^3t}\hat{q}_0^{(1)}\left(\frac{\alpha k}{\sigma_1}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^{(5)}} \frac{\sigma_2+\alpha\sigma_2-\sigma_1}{2\pi\alpha\sigma_1(\sigma_1+\sigma_2)} e^{\frac{ikx}{\sigma_1}-ik^3t}\hat{q}_0^{(1)}\left(\frac{\alpha^2 k}{\sigma_1}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^{(5)}} \frac{\sigma_1(2+\alpha)}{2\pi\alpha\sigma_2(\sigma_1+\sigma_2)} e^{\frac{ikx}{\sigma_1}-ik^3t}\hat{q}_0^{(2)}\left(\frac{\alpha k}{\sigma_2}\right)\,\mathrm{d}k \\
&- \int_{\partial D_R^{(5)}} \frac{\sigma_1(2+\alpha)}{2\pi\alpha\sigma_2(\sigma_1+\sigma_2)} e^{\frac{ikx}{\sigma_1}-ik^3t}\hat{q}_0^{(2)}\left(\frac{\alpha^2 k}{\sigma_2}\right)\,\mathrm{d}k,
\end{aligned}
$$

$$
\begin{aligned}
q^{(2)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{q}_0^{(2)}(k)\,\mathrm{d}k - \int_{\partial D_R^{(5)}} \frac{\sigma_2(2+\alpha)}{2\pi\alpha\sigma_1(\sigma_1+\sigma_2)} e^{\frac{ikx}{\sigma_2}-ik^3t}\hat{q}_0^{(1)}\left(\frac{\alpha k}{\sigma_1}\right)\,\mathrm{d}k \\
&+ \int_{\partial D_R^{(5)}} \frac{\sigma_2(2+\alpha)}{2\pi\alpha\sigma_1(\sigma_1+\sigma_2)} e^{\frac{ikx}{\sigma_2}-ik^3t}\hat{q}_0^{(1)}\left(\frac{\alpha^2 k}{\sigma_1}\right)\,\mathrm{d}k \\
&- \int_{\partial D_R^{(5)}} \frac{\sigma_2+\alpha\sigma_2-\sigma_1}{2\pi\alpha\sigma_2(\sigma_1+\sigma_2)} e^{\frac{ikx}{\sigma_2}-ik^3t}\hat{q}_0^{(2)}\left(\frac{\alpha k}{\sigma_2}\right)\,\mathrm{d}k \\
&- \int_{\partial D_R^{(5)}} \frac{\sigma_1+\alpha\sigma_1-\sigma_2}{2\pi\alpha\sigma_2(\sigma_1+\sigma_2)} e^{\frac{ikx}{\sigma_2}-ik^3t}\hat{q}_0^{(2)}\left(\frac{\alpha^2 k}{\sigma_2}\right)\,\mathrm{d}k.
\end{aligned}
$$

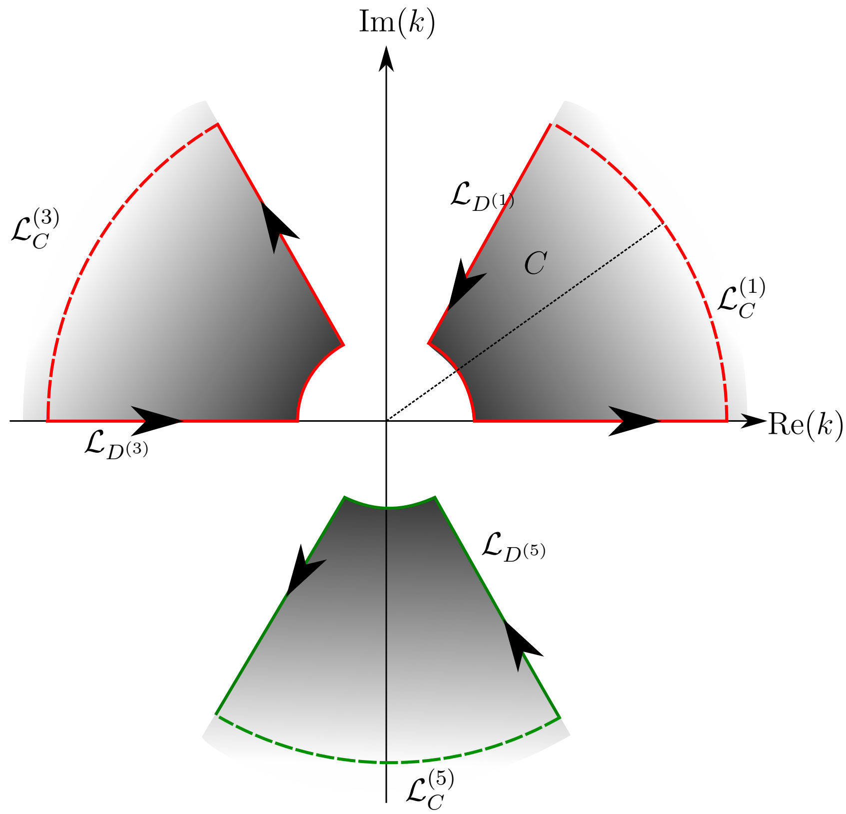

Figure 6.7: The contours $\mathcal{L}_{D^{(1)}}$ and $\mathcal{L}_{D^{(3)}}$ are shown as red solid lines and the contours $\mathcal{L}_{C^{(1)}}$ and $\mathcal{L}_{C^{(3)}}$ are shown as red dashed lines. The contour $\mathcal{L}_{D^{(5)}}$ is shown as a green solid line and the contour $\mathcal{L}_{C^{(5)}}$ is shown as a green dashed line. An application of Cauchy's Integral Theorem [1] using these contours allows elimination of the contribution of $\hat{q}^{(1)}(\cdot, t)$ and $\hat{q}^{(2)}(\cdot, t)$ from the integral expressions (6.22).

Chapter 7

Initial to Interface Maps

The construction of a Dirichlet to Neumann map, that is, determining the boundary values that are not prescribed as boundary conditions in terms of the initial and boundary conditions, is important in the study of PDEs and particularly inverse problems [24, 61]. In what follows we construct a similar map between the initial values of the PDE and the function (and some number of spatial derivatives) evaluated at the interface. This map allows for an alternative to the approach of finding simultaneous solutions to interface problems as presented in earlier chapters. Given the initial conditions one could find the value of the function and its derivatives at the interface(s). This changes the problem at hand from an interface problem to a BVP. At this point, the BVP could be solved using any number of methods appropriate for a particular problem.

7.1 Heat equation on an infinite domain with n interfaces

Consider

$$u_t = \sigma(x)u_{xx}, \tag{7.1}$$

together with the initial condition $u_0(x) = u(x,0)$ and the asymptotic conditions $\lim_{|x|\to\infty} u(x,t) = 0$, where $-\infty < x < \infty$, $0 < t < T$, and

$$\sigma(x) = \begin{cases} \sigma_1^2, & x < x_1, \\ \sigma_2^2, & x_1 < x < x_2, \\ \vdots & \\ \sigma_n^2, & x_{n-1} < x < x_n, \\ \sigma_{n+1}^2, & x > x_n. \end{cases}$$

We can rewrite (7.1) as the set of equations

$$u_t^{(j)} = \sigma_j^2 u_{xx}^{(j)}, \qquad x_{j-1} < x < x_j, \;\; 0 < t < T, \tag{7.2}$$

for $1 \le j \le n+1$ where $x_0 = -\infty$ and $x_{n+1} = \infty$. We impose the continuity interface conditions

$$\begin{aligned} u^{(j)}(x_j,t) &= u^{(j+1)}(x_j,t), & t > 0, \\ \sigma_j^2 u_x^{(j)}(x_j,t) &= \sigma_{j+1}^2 u_x^{(j+1)}(x_j,t), & t > 0, \end{aligned}$$

for $1 \le j \le n$. Since $u^{(j)}(x,t)$ is defined on the open interval $x_{j-1} < x < x_j$, when we write $u^{(j)}(x_j,t)$ we mean $\lim_{x\to x_j^-} u^{(j)}(x,t)$. Similarly, we denote $\lim_{x\to x_j^+} u^{(j+1)}(x,t)$ by $u^{(j+1)}(x_j,t)$. Without loss of generality we shift the problem so that $x_1 = 0$. Using the usual steps of the Fokas method we have the local relations

$$(e^{-ikx+\omega_j t} u^{(j)}(x,t))_t = (\sigma_j^2 e^{-ikx+\omega_j(k)t}(u_x^{(j)}(x,t) + iku^{(j)}(x,t)))_x, \tag{7.3}$$

where $\omega_j(k) = (\sigma_j k)^2$. These relations are a one-parameter family obtained by rewriting (7.2).

Integrating over the appropriate cells of the domain (see Figure 7.1) and applying Green's

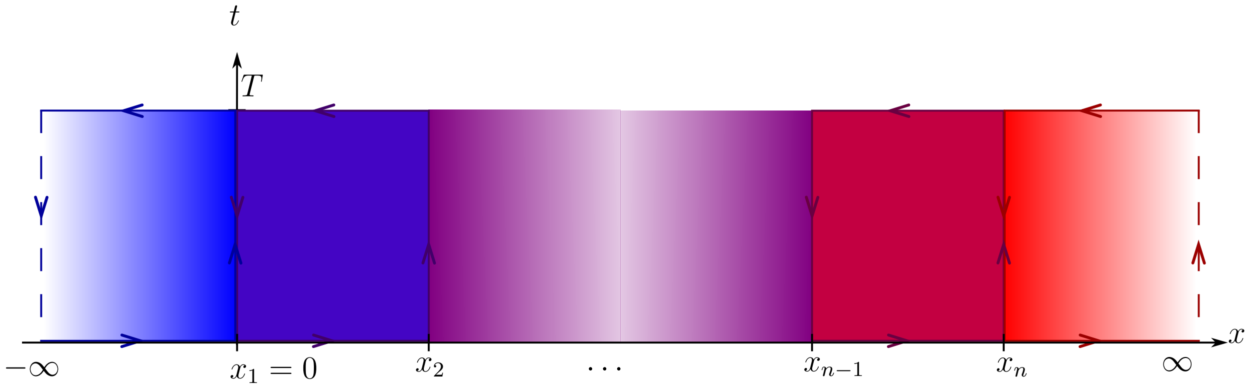

Figure 7.1: Domains for the application of Green's Theorem in the case of an infinite domain with n interfaces.

Theorem we find the global relations:

$$\begin{aligned} 0 = &\int_{x_{j-1}}^{x_j} e^{-ikx} u_0^{(j)}(x)\,\mathrm{d}x - \int_{x_{j-1}}^{x_j} e^{-ikx+\omega_j(k)T} u^{(j)}(x,T)\,\mathrm{d}x \\ &+ \int_0^T \sigma_j^2 e^{-ikx_j+\omega_j(k)s}(u_x^{(j)}(x_j,s) + iku^{(j)}(x_j,s))\,\mathrm{d}s \\ &- \int_0^T \sigma_j^2 e^{-ikx_{j-1}+\omega_j(k)s}(u_x^{(j)}(x_{j-1},s) + iku^{(j)}(x_{j-1},s))\,\mathrm{d}s, \end{aligned} \tag{7.4}$$

for $1 \le j \le n+1$. Define $D = \{k \in \mathbb{C} : \mathrm{Re}(\omega_j(k)) < 0\}$, $D_R = \{k \in D : |k| < R\}$ and $D_R^+ = \{k \in D_R : \mathrm{Im}(k) > 0\}$ as in Figure 2.6 where $R > 0$ is an arbitrary finite constant. When $j = 1$ (7.4) is valid for $k \in \mathbb{C}^+ \setminus D$. Similarly, for $j = n+1$, (7.4) is valid for $k \in \mathbb{C}^- \setminus D$. For $2 \le j \le n$, (7.4) is valid for $k \in \mathbb{C} \setminus D$. The dispersion relation $\omega_j(k) = (\sigma_j k)^2$ is invariant under the symmetry $k \to -k$. We supplement the $n+1$ global relations above with their evaluation at $-k$, namely,

$$\begin{aligned} 0 = &\int_{x_{j-1}}^{x_j} e^{ikx} u_0^{(j)}(x)\,\mathrm{d}x - \int_{x_{j-1}}^{x_j} e^{ikx+\omega_j(k)T} u^{(j)}(x,T)\,\mathrm{d}x \\ &+ \int_0^T \sigma_j^2 e^{ikx_j+\omega_j(k)s}(u_x^{(j)}(x_j,s) - iku^{(j)}(x_j,s))\,\mathrm{d}s \\ &- \int_0^T \sigma_j^2 e^{ikx_{j-1}+\omega_j(k)s}(u_x^{(j)}(x_{j-1},s) - iku^{(j)}(x_{j-1},s))\,\mathrm{d}s, \end{aligned} \tag{7.5}$$

for $1 \le j \le n+1$. When $j=1$, (7.5) is valid for $k \in \mathbb{C}^- \setminus D$. Similarly, for $j=n+1$, (7.5) is valid for $k \in \mathbb{C}^+ \setminus D$. For $2 \le j \le n$, (7.5) is valid for all $k \in \mathbb{C} \setminus D$. Without loss of generality we choose to work with the equations valid in the upper half plane. Define

$$\begin{aligned}
g_0^{(j)}(\omega,t) &= \int_0^t e^{\omega s} u^{(j)}(x_j,s)\,\mathrm{d}s = \int_0^t e^{\omega s} u^{(j+1)}(x_j,s)\,\mathrm{d}s,\\
g_1^{(j)}(\omega,t) &= \int_0^t e^{\omega s} u_x^{(j)}(x_j,s)\,\mathrm{d}s = \frac{\sigma_2^2}{\sigma_1^2}\int_0^t e^{\omega s} u_x^{(j+1)}(x_j,s)\,\mathrm{d}s,\\
\hat{u}^{(j)}(k,t) &= \int_{x_{j-1}}^{x_j} e^{-ikx} u^{(j)}(x,t)\,\mathrm{d}x,\\
\hat{u}_0^{(j)}(k) &= \int_{x_{j-1}}^{x_j} e^{-ikx} u_0^{(j)}(x)\,\mathrm{d}x,
\end{aligned}$$

for $1 \le j \le n$. Using the change of variables $k = \kappa/\sigma_j$ on the j^{th} equation, the global relations valid in the upper-half plane are

$$e^{\kappa^2 T}\hat{u}^{(1)}\left(\frac{\kappa}{\sigma_1},T\right) - \hat{u}_0^{(1)}\left(\frac{\kappa}{\sigma_1}\right) = e^{-i\kappa x_1/\sigma_1}\left(\frac{i\kappa}{\sigma_1}g_0^{(1)}(\kappa^2,T) + g_1^{(1)}(\kappa^2,T)\right), \tag{7.6a}$$

$$\begin{aligned}
e^{\kappa^2 T}\hat{u}^{(j)}\left(\frac{\kappa}{\sigma_j},T\right) - \hat{u}_0^{(j)}\left(\frac{\kappa}{\sigma_j}\right) =& e^{\frac{-i\kappa x_j}{\sigma_j}}\left(\frac{i\kappa}{\sigma_j}g_0^{(j)}(\kappa^2,T) + g_1^{(j)}(\kappa^2,T)\right)\\
&- e^{\frac{-i\kappa x_{j-1}}{\sigma_j}}\left(\frac{i\kappa}{\sigma_j}g_0^{(j-1)}(\kappa^2,T) + \frac{\sigma_{j-1}^2}{\sigma_j^2}g_1^{(j-1)}(\kappa^2,T)\right),
\end{aligned} \tag{7.6b}$$

$$\begin{aligned}
e^{\kappa^2 T}\hat{u}^{(j)}\left(\frac{-\kappa}{\sigma_j},T\right) - \hat{u}_0^{(j)}\left(\frac{-\kappa}{\sigma_j}\right) =& e^{\frac{i\kappa x_j}{\sigma_j}}\left(\frac{-i\kappa}{\sigma_j}g_0^{(j)}(\kappa^2,T) + g_1^{(j)}(\kappa^2,T)\right)\\
&+ e^{\frac{i\kappa x_{j-1}}{\sigma_j}}\left(\frac{i\kappa}{\sigma_j}g_0^{(j-1)}(\kappa^2,T) - \frac{\sigma_{j-1}^2}{\sigma_j^2}g_1^{(j-1)}(\kappa^2,T)\right),
\end{aligned} \tag{7.6c}$$

$$e^{\kappa^2 T}\hat{u}^{(n+1)}\left(\frac{-\kappa}{\sigma_{n+1}},T\right) - \hat{u}_0^{(n+1)}\left(\frac{-\kappa}{\sigma_{n+1}}\right) = e^{\frac{i\kappa x_n}{\sigma_{n+1}}}\left(\frac{i\kappa}{\sigma_{n+1}}g_0^{(n)}(\kappa^2,T) - \frac{\sigma_n^2}{\sigma_{n+1}^2}g_1^{(n)}(\kappa^2,T)\right), \tag{7.6d}$$

for $2 \le j \le n$. Equation (7.6) can be written as a linear system for the interface values:

$$\mathcal{A}(\kappa)X(\kappa^2, T) = Y(\kappa) + \mathcal{Y}(\kappa, T),$$

where

$$X(\kappa^2, T) = \left(g_0^{(1)}, g_0^{(2)}, \ldots, g_0^{(n)}, g_1^{(1)}, g_1^{(2)}, \ldots, g_1^{(n)}\right)^\top,$$

$$Y(\kappa) = -\left(\hat{u}_0^{(1)}\left(\frac{\kappa}{\sigma_1}\right), \ldots, \hat{u}_0^{(n)}\left(\frac{\kappa}{\sigma_n}\right), \hat{u}_0^{(2)}\left(\frac{-\kappa}{\sigma_2}\right), \ldots, \hat{u}_0^{(n+1)}\left(\frac{-\kappa}{\sigma_{n+1}}\right)\right)^\top,$$

$$\mathcal{Y}(\kappa, T) = e^{\kappa^2 T}\left(\hat{u}^{(1)}\left(\frac{\kappa}{\sigma_1}, T\right), \ldots, \hat{u}^{(n)}\left(\frac{\kappa}{\sigma_n}, T\right), \hat{u}^{(2)}\left(\frac{-\kappa}{\sigma_2}, T\right), \ldots, \hat{u}^{(n+1)}\left(\frac{-\kappa}{\sigma_{n+1}}, T\right)\right)^\top,$$

and

$$\mathcal{A}(\kappa) =$$

$$\left(\begin{array}{ccc|ccc}
\frac{i\kappa}{\sigma_1} e^{-i\frac{\kappa x_1}{\sigma_1}} & & & e^{-i\frac{\kappa x_1}{\sigma_1}} & & \\
\frac{-i\kappa}{\sigma_2} e^{-i\frac{\kappa x_1}{\sigma_2}} & \frac{i\kappa}{\sigma_2} e^{-i\frac{\kappa x_2}{\sigma_2}} & & \frac{-\sigma_1^2}{\sigma_2^2} e^{-i\frac{\kappa x_1}{\sigma_2}} & e^{-i\frac{\kappa x_2}{\sigma_2}} & \\
\ddots & \ddots & & \ddots & \ddots & \\
 & \frac{-i\kappa}{\sigma_n} e^{-i\frac{\kappa x_{n-1}}{\sigma_n}} & \frac{i\kappa}{\sigma_n} e^{-i\frac{\kappa x_n}{\sigma_n}} & & \frac{-\sigma_{n-1}^2}{\sigma_n^2} e^{-i\frac{\kappa x_{n-1}}{\sigma_n}} & e^{-i\frac{\kappa x_n}{\sigma_n}} \\
\hline
\frac{i\kappa}{\sigma_2} e^{i\frac{\kappa}{\sigma_2} x_1} & \frac{-i\kappa}{\sigma_2} e^{i\frac{\kappa x_2}{\sigma_2}} & & \frac{-\sigma_1^2}{\sigma_2^2} e^{i\frac{\kappa x_1}{\sigma_2}} & e^{i\frac{\kappa x_2}{\sigma_2}} & \\
\ddots & \ddots & & \ddots & \ddots & \\
 & \frac{i\kappa}{\sigma_{n-1}} e^{i\frac{\kappa x_{n-1}}{\sigma_n}} & \frac{-i\kappa}{\sigma_n} e^{i\frac{\kappa x_n}{\sigma_n}} & & \frac{-\sigma_{n-1}^2}{\sigma_n^2} e^{i\frac{\kappa x_{n-1}}{\sigma_n}} & e^{i\frac{\kappa x_n}{\sigma_n}} \\
 & & \frac{i\kappa}{\sigma_{n+1}} e^{i\frac{\kappa x_n}{\sigma_{n+1}}} & & & \frac{-\sigma_n^2}{\sigma_{n+1}^2} e^{i\frac{\kappa x_n}{\sigma_{n+1}}}
\end{array}\right).$$

The matrix $\mathcal{A}(\kappa)$ consists of four $n \times n$ blocks as indicated by the dashed lines. The two blocks in the upper half of $\mathcal{A}(\kappa)$ are zero except for entries on the main and -1 diagonals. The lower two blocks of $\mathcal{A}(\kappa)$ only have nonzero entries on the main and $+1$ diagonals. The matrix $\mathcal{A}(\kappa)$ is singular for isolated values of κ. Asymptotically, for large $|\kappa|$, the zeros of $\det(\mathcal{A}(\kappa))$ are on the real line [42]. Since asymptotically there are no zeros in D_R^+, a sufficiently large R may be chosen such that $\mathcal{A}(\kappa)$ is nonsingular for every $\kappa \in D_R^+$ and $\det(\mathcal{A}(\kappa)) \neq 0$.

Remark. We have been unable to construct physical examples where the zeros of $\det(\mathcal{A}(\kappa))$ are in D^+ and are different from 0. However, if nonphysical values of the parameters are chosen (*e.g.*, σ_j imaginary), then $\det(\mathcal{A}(\kappa))$ has zeros in D^+.

Using Cramer's Rule to solve this system, we have

$$g_0^{(j)}(\kappa^2,T) = \frac{\det(\mathcal{A}_j(\kappa,T))}{\det(\mathcal{A}(\kappa))}, \tag{7.7a}$$

$$g_1^{(j)}(\kappa^2,T) = \frac{\det(\mathcal{A}_{j+n}(\kappa,T))}{\det(\mathcal{A}(\kappa))}, \tag{7.7b}$$

where $1 \le j \le n$ and $\mathcal{A}_j(\kappa,T)$ is the matrix $\mathcal{A}(\kappa)$ with the j^{th} column replaced by $Y+\mathcal{Y}$. This does not give an effective initial-to-interface map because (7.7) depends on the solutions $\hat{u}^{(j)}(\cdot,T)$. To eliminate this dependence we multiply (7.7) by $\kappa e^{-\kappa^2 t}$ and integrate around D_R^+, as is typical in the construction of Dirichlet-to-Neumann maps [25]. Switching the order of integration we have

$$\int_0^T u^{(j)}(x_j,s) \int_{\partial D_R^+} \kappa e^{\kappa^2(s-t)}\,\mathrm{d}\kappa\,\mathrm{d}s = \int_{\partial D_R^+} e^{-\kappa^2 t} \frac{\kappa \det(\mathcal{A}_j(\kappa,T))}{\det(\mathcal{A}(\kappa))}\,\mathrm{d}\kappa, \tag{7.8a}$$

$$\int_0^T u_x^{(j)}(x_j,s) \int_{\partial D_R^+} \kappa e^{\kappa^2(s-t)}\,\mathrm{d}\kappa\,\mathrm{d}s = \int_{\partial D_R^+} e^{-\kappa^2 t} \frac{\kappa \det(\mathcal{A}_{j+n}(\kappa,T))}{\det(\mathcal{A}(\kappa))}\,\mathrm{d}\kappa. \tag{7.8b}$$

Using the change of variables $i\ell = \kappa^2$ and the classical Fourier transform formula for the delta function we have

$$u^{(j)}(x_j,t) = \frac{1}{i\pi} \int_{\partial D_R^+} e^{-\kappa^2 t} \frac{\kappa \det(\mathcal{A}_j(\kappa,T))}{\det(\mathcal{A}(\kappa))}\,\mathrm{d}\kappa, \tag{7.9a}$$

$$u_x^{(j)}(x_j,t) = \frac{1}{i\pi} \int_{\partial D_R^+} e^{-\kappa^2 t} \frac{\kappa \det(\mathcal{A}_{j+n}(\kappa,T))}{\det(\mathcal{A}(\kappa))}\,\mathrm{d}\kappa. \tag{7.9b}$$

To examine the right-hand-side of (7.9) we factor the matrix $\mathcal{A}(\kappa)$ as $\mathcal{A}^L(\kappa)\mathcal{A}^M(\kappa)$ where

$$\mathcal{A}^L(\kappa) = \left(\begin{array}{cccc|cccc} e^{-i\frac{\kappa}{\sigma_1}x_1} & & & & & & & \\ & e^{-i\frac{\kappa}{\sigma_2}x_2} & & & & & & \\ & & \ddots & & & & & \\ & & & e^{-i\frac{\kappa}{\sigma_n}x_n} & & & & \\ \hline & & & & e^{i\frac{\kappa}{\sigma_2}x_1} & & & \\ & & & & & e^{i\frac{\kappa}{\sigma_3}x_2} & & \\ & & & & & & \ddots & \\ & & & & & & & e^{i\frac{\kappa}{\sigma_{n+1}}x_n} \end{array}\right)$$

is a diagonal matrix. The elements of $\mathcal{A}^M(\kappa)$ are either 0, $\mathcal{O}(\kappa)$, or decaying exponentially fast for $\kappa \in D_R^+$. Hence,

$$\det(\mathcal{A}^M(\kappa)) = c(\kappa) = \mathcal{O}(\kappa^n),$$

for large κ in D_R^+. Now, $\det(\mathcal{A}(\kappa)) = c(\kappa)\det(\mathcal{A}^L(\kappa))$ as $\kappa \to \infty$ for $\kappa \in D_R^+$. Similarly, factor $\mathcal{A}_j(\kappa, T) = \mathcal{A}^L(\kappa)\mathcal{A}_j^M(\kappa, T)\mathcal{A}_j^R(\kappa, T)$ where $\mathcal{A}_j^R(\kappa, T)$ is the $2n \times 2n$ identity matrix with the (j, j) component replaced by $e^{\kappa^2 T}$. Then $\det(\mathcal{A}_j(\kappa, T)) = e^{\kappa^2 T}\det(\mathcal{A}^L(\kappa))\det(\mathcal{A}_j^M(\kappa, T))$. Thus, the integrand we are considering in (7.9) is

$$\int_{\partial D_R^+} e^{-\kappa^2 t}\frac{\kappa\det(\mathcal{A}_j(\kappa, T))}{\det(\mathcal{A})}\,\mathrm{d}\kappa = \int_{\partial D_R^+} e^{\kappa^2(T-t)}\frac{\kappa\det(\mathcal{A}_j^M(\kappa, T))}{c(\kappa)}\,\mathrm{d}\kappa.$$

The elements of $\mathcal{A}_j^M(\kappa, T)$ are the same as those in $\mathcal{A}^M(\kappa)$ except in the j^{th} column. Expanding the determinant of $\mathcal{A}_j^M(\kappa, T)$ along the j^{th} column we see that

$$\begin{aligned} e^{\kappa^2(T-t)}\frac{\kappa\det(\mathcal{A}_j^M(\kappa, T))}{c(\kappa)} &= \sum_{\ell=1}^{n}\left(c_\ell(\kappa)\left(e^{\frac{i\kappa x_\ell}{\sigma_\ell}+\kappa^2(T-t)}\hat{u}^{(\ell)}\left(\frac{\kappa}{\sigma_\ell}, T\right) - e^{-\kappa^2 t+\frac{i\kappa x_\ell}{\sigma_\ell}}\hat{u}_0^{(\ell)}\left(\frac{\kappa}{\sigma_\ell}\right)\right)\right. \\ &\left. + c_{\ell+n}(\kappa)\left(e^{\frac{-i\kappa x_\ell}{\sigma_{\ell+1}}+\kappa^2(T-t)}\hat{u}^{(\ell+1)}\left(\frac{-\kappa}{\sigma_{\ell+1}}, T\right) - e^{-\kappa^2 t-\frac{i\kappa x_\ell}{\sigma_{\ell+1}}}\hat{u}_0^{(\ell+1)}\left(\frac{-\kappa}{\sigma_{\ell+1}}\right)\right)\right), \end{aligned} \tag{7.10}$$

where $c_\ell = \mathcal{O}(\kappa^0)$ and $c_{\ell+n} = \mathcal{O}(\kappa)$ for $1 \le \ell \le n$. The terms involving $\hat{u}^{(\ell)}(\cdot, T)$, the solutions of our equation, are decaying exponentially for $\kappa \in D_R^+$. Thus, by Jordan's Lemma [1], the integral of this term along a closed, bounded curve in $\mathbb{C}^+$ vanishes. In particular we consider the closed curve $\mathcal{L}^+ = \mathcal{L}_{D_R^+} \cup \mathcal{L}_C^+$ where $\mathcal{L}_{D_R^+} = \partial D_R^+ \cap \{k : |k| < C\}$ and $\mathcal{L}_C^+ = \{k \in D_R^+ : |k| = C\}$, see Figure 1.3. Since the integral along $\mathcal{L}_C^+$ vanishes for large C, (7.10) must vanish since the contour $\mathcal{L}_{D_R^+}$ becomes ∂D_R^+ as $C \to \infty$.

Since the terms involving the elements of $\mathcal{Y}(\kappa, T)$ evaluate to zero in the solution expression we have the solution

$$u^{(j)}(x_j, t) = \frac{1}{i\pi} \int_{\partial D_R^+} e^{-\kappa^2 t} \frac{\kappa \det(A_j(\kappa))}{\det(\mathcal{A}(\kappa))} \, \mathrm{d}\kappa, \tag{7.11a}$$

$$u_x^{(j)}(x_j, t) = \frac{1}{i\pi} \int_{\partial D_R^+} e^{-\kappa^2 t} \frac{\kappa \det(A_{j+n}(\kappa))}{\det(\mathcal{A}(\kappa))} \, \mathrm{d}\kappa, \tag{7.11b}$$

where $A_j(\kappa)$ is the matrix $\mathcal{A}(\kappa)$ with the j^{th} column replaced by $Y(\kappa)$. Equation 7.11 is an effective map between the values of the function at the interface and the given initial conditions.

Remark. Note that since the problem is linear, one could have assumed the initial condition was zero for x outside the region $x_{\ell-1} < x < x_\ell$. Then, the map would be in terms of just $u_0^{(\ell)}(\cdot)$. Summing over $1 \leq \ell \leq n+1$ would give the complete map for a general initial condition.

As an example of a specific initial-to-interface map we consider the equation (7.1) with $n = 1$. Using (7.11) we have

$$\sigma_1^2 u_x^{(1)}(0, t) = \frac{i\sigma_1\sigma_2}{\pi(\sigma_1 + \sigma_2)} \int_{\partial D_R^+} \kappa e^{-\kappa^2 t} \left(\sigma_1 \hat{u}_0^{(1)} \left(\frac{\kappa}{\sigma_1} \right) - \sigma_2 \hat{u}_0^{(2)} \left(\frac{-\kappa}{\sigma_2} \right) \right) \mathrm{d}\kappa,$$

$$u^{(1)}(0, t) = \frac{1}{\pi(\sigma_1 + \sigma_2)} \int_{\partial D_R^+} e^{-\kappa^2 t} \left(\sigma_1^2 \hat{u}_0^{(1)} \left(\frac{\kappa}{\sigma_1} \right) + \sigma_2^2 \hat{u}_0^{(2)} \left(\frac{-\kappa}{\sigma_2} \right) \right) \mathrm{d}\kappa.$$

In this case we can deform D_R^+ back to the real line easily. For general n this is not the case.

Switching the order of integration and evaluating the κ integral we have

$$\sigma_1^2 u_x^{(1)}(0,t) = \frac{\sigma_1\sigma_2}{2t^{3/2}\sqrt{\pi}(\sigma_1+\sigma_2)}\left(\int_{-\infty}^0 ye^{-y^2/(4t\sigma_1^2)}u_0^{(1)}(y)\,\mathrm{d}y + \int_0^\infty ye^{-y^2/(4t\sigma_2^2)}u_0^{(2)}(y)\,\mathrm{d}y\right), \tag{7.12a}$$

$$u^{(1)}(0,t) = \frac{1}{\sqrt{\pi t}(\sigma_1+\sigma_2)}\left(\sigma_1^2\int_{-\infty}^0 e^{-y^2/(4t\sigma_1^2)}u_0^{(1)}(y)\,\mathrm{d}y + \sigma_2^2\int_0^\infty e^{-y^2/(4t\sigma_2^2)}u_0^{(2)}(y)\,\mathrm{d}y\right), \tag{7.12b}$$

which is an explicit map from the initial data to the value of the temperature and its associated flux at the interface, $x = 0$. If one allows $\sigma_1 = \sigma_2$ the problem is simply that of the heat equation on the whole line. Equation (7.12) with $\sigma_1 = \sigma_2$ is exactly the Green's Function solution of the whole line problem evaluated at $x = 0$ [38].

7.2 Heat equation on a finite domain with n interfaces

Consider (7.1) on a finite domain, $x_0 \le x \le x_{n+1}$, with the boundary conditions

$$\beta_1 u^{(1)}(x_0,t) + \beta_2 u_x^{(1)}(x_0,t) = f_1(t), \qquad t > 0, \tag{7.13a}$$

$$\beta_3 u^{(n+1)}(x_{n+1},t) + \beta_4 u_x^{(n+1)}(x_{n+1},t) = f_2(t), \qquad t > 0. \tag{7.13b}$$

As before, we rewrite (7.1) as the set of equations

$$u_t^{(j)} = \sigma_j^2 u_{xx}^{(j)}, \qquad x_{j-1} < x < x_j,\ \ 0 < t < T,$$

for $1 \le j \le n+1$ subject to the continuity interface conditions

$$u^{(j)}(x_j,t) = u^{(j+1)}(x_j,t), \qquad t > 0,$$

$$\sigma_j^2 u_x^{(j)}(x_j,t) = \sigma_{j+1}^2 u_x^{(j+1)}(x_j,t), \qquad t > 0,$$

for $1 \le j \le n$. Without loss of generality shift the problem so $x_0 = 0$.

The following steps are very similar to those presented in the previous section. In what follows we give a brief outline of the changes needed to solve on a finite domain.

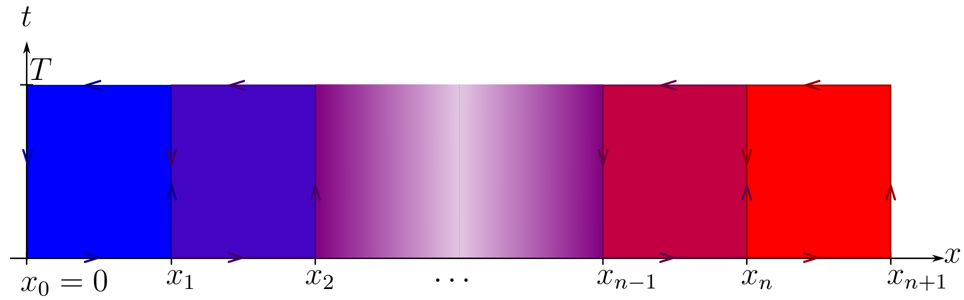

Figure 7.2: Domains for the application of Green's Theorem in the case of a finite domain with n interfaces.

Integrating the local relations (7.3) around the appropriate domain (see Figure 7.1) and applying Green's Theorem we find the global relations (7.4) and their evaluation at $-k$ (7.5). In contrast to Section 7.1, these $2n+2$ global relations are all valid for $k \in \mathbb{C} \setminus D$.

Without loss of generality we choose to work with the equations valid in the upper-half plane. In addition to the definitions in Section 7.1 we define

$$\begin{aligned}
g_0^{(0)}(\omega, t) &= \int_0^t e^{\omega s} u^{(1)}(x_0, s)\,\mathrm{d}s, \\
g_0^{(n+1)}(\omega, t) &= \int_0^t e^{\omega s} u^{(n+1)}(x_{n+1}, s)\,\mathrm{d}s, \\
g_1^{(0)}(\omega, t) &= \int_0^t e^{\omega s} u_x^{(1)}(x_0, s)\,\mathrm{d}s, \\
g_1^{(n+1)}(\omega, t) &= \int_0^t e^{\omega s} u_x^{(n+1)}(x_{n+1}, s)\,\mathrm{d}s, \\
\tilde{f}_m(\omega, t) &= \int_0^t e^{\omega s} f_m(s)\,\mathrm{d}s,
\end{aligned}$$

for $m = 1, 2$. Using the change of variables $k = \kappa/\sigma_j$, the global relations valid in the upper-half plane are

$$
\begin{aligned}
e^{\kappa^2 t}\hat{u}^{(j)}\left(\frac{\kappa}{\sigma_j},T\right)-\hat{u}_0^{(j)}\left(\frac{\kappa}{\sigma_j}\right)=\;& e^{\frac{-i\kappa x_j}{\sigma_j}}\left(\frac{i\kappa}{\sigma_j}g_0^{(j)}(\kappa^2,T)+g_1^{(j)}(\kappa^2,T)\right)\\
&-e^{\frac{-i\kappa x_{j-1}}{\sigma_j}}\left(\frac{i\kappa}{\sigma_j}g_0^{(j-1)}(\kappa^2,T)+\frac{\sigma_{j-1}^2}{\sigma_j^2}g_1^{(j-1)}(\kappa^2,T)\right),
\end{aligned}
\tag{7.14a}
$$

$$
\begin{aligned}
e^{\kappa^2 t}\hat{u}^{(j)}\left(\frac{-\kappa}{\sigma_j},T\right)-\hat{u}_0^{(j)}\left(\frac{-\kappa}{\sigma_j}\right)=\;& e^{\frac{i\kappa x_j}{\sigma_j}}\left(\frac{-i\kappa}{\sigma_j}g_0^{(j)}(\kappa^2,T)+g_1^{(j)}(\kappa^2,T)\right)\\
&+e^{\frac{i\kappa x_{j-1}}{\sigma_j}}\left(\frac{i\kappa}{\sigma_j}g_0^{(j-1)}(\kappa^2,T)-\frac{\sigma_{j-1}^2}{\sigma_j^2}g_1^{(j-1)}(\kappa^2,T)\right),
\end{aligned}
\tag{7.14b}
$$

for $1 \le j \le n+1$ where we define $\sigma_0=\sigma_1$ for convenience. These equations, together with the boundary values (7.13), can be written as a linear system for the interface values

$$
\mathcal{A}^F X^F = Y^F + \mathcal{Y}^F,
$$

where

$$
X^F(\kappa^2,T)=\left(g_0^{(0)},g_0^{(1)},\ldots,g_0^{(n+1)},g_1^{(0)},g_1^{(1)},\ldots,g_1^{(n+1)}\right)^\top,
$$

$$
Y^F(\kappa,T)=-\left(-\tilde{f}_1(i\kappa^2,T),\hat{u}_0^{(1)}\left(\frac{\kappa}{\sigma_1}\right),\ldots,\hat{u}_0^{(n+1)}\left(\frac{\kappa}{\sigma_n}\right),\hat{u}_0^{(1)}\left(\frac{-\kappa}{\sigma_1}\right),\ldots,\hat{u}_0^{(n+1)}\left(\frac{-\kappa}{\sigma_{n+1}}\right),-\tilde{f}_2(i\kappa^2,T)\right)^\top,
$$

$$
\mathcal{Y}^F(\kappa,T)=e^{\kappa^2 T}\left(0,\hat{u}^{(1)}\left(\frac{\kappa}{\sigma_1},T\right),\ldots,\hat{u}^{(n+1)}\left(\frac{\kappa}{\sigma_n},T\right),\hat{u}^{(1)}\left(\frac{-\kappa}{\sigma_1},T\right),\ldots,\hat{u}^{(n+1)}\left(\frac{-\kappa}{\sigma_{n+1}},T\right),0\right)^\top,
$$

and

$$
\mathcal{A}^F(\kappa)=
$$

$$
\left(\begin{array}{ccc:ccc}
\beta_1 & & & \beta_2 & & \\
\frac{-i\kappa}{\sigma_1}e^{-i\frac{\kappa x_0}{\sigma_1}} & \frac{i\kappa}{\sigma_1}e^{-i\frac{\kappa x_1}{\sigma_1}} & & -\frac{\sigma_0^2}{\sigma_1^2}e^{-i\frac{\kappa x_0}{\sigma_1}} & e^{-i\frac{\kappa x_1}{\sigma_1}} & \\
\ddots & \ddots & & \ddots & \ddots & \\
 & \frac{-i\kappa}{\sigma_{n+1}}e^{-i\frac{\kappa x_n}{\sigma_{n+1}}} & \frac{i\kappa}{\sigma_{n+1}}e^{-i\frac{\kappa x_{n+1}}{\sigma_{n+1}}} & & \frac{-\sigma_n^2}{\sigma_{n+1}^2}e^{-i\frac{\kappa x_n}{\sigma_{n+1}}} & e^{-i\frac{\kappa x_{n+1}}{\sigma_{n+1}}} \\
\hdashline
\frac{i\kappa}{\sigma_1}e^{i\frac{\kappa x_0}{\sigma_1}} & \frac{-i\kappa}{\sigma_1}e^{i\frac{\kappa x_1}{\sigma_1}} & & \frac{-\sigma_0^2}{\sigma_1^2}e^{i\frac{\kappa x_0}{\sigma_1}} & e^{i\frac{\kappa x_1}{\sigma_1}} & \\
\ddots & \ddots & & \ddots & \ddots & \\
 & \frac{i\kappa}{\sigma_{n+1}}e^{i\frac{\kappa x_{n+1}}{\sigma_{n+1}}} & \frac{-i\kappa}{\sigma_{n+1}}e^{i\frac{\kappa x_{n+1}}{\sigma_{n+1}}} & & \frac{-\sigma_n^2}{\sigma_{n+1}^2}e^{i\frac{\kappa x_n}{\sigma_{n+1}}} & e^{i\frac{\kappa x_{n+1}}{\sigma_{n+1}}} \\
 & & \beta_3 & & & \beta_4
\end{array}\right).
$$

The matrix $\mathcal{A}^F(\kappa)$ is made up of four $(n+2)\times(n+2)$ blocks as indicated by the dashed lines. The two blocks in the upper half of $\mathcal{A}^F(\kappa)$ are zero except for entries on the main and -1 diagonals. The lower two blocks of $\mathcal{A}^F(\kappa)$ only have entries on the main and $+1$ diagonals.

As before we use Cramer's Rule to solve this system. After multiplying the solutions by $\kappa e^{-\kappa^2 t}$, integrating around D_R^+, and simplifying as in the previous section we follow a similar process to show the terms from $\mathcal{Y}^F(\kappa, T)$ do not contribute to our solution formula using Jordan's Lemma and Cauchy's Theorem. One can show that $A_j^F(\kappa, T)$ can be replaced by $A_j^F(\kappa, t)$ by writing $\int_0^T \cdot \, \mathrm{d}s$ as $\int_0^t \cdot \, \mathrm{d}s + \int_t^T \cdot \, \mathrm{d}s$ and noticing where the function in analytic and decaying. If the boundary conditions (7.13) are time-independent then so is A_j^F.

In general, the initial-to-interface map for the heat equation on a finite domain with n interfaces is given by

$$u^{(j)}(x_j, t) = \int_{\partial D_R^+} e^{-\kappa^2 t} \frac{\kappa \det(A_j^F(\kappa, t))}{i\pi \det(\mathcal{A}^F(\kappa))} \, \mathrm{d}\kappa, \tag{7.16a}$$

$$u_x^{(j)}(x_j, t) = \int_{\partial D_R^+} e^{-\kappa^2 t} \frac{\kappa \det(A_{j+n}^F(\kappa, t))}{i\pi \det(\mathcal{A}^F(\kappa))} \, \mathrm{d}\kappa. \tag{7.16b}$$

where $A_j^F(\kappa, t)$ is the matrix $\mathcal{A}^F(\kappa, t)$ with the j^{th} column replaced by $Y^F(\kappa, t)$.

Chapter 8

The Stefan problem for the heat equation

A Stefan problem is a BVP for a PDE with a phase boundary that can move with time. In what follows, we consider the temperature distribution in a homogeneous medium undergoing a phase change, *i.e.*, ice melting into water. We solve the heat equation on the time-dependent domain and impose an initial temperature distribution as well as the "Stefan condition." This condition expresses the local velocity of a moving boundary as a function of quantities evaluated at both sides of the boundary. This is derived as usual by imposing conservation of energy. These problems take their name from Jožef Stefan [60] but were first considered by Lamé and Clapeyron [40]. In the case when the dependence of the boundary on time is known, this problem has been studied using the Fokas method in [26, 28, 29, 52]. This work is part of an ongoing collaboration with B. Deconinck, J. Lenells, B. Pelloni, and V. Vasan.

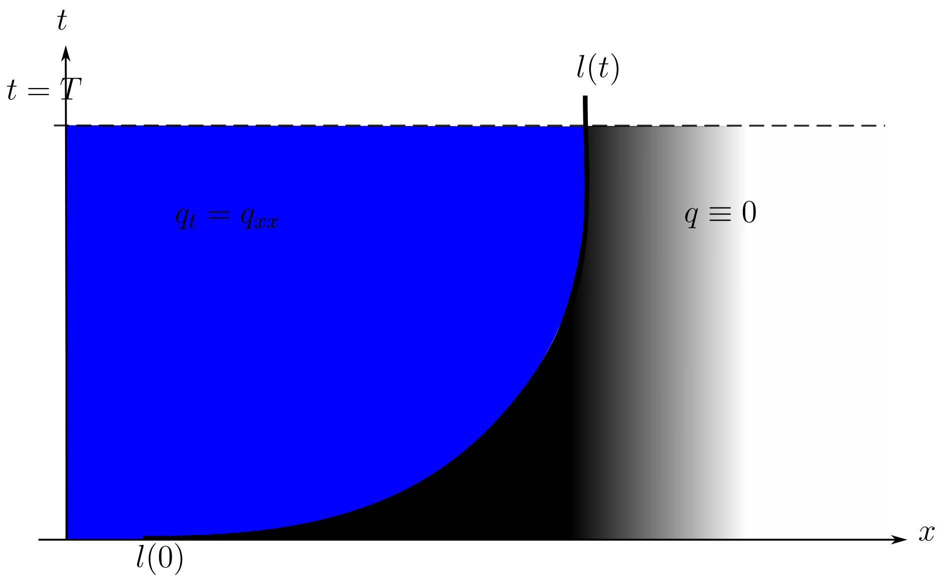

Figure 8.1: The domain under consideration in (8.1) including the moving boundary $l(t)$.

8.1 The one-phase Stefan problem on a semi-infinite domain

We consider the heat equation in one time-dependent domain as shown in Figure 8.1,

$$q_t - q_{xx} = 0, \qquad 0 < x < l(t), \quad t \geq 0, \tag{8.1}$$

with $l(t)$ a monotonic unknown function. The given initial, boundary, and interface conditions are

$$q(x,0) = q_0(x), \qquad 0 < x < l(t), \tag{8.2a}$$

$$q_x(0,t) = f(t), \qquad 0 < t < T, \tag{8.2b}$$

$$q(l(t),t) = 0, \qquad 0 < t < T, \tag{8.2c}$$

$$q_x(l(t),t) = -l'(t), \qquad 0 < t < T,. \tag{8.2d}$$

8.1.1 A similarity solution

Consider (8.1) and (8.2) and choose $\zeta = x/\sqrt{t}$. We look for a similarity solution $q(x,t) = Q(\zeta)$. The heat equation in the semi-infinite region $l(0) < x < l(t)$ for $t > 0$ becomes the ordinary differential equation

$$2Q''(\zeta) + \zeta Q'(\zeta) = 0, \tag{8.3}$$

where $'$ denotes a derivative with respect to ζ. Solving for $Q(\zeta)$ we have

$$Q(\zeta) = c_1 + c_2\sqrt{\pi}\,\mathrm{erf}\left(\frac{\zeta}{2}\right),$$

where c_1 and c_2 are constants of integration and $\mathrm{erf}(y) = 2/\sqrt{\pi}\int_0^y e^{-s^2}\,\mathrm{d}s$. To find c_2 we impose $q(0,t) = f(t)$. This gives $c_2 = \sqrt{t}f(t)$. However, c_2 must be constant with respect to t so $f(t) = \frac{c_3}{\sqrt{t}}$ where c_3 is constant. To solve for c_1 we use the condition of continuity at the interface, $q(l(t),t) = 0$. Then

$$c_1 = -c_3\sqrt{\pi}\,\mathrm{erf}\left(\frac{l(t)}{2\sqrt{t}}\right).$$

Again, c_1 must be a constant (independent of t) so $l(t) = c_4\sqrt{t}$ where c_4 is a constant. Hence, our similarity solution is

$$Q(\zeta) = c_3\sqrt{\pi}\left(\mathrm{erf}\left(\frac{\zeta}{2}\right) - \mathrm{erf}\left(\frac{c_4}{2}\right)\right). \tag{8.4}$$

Imposing $q_x(l(t),t) = -l'(t)$ we find

$$c_4 + 2c_3 e^{-\frac{c_4^2}{4}} = 0, \tag{8.5}$$

which can be solved numerically for c_4.

8.1.2 A generalized Dirichlet to Neumann map using the Fokas method

Consider (8.1) with the initial and boundary conditions (8.2). We begin with the local relation, a one-parameter family rewrite of (8.1):

$$(e^{-ikx+\omega(k)t}q(x,t))_t = (e^{-ikx+\omega(k)t}(q_x(x,t)+ikq(x,t)))_x, \tag{8.6}$$

where $\omega(k)=k^2$. Integrating around the domain $(0,l(t))\times(0,t)$ (see Figure 8.2), assuming continuity, and applying Green's Theorem we have the global relation

$$\begin{aligned}0 = &\int_0^{l(0)} e^{-ikx}q_0(x)\,\mathrm{d}x - \int_0^t e^{-ikl(s)+\omega(k)s}l'(s)\,\mathrm{d}s\\ &-\int_0^{l(t)} e^{-ikx+\omega(k)t}q(x,t)\,\mathrm{d}x - \int_0^t e^{\omega(k)s}(f(s)+ikq(0,s))\,\mathrm{d}s.\end{aligned} \tag{8.7}$$

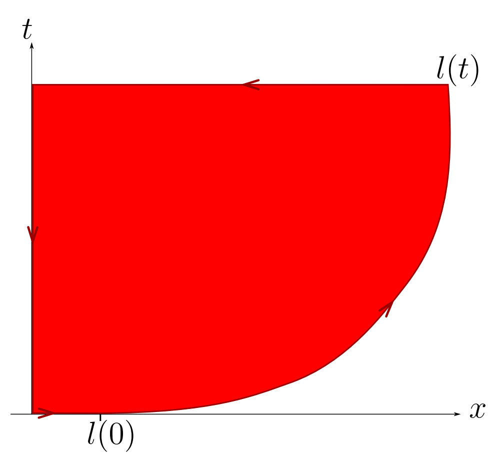

Figure 8.2: Domain for the application of Green's Theorem for the one-phase Stefan problem.

Let $D=\{k\in\mathbb{C}:\mathrm{Re}(\omega(k))<0\}=D^+\cup D^-$. The region D is shown in Figure 8.3.

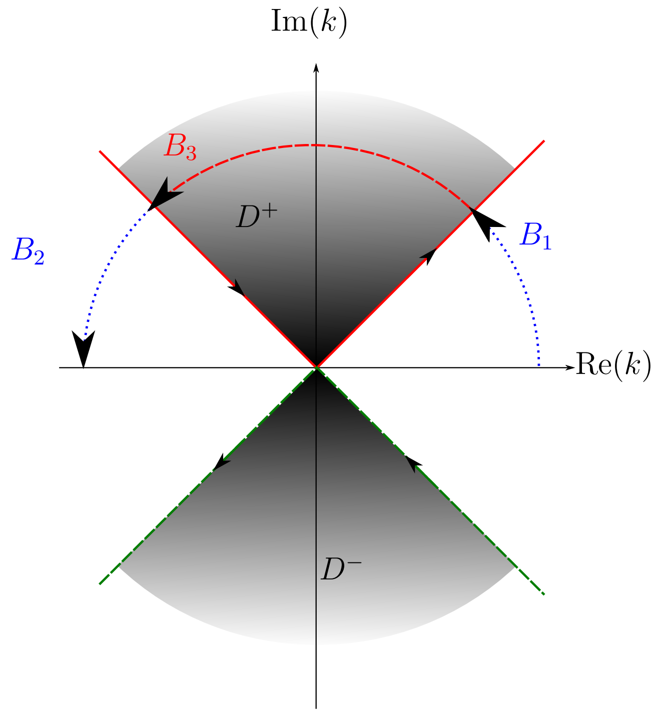

Figure 8.3: The domains D^+ and D^- for the Stefan problem.

Since the dispersion relation $\omega(k)$ is invariant under $k \to -k$ we supplement (8.7) with

$$\begin{aligned} 0 = &\int_0^{l(0)} e^{ikx} q_0(x)\,\mathrm{d}x - \int_0^t e^{ikl(s)+\omega(k)s} l'(s)\,\mathrm{d}s \\ &- \int_0^{l(t)} e^{ikx+\omega(k)t} q(x,t)\,\mathrm{d}x - \int_0^t e^{\omega(k)s}(f(s) - ikq(0,s))\,\mathrm{d}s. \end{aligned} \tag{8.8}$$

To eliminate the integral containing the unknown $q(0,s)$ we add (8.7) and (8.8) to find

$$\begin{aligned} -\int_0^{l(t)} e^{\omega(k)t} q(x,t)(e^{-ikx} + e^{ikx})\,\mathrm{d}x = 2\int_0^t e^{\omega(k)s} f(s)\,\mathrm{d}s \\ + \int_0^t e^{\omega(k)s} l'(s)(e^{-ikl(s)} + e^{ikl(s)})\,\mathrm{d}s - \int_0^{l(0)} q_0(x)(e^{-ikx} + e^{ikx})\,\mathrm{d}x. \end{aligned} \tag{8.9}$$

In analog to inverting the Fourier transform we multiply (8.9) by $ke^{ikl(t)-\omega(k)t}$ and integrate over ∂D^+:

$$\begin{aligned}
&-\int_{\partial D^+}\int_0^{l(t)} kq(x,t)(e^{ik(l(t)-x)}+e^{ik(l(t)+x)})\,\mathrm{d}x\,\mathrm{d}k\\
=&2\int_{\partial D^+}\int_0^{t} ke^{\omega(k)(s-t)+ikl(t)}f(s)\,\mathrm{d}s\,\mathrm{d}k\\
&+\int_{\partial D^+}\int_0^{t} ke^{\omega(k)(s-t)}l'(s)(e^{ik(l(t)-l(s))}+e^{ik(l(t)+l(s))})\,\mathrm{d}s\,\mathrm{d}k\\
&-\int_{\partial D^+}\int_0^{l(0)} kq_0(x)e^{-\omega(k)t}(e^{ik(l(t)-x)}+e^{ik(l(t)+x)})\,\mathrm{d}x\,\mathrm{d}k.
\end{aligned} \tag{8.10}$$

We simplify the terms in (8.10) one by one. In the term on the left-hand-side of (8.10), the integral $\int_{\partial D^+}$ can be closed in the upper-half plane. In order for this to be valid we must assume $l(t) > 0$ for all t. Thus, $\int_{\partial D^+}$ becomes $-\int_{B_3}$ where B_3 is the dashed blue path at infinity in Figure 8.3. Then, integrating by parts twice we find

$$\begin{aligned}
&-\int_{\partial D^+}\int_0^{l(t)} kq(x,t)(e^{ik(l(t)-x)}+e^{ik(l(t)+x)})\,\mathrm{d}x\,\mathrm{d}k\\
=&\int_{B_3}\left(\frac{-l'(t)}{k}(e^{2ikl(t)}+1)-\frac{2}{k}f(t)e^{ikl(t)}+\mathcal{O}(k^{-2})\right)\,\mathrm{d}k.
\end{aligned} \tag{8.11}$$

Consider the last term of (8.11). We note that

$$\int_{B_3}\frac{\mathrm{d}k}{k^2}=\lim_{R\to\infty}\int_{\pi/4}^{3\pi/4}\frac{i}{Re^{i\theta}}\,\mathrm{d}\theta=0.$$

Similarly

$$\int_{B_3}\frac{\mathrm{d}k}{k^n}=\lim_{R\to\infty}\int_{\pi/4}^{3\pi/4}\frac{i}{R^{(n-1)}e^{i\theta(n-1)}}\,\mathrm{d}\theta=0,$$

for $n \geq 2$. Thus, we eliminate the $\mathcal{O}(k^{-2})$ terms. Equation (8.11) becomes

$$\begin{aligned}
&\int_{B_3}\left(\frac{-l'(t)}{k}(e^{2ikl(t)}+1)-\frac{2}{k}f(t)e^{ikl(t)}\right)\,\mathrm{d}k\\
=&-\frac{i\pi}{2}l'(t)+\int_{B_3}\left(\frac{-l'(t)}{k}e^{2ikl(t)}-\frac{2}{k}f(t)e^{ikl(t)}\right)\,\mathrm{d}k\\
=&-\frac{i\pi}{2}l'(t)
\end{aligned} \tag{8.12}$$

where the last equality can be seen by noting the exponential decay in the integral and using Jordan's Lemma. Thus, (8.10) becomes

$$\begin{aligned}
-\frac{i\pi}{2}l'(t) =& 2\int_{\partial D^+}\int_0^t k e^{\omega(k)(s-t)+ikl(t)} f(s)\,\mathrm{d}s\,\mathrm{d}k \\
&+ \int_{\partial D^+}\int_0^t k e^{\omega(k)(s-t)} l'(s)(e^{ik(l(t)-l(s))} + e^{ik(l(t)+l(s))})\,\mathrm{d}s\,\mathrm{d}k \\
&- \int_{\partial D^+}\int_0^{l(0)} k q_0(x) e^{-\omega(k)t}(e^{ik(l(t)-x)} + e^{ik(l(t)+x)})\,\mathrm{d}x\,\mathrm{d}k.
\end{aligned} \tag{8.13}$$

Considering the decay and analyticity properties of the first term on the right-hand side of (8.13), we deform $\int_{\partial D^+}$ to $\int_{-\infty}^{\infty} + \int_{B_1} + \int_{B_2}$ where B_1 and B_2 are the dashed blue paths at infinity in Figure 8.3:

$$\begin{aligned}
2\int_{\partial D^+}\int_0^t k e^{\omega(k)(s-t)+ikl(t)} f(s)\,\mathrm{d}s\,\mathrm{d}k =& 2\int_{-\infty}^{\infty}\int_0^t k e^{\omega(k)(s-t)+ikl(t)} f(s)\,\mathrm{d}s\,\mathrm{d}k \\
&+ 2\int_{B_1+B_2}\int_0^t k e^{\omega(k)(s-t)+ikl(t)} f(s)\,\mathrm{d}s\,\mathrm{d}k.
\end{aligned} \tag{8.14}$$

Switching the order of integration and integrating the first term in (8.14) and integrating by parts twice for the integral over $B_1 + B_2$, the right-hand-side of (8.14) is

$$i\sqrt{\pi}l(t)\int_0^t \frac{e^{-\frac{l(t)^2}{4(t-s)}} f(s)}{(t-s)^{3/2}}\,\mathrm{d}s + 2\int_{B_1+B_2}\left(e^{ikl(t)}\left(\frac{f(t)}{k} - \frac{f(0)}{k}e^{-\omega(k)t}\right) - \int_0^t \frac{f'(s)}{k}e^{\omega(k)(s-t)}\,\mathrm{d}s\right)\mathrm{d}k.$$

The second integral evaluates to zero using integration by parts and Jordan's Lemma. Hence, (8.13) becomes

$$\begin{aligned}
-\frac{i\pi}{2}l'(t) =& i\sqrt{\pi}l(t)\int_0^t \frac{e^{-\frac{l(t)^2}{4(t-s)}} f(s)}{(t-s)^{3/2}}\,\mathrm{d}s \\
&+ \int_{\partial D^+}\int_0^t k e^{\omega(k)(s-t)} l'(s)(e^{ik(l(t)-l(s))} + e^{ik(l(t)+l(s))})\,\mathrm{d}s\,\mathrm{d}k \\
&- \int_{\partial D^+}\int_0^{l(0)} k q_0(x) e^{-\omega(k)t}(e^{ik(l(t)-x)} + e^{ik(l(t)+x)})\,\mathrm{d}x\,\mathrm{d}k.
\end{aligned} \tag{8.15}$$

Next, we consider the second term on the right-hand side of (8.15). Similar to above we deform $\int_{\partial D^+}$ to $\int_{-\infty}^{\infty} + \int_{B_1} + \int_{B_2}$. Then, switching the order of integration in the first integral and integrating by parts we find

$$
\begin{aligned}
&\int_0^t \int_{-\infty}^{\infty} k e^{\omega(k)(s-t)} l'(s)(e^{ik(l(t)-l(s))} + e^{ik(l(t)+l(s))})\,\mathrm{d}k\,\mathrm{d}s \\
&+ \int_{B_1+B_2} \int_0^t k e^{\omega(k)(s-t)} l'(s)(e^{ik(l(t)-l(s))} + e^{ik(l(t)+l(s))})\,\mathrm{d}s\,\mathrm{d}k \\
=&\frac{i\sqrt{\pi}}{2} \int_0^t \frac{l'(s)}{(t-s)^{3/2}} \left(e^{-\frac{(l(t)+l(s))^2}{4(t-s)}} (l(t)+l(s)) + e^{-\frac{(l(t)-l(s))^2}{4(t-s)}} (l(t)-l(s)) \right) \mathrm{d}s \\
&+ \int_{B_1+B_2} \left[\frac{l'(t)}{k}(1+e^{2ikl(t)}) - \frac{l'(0)}{k} e^{-\omega(k)t}(e^{ik(l(t)-l(0))} + e^{ik(l(t)+l(0))}) \right. \\
&\quad \left. - \int_0^t e^{\omega(k)(s-t)} \left(il'(s)^2 (e^{ik(l(t)+l(s))} - e^{ik(l(t)-l(s))}) + l''(s)(e^{ik(l(t)+l(s))} + e^{ik(l(t)-l(s))}) \right) \mathrm{d}s \right] \mathrm{d}k.
\end{aligned}
$$

Assuming $l(t)$ is monotone and noticing that continuing to do integration by parts leads to terms with decaying exponentials, the expression above becomes

$$
\begin{aligned}
&\frac{i\sqrt{\pi}}{2} \int_0^t \frac{l'(s)}{(t-s)^{3/2}} \left(e^{-\frac{(l(t)+l(s))^2}{4(t-s)}} (l(t)+l(s)) + e^{-\frac{(l(t)-l(s))^2}{4(t-s)}} (l(t)-l(s)) \right) \mathrm{d}s + \int_{B_1+B_2} \frac{l'(t)}{k}\,\mathrm{d}k \\
=&\frac{i\sqrt{\pi}}{2} \int_0^t \frac{l'(s)}{(t-s)^{3/2}} \left(e^{-\frac{(l(t)+l(s))^2}{4(t-s)}} (l(t)+l(s)) + e^{-\frac{(l(t)-l(s))^2}{4(t-s)}} (l(t)-l(s)) \right) \mathrm{d}s + \frac{i\pi}{2} l'(t).
\end{aligned}
$$

Hence, (8.15) is

$$
\begin{aligned}
-i\pi l'(t) =& i\sqrt{\pi} l(t) \int_0^t \frac{e^{-\frac{l(t)^2}{4(t-s)}} f(s)}{(t-s)^{3/2}}\,\mathrm{d}s \\
&+ \int_0^t \frac{i\sqrt{\pi} l'(s)}{2(t-s)^{3/2}} \left(e^{-\frac{(l(t)+l(s))^2}{4(t-s)}} (l(t)+l(s)) + e^{-\frac{(l(t)-l(s))^2}{4(t-s)}} (l(t)-l(s)) \right) \mathrm{d}s \qquad (8.16) \\
&- \int_{\partial D^+} \int_0^{l(0)} k q_0(x) e^{-\omega(k)t} (e^{ik(l(t)-x)} + e^{ik(l(t)+x)})\,\mathrm{d}x\,\mathrm{d}k.
\end{aligned}
$$

Finally, we consider the last term on the right-hand side of (8.16). As before, we deform $\int_{\partial D^+}$ to $\int_{-\infty}^{\infty} + \int_{B_1} + \int_{B_2}$, switch the order of integration for the first integral and in the second integral we integrate by parts and recognize that every term in the integral over B_1

and B_2 have decaying exponentials.

$$
\begin{aligned}
&-\int_0^{l(0)}\int_{-\infty}^{\infty} kq_0(x)e^{-\omega(k)t}(e^{ik(l(t)-x)}+e^{ik(l(t)+x)})\,\mathrm{d}x\,\mathrm{d}k\\
&-\int_{B_1+B_2}\int_0^{l(0)} kq_0(x)e^{-\omega(k)t}(e^{ik(l(t)-x)}+e^{ik(l(t)+x)})\,\mathrm{d}k\,\mathrm{d}x\\
=&-\int_0^{l(0)}\frac{i\sqrt{\pi}}{2t^{3/2}}q_0(x)\left(e^{-\frac{(l(t)+x)^2}{4t}}(l(t)+x)+e^{-\frac{(l(t)-x)^2}{4t}}(l(t)-x)\right)\mathrm{d}x.
\end{aligned}
$$

Finally (8.16) becomes

$$
\begin{aligned}
l'(t)=&-\frac{l(t)}{\sqrt{\pi}}\int_0^t\frac{e^{-\frac{l(t)^2}{4(t-s)}}f(s)}{(t-s)^{3/2}}\,\mathrm{d}s\\
&-\frac{1}{2\sqrt{\pi}}\int_0^t\frac{l'(s)}{(t-s)^{3/2}}\left(e^{-\frac{(l(t)+l(s))^2}{4(t-s)}}(l(t)+l(s))+e^{-\frac{(l(t)-l(s))^2}{4(t-s)}}(l(t)-l(s))\right)\mathrm{d}s\\
&+\frac{1}{2\sqrt{\pi}}\int_0^{l(0)}\frac{q_0(x)}{t^{3/2}}\left(e^{-\frac{(l(t)+x)^2}{4t}}(l(t)+x)+e^{-\frac{(l(t)-x)^2}{4t}}(l(t)-x)\right)\mathrm{d}x.
\end{aligned}
\tag{8.17}
$$

Defining the kernel $K(x,r;l(t))=\frac{1}{2r^{3/2}\sqrt{\pi}}\left(e^{-\frac{(l(t)+x)^2}{4r}}(l(t)+x)+e^{-\frac{(l(t)-x)^2}{4r}}(l(t)-x)\right)$, Equation (8.17) is

$$
l'(t)=-\int_0^t K(0,t-s;l(t))f(s)\,\mathrm{d}s-\int_0^t l'(s)K(l(s),t-s;l(t))\,\mathrm{d}s+\int_0^{l(0)}q_0(x)K(x,t;l(t))\,\mathrm{d}x.
\tag{8.18}
$$

An analogous result for Dirichlet boundary conditions is presented in Section 12-5 of [32]. Their result is obtained using "potential theoretic methods" rather than the Fokas method.

8.1.3 The Fokas Method for interface problems

In this section we find $l'(t)$ using another method. Once again consider (8.1) subject to the initial and boundary conditions (8.2). We begin with the global relation (8.7) and its evaluation at $-k$, Equation (8.8). Our goal is to solve for $q(x,t)$ using the Fokas method and then use the Stefan condition ($q_x(l(t),t)=-l'(t)$) to get an equation for $l'(t)$.

Inverting the x-transform in (8.7) we have

$$
\begin{aligned}
q(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{l(0)} e^{ik(x-y)-\omega(k)t} q_0(y)\,\mathrm{d}y\,\mathrm{d}k - \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{t} e^{ik(x-l(s))+\omega(k)(s-t)} l'(s)\,\mathrm{d}s\,\mathrm{d}k\\
& - \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{t} e^{ikx+\omega(k)(s-t)} f(s)\,\mathrm{d}s\,\mathrm{d}k - \frac{i}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{t} k e^{ikx+\omega(k)(s-t)} q(0,s)\,\mathrm{d}s\,\mathrm{d}k,
\end{aligned}
$$

for $0 < x < l(t)$ and $t > 0$. The integrand of the fourth k-integral is entire and decays as $k \to \infty$ for $k \in \mathbb{C}^+ \setminus D^+$. Using the analyticity of the integrand and applying Jordan's Lemma we can replace the contour of integration of the fourth integral by $\int_{\partial D^+}$:

$$
\begin{aligned}
q(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{l(0)} e^{ik(x-y)-\omega(k)t} q_0(y)\,\mathrm{d}y\,\mathrm{d}k\\
& - \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{t} e^{ik(x-l(s))+\omega(k)(s-t)} l'(s)\,\mathrm{d}s\,\mathrm{d}k\\
& - \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{t} e^{ikx+\omega(k)(s-t)} f(s)\,\mathrm{d}s\,\mathrm{d}k\\
& - \frac{i}{2\pi}\int_{\partial D^+}\int_{0}^{t} k e^{ikx+\omega(k)(s-t)} q(0,s)\,\mathrm{d}s\,\mathrm{d}k.
\end{aligned}
\tag{8.19}
$$

Equation (8.19) for $q(x,t)$ depends on the unknown function $q(0,s)$. We use (8.8) to solve for $q(0,s)$ and substitute this into (8.19):

$$
\begin{aligned}
q(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{l(0)} e^{-\omega(k)t} q_0(y)\left(e^{ik(x-y)} + e^{ik(x+y)}\right)\,\mathrm{d}y\,\mathrm{d}k\\
& - \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{t} e^{\omega(k)(s-t)} l'(s)\left(e^{ik(x-l(s))} + e^{ik(x+l(s))}\right)\,\mathrm{d}s\,\mathrm{d}k\\
& - \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{0}^{t} e^{ikx+\omega(k)(s-t)} f(s)\,\mathrm{d}s\,\mathrm{d}k - \frac{1}{2\pi}\int_{\partial D^+}\int_{0}^{l(t)} e^{ik(x+y)} q(y,t)\,\mathrm{d}y\,\mathrm{d}k.
\end{aligned}
\tag{8.20}
$$

where we have interchanged $\int_{\partial D^+}$ and $\int_{-\infty}^{\infty}$ whenever convenient by employing Cauchy's Theorem and Jordan's Lemma. The final term in the expression above depends on $q(y,t)$ which is unknown. However, the integrand is analytic for all $k \in \mathbb{C}^+$ and $\int_0^{l(t)} e^{iky} q(y,t)\,\mathrm{d}y$ decays for $k \to \infty$ for $k \in \mathbb{C}^+$. Thus, by Jordan's Lemma, the integral of $\int_0^{l(t)} e^{ik(x+y)} q(y,t)\,\mathrm{d}y$ along a closed, bounded curve in $\mathbb{C}^+$ vanishes. In particular we consider the closed curve

$\mathcal{L}^+ = \mathcal{L}_{D^+} \cup \mathcal{L}_C^+$ where $\mathcal{L}_{D^+} = \partial D^+ \cap \{k : |k| < C\}$ and $\mathcal{L}_C^+ = \{k \in D^+ : |k| = C\}$, see Figure 1.3.

Since the integral along $\mathcal{L}_C^+$ vanishes for large C, the fourth integral on the right-hand side of (8.20) must vanish since the contour $\mathcal{L}_{D^+}$ becomes ∂D^+ as $C \to \infty$. The uniform decay of $\int_0^{l(t)} e^{ik(x+y)} q(y,t)\,\mathrm{d}y$ for large k is exactly the condition required for the integral to vanish, using Jordan's Lemma. Now, we have an explicit representation for $q(x,t)$ in terms of initial conditions, boundary conditions, and the moving boundary $l(t)$:

$$
\begin{aligned}
q(x,t) = &\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^{l(0)} e^{-\omega(k)t} q_0(y)\left(e^{ik(x-y)} + e^{ik(x+y)}\right)\,\mathrm{d}y\,\mathrm{d}k \\
&- \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^{t} e^{\omega(k)(s-t)} l'(s)\left(e^{ik(x-l(s))} + e^{ik(x+l(s))}\right)\,\mathrm{d}s\,\mathrm{d}k \\
&- \frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^{t} e^{ikx+\omega(k)(s-t)} f(s)\,\mathrm{d}s\,\mathrm{d}k.
\end{aligned}
\tag{8.21}
$$

Switching the order of integration and evaluating the k integrals we have

$$
\begin{aligned}
q(x,t) = &\frac{1}{2\sqrt{\pi}}\int_0^{l(0)} \frac{q_0(y)}{\sqrt{t}}\left(e^{-\frac{(x-y)^2}{4t}} + e^{-\frac{(x+y)^2}{4t}}\right)\mathrm{d}y \\
&- \frac{1}{2\sqrt{\pi}}\int_0^{t} \frac{l'(s)}{\sqrt{t-s}}\left(e^{-\frac{(x-l(s))^2}{4(t-s)}} + e^{-\frac{(x+l(s))^2}{4(t-s)}}\right)\mathrm{d}s \\
&- \frac{1}{\sqrt{\pi}}\int_0^{t} \frac{e^{-\frac{x^2}{4(t-s)}} f(s)}{\sqrt{t-s}}\,\mathrm{d}s.
\end{aligned}
\tag{8.22}
$$

In order to find $l(t)$ we take an x derivative of (8.22) and evaluate at $(x,t) = (l(t),t)$. Since we are evaluating the inverse Fourier transform at the edge of the domain ($x = l(t)$) where there is a discontinuity we are actually reconstructing the average across the discontinuity which is $q_x(l(t),t)/2$. Thus, we take the derivative of (8.22) and evaluate at $(x,t) = (l(t),t)$ and multiply by 2 to find

$$
\begin{aligned}
l'(t) = &\frac{1}{2\sqrt{\pi}}\int_0^{l(0)} \frac{q_0(y)}{t^{3/2}}\left(e^{-\frac{(l(t)+y)^2}{4t}}(l(t)+y) + e^{-\frac{(l(t)-y)^2}{4t}}(l(t)-y)\right)\mathrm{d}y \\
&- \frac{1}{2\sqrt{\pi}}\int_0^{t} \frac{l'(s)}{(t-s)^{3/2}}\left(e^{-\frac{(l(t)-l(s))^2}{4(t-s)}}(l(t)-l(s)) + e^{-\frac{(l(t)+l(s))^2}{4(t-s)}}(l(t)+l(s))\right)\mathrm{d}s \\
&- \frac{l(t)}{\sqrt{\pi}}\int_0^{t} \frac{e^{-\frac{l(t)^2}{4(t-s)}} f(s)}{(t-s)^{3/2}}\,\mathrm{d}s.
\end{aligned}
\tag{8.23}
$$

This is the same expression as (8.17) in the previous section.

Remarks:

- If $l(t)$ is constant then $l(0) = \infty$, $l'(s) = 0$, and (8.21) becomes the solution to the heat equation on the half line with Neumann boundary conditions.

- Plugging $q_x(0,t) = f(t) = \frac{c_3}{\sqrt{t}}$, and $l(t) = c_4\sqrt{t}$ where c_4 is determined implicitly by plugging (8.5) into (8.22) and doing a change of variables one can easily see that the equation is dependent only on terms of the form $\zeta = \frac{x}{\sqrt{t}}$. This is consistent with the similarity solution given by (8.4).

- Taking derivatives of $q_x(l(t),t) = -l'(t)$ with respect to t and evaluating at $t = 0$ gives relationships between $q_0(l(0))$ and its derivates and $l(0)$ and its derivatives. Namely

$$\begin{aligned} {q_0}'(l(0)) =&0, \\ q_0''(l(0))l'(0) + q_0'''(l(0)) =&-l''(0), \\ q_0''(l(0)l''(0) + q_0'''(l(0))l'(0)^2 + 2q''''(l(0))l'(0) + q'''''(l(0)) =&-l'''(0), \\ \vdots& \end{aligned} \tag{8.24}$$

  for as many derivatives as one would care to take. Similarly, if we begin with $q(l(t),t) = 0$ and take derivatives with respect to t we have

$$\begin{aligned} q_0(l(0)) =&0, \\ q_0''(l(0)) + q_0'(l(0))l'(0) =&0, \\ q_0''''(l(0)) + q_0'''(l(0))(l'(0)+1) + q_0''(l(0))l'(0)^2 + q_0'(l(0))l''(0) =&0, \\ \vdots& \end{aligned} \tag{8.25}$$

  where again we can take as many derivatives as we would like.

  If $l(t)$ is analytic, then it has a Taylor expansion [1]. Assuming $l(t) = \sum_{j=0}^{\infty} l_j t^j$ and observing that the exponential decay in $K(x,t;l(t))$ as $t \to 0$ we know $l_j = 0$ for all

$j \geq 1$. An easy calculation shows that $l(t)$ is a spectrally small (beyond all orders) function of t. That is, any derivative of $l(t)$ evaluated at $t = 0$ is 0. Using this in (8.24) and (8.25) we find that $q_0(l(0))$ and all its derivatives must be exactly 0. This is obviously not a physically relevant problem. It follows that $l(t)$ is not analytic.

- It is not clear that we could analytically find $l(t)$. A numerical fixed point method for solving (8.23) is suggested in [32]. We implemented this in *Mathematica* by numerically approximating all functions by third order polynomials and considered the method converged at each step when the values were within 10^{-6} of the previous iteration. For the results plotted in 8.4 we assumed $l(0) = 1$, $q_0(x) = x - 1$, and $f(t) = \cos(t)$. We used a step size of .0001 in time.

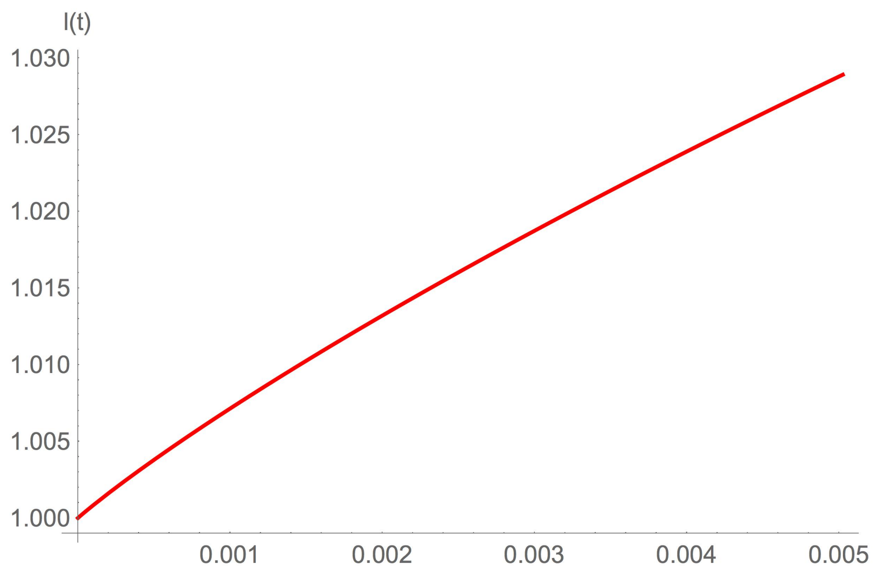

Figure 8.4: A numerical solution for $l(t)$.

8.2 The two-phase Stefan problem on an infinite domain

We consider the heat equation in two time-dependent domains, ice and water:

$$q_t^W - \sigma_W^2 q_{xx}^W = 0, \qquad x < l(t), \quad t \geq 0, \tag{8.26a}$$

$$q_t^I - \sigma_I^2 q_{xx}^I = 0, \qquad x > l(t), \quad t \geq 0, \tag{8.26b}$$

with $l(t)$ a monotonic unknown function. The given initial, boundary, and interface conditions are

$$q^W(x,0) = q_0^W(x) \tag{8.27a}$$

$$q^I(x,0) = q_0^I(x) \tag{8.27b}$$

$$\lim_{x\to-\infty} q^W(x,t) = \gamma^W, \tag{8.27c}$$

$$\lim_{x\to\infty} q^I(x,t) = \gamma^I, \tag{8.27d}$$

$$q^W(l(t),t) = q^I(l(t),t) = 0, \tag{8.27e}$$

$$\sigma_I^2 q_x^I(l(t),t) - \sigma_W^2 q_x^W(l(t),t) = l'(t), \tag{8.27f}$$

where $\gamma_I < 0$ and $\gamma_W > 0$. The domain is as pictured in Figure 8.5.

8.2.1 A similarity solution

Consider (8.26) subject to the conditions (8.27) and choose $\zeta = x/\sqrt{t}$. We look for a similarity solution $q^W(x,t) = Q^W(\zeta)$ and $q^I(x,t) = Q^I(\zeta)$. Our equation becomes the set of ordinary differential equations

$$\zeta Q_\zeta^W + 2\sigma_W^2 Q_{\zeta\zeta}^W = 0, \qquad \zeta < \frac{l(t)}{\sqrt{t}}, \quad t > 0, \tag{8.28a}$$

$$\zeta Q_\zeta^I + 2\sigma_I^2 Q_{\zeta\zeta}^I = 0, \qquad x > \frac{l(t)}{\sqrt{t}}, \quad t > 0. \tag{8.28b}$$

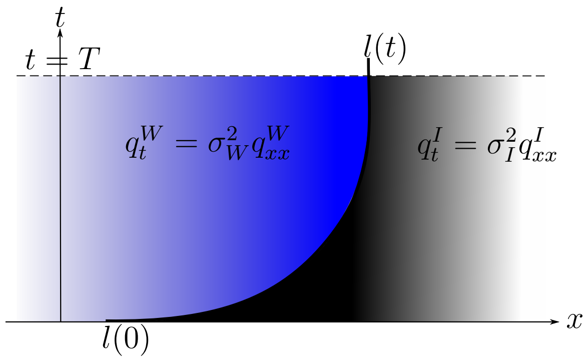

Figure 8.5: The domain over which one solves the heat equation including the moving boundary $l(t)$ for the Stefan problem.

Our boundary and interface conditions transform similarly to become

$$
\begin{aligned}
\lim_{\zeta\to-\infty} Q^W(\zeta) =& \gamma^W,\\
\lim_{\zeta\to\infty} Q^I(\zeta) =& \gamma^I,\\
Q^W\left(\frac{l(t)}{\sqrt{t}}\right) = Q^I\left(\frac{l(t)}{\sqrt{t}}\right) =& 0,\\
\frac{\sigma_I^2}{\sqrt{t}} Q_\zeta^I\left(\frac{l(t)}{\sqrt{t}}\right) - \frac{\sigma_W^2}{\sqrt{t}} Q_\zeta^W\left(\frac{l(t)}{\sqrt{t}}\right) =& l'(t).
\end{aligned}
$$

Solving (8.28) and imposing the asymptotic conditions (8.27c) and (8.27d) we find

$$
\begin{aligned}
Q^W(\zeta) =& c_1 + (c_1 - \gamma^W)\operatorname{erf}\left(\frac{\zeta}{2\sigma_W}\right),\\
Q^I(\zeta) =& c_2 + (\gamma^I - c_2)\operatorname{erf}\left(\frac{\zeta}{2\sigma_I}\right),
\end{aligned}
\tag{8.29}
$$

where c_1 and c_2 are constants of integration. To find c_1 and c_2 we impose the conditions at the interface. This requires

$$
\begin{aligned}
c_1 =& \frac{\gamma_W \operatorname{erf}\left(\frac{l(t)}{2\sigma_W\sqrt{t}}\right)}{\operatorname{erf}\left(\frac{l(t)}{2\sigma_W\sqrt{t}}\right) + 1},\\
c_2 =& \frac{\gamma_I \operatorname{erf}\left(\frac{l(t)}{2\sigma_I\sqrt{t}}\right)}{\operatorname{erf}\left(\frac{l(t)}{2\sigma_I\sqrt{t}}\right) - 1}.
\end{aligned}
$$

However c_1 and c_2 are constants independent of $\zeta(x,t)$. Thus, $l(t) = 2c_3\sigma_I\sigma_W\sqrt{t}$ where c_3 is a constant. Hence, our similarity solution is

$$\begin{aligned} Q^W(\zeta) =& \frac{\gamma_W \left(\text{erf}\,(c_3\sigma_I) - \text{erf}\left(\frac{\zeta}{2\sigma_W}\right)\right)}{\text{erf}\,(c_3\sigma_I) + 1}, \\ Q^I(\zeta) =& \frac{\gamma_I \left(\text{erf}\,(c_3\sigma_W) - \text{erf}\left(\frac{\zeta}{2\sigma_I}\right)\right)}{\text{erf}\,(c_3\sigma_W) - 1}. \end{aligned} \tag{8.30}$$

Imposing the Stefan condition we have an implicit equation for c_3

$$c_3\sigma_I\sigma_W\sqrt{\pi} = \frac{e^{-c_3^2\sigma_I^2}\gamma_W\sigma_W}{\text{erf}(c_3\sigma_I) + 1} - \frac{e^{-c_3^2\sigma_W^2}\gamma_I\sigma_I}{\text{erf}(c_3\sigma_W) - 1}. \tag{8.31}$$

Choosing the parameter values $\sigma_I = 2, \sigma_W = 3, \gamma_W = 6$, and $\gamma_I = -1$ and numerically solving for c_3 one can plot the solution as in Figure 8.6.

Remarks:

- The two-phase problem is clearly more physically relevant than the one-phase problem. As far as we know there are no solutions to this problem in the literature.

- The solution to this problem using an analog to the methods presented in Sections 8.1.2 and 8.1.3 is currently under investigation although the addition of a non-constant second domain introduces significant challenges.

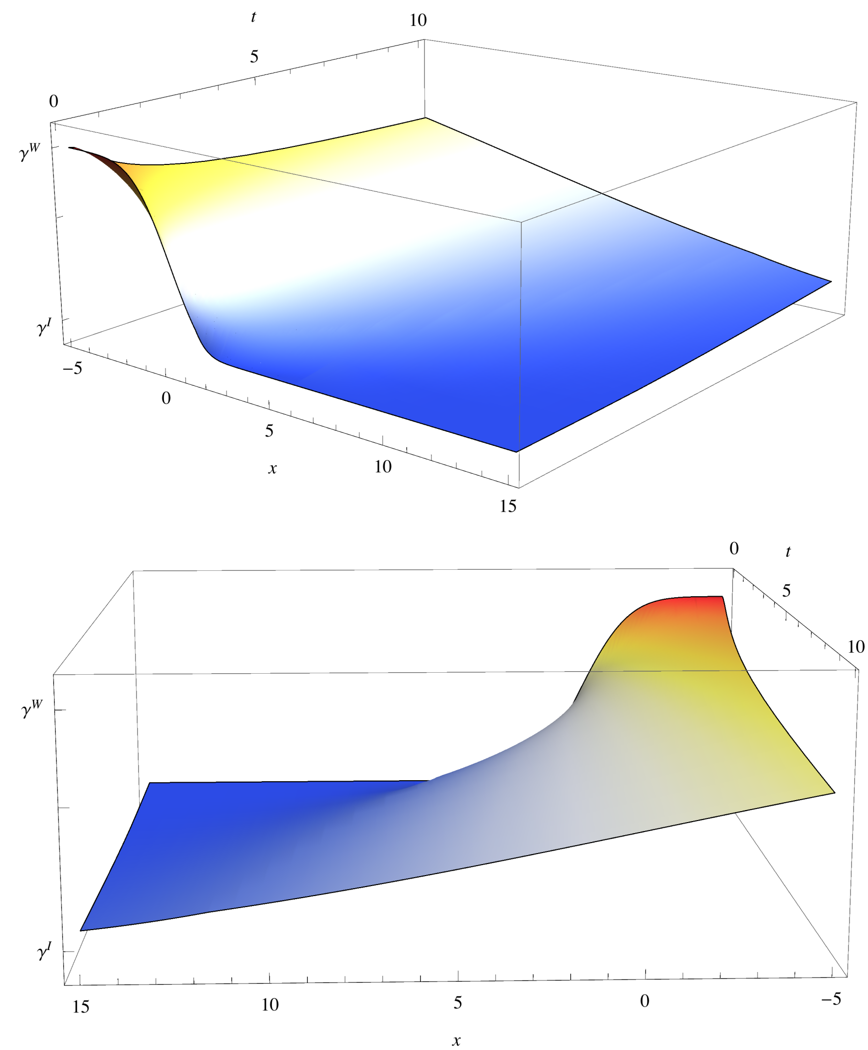

Figure 8.6: The solution to (8.28) with $\sigma_I = 2, \sigma_W = 3, \gamma_W = 6, \gamma_I = -1$, and c_3 the numerical solution to (8.31).

Appendix A

The transport and heat equations

We wish to find $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ satisfying

$$\begin{aligned} u_t^{(1)}(x,t) =& -c_1 u_x^{(1)}(x,t), \qquad x<0, \quad t>0, \\ u_t^{(2)}(x,t) =& \sigma_2^2 u_{xx}^{(2)}(x,t), \qquad x>0, \quad t>0, \end{aligned} \tag{A.1}$$

subject to the asymptotic conditions

$$\begin{aligned} \lim_{x\to-\infty} u^{(1)}(x,t) =& 0, \quad t>0, \\ \lim_{x\to\infty} u^{(2)}(x,t) =& 0, \quad t>0, \end{aligned} \tag{A.2}$$

the initial conditions

$$\begin{aligned} u^{(1)}(x,0) =& u_0^{(1)}(x), \qquad x<0, \\ u^{(2)}(x,0) =& u_0^{(2)}(x), \qquad x>0, \end{aligned} \tag{A.3}$$

and the interface condition

$$\rho_1 u^{(1)}(0,t) = \rho_2 u^{(2)}(0,t), \qquad t>0, \tag{A.4}$$

where c_1, σ_2, ρ_1 and ρ_2 are t-independent nonzero constants. The sub- and super-indices 1 and 2 denote the left and right domain, respectively as in Figure A.1. In what follows we assume that c_1 and σ_2 are both positive for convenience.

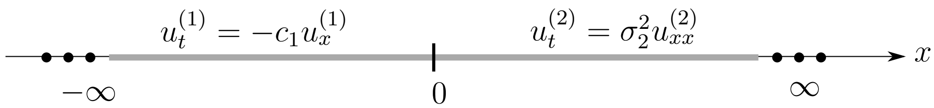

Figure A.1: The transport equation on the left semi-infinite domain and heat equation on the right semi-infinite domain.

We follow the standard steps in the application of the Fokas Method beginning with the local relations

$$\left(e^{-ikx+\omega_1 t}u^{(1)}(x,t)\right)_t = \left(-c_1 e^{-ikx+\omega_1 t}u^{(1)}(x,t)\right)_x, \tag{A.5a}$$

$$\left(e^{-ikx+\omega_2 t}u^{(2)}(x,t)\right)_t = \left(\sigma_2^2 e^{-ikx+\omega_2 t}(u_x^{(2)}(x,t) + iku^{(2)}(x,t))\right)_x. \tag{A.5b}$$

These are one parameter family relations obtained by rewriting (A.1) where $\omega_1(k) = ikc_1$ and $\omega_2(k) = (\sigma_2 k)^2$. Applying Green's Theorem in the strips $(-\infty, 0) \times (0, t)$ and $(0, \infty) \times (0, t)$ respectively (see Figure 2.2), we find the global relations

$$0 = \int_{-\infty}^0 e^{-ikx}u_0^{(1)}(x)\,\mathrm{d}x - \int_{-\infty}^0 e^{-ikx+\omega_1 t}u^{(1)}(x,t)\,\mathrm{d}x - \int_0^t c_1 e^{\omega_1 s}u^{(1)}(0,s)\,\mathrm{d}s, \tag{A.6a}$$

$$\begin{aligned}0 = &\int_0^\infty e^{-ikx}u_0^{(2)}(x)\,\mathrm{d}x - \int_0^\infty e^{-ikx+\omega_2 t}u^{(2)}(x,t)\,\mathrm{d}x \\ &- \int_0^t \sigma_2^2 e^{\omega_2 s}\left(u_x^{(2)}(0,s) + iku^{(2)}(0,s)\right)\,\mathrm{d}s.\end{aligned} \tag{A.6b}$$

For $k \in \mathbb{C}$, we define the following

$$\hat{u}^{(1)}(k,t) = \int_{-\infty}^0 e^{-ikx}u^{(1)}(x,t)\,\mathrm{d}x, \qquad \hat{u}_0^{(1)}(k) = \int_{-\infty}^0 e^{-ikx}u_0^{(1)}(x)\,\mathrm{d}x,$$

$$\hat{u}^{(2)}(k,t) = \int_0^\infty e^{-ikx}u^{(2)}(x,t)\,\mathrm{d}x, \qquad \hat{u}_0^{(2)}(k) = \int_0^\infty e^{-ikx}u_0^{(2)}(x)\,\mathrm{d}x.$$

Using these definitions and the interface condition, the global relations (A.6) are rewritten as

$$0 =\hat{u}_0^{(1)}(k) - e^{\omega_1 t}\hat{u}^{(1)}(k,t) - \int_0^t c_1 e^{\omega_1 s} u^{(1)}(0,s)\,\mathrm{d}s, \tag{A.7a}$$

$$0 =\hat{u}_0^{(2)}(k) - e^{\omega_2 t}\hat{u}^{(2)}(k,t) - \int_0^t \sigma_2^2 e^{\omega_2 s}\left(u_x^{(2)}(0,s) + \frac{ik\rho_1}{\rho_2}u^{(1)}(0,s)\right)\mathrm{d}s. \tag{A.7b}$$

Inverting the Fourier transforms in (A.7) we have the solution formulas

$$u^{(1)}(x,t) =\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{u}_0^{(1)}(k)\,\mathrm{d}k - \frac{c_1}{2\pi}\int_{-\infty}^{\infty}\int_0^t e^{ikx-\omega_1(t-s)}u^{(1)}(0,s)\,\mathrm{d}s\,\mathrm{d}k, \tag{A.8a}$$

$$\begin{aligned} u^{(2)}(x,t) =&\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{u}_0^{(2)}(k)\,\mathrm{d}k \\ &- \frac{\sigma_2^2}{2\pi}\int_{-\infty}^{\infty}\int_0^t e^{ikx-\omega_2(t-s)}\left(u_x^{(2)}(0,s) + \frac{ik\rho_1}{\rho_2}u^{(1)}(0,s)\right)\mathrm{d}s\,\mathrm{d}k. \end{aligned} \tag{A.8b}$$

Switching the order of integration, the integrand of the second term in equation (A.8a) is entire and decays for $k \in \mathbb{C}^-$ since c_1 is positive. Thus, using Jordan's Lemma and Cauchy's Theorem that term is zero. In (A.8b) the integrand of the second term decays and is entire for $\mathbb{C}^-/D^-$. Hence, we have closed-form solutions depending only on initial conditions:

$$u^{(1)}(x,t) =\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{u}_0^{(1)}(k)\,\mathrm{d}k, \tag{A.9a}$$

$$\begin{aligned} u^{(2)}(x,t) =&\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{u}_0^{(2)}(k)\,\mathrm{d}k \\ &+ \frac{\sigma_2^2}{2\pi}\int_{\partial D^-}\int_0^t e^{ikx-\omega_2(t-s)}\left(u_x^{(2)}(0,s) + \frac{ik\rho_1}{\rho_2}u^{(1)}(0,s)\right)\mathrm{d}s\,\mathrm{d}k. \end{aligned} \tag{A.9b}$$

Switching the order of integration in the first integral and evaluating the integral with respect to k we find the familiar d'Alembert form of the solution to the transport equation [51]

$$u^{(1)}(x,t) =u_0^{(1)}(x - c_1 t), \tag{A.10a}$$

$$\begin{aligned} u^{(2)}(x,t) =&\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_2 t}\hat{u}_0^{(2)}(k)\,\mathrm{d}k \\ &+ \frac{\sigma_2^2}{2\pi}\int_{\partial D^-}\int_0^t e^{ikx-\omega_2(t-s)}\left(u_x^{(2)}(0,s) + \frac{ik\rho_1}{\rho_2}u^{(1)}(0,s)\right)\mathrm{d}s\,\mathrm{d}k. \end{aligned} \tag{A.10b}$$

Of course, this example is trivial since one could easily solve the transport equation on the left and use the solution to impose boundary conditions for the heat equation on the right. We include this problem in an appendix as a proof of concept. This simple example shows how the Fokas Method might be applied to problems with different equations in adjacent domains. We hope to apply this method to more complex problems in the future.

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