# The analytic extension of solutions to initial-boundary value problems outside their domain of definition 

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#### Abstract

We examine the analytic extension of solutions of linear, constant-coefficient initialboundary value problems outside their spatial domain of definition. We use the Unified Transform Method or Method of Fokas, which gives a representation for solutions to half-line and finite-interval initial-boundary value problems as integrals of kernels with explicit spatial and temporal dependence. These solution representations are defined within the spatial domain of the problem. We obtain the extension of these representation formulae via Taylor series outside these spatial domains and find the extension of the initial condition that gives rise to a whole-line initial-value problem solved by the extended solution. In general, the extended initial condition is not differentiable or continuous unless the boundary and initial conditions satisfy compatibility conditions. We analyze dissipative and dispersive problems, and problems with continuous and discrete spatial variables.


## 1 Introduction

We examine the analytic continuation, for $x \in \mathbb{R}$, of solutions of linear, constant-coefficient initialboundary value problems (IBVPs) outside their spatial domain of definition. This is a classical question: for the heat equation on $x>0$ with Dirichlet boundary data,

$$
\begin{align*}
u_{t} & =u_{x x}, & & x>0, t>0  \tag{1a}\\
u(x, 0) & =u_{0}(x), & & x>0,  \tag{1b}\\
u(0, t) & =f_{0}(t), & & t>0, \tag{1c}
\end{align*}
$$

the solution can be written using the Fourier sine transform, see (10). This leads immediately to an odd extension of the solution for $x<0$, which cannot provide an analytic extension of the solution unless, for starters, the Dirichlet data $f_{0}(t) \equiv 0$. To illustrate our goal, consider the Fourier cosine series of the function $f(x)=(x-1 / 2)^{2}$,

$$
\begin{equation*}
f_{\mathrm{FS}}(x)=\frac{1}{12}+\sum_{n=1}^{\infty} \frac{\cos (2 n \pi x)}{n^{2} \pi^{2}} \tag{2}
\end{equation*}
$$



Figure 1: The function $f(x)=(x-1 / 2)^{2}$ shown in red and its Fourier cosine series $f_{\text {FS }}(x)(2)$, shown in dashed blue.
shown in Figure 1. While $f_{\mathrm{FS}}(x)$ converges uniformly to $f(x)$ in $[0,1]$, outside the interval, it converges to the even, periodic extension. There is no clear way to analytically extend the Fourier cosine series in $(0,1)$ to obtain $(x-1 / 2)^{2}$ outside of $(0,1)$, since using $(2)$ the higher-order derivatives are not defined.

In the rest of this introduction we use the IBVP (1) to fix the notation, but we examine problems far more general in this manuscript, including problems of higher order and problems with discrete spatial variables. For those situations where the question of an analytic extension is reasonable (e.g., for (1), $u_{0}(x)$ and $f_{0}(t)$ are analytic for $x>0$ and have sufficient decay, see Theorem 1 ), we derive explicit representations for the analytic extension $u_{\mathrm{ac}}(x, t), x \in \mathbb{R}$, of the solutions. Here $u_{\mathrm{ac}}(x, t) \equiv u(x, t)$ for $x>0$ in (1). Some of these extensions are obtained through expressions relating $u(x, t), x>0$ and $u_{\mathrm{ac}}(x, t), x<0$. Others are fully explicit in that they give the analytic extension $u_{\text {ac }}(x, t)$ directly in terms of the given initial and boundary data.

Our approach uses the Unified Transform Method (UTM) or Method of Fokas [6, 8], as this method allows for the solution of problems with continuous and discrete spatial dependence, of arbitrary order. Further, it results in solution expressions with well-defined derivatives in their spatial domain. Even using the UTM, there is confusion about the solution outside its domain of definition, see for instance [16, Section 115]. During the first steps of the method, a Fourier transform is used where the solution is assumed to be zero outside its domain of definition. However, when the expression for the representation of the solution is obtained, the solution no longer satisfies this assumption. This is a consequence of the elimination (using Jordan's lemma) of integral contributions in the solution expression that are identically zero in the IBVP's domain of definition, but not so outside of it. A secondary aim of this manuscript is to clarify this confusion.

We have several reasons for wanting to extend the solution to outside its original domain of definition. The first one arises from numerical analysis: spatial finite-difference methods remain among the most popular methods for numerically solving partial differential equations. However, the application of finite-difference stencils near boundaries often requires the value of the solution at so-called ghost points, grid points outside of the physical domain of definition (e.g., [13]). The extended solutions calculated here provide an answer to this, as one can simply evaluate $u_{\mathrm{ac}}(x, t)$ at
the desired negative values of $x$. Second, when testing numerical or other methods for IBVPs, it is common to start from a whole-line problem (where exact solutions may be more readily available) and restrict it to a smaller domain, using as boundary conditions the values of the whole-line solution on the boundaries. We can reverse this: using our approach, we can ask what the whole-line problem is whose restriction is the solution of the original IBVP. An example of a physical extension question one could ask is the following: suppose we have a bi-infinite rod, whose temperature at $t=0$ is known for $x>0$, and whose temperature at $x=0$ is measured for all $t>0$. What is the temperature for $x<0$ ? Indeed, for the heat equation, our formulas have been derived before, see [3].

It should be noted that the whole-line problem whose restriction is the original problem may not always be of physical interest. For instance, extending from $x>0$ to $x \in \mathbb{R}$, it may be the case that the extended solution grows as $x \rightarrow-\infty$ or even blows up at a finite $x_{0}<0$, see Figure 1 . Further, if the original initial and boundary conditions are incompatible, $\lim _{t \rightarrow 0^{+}} u_{\mathrm{ac}}(x, t)$ will differ from the analytic continuation of the original initial condition. Indeed, an analytic continuation of the original initial condition can only lead to a whole-line problem whose restriction would have compatible initial and boundary data.

Our approach may be thought of as a generalization of the Method of Images (see [10, 11], for instance). The Method of Images uses a whole-line (or whole-plane/space, for multi-dimensional settings) problem that reduces to the given half-line problem with the given boundary conditions. For instance, the method of images for a homogeneous, Dirichlet (or Neumann) half-line problem uses a whole-line "image" problem with odd (or even) extension of the initial condition. For this paper's boundary-to-initial maps, we wish to construct a whole-line problem that restricts to the given half-line problem with the corresponding nonhomogeneous boundary conditions. It is perhaps especially surprising to see such formulas for third-order problems or other problems that do not have the symmetry $x \rightarrow-x$.

As is often the case in function theory, it is important to distinguish between the abstract concept of a solution and its (different) explicit representations. However, in context it is often clear what is meant, and we may blur the line between solutions, extensions, and their representations frequently, to avoid introducing extra notation.

In Section 2, we consider extensions of the solutions of the half-line Dirichlet and Neumann IBVPs for the heat equation, as well as an example of a finite-interval IBVP. The advected heat equation is treated in Section 3. The half-line Dirichlet IBVPs for the linear KdV equations are treated in Sections 4 (one boundary condition) and 5 (two boundary conditions), respectively. Sections 6, 7 and the Appendix deal with spatially discrete IBVPs for the advection equation and the heat equation. It should be noted that the analytic continuation of solutions of spatially discrete IBVPs can be done in many ways, as the sole requirement is that the analytic continuation interpolates the solution at the fixed grid points, with a well-defined continuous limit as the grid spacing vanishes. Our approach is to consider functions as analytically depending on a discrete variable, a popular approach in this context, see [12]. We conclude with a summary of the extension formulae obtained, collected in one convenient location.

## 2 The heat equation

For problems with a continuous spatial variable, we may start by considering second-order problems. Indeed, for IBVPs involving the first-order transport equation,

$$
\begin{align*}
u_{t}+c u_{x} & =0, & & x>0, t>0  \tag{3a}\\
u(x, 0) & =u_{0}(x), & & x>0  \tag{3~b}\\
u(0, t) & =f_{0}(t), & & t>0 \tag{3c}
\end{align*}
$$

(the boundary condition in the last line is omitted if $c<0$ ), the solution is analytically continued trivially using the d'Alembert form [10] of the solution,

$$
u(x, t)=\left\{\begin{align*}
u_{0}(x-c t), & t<x / c, x>0  \tag{4}\\
f_{0}(t-x / c), & t>x / c, x>0
\end{align*}\right.
$$

for $c>0$, and

$$
\begin{equation*}
u(x, t)=u_{0}(x-c t), \quad x>0, t>0 \tag{5}
\end{equation*}
$$

for $c<0$. If $f_{0}(t)\left(u_{0}(x)\right)$ is analytic for $c>0(c<0)$, then the solution is trivially extended for negative values of $x$. If these functions are not analytic, then no analytic extension exists.

### 2.1 Dirichlet boundary conditions

Consider the heat equation on the half line with Dirichlet boundary conditions (1). Using Fokas' Unified Transform Method (UTM) [8], its solution is written as

$$
\begin{equation*}
u(x, t)=I_{0}(x, t)+I_{f_{0}}(x, t) \tag{6}
\end{equation*}
$$

with

$$
\begin{align*}
I_{0}(x, t) & =\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{i k x-k^{2} t} \hat{u}_{0}(k) d k-\frac{1}{2 \pi} \int_{\partial \Omega} e^{i k x-k^{2} t} \hat{u}_{0}(-k) d k  \tag{7}\\
I_{f_{0}}(x, t) & =\frac{1}{i \pi} \int_{\partial \Omega} k e^{i k x-k^{2} t} F_{0}\left(k^{2}, t\right) d k \tag{8}
\end{align*}
$$

and

$$
\begin{equation*}
\hat{u}_{0}(k)=\int_{0}^{\infty} e^{-i k y} u_{0}(y) d y, \quad F_{0}\left(k^{2}, t\right)=\int_{0}^{t} e^{k^{2} s} f_{0}(s) d s \tag{9}
\end{equation*}
$$

The region $\Omega=\{k \in \mathbb{C}:|k|>r$, and $\pi / 4<\operatorname{Arg}(k)<3 \pi / 4\}$ for some $r>0$ is shown in Figure 2.
A simple contour deformation followed by a combination of terms allows for the rewriting of the solution using the classical Fourier sine transform

$$
\begin{equation*}
u(x, t)=\frac{2}{\pi} \int_{0}^{\infty} d k e^{-k^{2} t} \sin (k x)\left[\int_{0}^{\infty} u_{0}(y) \sin (k y) d y+k \int_{0}^{t} e^{k^{2} s} f_{0}(s) d s\right] \tag{10}
\end{equation*}
$$

from which it is immediately clear that $u(-x, t)=-u(x, t)$. Thus, an analytic continuation of the solution for $x<0$ is not obtained from considering negative values of $x$ in (10), unless $f_{0}(t) \equiv 0$. We pursue a different approach.

The part $I_{0}(x, t)$ of the solution containing the initial condition, is entire in $x$ by Theorem 1 .


Figure 2: The region $\Omega$ for the heat equation on the half-line.

Theorem 1. If $u_{0} \in L^{1}\left(\mathbb{R}^{+}\right)$, then $I_{0}(x, t)$ is entire in $x$ for $t>0$. If $f_{0} \in L^{1}(0, T)$ for some $T>0$, then $I_{f_{0}}(x, t)$ is analytic for $|\operatorname{Im}(x)|<\operatorname{Re}(x)$ for $0 \leq t \leq T$.

Proof. Let $\Gamma$ be any closed, piecewise smooth contour in the complex $x$-plane. Then,

$$
\oint_{\Gamma} d x \int_{-\infty}^{\infty} e^{i k x-k^{2} t} \hat{u}_{0}(k) d k=\int_{-\infty}^{\infty} d k e^{-k^{2} t} \hat{u}_{0}(k) \oint_{\Gamma} e^{i k x} d x=0
$$

and, since $\hat{u}_{0}(-k)$ is analytic for $\operatorname{Im}(k)>0$,
$\oint_{\Gamma} d x \int_{\partial \Omega} e^{i k x-k^{2} t} \hat{u}_{0}(-k) d k=\oint_{\Gamma} d x \int_{-\infty}^{\infty} e^{i k x-k^{2} t} \hat{u}_{0}(-k) d k=\int_{-\infty}^{\infty} d k e^{-k^{2} t} \hat{u}_{0}(-k) \oint_{\Gamma} e^{i k x} d x=0$, by Cauchy's theorem [1]. The order of integration may be switched using Fubini's theorem [9], since

$$
\oint_{\Gamma}|d x| \int_{-\infty}^{\infty}\left|e^{i k x-k^{2} t} \hat{u}_{0}( \pm k)\right| d k \leq \ell(\Gamma)\left\|u_{0}\right\|_{1} \sqrt{\frac{\pi}{t}} \max _{x \in \Gamma} e^{|x|^{2} / 4 t}<\infty
$$

where $\ell(\Gamma)$ is the arclength of $\Gamma$. By Morera's theorem [1], $I_{0}(x, t)$ is entire in $x$.
We proceed similarly for $I_{f_{0}}(x, t)$. We have

$$
\oint_{\Gamma} d x \int_{\partial \Omega} k e^{i k x-k^{2} t} F_{0}\left(k^{2}, t\right) d k=\int_{\partial \Omega} d k k e^{-k^{2} t} F_{0}\left(k^{2}, t\right) \oint_{\Gamma} e^{i k x} d x=0
$$

so that $I_{f_{0}}(x, t)$ is analytic in $x$ wherever Fubini's theorem can be applied to switch the order of integration. Since for $k \in \partial \Omega$,

$$
\left|e^{-k^{2} t} F_{0}\left(k^{2}, t\right)\right|=\left|\int_{0}^{t} e^{-k^{2}(t-s)} f_{0}(s) d s\right| \leq \int_{0}^{T}\left|f_{0}(s)\right| d s=\left\|f_{0}\right\|_{1}
$$

it follows that, with $\Gamma$ in the region $|\operatorname{Im}(x)|<\operatorname{Re}(x)$,

$$
\begin{aligned}
\oint_{\Gamma}|d x| \int_{\partial \Omega}\left|k e^{i k x-k^{2} t} F_{0}\left(k^{2}, t\right)\right||d k| & \leq \ell(\Gamma)\left\|f_{0}\right\|_{1} \max _{x \in \Gamma} \int_{\partial \Omega}\left|k e^{i k x}\right||d k| \\
& \leq \ell(\Gamma)\left\|f_{0}\right\|_{1} \max _{x \in \Gamma} \frac{4|x|^{2}}{\left(\operatorname{Re}(x)^{2}-\operatorname{Im}(x)^{2}\right)^{2}}<\infty
\end{aligned}
$$

so that $I_{f_{0}}(x, t)$ is an analytic function of $x$ for $|\operatorname{Im}(x)|<\operatorname{Re}(x)$.
Returning to (6), for $I_{f_{0}}(x, t)$ we can switch the order of integration and integrate over $k$ to find

$$
\begin{equation*}
I_{f_{0}}(x, t)=\frac{1}{i \pi} \int_{0}^{t} d s f_{0}(s) \int_{\partial \Omega} k e^{i k x-k^{2}(t-s)} d k=\frac{x}{2 \sqrt{\pi}} \int_{0}^{t} \frac{f_{0}(s)}{(t-s)^{\frac{3}{2}}} e^{-\frac{x^{2}}{4(t-s)}} d s \tag{11}
\end{equation*}
$$

which is analytic for $x>0$ by Theorem 1 , but is discontinuous at $x=0$, unless $f_{0}(t) \equiv 0$, which is confirmed below, see (21). In fact [8],

$$
\begin{equation*}
\lim _{x \rightarrow 0^{+}} I_{f_{0}}(x, t)=f_{0}(t), \quad I_{f_{0}}(0, t)=0, \quad \lim _{x \rightarrow 0^{-}} I_{f_{0}}(x, t)=-f_{0}(t) \tag{12}
\end{equation*}
$$

as is easily seen from (10) or (11).
We extend $I_{f_{0}}(x, t)$ for $x>0$ to an entire function by constructing its Taylor series about $x=0$. If we do so,

$$
\begin{equation*}
I_{f_{0}}(x, t)=\sum_{n=0}^{\infty} a_{2 n}(t) x^{2 n}+\sum_{n=0}^{\infty} a_{2 n+1}(t) x^{2 n+1} \tag{13}
\end{equation*}
$$

which is valid for $x>0$, but trivially extended to negative $x$. With $x \rightarrow-x$, we find

$$
\begin{equation*}
I_{f_{0}}(-x, t)=\sum_{n=0}^{\infty} a_{2 n}(t) x^{2 n}-\sum_{n=0}^{\infty} a_{2 n+1}(t) x^{2 n+1} \tag{14}
\end{equation*}
$$

so that

$$
\begin{equation*}
I_{f_{0}}(x, t)=2 \sum_{n=0}^{\infty} a_{2 n}(t) x^{2 n}-I_{f_{0}}(-x, t) \tag{15}
\end{equation*}
$$

which relates the values for $x<0$ to the values for $x>0$. We have chosen to extend $I_{f_{0}}(x, t)$ to negative values of $x$ using the even terms in the Taylor series because $I_{f_{0}}(x, t)$ in (11) is odd and the Dirichlet condition immediately gives $a_{0}(t)=f_{0}(t)$.

We cannot find the Taylor series for $I_{f_{0}}(x, t)$ by finding the Taylor series for the kernel in (11), since the resulting integrals diverge. It is possible to transform the $s$ variable and find the Taylor series, but it is easier and more generalizable to use the unevaluated contour integrals (since their evaluation is not possible, in general):

$$
\begin{equation*}
\left.\frac{\partial^{2 n} I_{f_{0}}}{\partial x^{2 n}}\right|_{x=0}=\frac{(-1)^{n}}{i \pi} \int_{0}^{t} d s f_{0}(s) \int_{\partial \Omega} k^{2 n+1} e^{-k^{2}(t-s)} d k . \tag{16}
\end{equation*}
$$

Using [8],

$$
\begin{equation*}
\frac{\partial^{n}}{\partial s^{n}} \delta(s-t)=\frac{1}{i \pi} \int_{\partial \Omega} k^{2 n+1} e^{-k^{2}(t-s)} d k \tag{17}
\end{equation*}
$$

we obtain

$$
\begin{equation*}
\left.\frac{\partial^{2 n} I_{f_{0}}}{\partial x^{2 n}}\right|_{x=0}=(-1)^{n} \int_{0}^{t} f_{0}(s) \delta^{(n)}(s-t) d s=\int_{0}^{t} f_{0}^{(n)}(s) \delta(s-t) d s=f_{0}^{(n)}(t) \tag{18}
\end{equation*}
$$

We obtain this same result rigorously using small $x$ asymptotics [2], see (28), but it is convenient to use the delta function when possible.

Then, for $x<0$,

$$
\begin{equation*}
I_{f_{0}}(x, t)=2 \sum_{n=0}^{\infty} \frac{x^{2 n}}{(2 n)!} f_{0}^{(n)}(t)-I_{f_{0}}(-x, t)=\tilde{f}_{0}(x, t)-I_{f_{0}}(-x, t) \tag{19}
\end{equation*}
$$

where we have defined

$$
\begin{equation*}
\tilde{f}_{0}(x, t)=2 \sum_{n=0}^{\infty} \frac{x^{2 n}}{(2 n)!} f_{0}^{(n)}(t) \tag{20}
\end{equation*}
$$

This equation has been derived by Burggrag, see [3]. It follows that

$$
I_{f_{0}}^{\text {ext }}(x, t)= \begin{cases}I_{f_{0}}(x, t), & x>0  \tag{21}\\ f_{0}(t), & x=0 \\ \tilde{f}_{0}(x, t)-I_{f_{0}}(-x, t), & x<0\end{cases}
$$

is an analytic function for $x \in \mathbb{R}$ (see Theorem 2 below). This illustrates that for $I_{f_{0}}^{\text {ext }}(x, t)$ to equal the odd function $I_{f_{0}}(x, t)$ in (11), we need $f_{0}(t) \equiv 0$.

In order to obtain all coefficients of the Taylor series for $I_{f_{0}}(x, t)$, we start by deforming $\Omega$ down to $\gamma$, see Figure 2. Using repeated integration by parts,

$$
\begin{align*}
I_{f_{0}}(x, t) & =\frac{1}{i \pi} \int_{\gamma} d k k e^{i k x-k^{2} t} \int_{0}^{t} e^{k^{2} s} f_{0}(s) d s \\
& =\frac{1}{i \pi} \int_{\gamma} d k k e^{i k x-k^{2} t}\left[\sum_{m=1}^{n} \frac{(-1)^{m} f_{0}^{(m-1)}(0)}{k^{2 m}}+\frac{(-1)^{n}}{k^{2 n}} \int_{0}^{t} e^{k^{2} s} f_{0}^{(n)}(s) d s\right] \\
& =\sum_{m=1}^{n} \phi_{m}(x, t) f_{0}^{(m-1)}(0)+\int_{0}^{t} f_{0}^{(n)}(s) \phi_{n}(x, t-s) d s \tag{22}
\end{align*}
$$

where the integral around $\gamma$ of the terms involving $f_{0}^{(m-1)}(t) e^{k^{2} t}$ is zero. We define

$$
\begin{equation*}
\phi_{m}(x, t)=\frac{(-1)^{m}}{i \pi} \int_{\gamma} \frac{k e^{i k x-k^{2} t}}{k^{2 m}} d k \tag{23}
\end{equation*}
$$

Switching the order of integration above is allowed since, assuming analyticity of $f_{0}(t)$, for $x>0$,

$$
\begin{equation*}
\int_{0}^{t} d s\left|f_{0}^{(n)}(s)\right| \int_{\gamma}\left|\frac{k e^{i k x-k^{2}(t-s)}}{k^{2 m}}\right||d k|<\infty \tag{24}
\end{equation*}
$$

Differentiating (23) $(2 n-q)$-times with respect to $x(q=0,1)$, setting $x=0$, and evaluating the integral, we have

$$
\begin{equation*}
\phi_{m}^{(2 n-q)}(0, t)=\frac{(-1)^{m}}{i \pi} \int_{\gamma} \frac{(i k)^{2 n-q} k e^{-k^{2} t}}{k^{2 m}} d k=-\delta_{q, 1} \frac{(-1)^{n-m} \Gamma\left(n-m+\frac{1}{2}\right)}{\pi t^{n-m+\frac{1}{2}}}, \tag{25}
\end{equation*}
$$

where $\delta_{q, 1}=1$ if $q=1$ and 0 otherwise. We can switch the order of differentiation, integration, and taking limits because of absolute integrability and the boundedness in $x$. This is why we deform to $\gamma$. Therefore,

$$
\begin{equation*}
\left.\frac{\partial^{2 n-q} I_{f_{0}}}{\partial x^{2 n-q}}\right|_{x=0}=\sum_{m=1}^{n} \phi_{m}^{(2 n-q)}(x, t)+\int_{0}^{t} f_{0}^{(n)}(s) \phi_{n}^{(2 n-q)}(x, t-s) d s \tag{26}
\end{equation*}
$$

It follows that for the Taylor coefficients of the even powers, we have

$$
\begin{equation*}
\left.\frac{\partial^{2 n} I_{f_{0}}}{\partial x^{2 n}}\right|_{x=0}=\lim _{x \rightarrow 0^{+}} \int_{0}^{t} d s f_{0}^{(n)}(s) \frac{1}{i \pi} \int_{\partial \Omega} k e^{i k x-k^{2} t} d k=\lim _{x \rightarrow 0^{+}} \frac{x}{2 \sqrt{\pi}} \int_{0}^{t} \frac{f_{0}^{(n)}(s)}{(t-s)^{\frac{3}{2}}} e^{-\frac{x^{2}}{4(t-s)}} d s \tag{27}
\end{equation*}
$$

Note that the non-integral terms from (26) vanish. Substituting $z=x^{2} /(4(t-s))$,

$$
\begin{equation*}
\left.\frac{\partial^{2 n} I_{f_{0}}}{\partial x^{2 n}}\right|_{x=0}=\lim _{x \rightarrow 0^{+}} \frac{1}{\sqrt{\pi}} \int_{\frac{x^{2}}{4 t}}^{\infty} f_{0}^{(n)}\left(t-\frac{x^{2}}{4 z^{2}}\right) \frac{e^{-z}}{\sqrt{z}} d z=\frac{f_{0}^{(n)}(t)}{\sqrt{\pi}} \int_{0}^{\infty} \frac{e^{-z}}{\sqrt{z}} d z=f_{0}^{(n)}(t) \tag{28}
\end{equation*}
$$

We can take the limit inside the integral by the dominated convergence theorem [9]. This more rigorous derivation confirms the result (18) obtained above using delta functions. Due to the simplicity of the delta function approach, we prefer it in what follows.

Repeating this line of thought for the coefficients of the odd powers in the Taylor series, we get

$$
\begin{equation*}
a_{2 n-1}(t)=-\frac{\Gamma\left(\frac{1}{2}\right)}{\pi(2 n-1)!}\left[\sum_{m=1}^{n} \frac{(-1)^{n-m} \Gamma\left(n-m+\frac{1}{2}\right)}{t^{n-m+\frac{1}{2}} \Gamma\left(\frac{1}{2}\right)} f_{0}^{(m-1)}(0)+\int_{0}^{t} \frac{f_{0}^{(n)}(s)}{(t-s)^{\frac{1}{2}}} d s\right] \tag{29}
\end{equation*}
$$

These results may be combined to write

$$
\begin{equation*}
I_{f_{0}}(x, t)=\sum_{n=0}^{\infty} \frac{(-1)^{n} x^{n}}{n!} f_{0}^{\left(\frac{n}{2}\right)}(t) \tag{30}
\end{equation*}
$$

where the coefficients are interpreted as Riemann-Liouville fractional derivatives [15]. For nonmonomial dispersion relations $W(k)$, a concise notation using fractional derivatives is not possible. This representation of $I_{f_{0}}(x, t)$ is trivially analytically continued for $x \in \mathbb{C}$. Thus the right-hand side of (30) is a representation for $I_{f_{0}}^{\text {ext }}(x, t)$ for all $x \in \mathbb{C}$, which we prove below in Theorem 2 , provided $f_{0}(t)$ is analytic in a neighborhood of the positive real $t$-axis.

If an extension is required close to the boundary only (for instance to set up a numerical scheme), the series representation (30) may be more convenient than the more global extension provided by (21). In this section, we have been careful to distinguish between a function and its representation in a part of the complex plane. For the sake of brevity, less care is used below. We expect the distinction to be clear in context.

Theorem 2. Define $\mathcal{D}=\{t \in \mathbb{C}: \operatorname{dist}(t,[0, T]) \leq r\}$, for some $r>0, T>0$, a domain in the complex t-plane containing the interval $[0, T]$. If $f_{0}(t)$ is analytic in $\mathcal{D}$, then the series representation (30) is entire in $x$ for $0 \leq t \leq T$.

Proof. Since $f_{0}(t)$ is analytic in $\mathcal{D}$, then for all $0 \leq s \leq t \leq T, f_{0}(\tau)$ is analytic in $|\tau-s| \leq r$. By Cauchy's integral formula [1], we have

$$
\left|f_{0}^{(n)}(s)\right| \leq \frac{n!}{r^{n}} \sup _{|\tau-s|=r}\left|f_{0}(\tau)\right| \leq \frac{n!}{r^{n}}\left\|f_{0}\right\|_{\infty}
$$

where $\left\|f_{0}\right\|_{\infty}=\sup _{t \in \mathcal{D}}\left|f_{0}(t)\right|$, which exists, since $f_{0}(t)$ is analytic in $\mathcal{D}$. Then,

$$
\left|a_{2 n}(t)\right|=\frac{\left|f_{0}^{(n)}(t)\right|}{(2 n)!} \leq \frac{n!}{r^{n}(2 n)!}\left\|f_{0}\right\|_{\infty}
$$

so that the series of the even terms converges absolutely for all $x$. We also have

$$
\begin{aligned}
\left|a_{2 n-1}(t)\right| & =\frac{1}{\sqrt{\pi}(2 n-1)!}\left|\sum_{m=1}^{n} \frac{(-1)^{n-m} \Gamma\left(n-m+\frac{1}{2}\right)}{\sqrt{\pi} t^{n-m+\frac{1}{2}}} f_{0}^{(m-1)}(0)+\int_{0}^{t} \frac{f_{0}^{(n)}(s)}{\sqrt{t-s}} d s\right| \\
& \leq \frac{\left\|f_{0}\right\|_{\infty}}{\sqrt{\pi}(2 n-1)!}\left[\sum_{m=1}^{n} \frac{\Gamma\left(n-m+\frac{1}{2}\right)(m-1)!}{\sqrt{\pi} t^{n-m+\frac{1}{2}} r^{m-1}}+\frac{n!}{r^{n}} \int_{0}^{t} \frac{d s}{\sqrt{t-s}}\right] \\
& \leq \frac{2 n!\left\|f_{0}\right\|_{\infty} \sqrt{t}}{\sqrt{\pi}(2 n-1)!r^{n}}\left[1+\sum_{m=1}^{n} \frac{\Gamma\left(n-m+\frac{1}{2}\right)(m-1)!}{2 \sqrt{\pi} n!}\left(\frac{r}{t}\right)^{n-m+1}\right]
\end{aligned}
$$

so that for $t>r$,

$$
\left|a_{2 n-1}(t)\right| \leq \frac{3 n!\left\|f_{0}\right\|_{t} \sqrt{t}}{\sqrt{\pi}(2 n-1)!r^{n}}
$$

while for $t<r$,

$$
\left|a_{2 n-1}(t)\right| \leq \frac{3 n!\left\|f_{0}\right\|_{t}}{\sqrt{\pi}(2 n-1)!t^{n-\frac{1}{2}}}
$$

where we used

$$
\sum_{m=1}^{n} \frac{\Gamma\left(n-m+\frac{1}{2}\right)(m-1)!}{2 \sqrt{\pi} n!} \leq \frac{1}{2}
$$

which is proved by showing the left-hand side is a decreasing function of $n$. Thus, the series of the odd terms also converges absolutely for all $x \in \mathbb{C}$ and for any $0 \leq t \leq T$.

### 2.1.1 Boundary-to-Initial Map

Consider (20). It follows that $\tilde{f}_{0 t}=\tilde{f}_{0 x x}$, so that

$$
w(x, t)=u_{\mathrm{ac}}(x, t)= \begin{cases}u(x, t), & x \geq 0  \tag{31}\\ \tilde{f}_{0}(x, t)-u(-x, t), & x<0\end{cases}
$$

is an analytic solution to the heat equation on the whole line with initial condition

$$
w_{0}(x)= \begin{cases}u_{0}(x), & x \geq 0  \tag{32}\\ \tilde{f}_{0}(x, 0)-u_{0}(-x), & x<0\end{cases}
$$

This initial condition is analytic when the compatibility conditions $u_{0}^{(2 n)}(0)=f_{0}^{(n)}(0)$, for $n \geq 0$ are satisfied [14]. If $u_{0}(x)$ is analytic and these compatibility conditions are satisfied, then $w_{0}(x)$ equals the analytic extension of $u_{0}(x)$ for negative values of $x$. Note that for the homogeneous boundary condition $f_{0}(t)=0$, we recover the method of images' odd extension of the initial condition.


Figure 3: (a) The initial condition $w_{0}(x)$, given by (32), leading to the (analytic) solution on the right, shown with the UTM solution at $t=0$ and the whole-line initial condition, $u_{0}(x)=u_{\mathbb{R}}(x, 0)$. (b) The solution $u_{\text {ac }}(x, t)$ at $t=1$ obtained through analytic continuation, overlaid with the solution $u_{\mathbb{R}}(x, t)(34)$ and the discontinuous extension $u_{\mathrm{UTM}}(x, t)$ resulting from evaluating the UTM solution for negative $x$ values.

It is noteworthy that $w_{0}(x)$ may be unbounded for $x<0$. For example, if $f_{0}(t)=t e^{-t}$, then $f_{0}^{(n)}(0)=-(-1)^{n} n$, and

$$
\begin{equation*}
\tilde{f}_{0}(x, 0)=-\sum_{n=1}^{\infty} \frac{(-1)^{n} x^{2 n}}{(2 n-1)!}=x \sin (x) \tag{33}
\end{equation*}
$$

Thus, the corresponding whole-line problem may not be of physical interest. Nevertheless, the expressions obtained can be used to estimate the solution outside but near the physical domain to be used in numerical schemes requiring information at so-called ghost points.

### 2.1.2 Examples

We demonstrate our results using two examples. Our first example starts from a whole-line solution,

$$
\begin{equation*}
u_{\mathbb{R}}(x, t)=\frac{e^{-\frac{(x-1)^{2}}{4 t+1}}}{\sqrt{4 t+1}} \tag{34}
\end{equation*}
$$

from which we construct the initial and boundary conditions for a half-line problem: $u_{0}(x)=$ $u_{\mathbb{R}}(x, 0)$ for $x>0$, and $f_{0}(t)=u_{\mathbb{R}}(0, t)$. Next, we use our analytic continuation result $u_{\text {ac }}(x, t)$ to reconstruct the solution $u_{\mathbb{R}}(x, t)$. The results are shown in Figure 3.

Next, we consider (1) with $f_{0}(t)=t e^{-t}$ and $u_{0}(x)=u_{\mathbb{R}}(x, 0)$ for $x>0$. Since $f_{0}^{(n)}(t)=$ $(-1)^{n} e^{-t}(t-n)$, we obtain

$$
\begin{equation*}
\tilde{f}_{0}(x, t)=2 e^{-t} \sum_{n=0}^{\infty} \frac{(-1)^{n} x^{2 n}(t-n)}{(2 n)!}=e^{-t}(2 t \cos (x)+x \sin (x)) \tag{35}
\end{equation*}
$$

Using (31) and (32), we find the analytic continuation for $x \in \mathbb{R}$. In this case, $w_{0}(x)$ is discontinuous at $x=0$, (since the first compatibility condition is not satisfied), see Figure 4.


Figure 4: ( $\mathrm{a}, \mathrm{c}$ ) The (discontinuous) initial condition $w_{0}(x)$ (32) leading to the (analytic) solution on the right, shown with $u_{\mathrm{UTM}}(x, 0)$ and the analytic continuation of $u_{0}(x)$. Note that (a) is a close-up of (c). (b) The solution $u_{\mathrm{ac}}(x, t)(31)$ at $t=0.001$ obtained through analytic continuation, shown with the discontinuous UTM solution $u_{\mathrm{UTM}}(x, t)$. Note that the jump in $u_{\mathrm{UTM}}(x, t), u_{\mathrm{UTM}}\left(0^{+}, t\right)-$ $u_{\mathrm{UTM}}\left(0^{-}, t\right)=2 f_{0}(t) \approx 0.002$. (d) The solution $u_{\mathrm{ac}}(x, t)$ at $t=1$ obtained through analytic continuation, shown with the discontinuous UTM solution $u_{\mathrm{UTM}}(x, t)$.

### 2.2 Neumann boundary conditions

The heat equation on the half line with Neumann boundary conditions,

$$
\begin{align*}
u_{t} & =u_{x x}, & & x>0, t>0,  \tag{36a}\\
u(x, 0) & =u_{0}(x), & & x>0,  \tag{36b}\\
u_{x}(0, t) & =f_{1}(t), & & t>0, \tag{36c}
\end{align*}
$$

has the UTM solution [8]

$$
\begin{equation*}
u(x, t)=I_{0}(x, t)+I_{f_{1}}(x, t) \tag{37}
\end{equation*}
$$

with

$$
\begin{align*}
I_{0}(x, t) & =\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{i k x-k^{2} t} \hat{u}_{0}(k) d k+\frac{1}{2 \pi} \int_{\partial \Omega} e^{i k x-k^{2} t} \hat{u}_{0}(-k) d k  \tag{38}\\
I_{f_{1}}(x, t) & =-\frac{1}{\pi} \int_{\partial \Omega} e^{i k x-k^{2} t} F_{1}\left(k^{2}, t\right) d k \tag{39}
\end{align*}
$$

and

$$
\begin{equation*}
\hat{u}_{0}(k)=\int_{0}^{\infty} e^{-i k y} u_{0}(y) d y, \quad F_{1}\left(k^{2}, t\right)=\int_{0}^{t} e^{k^{2} s} f_{1}(s) d s \tag{40}
\end{equation*}
$$

The region $\Omega$ remains as in Section 2.1, Figure 2.
As before, $I_{0}(x, t)$ is an entire function of $x \in \mathbb{C}$ for $t>0$ by Theorem 1. Again as before, the integration with respect to $k$ in $I_{f_{1}}(x, t)$ can be evaluated so that

$$
\begin{equation*}
I_{f_{1}}(x, t)=-\frac{1}{\pi} \int_{0}^{t} d s f_{1}(s) \int_{\partial \Omega} e^{i k x-k^{2}(t-s)} d k=-\frac{1}{\sqrt{\pi}} \int_{0}^{t} \frac{f_{1}(s)}{\sqrt{t-s}} e^{-\frac{x^{2}}{4(t-s)}} d s \tag{41}
\end{equation*}
$$

which is analytic for $x>0$ (using a result analogous to Theorem 1), but not at $x=0$, although it is continuous there. This is an even function of $x$ (immediately seen using the classical cosine transform representation or from (41)) and not analytic at $x=0$ unless $f_{1}(t) \equiv 0$, as we see below, (46). We extend $I_{f_{1}}(x, t)$ to an entire function by finding a Taylor series representation about $x=0$ :

$$
\begin{equation*}
I_{f_{1}}(x, t)=2 \sum_{n=0}^{\infty} b_{2 n+1}(t) x^{2 n+1}+I_{f_{1}}(-x, t) \tag{42}
\end{equation*}
$$

where we use the odd terms in the Taylor series to extend the function to $x<0$. This choice is convenient because the Neumann condition (36c) gives $b_{1}(t)=f_{1}(t)$. Proceeding as before, we have

$$
\begin{equation*}
\left.\frac{\partial^{2 n+1} I_{f_{1}}}{\partial x^{2 n+1}}\right|_{x=0}=\frac{(-1)^{n}}{i \pi} \int_{0}^{t} d s f_{1}(s) \int_{\partial \Omega} k^{2 n+1} e^{-k^{2}(t-s)} d k=f_{1}^{(n)}(t) \tag{43}
\end{equation*}
$$

Thus, for $x<0$,

$$
\begin{equation*}
I_{f_{1}}(x, t)=\tilde{f}_{1}(x, t)+I_{f_{1}}(-x, t) \tag{44}
\end{equation*}
$$

where we define

$$
\begin{equation*}
\tilde{f}_{1}(x, t)=2 \sum_{n=0}^{\infty} \frac{x^{2 n+1}}{(2 n+1)!} f_{1}^{(n)}(t) \tag{45}
\end{equation*}
$$

so that

$$
I_{f_{1}}^{\mathrm{ext}}(x, t)= \begin{cases}I_{f_{1}}(x, t), & x \geq 0  \tag{46}\\ \tilde{f}_{1}(x, t)+I_{f_{1}}(-x, t), & x<0\end{cases}
$$

is an analytic function for $x \in \mathbb{R}$. At this point, it is clear that in order for this to equal the even function $I_{f_{1}}(x, t)$, we need $f_{1}(t) \equiv 0$. If we want the full series for $I_{f_{1}}(x, t)$, we can use integration by parts in the same way as in Section 2.1, to get the coefficients of the even terms in the Taylor series as

$$
\begin{equation*}
b_{2 n}(t)=-\frac{\Gamma\left(\frac{1}{2}\right)}{\pi(2 n)!}\left[\sum_{m=1}^{n} \frac{(-1)^{n-m} \Gamma\left(n-m+\frac{1}{2}\right)}{t^{n-m+\frac{1}{2}} \Gamma\left(\frac{1}{2}\right)}+\int_{0}^{t} \frac{f_{1}^{(n)}(s)}{(t-s)^{\frac{1}{2}}} d s\right] \tag{47}
\end{equation*}
$$

allowing us to rewrite $I_{f_{1}}(x, t)$ as

$$
\begin{equation*}
I_{f_{1}}(x, t)=-\sum_{n=0}^{\infty} \frac{(-1)^{n} x^{n}}{n!} f_{1}^{\left(\frac{n-1}{2}\right)}(t) \tag{48}
\end{equation*}
$$

As before, the radius of convergence of this series is infinite under the assumptions of Theorem 2.


Figure 5: (a) The initial condition $w_{0}(x)$ given by (50), leading to the (analytic) solution on the right, shown with the UTM solution at $t=0$ and the whole-line initial condition $u_{0}(x)=u_{\mathbb{R}}(x, 0)$. (b) The solution $u_{\mathrm{ac}}(x, t)(49)$ at $t=1$ obtained through analytic continuation, overlaid with the solution $u_{\mathbb{R}}(x, t)(34)$ and the nondifferentiable extension $u_{\mathrm{UTM}}(x, t)$ resulting from evaluating the UTM solution for negative $x$ values.

### 2.2.1 Boundary-to-Initial Map

Considering (45), we see that $\tilde{f}_{1 t}=\tilde{f}_{1 x x}$. Therefore

$$
w(x, t)=u_{\mathrm{ac}}(x, t)= \begin{cases}u(x, t), & x \geq 0  \tag{49}\\ \tilde{f}_{1}(x, t)+u(-x, t), & x<0\end{cases}
$$

is an analytic solution to the heat equation on the whole line with initial condition

$$
w_{0}(x)= \begin{cases}u_{0}(x), & x \geq 0  \tag{50}\\ \tilde{f}_{1}(x, 0)+u_{0}(-x), & x<0\end{cases}
$$

This initial condition is analytic when the compatibility conditions $u_{0}^{(2 n+1)}(0)=f_{1}^{(n)}(0), n \geq 0$ are satisfied [14]. If $u_{0}(x)$ is analytic and these corner compatibility conditions are met, then $w_{0}(x)=$ $u_{0}(x)$. As before, $w_{0}(x)$ is not necessarily bounded for $x<0$. Note that for the homogeneous boundary condition $f_{1}(t)=0$, we recover the method of images' even extension of the initial condition.

### 2.2.2 Examples

As an example, we construct a half-line problem using (34) but with the appropriate Neumann boundary condition. Since the solution of the Neumann IBVP, as given by the UTM, is even if we allow $x<0$, we obtain a continuous, but non-differentiable function. Using the analytic continuation described above, $u_{\mathbb{R}}(x, t)$ is recovered, see Figure 5 .

Using instead the boundary function $f_{1}(t)=t e^{-t}$,

$$
\begin{equation*}
\tilde{f}_{1}(x, t)=e^{-t}((2 t-1) \sin (x)-x \cos (x)) \tag{51}
\end{equation*}
$$



Figure 6: (a) The (non-differentiable) initial condition $w_{0}(x)$ (50) leading to the (analytic) solution on the right, shown with the analytic continuation of $u_{0}(x)$.(b) The solution $u_{\mathrm{ac}}(x, t)(49)$ at $t=1$ obtained through analytic continuation, shown with the even extension of $u_{\mathrm{UTM}}(x, t)$.

In this case, the resulting analytic continuation is a smooth function, but the corresponding initial condition is not differentiable at $x=0$, see Figure 6

### 2.3 Finite interval with Dirichlet boundary conditions

The solution to the heat equation on a finite interval with Dirichlet boundary conditions,

$$
\begin{align*}
u_{t} & =u_{x x}, & & x \in(0, L), t>0,  \tag{52a}\\
u(x, 0) & =u_{0}(x), & & x \in(0, L),  \tag{52b}\\
u(0, t) & =f_{0}(t), & & t>0,  \tag{52c}\\
u(L, t) & =g_{0}(t), & & t>0, \tag{52d}
\end{align*}
$$

can be written as [8]

$$
\begin{equation*}
u(x, t)=I_{0}(x, t)+I_{f_{0}}(x, t)+I_{g_{0}}(x, t) \tag{53}
\end{equation*}
$$

with

$$
\begin{align*}
I_{0}(x, t) & =\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{i k x-k^{2} t} \hat{u}_{0}(k) d k-\frac{1}{2 \pi} \int_{\partial \Omega} e^{-k^{2} t} \frac{e^{i k(L+x)}-e^{i k(L-x)}}{\Delta(k)} \hat{u}_{0}(k) d k \\
& -\frac{1}{2 \pi} \int_{\partial \Omega} e^{-k^{2} t} \frac{e^{i k(L-x)}-e^{-i k(L-x)}}{\Delta(k)} \hat{u}_{0}(-k) d k  \tag{54}\\
I_{f_{0}}(x, t) & =\frac{1}{i \pi} \int_{\partial \Omega} k e^{-k^{2} t} \frac{e^{i k(L-x)}-e^{-i k(L-x)}}{\Delta(k)} F_{0}\left(k^{2}, t\right) d k  \tag{55}\\
I_{g_{0}}(x, t) & =\frac{1}{i \pi} \int_{\partial \Omega} k e^{-k^{2} t} \frac{e^{i k x}-e^{-i k x}}{\Delta(k)} G_{0}\left(k^{2}, t\right) d k \tag{56}
\end{align*}
$$

Here

$$
\begin{gather*}
\hat{u}_{0}(k)=\int_{0}^{L} e^{-i k y} u_{0}(y) d y,  \tag{57}\\
F_{0}\left(k^{2}, t\right)=\int_{0}^{t} e^{k^{2} s} f_{0}(s) d s,  \tag{58}\\
G_{0}\left(k^{2}, t\right)=\int_{0}^{t} e^{k^{2} s} g_{0}(s) d s
\end{gather*}
$$

and $\Omega=\{k \in \mathbb{C}:|k|>r$, and $\pi / 4<\operatorname{Arg}(k)<3 \pi / 4\}$, for some $r>0$, shown in Figure 2.
The integral $I_{0}(x, t)$ is an entire function of $x$ by Theorem 3 below.
Theorem 3. If $u_{0} \in L^{1}(0, L)$, then $I_{0}(x, t)$ is entire in $x$ for $t>0$. If $f_{0} \in L^{1}(0, T)$ for some $T>0$, then for $0 \leq t \leq T, I_{f_{0}}(x, t)$ is analytic for $x \in\{x \in C:|\operatorname{Im}(x)|<\operatorname{Re}(x)$ and $|\operatorname{Im}(2 L-x)|<$ $\operatorname{Re}(2 L-x)\}$. If $g_{0} \in L^{1}(0, T)$, then for $0 \leq t \leq T, I_{g_{0}}(x, t)$ is analytic for $x \in\{x \in C:|\operatorname{Im}(L+x)|<$ $\operatorname{Re}(L+x)$ and $|\operatorname{Im}(L-x)|<\operatorname{Re}(L-x)\}$.

Proof. The proof is similar to the proof of Theorem 1.

The function $I_{f_{0}}(x, t)$ is defined and analytic only for $x \in(0,2 L)$, as otherwise its exponential kernel is growing on $\partial \Omega$. Deforming $\partial \Omega$ to the real axis, picking up residue contributions from the singularities, we obtain the classical Fourier series:

$$
\begin{equation*}
I_{f_{0}}(x, t)=\sum_{n=1}^{\infty} \frac{2 n \pi}{L^{2}} e^{-\frac{n^{2} \pi^{2} t}{L^{2}}} F_{0}\left(\frac{n^{2} \pi^{2}}{L^{2}}, t\right) \sin \left(\frac{n \pi x}{L}\right) . \tag{59}
\end{equation*}
$$

This Fourier series is defined outside $x \in(0,2 L)$, but this periodic extension is not in general analytic, see Figure 7. Upon integrating by parts,

$$
\begin{equation*}
F_{0}\left(k^{2}, t\right)=\int_{0}^{t} e^{k^{2} s} f_{0}(s) d s=\frac{f_{0}(t) e^{k^{2} t}-f_{0}(0)}{k^{2}}-\frac{1}{k^{2}} \int_{0}^{t} e^{k^{2} s} f_{0}^{\prime}(s) d s \sim \mathcal{O}\left(k^{-2} e^{k^{2} t}\right) \tag{60}
\end{equation*}
$$

and, due to a lack of convergence when taking $x$-derivatives, a Taylor series cannot be obtained from this representation.

Using the contour integral representation, we can analytically continue $I_{f_{0}}(x, t)$ for $x<0$, by finding a Taylor series about $x=0$. As before,

$$
\begin{equation*}
I_{f_{0}}(x, t)=2 \sum_{n=0}^{\infty} a_{2 n}(t) x^{2 n}-I_{f_{0}}(-x, t)=\tilde{f}_{0}(x, t)-I_{f_{0}}(-x, t) \tag{61}
\end{equation*}
$$

which defines $\tilde{f}_{0}(x, t)$. Using

$$
\begin{equation*}
I_{f_{0}}(x, t)=\frac{1}{i \pi} \int_{0}^{t} d s f_{0}(s) \int_{\partial \Omega} k \frac{\sin (k(L-x))}{\sin (k L)} e^{-k^{2}(t-s)} d k \tag{62}
\end{equation*}
$$

if we take $2 n x$-derivatives, we find

$$
\begin{equation*}
\left.\frac{\partial^{2 n} I_{f_{0}}}{\partial x^{2 n}}\right|_{x=0}=\frac{(-1)^{n}}{i \pi} \int_{0}^{t} d s f_{0}(s) \int_{\partial \Omega} k^{2 n+1} e^{-k^{2}(t-s)} d k=f_{0}^{(n)}(t) \tag{63}
\end{equation*}
$$

It follows that, for $x \in(-2 L, 0)$, as before,

$$
\begin{equation*}
I_{f_{0}}(x, t)=\tilde{f}_{0}(x, t)-I_{f_{0}}(-x, t) \tag{64}
\end{equation*}
$$

Thus,

$$
I_{f_{0}}^{\mathrm{ext}}(x, t)= \begin{cases}-I_{f_{0}}(-x, t)+\tilde{f}_{0}(x, t), & -2 L<x<0  \tag{65}\\ f_{0}(t), & x=0 \\ I_{f_{0}}(x, t), & 0<x<2 L\end{cases}
$$

is an analytic function for $-2 L<x<2 L$. For the remainder of this section, we define $I_{f_{0}}(0, t)=$ $f_{0}(t)$ and $I_{g_{0}}(L, t)=g_{0}(t)$, so as to not write the boundary terms separately. Since for $0<x<2 L$, $I_{f_{0}}(2 L-x, t)=-I_{f_{0}}(x, t)$,

$$
I_{f_{0}}^{\mathrm{ext}}(x-2 L, t)= \begin{cases}I_{f_{0}}(x, t)+\tilde{f}_{0}(x-2 L, t), & 0<x<2 L  \tag{66}\\ I_{f_{0}}(x-2 L, t), & 2 L \leq x<4 L\end{cases}
$$

is an analytic function for $0<x<4 L$, so that

$$
I_{f_{0}}^{\text {ext }}(x, t)= \begin{cases}-I_{f_{0}}(-x, t)+\tilde{f}_{0}(x, t), & -2 L<x<0  \tag{67}\\ I_{f_{0}}(x, t), & 0 \leq x<2 L \\ I_{f_{0}}(x-2 L, t)-\tilde{f}_{0}(x-2 L, t), & 2 L \leq x<4 L\end{cases}
$$

is an analytic function for $-2 L<x<4 L$. Continuing this, we find

$$
I_{f_{0}}^{\mathrm{ext}}(x, t)= \begin{cases}\vdots & -4 L \leq x<-2 L  \tag{68}\\ I_{f_{0}}(x+4 L, t)+\tilde{f}_{0}(x, t)+\tilde{f}_{0}(x+2 L, t), & -2 L \leq x<0 \\ I_{f_{0}}(x+2 L, t)+\tilde{f}_{0}(x, t), & 0 \leq x<2 L \\ I_{f_{0}}(x, t), & 2 L \leq x<4 L \\ I_{f_{0}}(x-2 L, t)-\tilde{f}_{0}(x-2 L, t), \\ I_{f_{0}}(x-4 L, t)-\tilde{f}_{0}(x-2 L, t)-\tilde{f}_{0}(x-4 L, t), & 4 L \leq x<6 L \\ \vdots & \end{cases}
$$

We can repeat the analysis similar to what we did for the whole-line problem, by finding the odd coefficients. Alternatively, we can expand about a point other than $x=0$. For instance, denoting the Taylor series about $x=L$ as

$$
\begin{equation*}
I_{f_{0}}(x, t)=\sum_{n=1}^{\infty} A_{2 n-1}(t)(x-L)^{2 n-1} \tag{69}
\end{equation*}
$$

(the coefficients of the even terms vanish), we have

$$
\begin{equation*}
A_{2 n-1}(t)=\left.\frac{1}{(2 n-1)!} \frac{\partial^{2 n-1} I_{f_{0}}}{\partial x^{2 n-1}}\right|_{x=L}=\frac{2(-1)^{n}}{\pi(2 n-1)!} \int_{0}^{t} d s f_{0}(s) \int_{\partial \Omega} k^{2 n} \frac{e^{i k L}}{e^{2 i k L}-1} e^{-k^{2}(t-s)} d k \tag{70}
\end{equation*}
$$

which provides an analytic continuation of $I_{f_{0}}(x, t)$ for all $x \in \mathbb{C}$, as shown below.
Theorem 4. If $f_{0}(t)$ is analytic in a strip in the complex t-plane, containing part of the positive real axis, $\mathcal{D}=\{t \in \mathbb{C}: \operatorname{dist}(t,[0, T]) \leq r\}$, for some $r>0$ and $T>0$, then the series representation (69) is entire in $x$ for $0 \leq t \leq T$.

Proof. The proof is similar to the proof of Theorem 2.

We proceed for $I_{g_{0}}(x, t)$ in exactly the same way as for $I_{f_{0}}(x, t)$. First, note that $I_{g_{0}}(x, t)$ is analytic for $x \in(-L, L)$. We write

$$
\begin{equation*}
I_{g_{0}}(x, t)=\frac{1}{i \pi} \int_{0}^{t} d s g_{0}(s) \int_{\partial \Omega} k \frac{\sin (k x)}{\sin (k L)} e^{-k^{2}(t-s)} d k \tag{71}
\end{equation*}
$$

Taking $2 n x$-derivatives,

$$
\begin{equation*}
\left.\frac{\partial^{2 n} I_{g_{0}}}{\partial x^{2 n}}\right|_{x=L}=\frac{(-1)^{n}}{i \pi} \int_{0}^{t} d s g_{0}(s) \int_{\partial \Omega} k^{2 n+1} e^{-k^{2}(t-s)} d k=g_{0}^{(n)}(t) \tag{72}
\end{equation*}
$$

Since

$$
\begin{equation*}
I_{g_{0}}(x, t)=2 \sum_{n} B_{n}(t)(x-L)^{2 n}-I_{g_{0}}(2 L-x, t)=\tilde{g}_{0}(x, t)-I_{g_{0}}(2 L-x, t) \tag{73}
\end{equation*}
$$

which defines $\tilde{g}_{0}(x, t)$. This implies that for $x \in(L, 3 L)$,

$$
\begin{equation*}
I_{g_{0}}(x, t)=2 \sum_{n=0}^{\infty} \frac{(x-L)^{2 n}}{(2 n)!} g_{0}^{(n)}(t)-I_{g_{0}}(2 L-x, t)=\tilde{g}_{0}(x, t)-I_{g_{0}}(2 L-x, t) \tag{74}
\end{equation*}
$$

This process can be continued as before to get

$$
I_{g_{0}}^{\mathrm{ext}}(x, t)= \begin{cases}\vdots & -5 L<x \leq-3 L  \tag{75}\\ I_{g_{0}}(x+4 L, t)-\tilde{g}_{0}(x+2 L, t)-\tilde{g}_{0}(x+4 L, t), & -3 L<x \leq-L \\ I_{g_{0}}(x+2 L, t)-\tilde{g}_{0}(x+2 L, t), & -L<x \leq L \\ I_{g_{0}}(x, t), & L<x \leq 3 L \\ I_{g_{0}}(x-2 L, t)+\tilde{g}_{0}(x, t), & 3 L<x \leq 5 L \\ I_{g_{0}}(x-4 L, t)+\tilde{g}_{0}(x, t)+\tilde{g}_{0}(x-2 L, t), & \\ \vdots & \end{cases}
$$

As for $I_{f_{0}}(x, t)$, we can also find the odd terms in the Taylor series or write the Taylor series about $x=0$ instead.

### 2.3.1 Boundary-to-Initial Map

Using the definitions for $\tilde{f}_{0}(x, t)$ and $\tilde{g}_{0}(x, t)$ above,

$$
\begin{equation*}
w(x, t)=u_{\mathrm{ac}}(x, t)=I_{0}(x, t)+I_{f_{0}}^{\mathrm{ext}}(x, t)+I_{g_{0}}^{\mathrm{ext}}(x, t) \tag{76}
\end{equation*}
$$

is a solution to the heat equation on the full-line with initial condition

$$
\begin{equation*}
w_{0}(x)=I_{0}(x, 0)+I_{f_{0}}^{\mathrm{ext}}(x, 0)+I_{g_{0}}^{\mathrm{ext}}(x, 0), \tag{77}
\end{equation*}
$$



Figure 7: (a) The initial condition $w_{0}(x)$ given by (77), leading to the (analytic) solution on the right, shown with the Fourier Series solution at $t=0 u_{\mathrm{FS}}(x, 0)=u_{\mathrm{UTM}}(x, 0)$ and the whole-line initial condition $u_{0}(x)=u_{\mathbb{R}}(x, 0)$. (b) The solution $u_{\text {ac }}(x, t)$ (76) at $t=1$ obtained through analytic continuation, overlaid with the solution $u_{\mathbb{R}}(x, t)(34)$ and the periodic odd Fourier series solution, $u_{\mathrm{FS}}(x, t)=u_{\mathrm{UTM}}(x, t)$.
where, for integer $n$,

$$
\begin{align*}
I_{0}(x, 0) & = \begin{cases}u_{0}(x-2 n L), & 2 n L \leq x<(2 n+1) L, \\
-u_{0}(2(n+1) L-x), & (2 n+1) L \leq x<2(n+1) L,\end{cases}  \tag{78}\\
I_{f_{0}}^{\text {ext }}(x, 0) & =\left\{\begin{array}{lll}
\sum_{j=0}^{|n|-1} \tilde{f}_{0}(x+2 j L, 0), & 2 n L \leq x<2(n+1) L, & n<0, \\
-\sum_{j=1}^{n} \tilde{f}_{0}(x-2 j L, 0), & 2 n L \leq x<2(n+1) L, & n \geq 0,
\end{array}\right.  \tag{79}\\
I_{g_{0}}^{\mathrm{ext}}(x, 0) & =\left\{\begin{array}{lll}
-\sum_{j=1}^{|n|} \tilde{g}_{0}(x+2 j L, t), & (2 n-1) L<x \leq(2 n+1) L, & n<0, \\
\sum_{j=0}^{n-1} \tilde{g}_{0}(x-2 j L, t), & (2 n-1) L<x \leq(2 n+1) L, & n \geq 0
\end{array}\right. \tag{80}
\end{align*}
$$

### 2.3.2 Examples

Using (34) again, we consider the IBVP with $u_{0}(x)=u_{\mathbb{R}}(x, 0)$, for $x \in(0,1)$, and boundary data $f_{0}(t)=u_{\mathbb{R}}(0, t), g_{0}(t)=u_{\mathbb{R}}(1, t)$ for $t>0$. We recover $u_{\mathbb{R}}(x, t)$, as shown in Figure 7 .

As our second example, we change the boundary functions to $f_{0}(t)=t e^{-t}$ and $g_{0}(t)=1 /(1+t)$. We show the analytic continuation of the solution on the full line in Figure 8. As in our previous examples, it should be noted that the solution is not bounded on $\mathbb{R}$, nor is the extended initial condition continuous.

## 3 The advected heat equation

The advected heat equation on the half-line,

$$
\begin{align*}
u_{t} & =u_{x x}+c u_{x}, & & x>0, t>0  \tag{81a}\\
u(x, 0) & =u_{0}(x), & & x>0  \tag{81b}\\
u(0, t) & =f_{0}(t), & & t>0, \tag{81c}
\end{align*}
$$



Figure 8: (a) The (discontinuous) initial condition $w_{0}(x)$ (77) leading to the (analytic) solution on the right, shown with the analytic continuation of $u_{0}(x)$. (b) The solution $u_{\text {ac }}(x, t)(76)$ at $t=1$ obtained through analytic continuation.
is an interesting next example since its underlying operator is not self adjoint. It is also a first example whose solution is more conveniently obtained using Fokas' UTM than using classical techniques [8]. The solution is represented as

$$
\begin{equation*}
u(x, t)=I_{0}(x, t)+I_{f_{0}}(x, t) \tag{82}
\end{equation*}
$$

with

$$
\begin{align*}
I_{0}(x, t) & =\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{i k x-W t} \hat{u}_{0}(k) d k-\frac{1}{2 \pi} \int_{\partial \Omega} e^{i k x-W t} \hat{u}_{0}(-k+i c) d k  \tag{83}\\
I_{f_{0}}(x, t) & =-\frac{1}{2 \pi} \int_{\partial \Omega}(2 i k+c) e^{i k x-W t} F_{0}(W, t) d k \tag{84}
\end{align*}
$$

where $W=k^{2}-i k c$, and

$$
\begin{equation*}
\hat{u}_{0}(k)=\int_{0}^{\infty} e^{-i k y} u_{0}(y) d y, \quad F_{0}(W, t)=\int_{0}^{t} e^{W s} f_{0}(s) d s \tag{85}
\end{equation*}
$$

Here $\Omega=\{k \in \mathbb{C}: \pi / 4<\operatorname{Arg}(k)<3 \pi / 4$ and $|k|>r\}$ for some $r>|c|$ (due to $W(i c)=0$ ), shown in Figure 2.

The initial condition contribution $I_{0}(x, t)$ is entire, as in Theorem 1. The boundary contribution can be written as

$$
\begin{equation*}
I_{f_{0}}(x, t)=-\frac{1}{2 \pi} \int_{\partial \Omega} d k(2 i k+c) e^{i k x-W t} \int_{0}^{t} e^{W s} f_{0}(s) d s=\frac{x}{2 \sqrt{\pi}} \int_{0}^{t} \frac{f_{0}(s)}{(t-s)^{\frac{3}{2}}} e^{-\frac{(x+c(t-s))^{2}}{4(t-s)}} d s \tag{86}
\end{equation*}
$$

which is analytic for $x>0$ (again, as in Theorem 1), but discontinuous at $x=0$ since $I_{f_{0}}(0, t)=0$ but $\lim _{x \rightarrow 0^{+}} I_{f_{0}}(x, t)=f_{0}(t)$. Instead, we deform $\partial \Omega$ to $\gamma$ (see Figure 2), passing above $k=i|c|$
and integrate by parts $n+1$ times so that

$$
\begin{align*}
I_{f_{0}}(x, t) & =-\frac{1}{2 \pi} \int_{\gamma} d k(2 i k+c) e^{i k x-W t}\left[\sum_{m=1}^{n+1}(-1)^{m} \frac{f_{0}^{(m-1)}(0)}{W^{m}}-\frac{(-1)^{n}}{W^{n+1}} \int_{0}^{t} e^{W s} f_{0}^{(n+1)}(s) d s\right] \\
& =\sum_{m=0}^{n+1} f_{0}^{(m-1)}(0) \phi_{m}(x, t)+\int_{0}^{t} f_{0}^{(n+1)}(s) \phi_{n+1}(x, t-s) d s \tag{87}
\end{align*}
$$

with

$$
\begin{equation*}
\phi_{m}(x, t)=-\frac{(-1)^{m}}{2 \pi} \int_{\gamma} \frac{(2 i k+c) e^{i k x-W t}}{W^{m}} d k \tag{88}
\end{equation*}
$$

Switching the order of integration is allowed, due to absolute integrability. Taking $2 n+q$ derivatives ( $q=0$ or 1 ),

$$
\begin{equation*}
\frac{\partial^{2 n+q} I_{f_{0}}}{\partial x^{2 n+q}}=\sum_{m=1}^{n+1} f_{0}^{(m-1)}(0) \phi_{m}^{(2 n+q)}(x, t)+\int_{0}^{t} f_{0}^{(n+1)}(s) \phi_{n+1}^{(2 n+q)}(x, t-s) d s \tag{89}
\end{equation*}
$$

Using $k \mapsto k / \sqrt{t}$,

$$
\begin{align*}
\phi_{n+1}^{(2 n+q)}(0, t) & =\frac{(-1)^{n}}{2 \pi} \int_{\gamma} \frac{(i k)^{2 n+q}(2 i k+c) e^{-W t}}{W^{n+1}} d k=\frac{i^{q}}{2 \pi} \int_{\gamma} \frac{k^{2 n+q}(2 i k+c) e^{-W t}}{W^{n+1}} d k \\
& =\frac{i^{q}}{2 \pi t^{\frac{q}{2}}} \int_{\gamma} \frac{k^{2 n+q}(2 i k+c \sqrt{t}) e^{-k^{2}+i k c \sqrt{t}}}{\left(k^{2}-i k c \sqrt{t}\right)^{n+1}} d k=\mathcal{O}\left(t^{-\frac{q}{2}}\right) \tag{90}
\end{align*}
$$

Using the dominated convergence theorem, we get the coefficients of the Taylor series for $I_{f_{0}}(x, t)$ as $(n \in \mathbb{N})$,

$$
\begin{equation*}
a_{2 n}(t)=\frac{1}{(2 n)!}\left[\sum_{m=1}^{n+1} f_{0}^{(m-1)}(0) \phi_{m}^{(2 n)}(0, t)+\int_{0}^{t} f_{0}^{(n+1)}(s) \phi_{n+1}^{(2 n)}(0, t-s) d s\right] \tag{91}
\end{equation*}
$$

and

$$
\begin{equation*}
a_{2 n+1}(t)=\frac{1}{(2 n+1)!}\left[\sum_{m=1}^{n+1} f_{0}^{(m-1)}(0) \phi_{m}^{(2 n+1)}(0, t)+\int_{0}^{t} f_{0}^{(n+1)}(s) \phi_{n+1}^{(2 n+1)}(0, t-s) d s\right] \tag{92}
\end{equation*}
$$

As before, we can use the UTM integral representation (86) to reduce the Taylor series to even or odd terms. With $a_{0}(t)=f_{0}(t)$, even is preferred, so that

$$
I_{f_{0}}^{\text {ext }}(x, t)=\left\{\begin{array}{ll}
I_{f_{0}}(x, t), & x \geq 0,  \tag{93}\\
\tilde{f}_{0}(x, t)-I_{f_{0}}(-x, t), & x<0,
\end{array} \quad \text { where } \quad \tilde{f}_{0}(x, t)=2 \sum_{n=0}^{\infty} a_{2 n}(t) x^{2 n}\right.
$$

### 3.1 Boundary-to-Initial Map

Define

$$
\begin{equation*}
I_{f_{0}}^{\mathrm{ext}}(x, t)=\sum_{n=0}^{\infty} a_{n}(t) x^{n} \tag{94}
\end{equation*}
$$

where $a_{n}(t)$ is defined above in (91) and (92). Then

$$
\begin{equation*}
w(x, t)=u_{\mathrm{ac}}(x, t)=I_{0}(x, t)+I_{f_{0}}^{\mathrm{ext}}(x, t) \tag{95}
\end{equation*}
$$

is a whole-line solution to the heat equation with advection with initial condition

$$
\begin{equation*}
w_{0}(x)=I_{0}(x, 0)+I_{f_{0}}^{\mathrm{ext}}(x, 0) \tag{96}
\end{equation*}
$$

Since

$$
\begin{equation*}
I_{0}(x, t)=\frac{1}{2 \pi} \int_{0}^{\infty} d y u_{0}(y) \int_{-\infty}^{\infty} e^{i k(x-y)-W t} d k-\frac{1}{2 \pi} \int_{0}^{\infty} d y u_{0}(y) e^{c y} \int_{\partial \Omega} e^{i k(x+y)-W t} d k \tag{97}
\end{equation*}
$$

we have

$$
I_{0}(x, 0)= \begin{cases}u_{0}(x), & x \geq 0  \tag{98}\\ -e^{-c x} u_{0}(-x), & x<0\end{cases}
$$

For small $t>0$,

$$
\begin{align*}
\tilde{f}_{0}(x, t) & =2 \sum_{n=0}^{\infty} \frac{x^{2 n}}{(2 n)!}\left[\sum_{m=1}^{n+1} f_{0}^{(m-1)}(0) \phi_{m}^{(2 n)}(0, t)+\int_{0}^{t} f_{0}^{(n+1)}(s) \phi_{n+1}^{(2 n)}(0, t-s) d s\right] \\
& \sim 2 \sum_{m=0}^{\infty} f_{0}^{(m)}(0) \sum_{n=m}^{\infty} \frac{x^{2 n}}{(2 n)!} \phi_{m+1}^{(2 n)}(0, t) \\
& =\frac{1}{\pi} \sum_{m=0}^{\infty} f_{0}^{(m)}(0) \sum_{n=m}^{\infty} \frac{(-1)^{m+n} x^{2 n}}{(2 n)!} \int_{\gamma} \frac{(2 i k+c) k^{2 n-m-1} e^{-\left(k^{2}-i k c\right) t}}{(k-i c)^{m+1}} d k \tag{99}
\end{align*}
$$

Using $k=\kappa+i c / 2$,

$$
\begin{equation*}
\tilde{f}_{0}(x, t) \sim \frac{1}{\pi} \sum_{m=0}^{\infty} f_{0}^{(m)}(0) \sum_{n=m}^{\infty} \frac{(-1)^{m+n} x^{2 n}}{(2 n)!} \int_{\gamma^{\prime}} \frac{2 i \kappa\left(\kappa+\frac{i c}{2}\right)^{2 n-m-1} e^{-\left(\kappa^{2}+\frac{c^{2}}{4}\right) t}}{\left(\kappa+\frac{i c}{2}\right)^{m+1}} d \kappa \tag{100}
\end{equation*}
$$

where $\gamma^{\prime}$ is $\gamma$ shifted by $i c / 2$. If $n=m=0$, the integrand has a simple pole in the UHP at $k=i|c| / 2$. Otherwise it has a pole in the upper half plane if $c>0$ at $k=i c / 2$ of order $m+1$. Deforming down to the real axis, around these poles, and using the residue theorem gives

$$
\begin{equation*}
\tilde{f}_{0}(x, t) \sim \tilde{f}_{0}^{(i)}(x, t)+\tilde{f}_{0}^{(r)}(x, t) \tag{101}
\end{equation*}
$$

with integral part

$$
\begin{align*}
\tilde{f}_{0}^{(i)}(x, t) & =\frac{1}{\pi} \sum_{m=0}^{\infty} f_{0}^{(m)}(0) \sum_{n=m}^{\infty} \frac{(-1)^{m+n} x^{2 n}}{(2 n)!} \int_{-\infty}^{\infty} \frac{2 i \kappa\left(\kappa+\frac{i c}{2}\right)^{2 n-m-1} e^{-\left(\kappa^{2}+\frac{c^{2}}{4}\right) t}}{\left(\kappa+\frac{i c}{2}\right)^{m+1}} d \kappa \\
& \rightarrow \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{x^{2 m}}{(2 m)!} f_{0}^{(m)}(0) \int_{-\infty}^{\infty} \frac{2 i \kappa\left(\kappa+\frac{i c}{2}\right)^{m-1}}{\left(\kappa+\frac{i c}{2}\right)^{m+1}}{ }_{1} F_{2}\left(\begin{array}{c}
1 \\
m+\frac{1}{2}, m+1
\end{array} ;-\frac{\left(\kappa+\frac{i c}{2}\right)^{2} x^{2}}{4}\right) d \kappa \tag{102}
\end{align*}
$$

after switching the sum and the integral. Here ${ }_{1} F_{2}$ denotes the generalized hypergeometric function [7]. The residue part equals

$$
\begin{equation*}
\tilde{f}_{0}^{(r)}(x, t)=2 \sum_{m=0}^{\infty} f_{0}^{(m)}(0) \sum_{n=m}^{\infty} \frac{(-1)^{m+n} x^{2 n}}{(2 n)!} \operatorname{Res}_{\kappa=i|c| / 2}\left(\frac{2 \kappa\left(\kappa+\frac{i c}{2}\right)^{2 n-m-1} e^{-\left(\kappa^{2}+\frac{c^{2}}{4}\right) t}}{\left(\kappa+\frac{i c}{2}\right)^{m+1}}\right) \tag{103}
\end{equation*}
$$

For $(n, m)=(0,0)$,

$$
\begin{equation*}
\operatorname{ReS}_{\kappa=i|c| / 2}\left(\frac{2 \kappa e^{-\left(\kappa^{2}+\frac{c^{2}}{4}\right) t}}{\kappa^{2}+\frac{c^{2}}{4}}\right)=1 \tag{104}
\end{equation*}
$$

while for $(n, m) \neq(0,0)$,

$$
\begin{align*}
\frac{2 \kappa\left(\kappa+\frac{i c}{2}\right)^{2 n-m-1}}{\left(\kappa-\frac{i c}{2}\right)^{m+1}} & =\frac{2 \kappa\left(\kappa-\frac{i c}{2}+i c\right)^{2 n-m-1}}{\left(\kappa-\frac{i c}{2}\right)^{m+1}}=\sum_{j=0}^{2 n-m-1}\binom{2 n-m-1}{j} \frac{2 \kappa(i c)^{2 n-m-j-1}}{\left(\kappa-\frac{i c}{2}\right)^{m-j+1}} \\
& =\sum_{j=0}^{2 n-m-1}\binom{2 n-m-1}{j}\left[\frac{2(i c)^{2 n-m-j-1}}{\left(\kappa-\frac{i c}{2}\right)^{m-j}}+\frac{(i c)^{2 n-m-j}}{\left(\kappa-\frac{i c}{2}\right)^{m-j+1}}\right] \tag{105}
\end{align*}
$$

so that

$$
\begin{align*}
\operatorname{Res}_{\kappa=i|c| / 2}\left(\frac{2 \kappa\left(\kappa+\frac{i c}{2}\right)^{2 n-m-1} e^{-\left(\kappa^{2}+\frac{c^{2}}{4}\right) t}}{\left(\kappa+\frac{i c}{2}\right)^{m+1}}\right) & \sim 2\binom{2 n-m-1}{m-1}(i c)^{2 n-2 m}+\binom{2 n-m-1}{m}(i c)^{2 n-2 m} \\
& =\frac{(-1)^{n-m}(2 n)(2 n-m-1)!c^{2 n-2 m}}{m!(2 n-2 m)!} \tag{106}
\end{align*}
$$

Then,
$\tilde{f}_{0}^{(r)}(x, t) \sim \begin{cases}2 f_{0}(0), & c<0, \\ 2 f_{0}(0)+2 f_{0}(0) \sum_{n=1}^{\infty} \frac{c^{2 n} x^{2 n}}{(2 n)!}+2 \sum_{m=1}^{\infty} f_{0}^{(m)}(0) \sum_{n=m}^{\infty} \frac{(2 n-m-1)!c^{2 n-2 m} x^{2 n}}{(2 n-1)!m!(2 n-2 m)!}, & c>0,\end{cases}$
and

$$
\tilde{f}_{0}^{(r)}(x, 0)= \begin{cases}2 f_{0}(0), & c<0 \\ 2 f_{0}(0) \cosh (c x)+2 \sqrt{\pi} \cosh \left(\frac{c x}{2}\right) \sum_{m=1}^{\infty} \frac{|x|^{m+\frac{1}{2}}}{c^{m-\frac{1}{2}} m!} I_{m-\frac{1}{2}}\left(\frac{c x}{2}\right) f_{0}^{(m)}(0), & c>0\end{cases}
$$

which gives the boundary-to-initial map,

$$
w_{0}(x)= \begin{cases}u_{0}(x), & x \geq 0  \tag{109}\\ -e^{-c x} u_{0}(-x)+\tilde{f}_{0}(x, 0), & x<0\end{cases}
$$

It can be shown that these integrals and sums are convergent and that in the limit as $c \rightarrow 0, \tilde{f}_{0}(x, t)$ limits to (20).


Figure 9: With $c=1$ : (a) The initial condition $w_{0}(x)$ (109), leading to the (analytic) solution on the right, shown together with the UTM solution at $t=0$ and the whole-line initial condition $u_{0}(x)=u_{\mathbb{R}}(x+1,0)$. (b) The solution $u_{\text {ac }}(x, t)(95)$ at $t=1$ obtained through analytic continuation.

### 3.2 Examples

We start with the whole line solution $u_{\mathbb{R}}(x+c t+1, t)(34)$ with the corresponding boundary and initial conditions, equating $c= \pm 1$. The boundary integral is discontinuous, but the analytic continuation recovers the exact whole-line solution as shown in Figures 9 and 10. Considering the boundary condition $f_{0}(t)=t e^{-t}$, we find the analytic continuation of the solution shown in Figure 11.

## 4 The linear KdV equation, 1 boundary condition

Consider the linear Korteweg-de Vries (KdV) equation on the half-line with Dirichlet boundary conditions,

$$
\begin{align*}
u_{t}+u_{x x x} & =0, & & x>0, t>0,  \tag{110a}\\
u(x, 0) & =u_{0}(x), & & x>0,  \tag{110b}\\
u(0, t) & =f_{0}(t), & & t>0, \tag{110c}
\end{align*}
$$

which requires only one boundary condition [8], unlike the similar equation in the next section. Its solution is written as

$$
\begin{equation*}
u(x, t)=I_{0}(x, t)+I_{f_{0}}(x, t) \tag{111}
\end{equation*}
$$

with

$$
\begin{align*}
I_{0}(x, t) & =\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{i k x+i k^{3} t} \hat{u}_{0}(k) d k+\frac{1}{2 \pi} \int_{\partial \Omega} e^{i k x+i k^{3} t}\left(\alpha \hat{u}_{0}(\alpha k)+\alpha^{2} \hat{u}_{0}\left(\alpha^{2} k\right)\right) d k  \tag{112}\\
I_{f_{0}}(x, t) & =-\frac{1}{2 \pi} \int_{\partial \Omega} 3 k^{2} e^{i k x+i k^{3} t} F_{0}\left(-i k^{3}, t\right) d k \tag{113}
\end{align*}
$$

and

$$
\begin{equation*}
\hat{u}_{0}(k)=\int_{0}^{\infty} e^{-i k y} u_{0}(y) d y, \quad F_{0}\left(-i k^{3}, t\right)=\int_{0}^{t} e^{-i k^{3} s} f_{0}(s) d s \tag{114}
\end{equation*}
$$



Figure 10: with $c=-1$ : (a) The initial condition, $w_{0}(x)$, (109), leading to the (analytic) solution on the left, shown with the the UTM solution at $t=0$ and the whole-line initial condition $u_{0}(x)=$ $u_{\mathbb{R}}(x+1,0)$. (b) The solution $u_{\mathrm{ac}}(x, t)$ (95) at $t=1$ obtained using analytic continuation.
and $\alpha=\exp (2 i \pi / 3)$. The region $\Omega=\{k \in \mathbb{C}: \pi / 3<\operatorname{Arg}(k)<2 \pi / 3\}$ is shown in Figure 12 . Theorem 5 shows that after some contour deformations, $I_{0}(x, t)$ defines an entire function of $x$.

Theorem 5. If there exists an $\epsilon>0$ such that

$$
\left\|e^{\epsilon y} u_{0}(y)\right\|_{1}=\int_{0}^{\infty}\left|u_{0}(y)\right| e^{\epsilon y} d y<\infty
$$

then $I_{0}(x, t)$, is entire in $x$ for each $t>0 .{ }^{1}$ If $f_{0} \in L^{1}(0, T)$ for some $T>0$, then $I_{f_{0}}(x, t)$ is analytic for $|\operatorname{Im}(x)|<\sqrt{3} \operatorname{Re}(x)$ for $0<t<T$.
Proof. We deform the path for the first integral in $I_{0}(x, t)$ up to $\gamma=\gamma^{-} \cup \gamma^{+}$, see Figure 12, using the assumption on $u_{0}(y)$, still assuming $x>0$. Next, an integral over any closed contour $\Gamma$ in the complex $x$-plane is zero by Cauchy's theorem, where we can use Fubini's theorem to swap the order of integration since

$$
\begin{aligned}
\oint_{\Gamma}|d x| \int_{\gamma}\left|e^{i k x+i k^{3} t} \hat{u}_{0}(k)\right||d k| & \leq \ell(\Gamma)\left\|e^{\epsilon y} u_{0}(y)\right\|_{1} \max _{x \in \Gamma} \int_{-\infty}^{\infty} e^{\epsilon|x|+\kappa|x|+\left(\epsilon^{3}-3 \epsilon \kappa^{2}\right) t} d \kappa \\
& =\ell(\Gamma)\left\|e^{\epsilon y} u_{0}(y)\right\|_{1} \sqrt{\frac{\pi}{3 \epsilon t}} \max _{x \in \Gamma} e^{\epsilon|x|+\epsilon^{3} t+\frac{|x|^{2}}{12 \epsilon t}}<\infty
\end{aligned}
$$

where we parametrized the contour $\gamma$ with $k=\kappa+i \epsilon$. It follows that the first integral in $I_{0}(x, t)$, after the contour deformation, is entire by Morera's theorem.

For the second integral, we break the contour $\partial \Omega$ into two parts: the left part $\partial \Omega^{-}$and the right part $\partial \Omega^{+}$. We deform $\partial \Omega^{-}$down to $\gamma^{-}$, so that the corresponding integral over $\Gamma$ is zero by Cauchy's theorem, since

$$
\begin{aligned}
\oint_{\Gamma}|d x| \int_{\gamma^{-}}\left|e^{i k x+i k^{3} t} \hat{u}_{0}(\alpha k)\right||d k| & \leq \ell(\Gamma)\left\|u_{0}\right\|_{1} \max _{x \in \Gamma} \int_{0}^{\infty} e^{\epsilon|x|+\kappa|x|+\left(\epsilon^{3}-3 \epsilon \kappa^{2}\right) t} d \kappa \\
& \leq \ell(\Gamma)\left\|u_{0}\right\|_{1} \sqrt{\frac{\pi}{3 \epsilon t}} \max _{x \in \Gamma} e^{\epsilon|x|+\epsilon^{3} t+\frac{|x|^{2}}{3 \epsilon t}}<\infty,
\end{aligned}
$$

${ }^{1}$ In fact, less decay is needed. If $u_{0}(y)=\mathcal{O}\left(e^{-z^{p}}\right)$ for $p>1 / 2$, then $I_{0}(x, t)$ is analytic for $x \in \mathbb{R}$.


Figure 11: (a) The (discontinuous) initial condition $w_{0}(x)$ (109) leading to the (analytic) solution on the right, shown with the analytic continuation of $u_{0}(x)=u_{\mathbb{R}}(x+1,0)$. (b) The solution $u_{\text {ac }}(x, t)$ (95) at $t=1$ obtained through analytic continuation.


Figure 12: The region $\Omega$ for KdV and the contour $\gamma=\gamma^{-} \cup \gamma^{+}$.
where we used $1+\operatorname{erf}(y) \leq 2$ for $y \in \mathbb{R}$.
For the $\partial \Omega^{+}$part, we use the transformation $\kappa=\alpha k$ and deform up to $\gamma^{-}$, so that

$$
\int_{\partial \Omega^{+}} e^{i k x+i k^{3} t} \alpha \hat{u}_{0}(\alpha k) d k=\int_{\mathbb{R}^{-}} e^{i \alpha^{2} \kappa x+i \kappa^{3} t} \hat{u}_{0}(\kappa) d k=\int_{\gamma^{-}} e^{i \alpha^{2} \kappa x+i \kappa^{3} t} \hat{u}_{0}(\kappa) d k
$$

If we integrate this over $\Gamma$, we get zero by Cauchy's theorem, where we can use Fubini's theorem since

$$
\begin{aligned}
\oint_{\Gamma}|d x| \int_{\gamma^{-}}\left|e^{i \alpha^{2} \kappa x+i \kappa^{3} t} \hat{u}_{0}(\kappa)\right||d k| & \leq \ell(\Gamma)\left\|e^{\epsilon y} u_{0}(y)\right\|_{1} \max _{x \in \Gamma} \int_{0}^{\infty} e^{\sqrt{3} \epsilon|x|+\sqrt{3} \kappa|x|+\left(\epsilon^{3}-3 \epsilon \kappa^{2}\right) t} d \kappa \\
& \leq \ell(\Gamma)\left\|e^{\epsilon y} u_{0}(y)\right\|_{1} \sqrt{\frac{\pi}{3 \epsilon t}} \max _{x \in \Gamma} e^{\sqrt{3} \epsilon|x|+\epsilon^{2} t+\frac{|x|^{2}}{4 \epsilon t}}<\infty .
\end{aligned}
$$

The same holds for the third term in $I_{0}(x, t)$, thus $I_{0}(x, t)$ is entire. For $I_{f_{0}}(x, t)$, an integral over
a closed contour $\Gamma$ in the region $|\operatorname{Im}(x)|<\sqrt{3} \operatorname{Re}(x)$ is zero if

$$
\begin{aligned}
\oint|d x| \int_{\partial \Omega}\left|3 k^{2} e^{i k x+i k^{3} t} F_{0}\left(-i k^{3}, t\right)\right||d k| & \leq \ell(\Gamma)\left\|f_{0}\right\|_{1} \max _{x \in \Gamma} \int_{\partial \Omega} 3|k|^{2}\left|e^{i k x}\right||d k| \\
& \leq \ell(\Gamma)\left\|f_{0}\right\|_{1} \max _{x \in \Gamma} \frac{288 \sqrt{3}|x|^{3}}{\left(3 \operatorname{Re}(x)^{2}-\operatorname{Im}(x)^{2}\right)^{3}}<\infty
\end{aligned}
$$

so that $I_{f_{0}}(x, t)$ is an analytic function of $x$ for $|\operatorname{Im}(x)|<\sqrt{3} \operatorname{Re}(x)$.

If we swap the order of integration for $I_{f_{0}}(x, t)$ and integrate the $k$-integral, we find

$$
\begin{equation*}
I_{f_{0}}(x, t)=\frac{x}{\sqrt[3]{3}} \int_{0}^{t} \frac{f_{0}(s)}{(t-s)^{\frac{4}{3}}} \operatorname{Ai}\left(\frac{x}{\sqrt[3]{3(t-s)}}\right) d s \tag{115}
\end{equation*}
$$

where $\operatorname{Ai}(z)$ denotes the Airy function, see [7]. From this, it is clear this is not defined for $x<0$. Instead, we deform to a contour $\gamma$ lying under $\Omega$ (ensuring $\gamma$ passes above the origin) and above $\mathbb{R}$, see Figure 12:

$$
\begin{equation*}
I_{f_{0}}(x, t)=-\frac{1}{2 \pi} \int_{\gamma} d k 3 k^{2} e^{i k x+i k^{3} t} \int_{0}^{t} e^{-i k^{3} s} f_{0}(s) d s \tag{116}
\end{equation*}
$$

For the $x$-derivatives of order $3 n$, we use

$$
\begin{equation*}
\frac{\partial^{n}}{\partial s^{n}} \delta(s-t)=-\frac{(-i)^{n}}{2 \pi} \int_{\partial \Omega} 3 k^{3 n+2} e^{i k^{3}(t-s)} d k \tag{117}
\end{equation*}
$$

leading to

$$
\begin{equation*}
a_{3 n}(t)=\frac{1}{(3 n)!} \int_{0}^{t} f_{0}(s) \delta^{(n)}(s-t) d s=\frac{(-1)^{n} f_{0}^{(n)}(t)}{(3 n)!} \tag{118}
\end{equation*}
$$

For the other derivatives, we deform to $\gamma$ and integrate by parts $n$ times to find

$$
\begin{align*}
I_{f_{0}}(x, t) & =-\frac{1}{2 \pi} \int_{\gamma} d k 3 k^{2} e^{i k x+i k^{3} t}\left[\sum_{m=1}^{n} \frac{f_{0}^{(m-1)}(0)}{\left(i k^{3}\right)^{m}}+\frac{1}{\left(i k^{3}\right)^{n}} \int_{0}^{t} e^{-i k^{3} s} f_{0}^{(n)}(s) d s\right] \\
& =\sum_{m=1}^{n} f_{0}^{(m-1)}(0) \phi_{m}(x, t)+\int_{0}^{t} f_{0}^{(n)}(s) \phi_{n}(x, t-s) d s \tag{119}
\end{align*}
$$

with

$$
\begin{equation*}
\phi_{m}(x, t)=-\frac{1}{2 \pi} \int_{\gamma} \frac{3 k^{2} e^{i k x+i k^{3} t}}{\left(i k^{3}\right)^{m}} d k \tag{120}
\end{equation*}
$$

Taking $3 n-q$ derivatives $(q=1,2)$,

$$
\begin{equation*}
\frac{\partial^{3 n-q} I_{f_{0}}}{\partial x^{3 n-q}}=\sum_{m=1}^{n} f_{0}^{(m-1)}(0) \phi_{m}^{(3 n-q)}(x, t)+\int_{0}^{t} f_{0}^{(n)}(s) \phi_{n}^{(3 n-q)}(x, t-s) d s \tag{121}
\end{equation*}
$$

The contour deformation to $\gamma$ enables the differentiating under the integral sign and the use of the dominated convergence theorem. With

$$
\begin{align*}
\phi_{m}^{(3 n-q)}(0, t) & =-\frac{1}{2 \pi} \int_{\gamma} d k \frac{3(i k)^{3 n-q} k^{2} e^{i k^{3} t}}{\left(i k^{3}\right)^{m}}=\frac{3(-1)^{m-q}}{\pi} \sin \left(\frac{q \pi}{3}\right) \int_{0}^{\infty} d \rho \rho^{3(n-m)-q+2} e^{-\rho^{3} t} \\
& =\frac{(-1)^{m-q}}{\pi t^{n-m+1-\frac{q}{3}}} \sin \left(\frac{q \pi}{3}\right) \Gamma\left(n-m+1-\frac{q}{3}\right) \tag{122}
\end{align*}
$$

It follows that the coefficients of the Taylor series are

$$
\begin{equation*}
a_{3 n-2}(t)=\frac{\sqrt{3} \Gamma\left(\frac{1}{3}\right)}{2 \pi(3 n-2)!}\left[\sum_{m=1}^{n} \frac{(-1)^{m} \Gamma\left(n-m+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right) t^{n-m+\frac{1}{3}}} f_{0}^{(m-1)}(0)+(-1)^{n} \int_{0}^{t} d s \frac{f_{0}^{(n)}(s)}{(t-s)^{\frac{1}{3}}}\right] \tag{123}
\end{equation*}
$$

and

$$
\begin{equation*}
a_{3 n-1}(t)=-\frac{\sqrt{3} \Gamma\left(\frac{2}{3}\right)}{2 \pi(3 n-1)!}\left[\sum_{m=1}^{n} \frac{(-1)^{m} \Gamma\left(n-m+\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right) t^{n-m+\frac{2}{3}}} f_{0}^{(m-1)}(0)+(-1)^{n} \int_{0}^{t} d s \frac{f_{0}^{(n)}(s)}{(t-s)^{\frac{2}{3}}}\right] \tag{124}
\end{equation*}
$$

For $x<0$, we may write

$$
\begin{equation*}
I_{f_{0}}(x, t)=2 \sum_{n=0}^{\infty} a_{2 n}(t) x^{2 n}-I_{f_{0}}(-x, t)=\tilde{f}_{0}(x, t)-I_{f_{0}}(-x, t) \tag{125}
\end{equation*}
$$

so that

$$
I_{f_{0}}^{\mathrm{ext}}(x, t)= \begin{cases}I_{f_{0}}(x, t), & x>0  \tag{126}\\ \tilde{f}_{0}(x, t)-I_{f_{0}}(-x, t), & x<0\end{cases}
$$

and we may reduce the number of coefficients to be computed. As for the other monomial dispersion relations, the Taylor series may be written more compactly as

$$
\begin{equation*}
I_{f_{0}}^{\mathrm{ext}}(x, t)=\sum_{n=0}^{\infty} a_{n}(t) x^{n}=\sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!} f_{0}^{\left(\frac{n}{3}\right)}(t) x^{n} \tag{127}
\end{equation*}
$$

using Riemann-Liouville fractional derivatives.

### 4.1 Boundary-to-Initial Map

The function $I_{f_{0}}^{\text {ext }}(x, t)$, defined by (127), is easily seen to be a solution to the linear KdV equation, (110a). We have

$$
\begin{equation*}
I_{f_{0}}^{\mathrm{ext}}(x, t)=\sum_{n=0}^{\infty} a_{3 n}(t) x^{3 n}+\sum_{n=1}^{\infty} a_{3 n-1}(t) x^{3 n-1}+\sum_{n=1}^{\infty} a_{3 n-2}(t) x^{3 n-2} \tag{128}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{n=0}^{\infty} a_{3 n}(t) x^{3 n}=\sum_{n=0}^{\infty} \frac{(-1)^{n} x^{3 n}}{(3 n)!} f_{0}^{(n)}(t) \rightarrow \sum_{n=0}^{\infty} \frac{(-1)^{n} x^{3 n}}{(3 n)!} f_{0}^{(n)}(0) \tag{129}
\end{equation*}
$$

as $t \rightarrow 0^{+}$. For the other terms,

$$
\begin{equation*}
\sum_{n=1}^{\infty} a_{3 n-2}(t) x^{3 n-2} \sim \frac{\sqrt{3}}{2 \pi} \sum_{n=1}^{\infty} \frac{x^{3 n-2}}{(3 n-2)!} \sum_{m=1}^{n} \frac{(-1)^{m} \Gamma\left(n-m+\frac{1}{3}\right)}{t^{n-m+\frac{1}{3}}} f_{0}^{(m-1)}(0) \tag{130}
\end{equation*}
$$

since the integral terms approach zero as $t \rightarrow 0^{+}$. We switch sums, so that

$$
\begin{align*}
\sum_{n=1}^{\infty} a_{3 n-2}(t) x^{3 n-2} & \sim \frac{\sqrt{3}}{2 \pi} \sum_{m=1}^{\infty}(-1)^{m} f_{0}^{(m-1)}(0) \sum_{n=m}^{\infty} \frac{x^{3 n-2}}{(3 n-2)!} \frac{\Gamma\left(n-m+\frac{1}{3}\right)}{t^{n-m+\frac{1}{3}}} \\
& =\frac{\sqrt{3} \Gamma\left(\frac{1}{3}\right)}{2 \pi t^{\frac{1}{3}}} \sum_{m=1}^{\infty} \frac{(-1)^{m} x^{3 m-2}}{(3 m-2)!} f_{0}^{(m-1)}(0)_{2} F_{3}\left(\begin{array}{c}
\frac{1}{3}, 1 \\
m-\frac{1}{3}, m, m+\frac{1}{3}
\end{array} ; \frac{x^{3}}{27 t}\right) \tag{131}
\end{align*}
$$

Since [7]

$$
\frac{\sqrt{3} \Gamma\left(\frac{1}{3}\right)}{2 \pi t^{\frac{1}{3}}}{ }_{2} F_{3}\left(\begin{array}{c}
\frac{1}{3}, 1  \tag{132}\\
m-\frac{1}{3}, m, m+\frac{1}{3}
\end{array} ; \frac{x^{3}}{27 t}\right) \sim-\frac{(3 m-2)}{x}, \quad \text { as } t \rightarrow 0^{+}, \quad \text { for } x<0
$$

we obtain

$$
\begin{equation*}
\sum_{n=1}^{\infty} a_{3 n-2}(t) x^{3 n-2} \rightarrow \sum_{m=0}^{\infty} \frac{(-1)^{m} x^{3 m}}{(3 m)!} f_{0}^{(m)}(0), \quad \text { as } t \rightarrow 0^{+}, \quad \text { for } x<0 \tag{133}
\end{equation*}
$$

Similarly,

$$
\begin{equation*}
\sum_{n=1}^{\infty} a_{3 n-1}(t) x^{3 n-1} \rightarrow \sum_{m=0}^{\infty} \frac{(-1)^{m} x^{3 m}}{(3 m)!} f_{0}^{(m)}(0), \quad \text { as } t \rightarrow 0^{+}, \quad \text { for } x<0 \tag{134}
\end{equation*}
$$

Combining (125) and (128),

$$
\begin{equation*}
I_{f_{0}}(x, t) \rightarrow \tilde{f}(x, 0)=3 \sum_{n=0}^{\infty} \frac{(-1)^{n} x^{3 n}}{(3 n)!} f_{0}^{(n)}(0), \quad \text { as } t \rightarrow 0^{+}, \quad \text { for } x<0 \tag{135}
\end{equation*}
$$

For $I_{0}(x, t)$, we have

$$
\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{i k x+i k^{3} t} \hat{u}_{0}(k) d k \rightarrow \begin{cases}u_{0}(x), & x>0  \tag{136}\\ 0, & x<0\end{cases}
$$

and for $m=1,2$,

$$
\begin{align*}
\frac{\alpha^{m}}{2 \pi} \int_{0}^{\infty} d y u_{0}(y) \int_{\gamma} d k e^{i k\left(x-\alpha^{m} y\right)+i k^{3} t} & =\frac{1}{2 \pi} \int_{\alpha^{m} \mathbb{R}^{+}} d y u_{0}\left(\alpha^{-m} y\right) \int_{\gamma} d k e^{i k(x-y)+i k^{3} t} \\
& \rightarrow \begin{cases}0, & x>0 \\
-u_{0}\left(\alpha^{-m} x\right), & x<0\end{cases} \tag{137}
\end{align*}
$$

as $t \rightarrow 0^{+}$, so that

$$
w_{0}(x)= \begin{cases}u_{0}(x), & x>0  \tag{138}\\ \tilde{f}_{0}(x, 0)-u_{0}(\alpha x)-u_{0}\left(\alpha^{2} x\right), & x<0\end{cases}
$$

provides the boundary-to-initial map.


Figure 13: (a) The initial condition $w_{0}(x)$ (138), leading to the (analytic) solution on the right, shown with the UTM solution at $t=0$ for $x>0$ (it is not defined for $x<0$ ) and the whole-line initial condition $u_{0}(x)=u_{\mathbb{R}}(x, 0)$. (b) The solution $u_{\mathrm{ac}}(x, t)$ at $t=1$ obtained through analytic continuation, shown with the whole line solution, $u_{\mathbb{R}}(x, t)$, and the UTM solution for $x>0$ (it is not defined for $x<0$ ).

### 4.2 Examples

Our first example uses the whole-line solution,

$$
\begin{equation*}
u_{\mathbb{R}}(x, t)=2 e^{-(x+2 t)} \cos (x-2 t) \tag{139}
\end{equation*}
$$

restricted to $x>0$, with $u_{0}(x)=u_{\mathbb{R}}(x, 0)$ and $f_{0}(t)=u_{\mathbb{R}}(0, t)$. The solution obtained using UTM is no longer defined for $x<0$, but the analytic continuation recovers the exact solution on the whole line. The results are shown in Figure 13.

Next, we consider (110a) with $f_{0}(t)=t e^{-t}$ and $u_{0}(x)=u_{\mathbb{R}}(x, 0)$ for $x>0$. Since $f_{0}^{(n)}(0)=$ $-(-1)^{n} n$,

$$
\begin{equation*}
\tilde{f}_{0}(x, 0)=-\frac{1}{3} x e^{x}+\frac{2}{3} x e^{-\frac{x}{2}} \sin \left(\frac{\sqrt{3}}{2} x+\frac{\pi}{6}\right) \tag{140}
\end{equation*}
$$

and we find the analytic continuation of the solution and the corresponding initial condition shown in Figure 14.

## 5 The linear KdV equation, 2 boundary conditions

Consider the version of linear KdV with 2 boundary conditions

$$
\begin{align*}
u_{t}-u_{x x x} & =0, & & x>0, t>0,  \tag{141a}\\
u(x, 0) & =u_{0}(x), & & x>0,  \tag{141b}\\
u(0, t) & =f_{0}(t), & & t>0,  \tag{141c}\\
u_{x}(0, t) & =f_{1}(t), & & t>0 . \tag{141d}
\end{align*}
$$



Figure 14: (a) The (discontinuous) initial condition $w_{0}(x)$ (138) leading to the (analytic) solution on the right, shown with the analytic continuation of $u_{0}(x)=u_{\mathbb{R}}(x, 0)$. (b) The solution $u_{\mathrm{ac}}(x, t)$ at $t=1$ obtained through analytic continuation, and the UTM solution for $x>0$ (it is not defined for $x<0$ ).

The solution is given by [8]

$$
\begin{equation*}
u(x, t)=I_{0}(x, t)+I_{f_{0}}(x, t)+I_{f_{1}}(x, t) \tag{142}
\end{equation*}
$$

where

$$
\begin{align*}
I_{0}(x, t) & =\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{i k x-i k^{3} t} \hat{u}_{0}(k) d k-\frac{1}{2 \pi} \int_{\partial \Omega_{1}} e^{i k x-i k^{3} t} \hat{u}_{0}\left(\alpha^{2} k\right) d k-\frac{1}{2 \pi} \int_{\partial \Omega_{2}} e^{i k x-i k^{3} t} \hat{u}_{0}(\alpha k) d k  \tag{143}\\
I_{f_{0}}(x, t) & =\frac{1-\alpha}{2 \pi} \int_{\partial \Omega_{1}} e^{i k x-i k^{3} t} k^{2} F_{0}\left(i k^{3}, t\right) d k+\frac{1-\alpha^{2}}{2 \pi} \int_{\partial \Omega_{2}} e^{i k x-i k^{3} t} k^{2} F_{0}\left(i k^{3}, t\right) d k  \tag{144}\\
I_{f_{1}}(x, t) & =\frac{1-\alpha^{2}}{2 \pi i} \int_{\partial \Omega_{1}} e^{i k x-i k^{3} t} k F_{1}\left(i k^{3}, t\right) d k+\frac{1-\alpha}{2 \pi i} \int_{\partial \Omega_{2}} e^{i k x-i k^{3} t} k F_{1}\left(i k^{3}, t\right) d k \tag{145}
\end{align*}
$$

and

$$
\begin{equation*}
\hat{u}_{0}(k)=\int_{0}^{\infty} e^{-i k y} u_{0}(y) d y, \quad F_{m}\left(i k^{3}, t\right)=\int_{0}^{t} e^{i k^{3} s} f_{m}(s) d s, \quad m=0,1 \tag{146}
\end{equation*}
$$

and $\alpha=\exp (2 \pi i / 3)$.
For the initial condition parts, we deform into the green regions (see Figure 15) so that

$$
\begin{equation*}
I_{0}(x, t)=\frac{1}{2 \pi} \int_{\gamma_{0}} e^{i k x-i k^{3} t} \hat{u}_{0}(k) d k-\frac{1}{2 \pi} \int_{\gamma_{1}} e^{i k x-i k^{3} t} \hat{u}_{0}\left(\alpha^{2} k\right) d k-\frac{1}{2 \pi} \int_{\gamma_{2}} e^{i k x-i k^{3} t} \hat{u}_{0}(\alpha k) d k \tag{147}
\end{equation*}
$$

where $\gamma_{j}(j=1,2)$ is the deformation of $\partial \Omega_{j}$ into the green region (avoiding the origin) defined by $\operatorname{Re}\left(i k^{3}\right)>0$ and $\gamma_{0}$ is the similar deformation of the real axis into the lower half plane, as shown in Figure 15.

Theorem 6. If $u_{0} \in L^{1}\left(\mathbb{R}^{+}\right)$, then $I_{0}(x, t)$ is entire in $x$ for $t>0$.


Figure 15: The regions $\Omega_{1}$ and $\Omega_{2}$ for (141a) and the contours $\gamma_{0}, \gamma_{1}$, and $\gamma_{2}$.

Proof. If we integrate each of the integrals in $I_{0}(x, t)$ over a closed contour $\Gamma$ in the complex $x$ plane, we get zero by Cauchy's theorem after switching the order of integration. This is allowed by Fubini's theorem, since

$$
\oint_{\Gamma}|d x| \int_{\gamma_{0}}\left|e^{i k x-i k^{3} t} \hat{u}_{0}(k)\right||d k| \leq 2 \ell(\Gamma)\left\|u_{0}\right\|_{1} \max _{x \in \Gamma} \int_{0}^{\infty} e^{2|x| \rho-\rho^{3} t} d \rho<\infty,
$$

similarly for the other two terms. Thus, by Morera's theorem, $I_{0}(x, t)$ is entire in $x$.
It is interesting to compare this to Theorem 5 , where exponential decay is needed for analyticity. We defer the proof for $I_{f_{0}}(x, t)$ and $I_{f_{1}}(x, t)$ to Theorem 7 below.

Define

$$
\begin{equation*}
I_{f_{0}, j}(x, t)=\frac{1-\alpha^{j}}{2 \pi} \int_{\partial \Omega_{j}} k^{2} e^{i k x-i k^{3} t} F_{0}\left(i k^{3}, t\right) d k, \quad j=1,2, \tag{148}
\end{equation*}
$$

so that

$$
\begin{equation*}
I_{f_{0}}(x, t)=I_{f_{0}, 1}(x, t)+I_{f_{0}, 2}(x, t) . \tag{149}
\end{equation*}
$$

We integrate by parts $n$ times, resulting in

$$
\begin{equation*}
I_{f_{0}, j}(x, t)=\frac{1-\alpha^{j}}{2 \pi} \int_{\partial \Omega_{j}} d k k^{2} e^{i k x-i k^{3} t}\left[\sum_{m=1}^{n} \frac{f_{0}^{(m-1)}(0)}{\left(-i k^{3}\right)^{m}}+\frac{1}{\left(-i k^{3}\right)^{n}} \int_{0}^{t} e^{i k^{3} s} f_{0}^{(n)}(s) d s\right] . \tag{150}
\end{equation*}
$$

The terms involving $f_{0}^{(m)}(t) e^{i k^{3} t}$ integrate to zero around the contour $\partial \Omega_{j}$ by Jordan's lemma and Cauchy's theorem. We can swap the order of integration by Fubini's Theorem. Technically, we can only do this for $p \geq 2$, but we can also do it for $p=1$, if we do another integration by parts, switch the order, and undo the integration by parts. Then,

$$
\begin{equation*}
I_{f_{0}, j}(x, t)=\sum_{m=1}^{n} f_{0}^{(m-1)}(0) \phi_{m, j}(x, t)+\int_{0}^{t} f_{0}^{(n)}(s) \phi_{n, j}(x, t-s) d s \tag{151}
\end{equation*}
$$

with

$$
\begin{equation*}
\phi_{m, j}(x, t)=\frac{1-\alpha^{j}}{2 \pi} \int_{\partial \Omega_{j}} \frac{k^{2} e^{i k x-i k^{3} t}}{\left(-i k^{3}\right)^{m}} d k=\frac{1-\alpha^{j}}{2 \pi} \int_{\gamma_{j}} \frac{k^{2} e^{i k x-i k^{3} t}}{\left(-i k^{3}\right)^{m}} d k \tag{152}
\end{equation*}
$$

where we deform $\partial \Omega_{j}$ to $\gamma_{j}$ in the same way as before, which we can do since $m \geq 1$. It follows that $\phi_{m, j}(x, t)$ is entire in $x$. This expression for $I_{f_{0}, j}(x, t)$ is smooth up to the $3 n$-th $x$-derivative at $x=0$. Then,

$$
\begin{equation*}
\left.\frac{\partial^{3 n-q} I_{f_{0}, j}}{\partial x^{3 n-q}}\right|_{x=0}=\sum_{m=1}^{n} f_{0}^{(m-1)}(0) \phi_{m, j}^{(3 n-q)}(0, t)+\int_{0}^{t} f_{0}^{(n)}(s) \phi_{n, j}^{(3 n-q)}(0, t-s) d s \tag{153}
\end{equation*}
$$

for $q=1,2$. Note that we can swap the limit and integral since

$$
\begin{equation*}
\phi_{m, j}^{(3 n-q)}(0, t)=\frac{1-\alpha^{j}}{2 \pi} \int_{\gamma_{j}} \frac{(i k)^{3 n+2-q} e^{-i k^{3} t}}{\left(-i k^{3}\right)^{m}} d k=\frac{1-\alpha^{j}}{2 \pi t^{n-m+1-\frac{q}{3}}} \int_{\gamma_{j}} \frac{(i k)^{3 n+2-q} e^{-i k^{3}}}{\left(-i k^{3}\right)^{m}} d k \tag{154}
\end{equation*}
$$

so that $\phi_{n, j}^{(3 n-q)}(0, t)=\mathcal{O}\left(t^{-1+\frac{q}{3}}\right)$, and the integrand is absolutely integrable. For $j=2$, let $k \mapsto \alpha k$, so that $\gamma_{2} \mapsto \gamma_{1}$, and

$$
\begin{equation*}
\phi_{m, 2}^{(3 n-q)}(0, t)=\frac{\left(1-\alpha^{2}\right) \alpha^{-q}}{2 \pi} \int_{\gamma_{1}} \frac{(i k)^{3 n+2-q} e^{-i k^{3} t}}{\left(-i k^{3}\right)^{m}} d k \tag{155}
\end{equation*}
$$

Therefore, after parametrizing $\gamma_{1}$ and computing the integral,

$$
\begin{equation*}
\sum_{j=1}^{2} \phi_{m, j}^{(3 n-q)}(0, t)=\delta_{q, 1} \frac{-\sqrt{3}(-1)^{n-m} \Gamma\left(n-m+\frac{2}{3}\right)}{2 \pi t^{n-m+\frac{2}{3}}} \tag{156}
\end{equation*}
$$

where $\delta_{q, 1}=1$ if $q=1$ and 0 otherwise. With

$$
\begin{equation*}
I_{f_{0}}(x, t)=\sum_{n=0}^{\infty} a_{n}(t) x^{n}, \quad \text { and } \quad I_{f_{1}}(x, t)=\sum_{n=0}^{\infty} b_{n}(t) x^{n} \tag{157}
\end{equation*}
$$

the above leads to

$$
\begin{align*}
a_{3 n-1}(t) & =-\frac{\sqrt{3} \Gamma\left(\frac{2}{3}\right)}{2 \pi(3 n-1)!}\left[\sum_{m=1}^{n} \frac{(-1)^{n-m} \Gamma\left(n-m+\frac{2}{3}\right)}{t^{n-m+\frac{2}{3}} \Gamma\left(\frac{2}{3}\right)} f_{0}^{(m-1)}(0)+\int_{0}^{t} d s \frac{f_{0}^{(n)}(s)}{(t-s)^{\frac{2}{3}}}\right] \\
& =-\frac{f_{0}^{\left(n-\frac{1}{3}\right)}(t)}{(3 n-1)!}, \quad n=1,2, \ldots \tag{158}
\end{align*}
$$

and

$$
\begin{equation*}
a_{3 n-2}(t)=0, \quad n=1,2, \ldots \tag{159}
\end{equation*}
$$

If $q=0$, we cannot pass the limit under the integral sign to get (153). Using (156),

$$
\begin{align*}
\left.\frac{\partial^{3 n} I_{f_{0}}}{\partial x^{3 n}}\right|_{x=0} & =\int_{0}^{t} d s f_{0}^{(n)}(s) \sum_{j=1}^{2} \phi_{n, j}^{(3 n)}(x, t-s) \\
& =\frac{1}{2 \pi} \int_{0}^{t} d s f_{0}^{(n)}(s)\left[(1-\alpha) \int_{\gamma_{1}} k^{2} e^{i k x-i k^{3}(t-s)} d k+\left(1-\alpha^{2}\right) \int_{\gamma_{2}} k^{2} e^{i k x-i k^{3}(t-s)} d k\right] \tag{160}
\end{align*}
$$

Let $z=k^{3}$ so that $d z=3 k^{2} d k$, and

$$
\begin{align*}
a_{3 n}(t) & =\left.\frac{1}{(3 n)!} \frac{\partial^{3 n} I_{f_{0}}}{\partial x^{3 n}}\right|_{x=0} \\
& =\frac{1}{6 \pi(3 n)!} \int_{0}^{t} d s f_{0}^{(n)}(s)\left[(1-\alpha) \int_{-\infty}^{\infty} e^{-i z(t-s)} d z+\left(1-\alpha^{2}\right) \int_{-\infty}^{\infty} e^{-i z(t-s)} d z\right] \\
& =\frac{1}{(3 n)!} \int_{0}^{t} f_{0}^{(n)}(s) \delta(t-s) d s=\frac{f_{0}^{(n)}(t)}{(3 n)!}, \quad n=0,1, \ldots \tag{161}
\end{align*}
$$

which we can confirm rigorously using asymptotics. Similarly, we can find the coefficients for the Taylor series for $I_{f_{1}}(x, t)(n=1,2$,$) ,$

$$
\begin{array}{rlrl}
b_{3 n-1}(t) & =-\frac{\sqrt{3} \Gamma\left(\frac{1}{3}\right)}{2 \pi(3 n-1)!}\left[\sum_{m=1}^{n} \frac{(-1)^{n-m} \Gamma\left(n-m+\frac{1}{3}\right)}{t^{n-m+\frac{1}{3}} \Gamma\left(\frac{1}{3}\right)} f_{1}^{(m-1)}(0)+\int_{0}^{t} d s \frac{f_{1}^{(n)}(s)}{(t-s)^{\frac{1}{3}}}\right] \\
& =-\frac{f_{1}^{\left(n-\frac{2}{3}\right)}(t)}{(3 n-1)!}, & & n=1,2, \ldots \\
b_{3 n}(t) & =0, & n=0,1, \ldots \\
b_{3 n+1}(t) & =\frac{f_{1}^{(n)}(t)}{(3 n+1)!}, & & n=0,1, \ldots \tag{164}
\end{array}
$$

Then, the two series in (157) give analytic extensions for the two functions $I_{f_{0}}(x, t)$ and $I_{f_{1}}(x, t)$. To reduce the number of terms in the Taylor series, we may once again write

$$
\begin{align*}
I_{f_{0}}^{\text {ext }}(x, t) & =\left\{\begin{array}{ll}
I_{f_{0}}(x, t), & x \geq 0, \\
\tilde{f}_{0}(x, t)-I_{f_{0}}(-x, t), & x<0,
\end{array} \quad \text { where } \quad \tilde{f}_{0}(x, t)=2 \sum_{n=0}^{\infty} a_{2 n}(t) x^{2 n},\right.  \tag{165}\\
I_{f_{1}}^{\text {ext }}(x, t) & =\left\{\begin{array}{ll}
I_{f_{1}}(x, t), \\
\tilde{f}_{1}(x, t)+I_{f_{1}}(-x, t), & x \geq 0, \\
f_{1} & x<0,
\end{array} \quad \text { where } \quad \tilde{f}_{1}(x, t)=2 \sum_{n=1}^{\infty} b_{2 n-1}(t) x^{2 n-1} .\right. \tag{166}
\end{align*}
$$

Theorem 7. Define $\mathcal{D}=\{t \in \mathbb{C}: \operatorname{dist}(t,[0, T]) \leq r\}$, for some $r>0, T>0$, a domain in the complex t-plane containing the interval $[0, T]$. If $f_{0}(t)\left(f_{1}(t)\right)$ is analytic in $\mathcal{D}$ then $I_{f_{0}}(x, t)$ $\left(I_{f_{1}}(x, t)\right)$ is analytic for $x>0$ and $0 \leq t \leq T$, and the series representation (157) is entire in $x$ for $0 \leq t \leq T$.

Proof. The proof for the convergence of the series is similar to that of Theorem 2. Since the series is absolutely convergent, and it solves the corresponding initial and boundary value problem, by uniqueness, it converges to $I_{f_{0}}(x, t)\left(I_{f_{1}}(x, t)\right)$, so that $I_{f_{0}}(x, t)\left(I_{f_{1}}(x, t)\right)$ is an analytic function for $x>0$ and for $0 \leq t \leq T$.

The previously used combination of the Cauchy, Fubini, and Morera theorems is insufficient here. We need the stronger condition of analyticity of $f_{0}(t)\left(f_{1}(t)\right)$ to determine the analyticity of $I_{f_{0}}(x, t)\left(I_{f_{1}}(x, t)\right)$. This is to be compared to the case of the previous section (linear KdV, one boundary condition), where the stronger condition of exponential decay on $u_{0}(x)$ was needed to determine the analyticity of $I_{0}(x, t)$.

### 5.1 Boundary-to-Initial Map

The functions

$$
\begin{equation*}
\tilde{f}_{0}(x, t)=2 \sum_{n=0}^{\infty} a_{2 n}(t) x^{2 n}, \quad \text { and } \quad \tilde{f}_{1}(x, t)=2 \sum_{n=0}^{\infty} b_{2 n}(t) x^{2 n} \tag{167}
\end{equation*}
$$

are solutions to the linear KdV equation (141a), which is easily seen from the expressions for the Taylor coefficients in the previous section. Since

$$
\begin{equation*}
\sum_{n=0}^{\infty} a_{3 n}(t) x^{3 n}=\sum_{n=0}^{\infty} \frac{x^{3 n}}{(3 n)!} f_{0}^{(n)}(t) \rightarrow \sum_{n=0}^{\infty} \frac{x^{3 n}}{(3 n)!} f_{0}^{(n)}(0), \quad \text { as } t \rightarrow 0^{+} \tag{168}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{n=1}^{\infty} a_{3 n-1}(t) x^{3 n-1} \sim-\frac{\sqrt{3}}{2 \pi} \sum_{n=1}^{\infty} \frac{x^{3 n-1}}{(3 n-1)!} \sum_{m=1}^{n} \frac{(-1)^{n-m} \Gamma\left(n-m+\frac{2}{3}\right)}{t^{n-m+\frac{2}{3}}} f_{0}^{(m-1)}(0), \quad \text { as } t \rightarrow 0^{+} \tag{169}
\end{equation*}
$$

and the integral term approaches zero as $t \rightarrow 0^{+}$. Switching the order of summation and evaluating the inner sum, we find for $x<0$,

$$
\sum_{n=1}^{\infty} a_{3 n-1}(t) x^{3 n-1} \sim-\frac{\sqrt{3} \Gamma\left(\frac{2}{3}\right)}{2 \pi t^{\frac{2}{3}}} \sum_{m=0}^{\infty} \frac{x^{3 m+2}}{(3 m+2)!} f_{0}^{(m)}(0)_{2} F_{3}\left(\begin{array}{c}
\frac{2}{3}, 1  \tag{170}\\
m+1, m+\frac{4}{3}, m+\frac{5}{3}
\end{array} ; \frac{|x|^{3}}{27 t}\right),
$$

as $t \rightarrow 0^{+}$, and since

$$
\frac{\sqrt{3} \Gamma\left(\frac{2}{3}\right)}{2 \pi t^{\frac{2}{3}}}{ }_{2} F_{3}\left(\begin{array}{c}
\frac{2}{3}, 1  \tag{171}\\
m+1, m+\frac{4}{3}, m+\frac{5}{3}
\end{array} ; \frac{|x|^{3}}{27 t}\right) \sim \frac{3^{\frac{3 m}{2}+\frac{3}{4}}(3 m+2)!t^{\frac{3 m}{2}+\frac{1}{4}}}{\sqrt{\pi}|x|^{\frac{9 m}{2}+\frac{11}{4}}} e^{\frac{2|x| \frac{3}{2}}{3 \sqrt{3 t}}}, \quad \text { as } t \rightarrow 0^{+}
$$

we find for $x<0$,

$$
\begin{equation*}
\sum_{n=1}^{\infty} a_{3 n-1}(t) x^{3 n-1} \sim-e^{\frac{2 \left\lvert\, x x^{\frac{3}{2}}\right.}{3 \sqrt{3 t}}} \sum_{m=0}^{\infty} \frac{(-1)^{m} 3^{\frac{3 m}{2}+\frac{3}{4}} t^{\frac{3 m}{2}+\frac{1}{4}}}{\sqrt{\pi}|x|^{\frac{3 m}{2}+\frac{3}{4}}} f_{0}^{(m)}(0), \quad \text { as } t \rightarrow 0^{+} \tag{172}
\end{equation*}
$$

and, in general, $\left|I_{f_{0}}(x, t)\right| \rightarrow \infty$, as $t \rightarrow 0^{+}$. Similarly, for $x<0$, we have

$$
\begin{equation*}
\sum_{n=1}^{\infty} b_{3 n-1}(t) x^{3 n-1} \sim-e^{\frac{2|x|^{\frac{3}{2}}}{3 \sqrt{3 t}}} \sum_{m=0}^{\infty} \frac{3^{\frac{5}{4}+\frac{3 m}{2}} t^{\frac{3}{4}+\frac{3 m}{2}}}{2 \sqrt{\pi}|x|^{\frac{5}{4}+\frac{3 m}{2}}} f_{1}^{(m)}(0), \quad \text { as } t \rightarrow 0^{+} \tag{173}
\end{equation*}
$$

and $\left|I_{f_{1}}(x, t)\right| \rightarrow \infty$ as $t \rightarrow 0^{+}$as well.
We write

$$
\begin{equation*}
I_{0}(x, t)=I_{0}^{(0)}(x, t)+I_{0}^{(1)}(x, t)+I_{0}^{(2)}(x, t)=I_{0}^{(0)}(x, t)+2 \operatorname{Re}\left\{I_{0}^{(1)}(x, t)\right\} \tag{174}
\end{equation*}
$$

where we define

$$
I_{0}^{(0)}(x, t)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{i k x-i k^{3} t} \hat{u}_{0}(k) d k \rightarrow\left\{\begin{array}{ll}
u_{0}(x), & x>0,  \tag{175}\\
0, & x<0
\end{array} \quad \text { as } t \rightarrow 0^{+}\right.
$$



Figure 16: Contour deformation for the asymptotics of $I_{0}^{(1)}(x, t)$ for $x>0$ (blue, dashed) and for $x<0$ (red, solid).

However, the other terms in $I_{0}(x, t)$ diverge to infinity as $t \rightarrow 0^{+}$, as shown next. We have

$$
\begin{equation*}
I_{0}^{(1)}(x, t)=-\frac{1}{2 \pi} \int_{0}^{\infty} d y u_{0}(y) \int_{\gamma_{1}} e^{i k\left(x-\alpha^{2} y\right)-i k^{3} t} d k=\frac{\alpha}{\sqrt[3]{3 t}} \int_{0}^{\infty} \operatorname{Ai}\left(\frac{y-\alpha x}{\sqrt[3]{3 t}}\right) u_{0}(y) d y \tag{176}
\end{equation*}
$$

Using $z=(y-\alpha x) / \sqrt[3]{3 t}$, we find

$$
\begin{align*}
I_{0}^{(1)}(x, t) & =\alpha \int_{C} \operatorname{Ai}(z) u_{0}(\alpha x+z \sqrt[3]{3 t}) d z \\
& =\alpha \int_{0}^{\infty} \operatorname{Ai}(z) u_{0}(\alpha x+z \sqrt[3]{3 t}) d z-\alpha^{2} \int_{0}^{-x / \sqrt[3]{3 t}} \operatorname{Ai}(\alpha \rho) u_{0}(\alpha x+\alpha \rho \sqrt[3]{3 t}) d \rho \tag{177}
\end{align*}
$$

where the positive real axis is mapped to the contour $C$ as shown in Figure 16. As $t \rightarrow 0^{+}$, the first term limits to $\alpha u_{0}(\alpha x) / 3$. For $x>0$, the second term has the limit $-\alpha u_{0}(\alpha x) / 3$, so that $I_{0}(x, t) \rightarrow 0$ as $t \rightarrow 0^{+}$, as expected. For $x<0$,

$$
\begin{align*}
I_{0}^{(1)}(x, t) \sim & \frac{\alpha}{3} u_{0}(\alpha x)-\alpha^{2} \sum_{n=0}^{\infty} \frac{\alpha^{n}}{n!} u_{0}^{(n)}(0) \int_{0}^{|x| / \sqrt[3]{3 t}} \operatorname{Ai}(\alpha \rho)(x+\rho \sqrt[3]{3 t})^{n} d \rho \\
\sim & -\alpha^{2} \sum_{n=0}^{\infty} \frac{\alpha^{n}}{n!} u_{0}^{(n)}(0) \sum_{j=0}^{n}\binom{n}{j} x^{n-j}(3 t)^{\frac{j}{3}} \int_{0}^{|x| / \sqrt[3]{3 t}} \operatorname{Ai}(\alpha \rho) \rho^{j} d \rho \\
= & \sum_{n=0}^{\infty} \frac{\alpha^{n+2} x^{n+1}}{n!} u_{0}^{(n)}(0) \sum_{j=0}^{n}\binom{n}{j} \frac{(-1)^{j}}{3(j+1) \Gamma\left(\frac{2}{3}\right) t^{\frac{1}{3}}} 1 F_{2}\left(\begin{array}{c}
\frac{j+1}{3} \\
\frac{2}{3}, \frac{j+4}{3}
\end{array} ; \frac{|x|^{3}}{27 t}\right) \\
& +\sum_{n=0}^{\infty} \frac{\alpha^{n} x^{n+2}}{n!} u_{0}^{(n)}(0) \sum_{j=0}^{n}\binom{n}{j} \frac{(-1)^{j}}{3(j+2) \Gamma\left(\frac{1}{3}\right) t^{\frac{2}{3}}} 1 F_{2}\left(\begin{array}{c}
\frac{j+2}{3} \\
\frac{4}{3}, \frac{j+5}{3}
\end{array} \frac{|x|^{3}}{27 t}\right) \tag{178}
\end{align*}
$$

where the bounded function $\alpha u_{0}(\alpha x) / 3$ was omitted from the asymptotic series because it is not part of the leading-order behavior. The last integral is evaluated by integrating the Maclaurin series for the Airy function and manipulating the resulting series, obtaining the result in terms of hypergeometric functions. Using the asymptotic expansions for the generalized hypergeometric
functions ${ }_{p} F_{q}$ from [7, Section 16.11], we find

$$
\begin{equation*}
I_{0}(x, t) \sim e^{\frac{2|x|^{\frac{3}{2}}}{3 \sqrt{3 t}}} \sum_{n=0}^{\infty} \frac{(-1)^{n} 3^{\frac{3 n}{2}+\frac{3}{4}} t^{\frac{3 n}{2}+\frac{1}{4}}}{\sqrt{\pi}|x|^{\frac{3 n}{2}+\frac{3}{4}}} u_{0}^{(3 n)}(0)+e^{\frac{2|x|^{\frac{3}{2}}}{3 \sqrt{3 t}}} \sum_{n=0}^{\infty} \frac{(-1)^{n} 3^{\frac{3 n}{2}+\frac{5}{4}} t^{\frac{3 n}{2}+\frac{3}{4}}}{\sqrt{\pi}|x|^{\frac{3 n}{2}+\frac{5}{4}}} u_{0}^{(3 n+1)}(0) \tag{179}
\end{equation*}
$$

so that, in general, $\left|I_{0}(x, t)\right| \rightarrow \infty$, as $t \rightarrow 0^{+}$. It is clear that for some initial and boundary data $u_{0}(x), f_{0}(t)$, and $f_{1}(t)$, (for instance, $f_{0}(t)=f_{1}(t)=0, u_{0}(x) \neq 0$ ), a boundary-to-initial map does not exist. However, if the compatibility conditions,

$$
\begin{equation*}
f_{0}^{(n)}(0)=u_{0}^{(3 n)}(0), \quad \text { and } \quad f_{1}^{(n)}(0)=u_{0}^{(3 n+1)}(0), \quad n=0,1, \ldots \tag{180}
\end{equation*}
$$

are satisfied, then it can be shown that singular behavior as $t \rightarrow 0^{+}$cancels and

$$
\begin{equation*}
I_{0}(x, t)+I_{f_{0}}^{\text {ext }}(x, t)+I_{f_{1}}^{\text {ext }}(x, t) \rightarrow u_{0}(x), \quad \text { as } \quad t \rightarrow 0^{+} \tag{181}
\end{equation*}
$$

where $u_{0}(x)$ is the analytical continuation of $u_{0}(x)$ (which is defined for $x>0$ ) to the whole real line. Indeed, comparing (179) to (172) and (173) is suggestive of this, although a proof requires more work. It is an interesting question to isolate the conditions on the initial and boundary data for which a boundary-to-initial map exists, and the half-line problem can be viewed as the restriction of a whole-line problem, even if the whole-line initial condition is unbounded and discontinuous.

### 5.2 Examples

We demonstrate our results using two examples. Our first example uses the whole-line solution,

$$
\begin{equation*}
u_{\mathbb{R}}(x, t)=2 e^{-\sqrt{3} x} \cos (x+8 t) \tag{182}
\end{equation*}
$$

with the inferred boundary and initial conditions. The UTM solution is no longer defined for $x<0$, but the analytic continuation recovers the exact solution on the whole line. The results are shown in Figure 17a.

Next, we consider (141a) with $f_{0}(t)=t e^{-t}, u_{0}(x)=u_{\mathbb{R}}(x, 0)$, and $f_{1}(t)=\left.u_{\mathbb{R}, x}(x, t)\right|_{x=0}$ for $x>0$. We find the analytic continuation of the solution shown in Figure 17b.

## 6 The discretized advection equation

As discussed above, the analytic continuation of the solution for the continuous-in-space advection equation is immediate through the use of d'Alembert's formula. The situation is more complicated in the discrete space setting, as the discretization used affects how the analytic continuation is done.

### 6.1 Backward discretization

Consider, with wave speed $c>0$,

$$
\begin{align*}
u_{t}=-c u_{x}, & x>0, t>0  \tag{183a}\\
u(x, 0)=\phi(x), & x>0  \tag{183b}\\
u(0, t)=f_{0}(t), & t>0 \tag{183c}
\end{align*}
$$



Figure 17: (a) The solution $u_{\mathrm{ac}}(x, t)$ at $t=1$ obtained through analytic continuation, shown with the UTM solution $u_{\text {UTM }}(x, t)$ for $x>0$ (it is not defined for $x<0$ ) and the whole line solution $u_{\mathbb{R}}(x, t)$ given by (182). (b) The same plot zoomed in. (c) The solution $u_{\text {ac }}(x, t)$ at $t=1$, for the incompatible case, obtained through analytic continuation, shown with the UTM solution $u_{\mathrm{UTM}}(x, t)$ for $x>0$ (it is not defined for $x<0$ ). (d) The same plot zoomed in.

Discretizing the spacial derivative $u_{x}(x, t)$ using the standard backward stencil gives

$$
\begin{equation*}
\dot{u}_{n}(t)=c \frac{u_{n-1}(t)-u_{n}(t)}{h} \tag{184}
\end{equation*}
$$

with dispersion relation

$$
\begin{equation*}
W(k)=c \frac{1-e^{-i k h}}{h} \tag{185}
\end{equation*}
$$

Following [4], the solution to this semi-discrete IBVP (183) is

$$
\begin{equation*}
u_{n}(T)=\frac{1}{2 \pi} \int_{-\pi / h}^{\pi / h} e^{i k n h} e^{-W T} \hat{u}(k, 0) d k+\frac{c}{2 \pi} \int_{-\pi / h}^{\pi / h} e^{i k(n-1) h} e^{-W T} F_{0}(W, T) d k, \quad n \geq 1 \tag{186}
\end{equation*}
$$

where the Fourier transform

$$
\begin{equation*}
\hat{u}(k, 0)=h \sum_{n=1}^{\infty} e^{-i k n h} u_{n}(0), \quad \operatorname{Im}(k) \leq 0 \tag{187}
\end{equation*}
$$



Figure 18: The green region depicts where $\operatorname{Re}(-W) \leq 0$ and $e^{-W T}$ is decaying with the dispersion relation (185).
begins at $n=1$ since the Dirichlet boundary condition is given, and

$$
\begin{equation*}
F_{0}(W, T)=\int_{0}^{T} e^{W t} f_{0}(t) d t, \quad k \in \mathbb{C} \tag{188}
\end{equation*}
$$

Unlike the solution to the continuous advection equation, the semi-discrete solution (186) couples the initial and boundary conditions, and both contribute at every mesh point. According to Figure 18, substituting $n \rightarrow-n$ for $n \in \mathbb{Z}^{+}$allows both integral paths to be deformed below the real line where the integrands decay, so that $u_{-n}(T) \equiv 0$ for $n>0$ and for all $T$. In the continuum limit, i.e., as $h \rightarrow 0$, (186) converges to

$$
\begin{equation*}
u(x, T)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{i k x} e^{-\tilde{W} T} \hat{u}(k, 0) d k+\frac{c}{2 \pi} \int_{-\infty}^{\infty} e^{i k x} e^{-\tilde{W} T} \tilde{F}_{0}(\tilde{W}, T) d k, \quad x>0 \tag{189}
\end{equation*}
$$

with $\tilde{W}(k)=i c k, \hat{u}(k, 0)=\int_{0}^{\infty} e^{-i k x} u(x, 0) d x$, and $\tilde{F}_{0}(\tilde{W}, T)=\int_{0}^{T} e^{\tilde{W} t} u(0, t) d t$. It follows that $u(-x, T)=0$ for $x>0$ from this representation. In what follows, we examine how (186) may be analytically continued for $n<0$.

For the first integral in (186) determined by the initial condition, we substitute the definition of $\hat{u}(k, 0)$ :

$$
\begin{equation*}
\frac{1}{2 \pi} \int_{-\pi / h}^{\pi / h} e^{i k n h} e^{-W T} \hat{u}(k, 0) d k=\sum_{m=1}^{\infty}\left[\frac{h}{2 \pi} \int_{-\pi / h}^{\pi / h} e^{i k(n-m) h} e^{-W T} d k\right] u_{m}(0) \tag{190}
\end{equation*}
$$

Using $z=e^{i k h}$,

$$
\begin{align*}
\frac{h}{2 \pi} \int_{-\pi / h}^{\pi / h} e^{i k(n-m) h} e^{-W T} d k & =\frac{e^{-c T / h}}{2 \pi i} \oint_{|z|=1} z^{n-m-1} e^{c T / h z} d z=e^{-c T / h} \operatorname{Res}_{z=0}\left\{z^{n-m-1} e^{c T / h z}\right\} \\
& =e^{-c T / h}\left(\frac{c T}{h}\right)^{n-m} \frac{1}{(n-m)!} \tag{191}
\end{align*}
$$

Hence,

$$
\begin{equation*}
\frac{1}{2 \pi} \int_{-\pi / h}^{\pi / h} e^{i k n h} e^{-W T} \hat{u}(k, 0) d k=e^{-c T / h} \sum_{m=1}^{\infty}\left(\frac{c T}{h}\right)^{n-m} \frac{u_{m}(0)}{(n-m)!} \tag{192}
\end{equation*}
$$

We follow a similar procedure for the second integral of (186) by substituting the definition of $F_{0}(W, T)$. Denoting this integral by $I(n, t)$,

$$
\begin{align*}
I(n, t) & =\frac{c}{2 \pi} \int_{-\pi / h}^{\pi / h} e^{i k(n-1) h} e^{-W T} F_{0}(W, T) d k=c \int_{0}^{T}\left[\frac{1}{2 \pi} \int_{-\pi / h}^{\pi / h} e^{i k(n-1) h} e^{-W(T-t)} d k\right] f_{0}(t) d t \\
& =\frac{c}{h(n-1)!} \int_{0}^{T} e^{-c(T-t) / h}\left(\frac{c(T-t)}{h}\right)^{n-1} f_{0}(t) d t \tag{193}
\end{align*}
$$

using the same steps as above. Thus (186) is rewritten as

$$
\begin{equation*}
u_{n}(T)=e^{-c T / h} \sum_{m=1}^{\infty}\left(\frac{c T}{h}\right)^{n-m} \frac{u_{m}(0)}{(n-m)!}+I(n, T), \quad n \geq 1 \tag{194}
\end{equation*}
$$

In what follows, we explore how this expression can be analytically continued from $n>0$ to $n<0$. To this end, we manipulate the expression (194) to obtain a different representation which can be evaluated for negative $n$.

The first term is rewritten using $1 /(n-m)!=1 / \Gamma(n-m+1)$, which may be evaluated for $n<0$. Since $1 / \Gamma(\alpha)$ has simple zeros at nonpositive integers $\alpha$, the first term does not contribute for $n<0$. We focus on the second term. Substituting $s=\frac{c(T-t)}{h}$ and Taylor expanding about $h=0$ gives

$$
\begin{align*}
I(n, T) & =\frac{1}{(n-1)!} \int_{0}^{c T / h} e^{-s} s^{n-1} f_{0}\left(T-\frac{h}{c} s\right) d s \\
& =\frac{1}{(n-1)!} \int_{0}^{c T / h} e^{-s} s^{n-1} \sum_{\ell=0}^{\infty} \frac{f_{0}^{(\ell)}(T)(-1)^{\ell}}{\ell!}\left(\frac{h}{c}\right)^{\ell} s^{\ell} d s \\
& =\frac{1}{\Gamma(n)} \sum_{\ell=0}^{\infty} \frac{f_{0}^{(\ell)}(T)(-1)^{\ell}}{\ell!}\left(\frac{h}{c}\right)^{\ell} \gamma\left(n+\ell, \frac{c T}{h}\right), \tag{195}
\end{align*}
$$

where

$$
\begin{equation*}
\gamma(a, y)=\int_{0}^{y} t^{a-1} e^{-t} d t \tag{196}
\end{equation*}
$$

is the lower incomplete gamma function [7]. The solution (194) is written as

$$
\begin{equation*}
u_{n}(T)=e^{-c T / h} \sum_{m=1}^{\infty}\left(\frac{c T}{h}\right)^{n-m} \frac{u_{m}(0)}{(n-m)!}+\sum_{\ell=0}^{\infty} \frac{f_{0}^{(\ell)}(T)(-1)^{\ell}}{\ell!}\left(\frac{h}{c}\right)^{\ell} \frac{\gamma\left(n+\ell \frac{c T}{h}\right)}{\Gamma(n)} \tag{197}
\end{equation*}
$$

for $n \geq 1$. This representation is only valid for $\ell \geq 1-n$ and nonzero for $n \geq 1$ due to the ratio $\gamma(n+\ell, c T / h) / \Gamma(n)$. Recursively applying $\Gamma(k+1)=k \Gamma(k)$,

$$
\begin{equation*}
\Gamma(a-b)=(a-b-1)(a-b-2) \cdot \ldots \cdot(-b) \Gamma(-b)=\frac{(-1)^{a} \Gamma(b+1) \Gamma(-b)}{\Gamma(b-a+1)} \tag{198}
\end{equation*}
$$

Using the power series of $\gamma(a, y)$ [7],

$$
\begin{align*}
\frac{\gamma(n+\ell, y)}{\Gamma(n)} & =\frac{1}{\Gamma(n)} \sum_{k=0}^{\infty} \frac{(-1)^{k} y^{n+\ell+k} \Gamma(n+\ell+k)}{k!\Gamma(n+\ell+k+1)} \\
& =(-1)^{\ell} \sum_{k=0}^{\infty} \frac{y^{n+\ell+k} \Gamma(1-n)}{k!\Gamma(1+n+k+\ell) \Gamma(1-n-k-\ell)} \\
& =\frac{(-1)^{\ell} y^{n+\ell-n-\ell} \Gamma(1-n)}{(-n-\ell)!}=\frac{(-1)^{\ell} \Gamma(1-n)}{\Gamma(1-n-\ell)} \tag{199}
\end{align*}
$$

where the sum collapses since $\Gamma(1+n+k+l) \Gamma(1-n-k-l)$ is infinite unless $n+k+l=0$. This representation is valid for $n \geq 0$ and nonzero for $\ell \leq-n$. For any $n \in \mathbb{Z}$,

$$
\begin{align*}
u_{n}(T) & =\lim _{\alpha \rightarrow n} u_{\alpha}(T) \\
& =e^{-c T / h} \sum_{m=1}^{\infty}\left(\frac{c T}{h}\right)^{n-m} \frac{u_{m}(0)}{(n-m)!}+\sum_{\ell=0}^{\infty} \frac{f_{0}^{(\ell)}(T)(-1)^{\ell}}{\ell!}\left(\frac{h}{c}\right)^{\ell} \lim _{\alpha \rightarrow n} \frac{\gamma\left(\alpha+\ell, \frac{c T}{h}\right)}{\Gamma(\alpha)}, \tag{200}
\end{align*}
$$

where

$$
\lim _{\alpha \rightarrow n} \frac{\gamma\left(\alpha+\ell, \frac{c T}{h}\right)}{\Gamma(\alpha)}= \begin{cases}\frac{\gamma\left(n+\ell, \frac{c T}{h}\right)}{\Gamma(n)}, & \ell \geq 1-n \quad \text { and } n \geq 1  \tag{201}\\ \frac{(-1)^{\ell} \Gamma(1-n)}{\Gamma(1-n-\ell)}, & n \leq 0 \text { and } \ell \leq-n \\ 0, & \text { otherwise }\end{cases}
$$

so that

$$
I(n, T)= \begin{cases}\sum_{\ell=0}^{-n} \frac{f_{0}^{(\ell)}(T)}{\ell!}\left(\frac{h}{c}\right)^{\ell} \frac{\Gamma(1-n)}{\Gamma(1-n-\ell)}, & n \leq 0  \tag{202}\\ \sum_{\ell=0}^{\infty} \frac{f_{0}^{(\ell)}(T)(-1)^{\ell}}{\ell!}\left(\frac{h}{c}\right)^{\ell} \frac{\gamma\left(n+\ell, \frac{c T}{h}\right)}{\Gamma(n)}, & n \geq 1\end{cases}
$$

Thus, $u_{n}(T)$ with $n \geq 1$ is given by (197), whereas for $n \leq 0$,

$$
\begin{equation*}
u_{n}(T)=I(n, T)=\sum_{\ell=0}^{n} \frac{f_{0}^{(\ell)}(T)}{\ell!}\left(\frac{h}{c}\right)^{\ell} \frac{\Gamma(1-n)}{\Gamma(1-n-\ell)} \tag{203}
\end{equation*}
$$

Since (197) originates from (186), we combine (186) and (203) to create an expression valid for $n \in \mathbb{Z}$ as

$$
\begin{align*}
u_{n}(T)= & \frac{1}{2 \pi} \int_{-\pi / h}^{\pi / h} e^{i k n h} e^{-W T} \hat{u}(k, 0) d k+\frac{c}{2 \pi} \int_{-\pi / h}^{\pi / h} e^{i k(n-1) h} e^{-W T} F_{0} d k \\
& +\sum_{\ell=0}^{-n} \frac{f_{0}^{(\ell)}(T)}{\ell!}\left(\frac{h}{c}\right)^{\ell} \frac{\Gamma(1-n)}{\Gamma(1-n-\ell)} \tag{204}
\end{align*}
$$

where the integral terms only contribute for $n \geq 1$ and the sum only contributes for $n \leq 0$.

### 6.1.1 Continuum Limit

For the continuous case, (183),

$$
\begin{equation*}
u(-x, T)=f_{0}\left(T+\frac{x}{c}\right), \quad x>0 \tag{205}
\end{equation*}
$$

and the negative half-line solution (203) depends only on the boundary condition. The same is true in the semi-discrete case, see (203). We show that (203) limits to (205) by taking $h \rightarrow 0$.

Recursively applying $\Gamma(z+1)=z \Gamma(z)$,

$$
f(n, \ell)=\frac{\Gamma(1+n)}{\Gamma(1+n-\ell)}=\prod_{p=0}^{\ell-1}(n-p)= \begin{cases}1, & \ell=0  \tag{206}\\ \sum_{p=0}^{\ell-1} a_{p} n^{p+1}, & \ell \geq 1\end{cases}
$$

and $f(n, \ell)$ is a polynomial in $n$ of degree $\ell$ with leading coefficient $a_{\ell-1}=1$. Hence, with $n \geq 0$,

$$
\begin{equation*}
u_{-n}(T)=\sum_{\ell=0}^{n} \frac{f_{0}^{(\ell)}(T)}{\ell!}\left[\left(\frac{n h}{c}\right)^{\ell}+a_{\ell-2}\left(\frac{h}{c}\right)\left(\frac{n h}{c}\right)^{\ell-1}+\ldots\right] \tag{207}
\end{equation*}
$$

In the continuum limit $h \rightarrow 0$ with $\lim _{h \rightarrow 0} n h=x$,

$$
\begin{equation*}
u(-x, T)=\sum_{\ell=0}^{\infty} \frac{f_{0}^{(\ell)}(T)}{\ell!}\left(\frac{x}{c}\right)^{\ell}=f_{0}\left(T+\frac{x}{c}\right) \tag{208}
\end{equation*}
$$

### 6.1.2 Examples

Figure 19 depicts the semi-discrete UTM solutions for $n \in \mathbb{Z}$ using (186) and (203) on the left and (204) on the right for the IBVP (183) with, for $x, t>0$,

$$
\begin{equation*}
u(x, 0)=\phi(x)=\left(\frac{\sin (4 \pi x)+1}{2}\right) e^{-2 x}, \quad \text { and } \quad u(0, t)=f_{0}(t)=\frac{1}{2}+(1-2 \pi) t e^{-t} \tag{209}
\end{equation*}
$$



Figure 19: (a) For $h=0.05$, the solution $u_{n}^{\text {ac }}(t)$ given by (204) at $t=0.5$ and with $c=1$ obtained through analytic continuation, shown with the SDUTM solution $u_{n}^{\text {SDUTM }}(t)(197)$ and the continuum solution $u_{c}(x, t)$ given by (4). (b) The same plot for $h=0.01$.

### 6.2 A second-order backward discretization

The backward, second-order discretization applied to (183) is

$$
\begin{equation*}
\dot{u}_{n}(t)=c \frac{-u_{n-2}(t)+4 u_{n-1}(t)-3 u_{n}(t)}{2 h}, \quad n \geq 1, t>0 \tag{210}
\end{equation*}
$$

The details for the standard second-order centered discretization are presented in the Appendix, as they present peculiarities that are inherent to the equation and the discretization, but not the analytic continuation. Following the steps outlined in [4, 5], (210) has the solution

$$
\begin{align*}
u_{n}(T)= & \frac{1}{2 \pi} \int_{-\pi / h}^{\pi / h} e^{i k n h} e^{-W T} \hat{u}(k, 0) d k+\frac{c}{4 \pi} \int_{-\pi / h}^{\pi / h} e^{i k(n-1) h} e^{-W T}\left(3-e^{-i k h}\right) F_{0}(W, T) d k \\
& -\frac{c}{4 \pi} \int_{-\pi / h}^{\pi / h} e^{i k(n-1) h} e^{-W T}\left[\left(\frac{h}{c}\right) U_{1}+\left(\frac{h^{2}}{2 c^{2}}\right) U_{2}\right] d k, \quad n \geq 1, t>0 \tag{211}
\end{align*}
$$

with dispersion relation

$$
\begin{equation*}
W(k)=c \frac{e^{-2 i k h}-4 e^{-i k h}+3}{2 h} \tag{212}
\end{equation*}
$$

and $F_{0}$ as before, see (188). Further,

$$
\begin{equation*}
U_{1}(W, T)=\int_{0}^{T} e^{W t} \dot{f}_{0}(t) d t, \quad U_{2}(W, T)=\int_{0}^{T} e^{W t} \ddot{f}_{0}(t) d t \tag{213}
\end{equation*}
$$

The dots on $f_{0}(t)$ on the right-hand sides denote time derivatives. To derive (211), (183) has been used to derive boundary conditions from the Dirichlet data,

$$
\begin{equation*}
u_{x}(0, t)=-\dot{f}_{0}(t) / c, \quad u_{x x}(0, t)=\ddot{f}_{0}(t) / c^{2} \tag{214}
\end{equation*}
$$

These derivative conditions are discretized using centered second-order accurate stencils (to retain the desired second-order accuracy), followed by multiplying by $\exp (W t)$ and integrating:

$$
\begin{equation*}
\frac{F_{1}-F_{-1}}{2 h}=\frac{-U_{1}}{c}, \quad \frac{F_{1}-2 F_{0}+F_{-1}}{h^{2}}=\frac{U_{2}}{c^{2}} . \tag{215}
\end{equation*}
$$

As $h \rightarrow 0$, the semi-discrete solution (211) converges to (189), as the semi-discrete solution correctly loses dependence on the derivative boundary conditions in the continuum limit. As before, one shows that (211) gives $u_{-n}(T)=0$ when evaluated for $n \in \mathbb{Z}^{+}$, similar to how (189) gives $u(-x, T)=0$ for $x \in \mathbb{R}^{+}$. The analytical continuation of (211) for negative values of $n$ is constructed below.

We introduce

$$
\begin{equation*}
B_{m}(n, T, g)=\frac{1}{4 \pi} \int_{-\pi / h}^{\pi / h} e^{i k(n-m) h-W T} G d k, \quad G(W, T)=\int_{0}^{T} e^{W t} g(t) d t \tag{216}
\end{equation*}
$$

The semi-discrete solution is rewritten as

$$
\begin{align*}
u_{n}(T)= & \frac{1}{2 \pi} \int_{-\pi / h}^{\pi / h} e^{i k n h} e^{-W T} \hat{u}(k, 0) d k+3 c B_{1}\left(n, T, f_{0}\right)-c B_{2}\left(n, T, f_{0}\right)  \tag{217}\\
& -h B_{1}\left(n, T, \dot{f}_{0}\right)-\frac{h^{2}}{2 c} B_{1}\left(n, T, \ddot{f}_{0}\right)
\end{align*}
$$

Leaving the initial-condition as is, we consider $B_{m}(n, T, g)$. We have

$$
B_{m}(n, T, g)=\frac{1}{4 \pi} \int_{-\pi / h}^{\pi / h} e^{i k(n-m) h-W T}\left[\int_{0}^{T} e^{W t} g(t) d t\right] d k=\int_{0}^{T} I_{m}(n, T, t) g(t) d t
$$

with

$$
\begin{align*}
I_{m}(n, T, t) & =\frac{1}{4 \pi} \int_{-\pi / h}^{\pi / h} e^{i k(n-m) h-W(T-t)} d k=\frac{1}{4 \pi} \oint_{|z|=1} z^{n-m} e^{-W(T-t)} \frac{d z}{i h z} \\
& =\frac{e^{\frac{-3 c(T-t)}{2 h}}}{2 h} \operatorname{Res}_{z=0}\left\{z^{n-m-1} \exp \left[\left(-z^{-2}+4 z^{-1}\right) \frac{c(T-t)}{2 h}\right]\right\} \\
& =\frac{e^{\frac{-3 c(T-t)}{2 h}}}{2 h} \sum_{k=0}^{\frac{n-m}{2}} \frac{(-1)^{k}}{2^{2 k} k!(n-m-2 k)!}\left(\frac{2 c(T-t)}{h}\right)^{n-m-k} \tag{218}
\end{align*}
$$

where a tedious calculation shows that there is no contribution if $n-m$ is odd. Using the substitution $s=3 c(T-t) /(2 h)$ and expanding about $h=0$,

$$
\begin{align*}
& B_{m}(n, T, g)=\int_{0}^{T}\left[\frac{e^{\frac{-3 c(T-t)}{2 h}}}{2 h} \sum_{k=0}^{\frac{n-m}{2}} \frac{(-1)^{k}}{2^{2 k} k!(n-m-2 k)!}\left(\frac{2 c(T-t)}{h}\right)^{n-m-k}\right] g(t) d t \\
& =\frac{1}{3 c} \sum_{k=0}^{\frac{n-m}{2}} \frac{(-1)^{k}}{2^{2 k} k!(n-m-2 k)!}\left(\frac{4}{3}\right)^{n-m-k}\left[\int_{0}^{\frac{3 c T}{2 h}} e^{-s} s^{n-m-k} g\left(T-\frac{2 h}{3 c} s\right) d s\right] \\
& =\frac{1}{3 c} \sum_{k=0}^{\frac{n-m}{2}} \frac{(-1)^{k}}{2^{2 k} k!(n-m-2 k)!}\left(\frac{4}{3}\right)^{n-m-k} \sum_{p=0}^{\infty} \frac{g^{(p)}(T)(-1)^{p}}{p!}\left(\frac{2 h}{3 c}\right)^{p} \int_{0}^{\frac{3 c T}{2 h}} e^{-s} s^{n-m-k+p} d s \\
& =\frac{1}{3 c} \sum_{k=0}^{\frac{n-m}{2}} \frac{(-1)^{k}}{2^{2 k} k!(n-m-2 k)!}\left(\frac{4}{3}\right)^{n-m-k} \sum_{p=0}^{\infty} \frac{g^{(p)}(T)(-1)^{p}}{p!}\left(\frac{2 h}{3 c}\right)^{p} \gamma\left(n-m-k+p+1, \frac{3 c T}{2 h}\right) \tag{219}
\end{align*}
$$

Applying relations (198) and the power series of the lower incomplete gamma function [7],

$$
\lim _{\alpha \rightarrow n} \frac{\gamma\left(\alpha-m-k+p+1, \frac{3 c T}{2 h}\right)}{\Gamma(\alpha-m-2 k+1)}= \begin{cases}\frac{\gamma\left(n-m-k+p+1, \frac{3 c T}{2 h}\right)}{\Gamma(n-m-2 k+1)}, & p \geq k-n+m, k \leq \frac{n-m}{2}  \tag{220}\\ \frac{(-1)^{k+p} \Gamma(m+2 k-n)}{\Gamma(m+k-p-n)}, & k \geq \frac{1+n-m}{2}, p \leq m+k-n-1 \\ 0, & \text { otherwise. }\end{cases}
$$

Collecting the summands in $S(n, T, k, p)$ and relaxing the upper bound on the summation over $k$ (again by adding zero terms), we write

$$
\begin{align*}
B_{m}(n, T, g) & =\sum_{k=0}^{\infty} \sum_{p=0}^{\infty} S(n, T, k, p) \\
& =\left(\sum_{k=0}^{\frac{n-m}{2}}+\sum_{k=\max \left(\frac{1+n-m}{2}, 0\right)}^{\infty} S \sum_{p=0}^{m+k-n-1}+\sum_{p=\max (k-n+m, 0)}^{\infty} S(n, T, k, p)\right. \\
& =\sum_{k=0}^{\frac{n-m}{2}} \sum_{p=\max (k-n+m, 0)}^{\infty} S(n, T, k, p)+\sum_{k=\max \left(\frac{1+n-m}{2}, 0\right)}^{\infty} \sum_{p=0}^{m+k-n-1} S(n, T, k, p), \tag{221}
\end{align*}
$$

where the other terms are shown to vanish. For $n>0$, the second of the double sums above vanishes, and the first can be rewritten in the integral form from (211). For $n<0$, the first pair vanishes. We let $n \rightarrow-n$ for $n>0$ from here on. Relaxing the bounds (re-introducing zero contributions),

$$
\begin{align*}
& B_{m}(-n, T, g)=\sum_{k=\max \left(\frac{1-n-m}{2}, 0\right)}^{\infty} \sum_{p=0}^{m+k+n-1} S(-n, T, k, p) \\
& \quad=\frac{1}{3 c} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^{k}}{2^{2 k} k!}\left(\frac{4}{3}\right)^{-n-m-k} \frac{g^{(p)}(T)(-1)^{p}}{p!}\left(\frac{2 h}{3 c}\right)^{p} \frac{(-1)^{k+p} \Gamma(m+2 k+n)}{\Gamma(m+k-p+n)} \\
& \quad=\frac{1}{3 c}\left(\frac{3}{4}\right)^{n+m} \sum_{p=0}^{\infty} \frac{g^{(p)}(T)}{p!}\left(\frac{2 h}{3 c}\right)^{p} \sum_{k=0}^{\infty} \frac{1}{2^{2 k} k!}\left(\frac{3}{4}\right)^{k} \frac{\Gamma(m+2 k+n)}{\Gamma(m+k-p+n)} \\
& \quad=\frac{1}{3 c}\left(\frac{3}{4}\right)^{n+m} \sum_{p=0}^{\infty} \frac{g^{(p)}(T)}{p!}\left(\frac{2 h}{3 c}\right)^{p} \Gamma(m+n)_{2} \tilde{F}_{1}\left(\frac{m+n}{2}, \frac{m+n+1}{2} ; m+n-p ; \frac{3}{4}\right), \tag{222}
\end{align*}
$$

where ${ }_{2} \tilde{F}_{1}(a, b ; c ; z)={ }_{2} F_{1}(a, b ; c ; z) / \Gamma(c)$ is the regularized hypergeometric function [7]. In summary, the solution for $n>0$ is given by (211), while the analytic continuation for negative values of $n$,

$$
\begin{equation*}
u_{n}(T)=3 c B_{1}\left(n, T, f_{0}\right)-c B_{2}\left(n, T, f_{0}\right)-h B_{1}\left(n, T, \dot{f}_{0}\right)-\frac{h^{2}}{2 c} B_{1}\left(n, T, \ddot{f_{0}}\right), \quad n \leq 0 \tag{223}
\end{equation*}
$$

since the initial-condition integral vanishes. A careful limit calculation shows that the analytic continuation obtained converges to $u(-x, T)=f_{0}(T+x / c)$ as $h \rightarrow 0$.


Figure 20: (a) For $h=1 / 20$, the solution $u_{n}^{\mathrm{ac}}(t)$ given by (223) at $t=0.5$ and with $c=1$ obtained through analytic continuation, shown with the SDUTM solution $u_{n}^{\text {SDUTM }}(t)(211)$ and the continuum solution $u_{c}(x, t)$ given by (4). (b) The same plot for $h=1 / 150$.

### 6.2.1 Examples

Figure 20 illustrates the semi-discrete solutions for $n \in \mathbb{Z}$ for the IBVP (210) with boundary data given by (209).

## 7 The discretized heat equation, Dirichlet boundary conditions, centered discretization

We examine a discretization of (1c $)^{2}$. Discretizing the spacial derivative $u_{x x}$ with the standard centered stencil gives

$$
\begin{equation*}
\dot{u}_{n}(t)=\frac{u_{n-1}(t)-2 u_{n}(t)+u_{n+1}(t)}{h^{2}} \tag{224}
\end{equation*}
$$

with dispersion relation

$$
\begin{equation*}
W(k)=\frac{2-e^{i k h}-e^{-i k h}}{h^{2}} . \tag{225}
\end{equation*}
$$

The solution [4] to this semi-discrete problem is
$u_{n}(T)=\frac{1}{2 \pi} \int_{-\pi / h}^{\pi / h} e^{i k n h} e^{-W T}[\hat{u}(k, 0)-\hat{u}(-k, 0)] d k-\frac{i}{\pi h} \int_{-\pi / h}^{\pi / h} e^{i k n h} e^{-W T} \sin (k h) F_{0} d k, \quad n>0$,
where $\hat{u}(k, 0)$ and $F_{0}(W, T)$ are defined in (187) and (188) respectively. When evaluated for $n \leq 0$, this representation of the solution gives $u_{-n}(T)=-u_{n}(T)$ for $n \geq 1$ and $u_{0}(T)=0$. In the continuum limit, (226) converges to

$$
\begin{equation*}
u(x, T)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{i k x} e^{-\tilde{W} T}[\hat{u}(k, 0)-\hat{u}(-k, 0)] d k-\frac{i}{\pi} \int_{-\infty}^{\infty} e^{i k x} e^{-\tilde{W} T} k \tilde{F}_{0} d k \tag{227}
\end{equation*}
$$

[^0]with $\tilde{W}=k^{2}$ and $u(-x, T)=-u(x, T)$ for $x>0$.
We leave the first integral term as is, and introduce
\[

$$
\begin{equation*}
B(n, a):=\frac{1}{2 \pi i} \oint_{|z|=1} \exp \left[\frac{a}{2}\left(z+\frac{1}{z}\right)\right] \frac{1}{z^{n+1}} d z \equiv \sum_{\ell=0}^{\infty} \frac{1}{\ell!\Gamma(\ell+n+1)}\left(\frac{a}{2}\right)^{2 \ell+n} \tag{228}
\end{equation*}
$$

\]

the modified Bessel function of the first kind [7] with vital properties

$$
\begin{equation*}
B(-n, a)=B(n, a) \text { for } n \in \mathbb{Z}, \quad \text { and } \quad B(n, a)=\frac{2(n+1)}{a} B(n+1, a)+B(n+2, a) \tag{229}
\end{equation*}
$$

For the second term of (226), we have

$$
\begin{equation*}
\frac{i}{\pi h} \int_{-\pi / h}^{\pi / h} e^{i k n h} e^{-W T} \sin (k h) F_{0} d k=\frac{i}{\pi h} \int_{0}^{T} J(n, T-t) f_{0}(t) d t \tag{230}
\end{equation*}
$$

with

$$
\begin{equation*}
J(n, T)=\int_{-\pi / h}^{\pi / h} e^{i k n h} e^{-W T} \sin (k h) d k=\frac{\pi}{i h} I_{1}(n, T,-1)-\frac{\pi}{i h} I_{1}(n, T, 1) \tag{231}
\end{equation*}
$$

where

$$
\begin{align*}
I_{1}(n, T, m) & =\frac{h}{2 \pi} \int_{-\pi / h}^{\pi / h} e^{i k(n-m) h} e^{-W T} d k=\frac{1}{2 \pi i} \oint_{|z|=1} z^{n-m-1} \exp \left[-\left(\frac{2-z-z^{-1}}{h^{2}}\right) T\right] \\
& =e^{-2 T / h^{2}} B\left(m-n, \frac{2 T}{h^{2}}\right) \tag{232}
\end{align*}
$$

so that, using (229),

$$
\begin{equation*}
J(n, T)=\frac{\pi e^{-2 T / h^{2}}}{i h}\left[B\left(-1-n, \frac{2 T}{h^{2}}\right)-B\left(1-n, \frac{2 T}{h^{2}}\right)\right]=\frac{-n h \pi e^{-2 T / h^{2}}}{i T} B\left(n, \frac{2 T}{h^{2}}\right) \tag{233}
\end{equation*}
$$

This allows for a rewrite of the second term and (226) becomes

$$
\begin{equation*}
u_{n}(T)=\frac{1}{2 \pi} \int_{-\pi / h}^{\pi / h} e^{i k n h} e^{-W T}[\hat{u}(k, 0)-\hat{u}(-k, 0)] d k+n \int_{0}^{T} \frac{e^{-2(T-t) / h^{2}}}{T-t} B\left(n, \frac{2(T-t)}{h^{2}}\right) f_{0}(t) d t \tag{234}
\end{equation*}
$$

As before, (234) gives $u_{-n}(T)=-u_{n}(T)$ for $n \geq 1$ and $u_{0}(T)=0$.
We denote the second term of (234) by $K(n, T)$. Using the transformation $s=2(T-t) / h^{2}$ and
a Taylor series expansion about $h=0$,

$$
\begin{align*}
K(n, T) & =n \int_{0}^{2 T / h^{2}} \frac{e^{-s}}{s} B(n, s) f_{0}\left(T-\frac{h^{2}}{2} s\right) d s \\
& =n \sum_{p=0}^{\infty} \frac{f_{0}^{(p)}(T)(-1)^{p}}{p!}\left(\frac{h^{2}}{2}\right)^{p} \int_{0}^{2 T / h^{2}} e^{-s} B(n, s) s^{p-1} d s \\
& =n \sum_{p=0}^{\infty} \frac{f_{0}^{(p)}(T)(-1)^{p}}{p!}\left(\frac{h^{2}}{2}\right)^{p} \sum_{\ell=0}^{\infty} \frac{1}{\ell!\Gamma(\ell+n+1) 2^{2 \ell+n}} \int_{0}^{2 T / h^{2}} e^{-s} s^{2 \ell+n+p-1} d s \\
& =\frac{n}{2^{n}} \sum_{p=0}^{\infty} \frac{f_{0}^{(p)}(T)(-1)^{p}}{p!}\left(\frac{h^{2}}{2}\right)^{p} \sum_{\ell=0}^{\infty} \frac{1}{\ell!\Gamma(\ell+n+1) 2^{2 \ell}} \gamma\left(2 \ell+n+p, \frac{2 T}{h^{2}}\right) . \tag{235}
\end{align*}
$$

To evaluate at any $n \in \mathbb{Z}$, we consider the limit

$$
\begin{equation*}
K(n, T)=\frac{n}{2^{n}} \sum_{p=0}^{\infty} \frac{u^{(p)}(T)(-1)^{p}}{p!}\left(\frac{h^{2}}{2}\right)^{p} \sum_{\ell=0}^{\infty} \frac{1}{\ell!2^{2 \ell}} \lim _{\alpha \rightarrow n} \frac{\gamma\left(2 \ell+\alpha+p, \frac{2 T}{h^{2}}\right)}{\Gamma(\ell+\alpha+1)}, \tag{236}
\end{equation*}
$$

where (198) and the Taylor series of the incomplete gamma function [7] with $y=2 T / h^{2}$ give

$$
\begin{align*}
\frac{\gamma(2 \ell+n+p, y)}{\Gamma(\ell+n+1)} & =\frac{1}{\Gamma(\ell+n+1)} \sum_{k=0}^{\infty} \frac{(-1)^{k} y^{2 \ell+n+p+k}}{k!\Gamma(1+2 \ell+n+p+k)} \cdot \Gamma(2 \ell+n+p+k) \\
& =\frac{\Gamma(1-n-\ell-1)}{(-1)^{\ell+1} \Gamma(1-n) \Gamma(n)} \sum_{k=0}^{\infty} \frac{(-1)^{k} y^{2 \ell+n+p+k}}{k!\Gamma(1+2 \ell+n+p+k)} \cdot \frac{(-1)^{2 \ell+p+k} \Gamma(1-n) \Gamma(n)}{\Gamma(1-2 \ell-n-p-k)} \\
& =(-1)^{\ell+p+1} \Gamma(-n-\ell) \cdot \frac{y^{2 \ell+n+p-2 \ell-n-p}}{(-2 \ell-n-p)!}=\frac{(-1)^{\ell+p+1} \Gamma(-n-\ell)}{\Gamma(1-2 \ell-n-p)}, \tag{237}
\end{align*}
$$

so that

$$
\lim _{\alpha \rightarrow n} \frac{\gamma\left(2 \ell+\alpha+p, \frac{2 T}{h^{2}}\right)}{\Gamma(\ell+\alpha+1)}= \begin{cases}\frac{\gamma\left(2 \ell+n+p, \frac{2 T}{h^{2}}\right)}{\Gamma(\ell+n+1)}, & 2 \ell+p \geq 1-n \text { and } \ell \geq-n  \tag{238}\\ \frac{(-1)^{\ell+p+1} \Gamma(-n-\ell)}{\Gamma(1-n-2 \ell-p)}, & \ell \leq-n-1 \quad \text { and } 2 \ell+p \leq-n \\ 0, & \text { otherwise }\end{cases}
$$

For brevity, define

$$
\begin{equation*}
S(n, T, \ell, p)=\frac{f_{0}^{(p)}(T)(-1)^{p}}{p!}\left(\frac{h^{2}}{2}\right)^{p} \frac{1}{\ell!2^{2 \ell}} \lim _{\alpha \rightarrow n} \frac{\gamma\left(2 \ell+\alpha+p, \frac{2 T}{h^{2}}\right)}{\Gamma(\ell+\alpha+1)}, \tag{239}
\end{equation*}
$$

so that splitting the $\ell$ and $p$ sums gives

$$
\begin{align*}
K(n, T)= & \frac{n}{2^{n}} \sum_{p=0}^{\infty} \sum_{\ell=0}^{\infty} S(n, T, \ell, p) \\
= & \frac{n}{2^{n}} \sum_{\ell=0}^{-n-1} \sum_{p=0}^{-n-2 \ell} S(n, T, \ell, p)+\frac{n}{2^{n}} \sum_{\ell=\max (-n, 0)}^{\infty} \sum_{p=0}^{-n-2 \ell} S(n, T, \ell, p) \\
& +\frac{n}{2^{n}} \sum_{\ell=0}^{-n-1} \sum_{p=\max (-n-2 \ell+1,0)}^{\infty} S(n, T, \ell, p)+\frac{n}{2^{n}} \sum_{\ell=\max (-n, 0)}^{\infty} \sum_{p=\max (-n-2 \ell+1,0)}^{\infty} S(n, T, \ell, p) . \tag{240}
\end{align*}
$$

Consider $n<0$, so that the $\ell$-indexed sum from the second pair of sums begins at $\ell=-n$, from which the upper bound of the $p$-indexed sum is $-n-2 \ell=n<0$. Since the starting index is $p=0$, this second pair of sums does not contribute for $n<0$. Now consider $n \geq 0$, so that $\ell=0$ is the starting index for the first sum, from which $-n-2 \ell=-n \leq 0$ is the upper bound of the $p$-indexed sum. Thus, the second pair of sums does not contribute for any $n \in \mathbb{Z}$. The third pair of sums likewise vanishes for all $n \in \mathbb{Z}$, since $S(n, T, \ell, p)=0$ for these ranges of $\ell$ and $p$, regardless of $\lim _{\alpha \rightarrow n} \gamma\left(2 \ell+\alpha+p, 2 T / h^{2}\right) / \Gamma(\ell+\alpha+1)$. We have

$$
\begin{align*}
K(n, T)= & \frac{-n}{2^{n}} \sum_{\ell=0}^{-n-1} \frac{(-1)^{\ell}}{\ell!2^{2 \ell}} \sum_{p=0}^{-n-2 \ell} \frac{f_{0}^{(p)}(T)}{p!}\left(\frac{h^{2}}{2}\right)^{p} \frac{\Gamma(-n-\ell)}{\Gamma(1-n-2 \ell-p)} \\
& +\frac{n}{2^{n}} \sum_{\ell=\max (-n, 0)}^{\infty} \frac{1}{\ell!2^{2 \ell}} \sum_{p=\max (-n-2 \ell+1,0)}^{\infty} \frac{f_{0}^{(p)}(T)(-1)^{p}}{p!}\left(\frac{h^{2}}{2}\right)^{p} \frac{\gamma\left(2 \ell+n+p, \frac{2 T}{h^{2}}\right)}{\Gamma(\ell+n+1)} . \tag{241}
\end{align*}
$$

In the first coupled sum, we relax the upper bound of the $p$-indexed sum (adding zeros), allowing us to interchange the sums:

$$
\begin{align*}
& \frac{-n}{2^{n}} \sum_{\ell=0}^{-n-1} \frac{(-1)^{\ell}}{\ell!2^{2 \ell}} \sum_{p=0}^{\infty} \frac{f_{0}^{(p)}(T)}{p!}\left(\frac{h^{2}}{2}\right)^{p} \frac{\Gamma(-n-\ell)}{\Gamma(1-n-2 \ell-p)} \\
& =\frac{-n}{2^{n}} \sum_{p=0}^{\infty} \frac{f_{0}^{(p)}(T)}{p!}\left(\frac{h^{2}}{2}\right)^{p-n-1} \sum_{\ell=0} \frac{(-1)^{\ell}}{\ell!2^{2 \ell}} \frac{\Gamma(-n-\ell)}{\Gamma(1-n-2 \ell-p)}=\frac{-n}{2^{n}} \sum_{p=0}^{\infty} f_{0}^{(p)}(T)\left(\frac{h^{2}}{2}\right)^{p} L(n, p), \tag{242}
\end{align*}
$$

with

$$
\begin{align*}
L(n, p) & =\frac{1}{p!} \sum_{\ell=0}^{(-n-p) / 2} \frac{(-1)^{\ell}}{\ell!2^{2 \ell}} \frac{\Gamma(-n-\ell)}{\Gamma(1-n-2 \ell-p)}=\frac{2^{n+p}(-1)^{n+p} \Gamma(1-n) \Gamma(n) \Gamma(2 p-n-p)}{p!\Gamma(1-p) \Gamma(2 p) \Gamma(1-n-p)} \\
& =\frac{2^{n+p+1} \Gamma(p-n)}{\Gamma(1-n-p) \Gamma(2 p+1)} \tag{243}
\end{align*}
$$

where the last equality follows from (198). Substituting $L(n, p)$ and truncating the $p$-indexed sum
from $p=0$ to $p=-n$,

$$
\begin{align*}
K(n, T) & =-2 n \sum_{p=0}^{-n} \frac{f_{0}^{(p)}(T) h^{2 p} \Gamma(p-n)}{\Gamma(1-n-p) \Gamma(2 p+1)} \\
& +\frac{n}{2^{n}} \sum_{\ell=\max (-n, 0)}^{\infty} \frac{1}{\ell!2^{2 \ell}} \sum_{p=\max (-n-2 \ell+1,0)}^{\infty} \frac{f_{0}^{(p)}(T)(-1)^{p}}{p!}\left(\frac{h^{2}}{2}\right)^{p} \frac{\gamma\left(2 \ell+n+p, \frac{2 T}{h^{2}}\right)}{\Gamma(\ell+n+1)} \tag{244}
\end{align*}
$$

so that (234) is written as

$$
\begin{align*}
u_{n}(T)= & \frac{1}{2 \pi} \int_{-\pi / h}^{\pi / h} e^{i k n h} e^{-W T}[\hat{u}(k, 0)-\hat{u}(-k, 0)] d k-2 n \sum_{p=0}^{-n} \frac{f_{0}^{(p)}(T) h^{2 p} \Gamma(p-n)}{\Gamma(1-n-p) \Gamma(2 p+1)} \\
& +\frac{n}{2^{n}} \sum_{\ell=\max (-n, 0)}^{\infty} \frac{1}{\ell!2^{2 \ell}} \sum_{p=\max (-n-2 \ell+1,0)}^{\infty} \frac{f_{0}^{(p)}(T)(-1)^{p}}{p!}\left(\frac{h^{2}}{2}\right)^{p} \frac{\gamma\left(2 \ell+n+p, \frac{2 T}{h^{2}}\right)}{\Gamma(\ell+n+1)} . \tag{245}
\end{align*}
$$

For $n>0$, the second term above does not contribute and the remaining terms are rewritten as in the original representation (226). For $n \leq 0$,

$$
\begin{align*}
u_{n}(T)= & \frac{1}{2 \pi} \int_{-\pi / h}^{\pi / h} e^{i k n h} e^{-W T}[\hat{u}(k, 0)-\hat{u}(-k, 0)] d k-2 n \sum_{p=0}^{-n} \frac{f_{0}^{(p)}(T) h^{2 p} \Gamma(p-n)}{\Gamma(1-n-p) \Gamma(2 p+1)} \\
& +\frac{n}{2^{n}} \sum_{\ell=-n}^{\infty} \frac{1}{\ell!2^{2 \ell}} \sum_{p=\max (-n-2 \ell+1,0)}^{\infty} \frac{f_{0}^{(p)}(T)(-1)^{p}}{p!}\left(\frac{h^{2}}{2}\right)^{p} \frac{\gamma\left(2 \ell+n+p, \frac{2 T}{h^{2}}\right)}{\Gamma(\ell+n+1)} \\
= & \frac{-1}{2 \pi} \int_{-\pi / h}^{\pi / h} e^{-i k n h} e^{-W T}[\hat{u}(k, 0)-\hat{u}(-k, 0)] d k-2 n \sum_{p=0}^{-n} \frac{f_{0}^{(p)}(T) h^{2 p} \Gamma(p-n)}{\Gamma(1-n-p) \Gamma(2 p+1)} \\
& +\frac{n}{2^{-n}} \sum_{\ell=0}^{\infty} \frac{1}{\ell!2^{2 \ell}} \sum_{p=0}^{\infty} \frac{f_{0}^{(p)}(T)(-1)^{p}}{p!}\left(\frac{h^{2}}{2}\right)^{p} \frac{\gamma\left(2 \ell-n+p, \frac{2 T}{h^{2}}\right)}{\Gamma(\ell-n+1)} \tag{246}
\end{align*}
$$

or

$$
\begin{equation*}
u_{n}(T)=-2 n \sum_{p=0}^{-n} \frac{f_{0}^{(p)}(T) h^{2 p} \Gamma(p-n)}{\Gamma(1-n-p) \Gamma(2 p+1)}-u_{-n}(T), \quad n \leq 0 \tag{247}
\end{equation*}
$$

To recover the boundary condition at $n=0$, we extract the first term of the sum above:

$$
\begin{align*}
u_{n}(T) & =2 n\left[\frac{f_{0}(T)}{n}+\sum_{p=1}^{-n} \frac{f_{0}^{(p)}(T) h^{2 p} \Gamma(p-n)}{\Gamma(1-n-p) \Gamma(2 p+1)}\right]-u_{-n}(T) \\
& =2 f_{0}(T)+2 n \sum_{p=1}^{-n} \frac{f_{0}^{(p)}(T) h^{2 p} \Gamma(p-n)}{\Gamma(1-n-p) \Gamma(2 p+1)}-u_{-n}(T) \tag{248}
\end{align*}
$$

so that for $n=0, u_{0}(T)=2 f_{0}(T)-u_{0}(T)=f_{0}(T)$.

### 7.1 Continuum Limit

The analytic extension (21) of the continuous solution to the IBVP (1) is

$$
\begin{equation*}
u(x, T)=2 \sum_{p=0}^{\infty} \frac{f_{0}^{(p)}(T)}{(2 p)!} x^{2 p}-u(-x, T), \quad x>0 \tag{249}
\end{equation*}
$$

It is possible to recover (249) from the continuum limit of (247) by noting that

$$
f(n, p)=\frac{n \Gamma(p+n)}{\Gamma(1+n-p)}=\prod_{\ell=0}^{p-1}(n-\ell)(n+\ell)= \begin{cases}1, & p=0  \tag{250}\\ \sum_{\ell=1}^{p} a_{\ell} n^{2 \ell}, & p \geq 1\end{cases}
$$

and $f(n, p)$ is a polynomial in $n$ of degree $2 p$ with leading coefficient $a_{p}=1$. Hence,

$$
\begin{align*}
u_{n}(T) & =2 \sum_{p=0}^{-n} \frac{f_{0}^{(p)}(T) h^{2 p}}{(2 p)!} f(-n, p)-u_{-n}(T) \\
& =2 \sum_{p=0}^{-n} \frac{f_{0}^{(p)}(T)}{(2 p)!}\left[(n h)^{2 p}+a_{p-1} h^{2}(n h)^{2 p-2}+\ldots\right]-u_{-n}(T) \tag{251}
\end{align*}
$$

which gives the desired continuum limit.

### 7.1.1 Examples

We solve the IBVP

$$
\begin{cases}u_{t}=u_{x x}, & x>0, t>0  \tag{252}\\ u(x, 0)=3 x e^{-x}, & x>0 \\ u(0, t)=\sin (4 \pi t), & t>0\end{cases}
$$

for all $n \in \mathbb{Z}$. The results are presented in Figure 21.

## 8 Conclusion

We have demonstrated how the solutions of linear, constant-coefficient IBVPs can be analytically extended outside their original spatial domain of definition, using the UTM as a method for doing so. A number of representative examples were used to illustrate our approach. The results most useful for computational purposes for these examples are collected here. The reader should refer to the sections where these examples are treated in detail for additional results, and for the introduction of the notation used below.

- The heat equation on $x>0$ with Dirichlet boundary conditions, see (1). The solution for $x \in \mathbb{R}$ may be represented as

$$
\begin{equation*}
u(x, t)=I_{0}(x, t)+I_{f_{0}}^{\mathrm{ext}}(x, t) \tag{253}
\end{equation*}
$$

where $I_{0}(x, t)$ and $I_{f_{0}}^{\text {ext }}(x, t)$ are defined in (7) and (21), respectively.


Figure 21: (a) For $h=1 / 20$, the solution $u_{n}^{\text {ac }}(t)$ given by (247) at $t=0.5$ obtained through analytic continuation, shown with the SDUTM solution $u_{n}^{\operatorname{SDUTM}}(t)(226)$ and the continuum solution $u_{c}(x, t)$ given by (253). (b) The same plot for $h=1 / 100$.

- The heat equation on $x>0$ with Neumann boundary conditions, see (36). The solution for $x \in \mathbb{R}$ may be represented as

$$
\begin{equation*}
u(x, t)=I_{0}(x, t)+I_{f_{1}}^{\operatorname{ext}}(x, t) \tag{254}
\end{equation*}
$$

where $I_{0}(x, t)$ and $I_{f_{1}}^{\text {ext }}(x, t)$ are defined in (38) and (46), respectively.

- The heat equation on $x \in(0, L)$ with Dirichlet boundary conditions, see ( 52 ). The solution for $x \in \mathbb{R}$ may be represented as

$$
\begin{equation*}
u(x, t)=I_{0}(x, t)+I_{f_{0}}^{\mathrm{ext}}(x, t)+I_{g_{0}}^{\mathrm{ext}}(x, t) \tag{255}
\end{equation*}
$$

where $I_{0}(x, t), I_{f_{0}}^{\text {ext }}(x, t)$ and $I_{g_{0}}^{\text {ext }}(x, t)$ are defined in $(54),(68)$ and (75), respectively.

- The advected heat equation on $x>0$ with Dirichlet boundary conditions, see (81). The solution for $x \in \mathbb{R}$ may be represented as

$$
\begin{equation*}
u(x, t)=I_{0}(x, t)+I_{f_{0}}^{\mathrm{ext}}(x, t), \tag{256}
\end{equation*}
$$

where $I_{0}(x, t)$ and $I_{f_{0}}^{\text {ext }}(x, t)$ are defined in (83) and (93), respectively.

- The linear KdV equation $u_{t}+u_{x x x}=0$ on $x>0$ with Dirichlet boundary conditions, see (110). The solution for $x \in \mathbb{R}$ may be represented as

$$
\begin{equation*}
u(x, t)=I_{0}(x, t)+I_{f_{0}}^{\mathrm{ext}}(x, t), \tag{257}
\end{equation*}
$$

where $I_{0}(x, t)$ and $I_{f_{0}}^{\text {ext }}(x, t)$ are defined in (112) and (126), respectively.

- The linear KdV equation $u_{t}-u_{x x x}=0$ on $x>0$ with Dirichlet boundary conditions, see (141). The solution for $x \in \mathbb{R}$ may be represented as

$$
\begin{equation*}
u(x, t)=I_{0}(x, t)+I_{f_{0}}^{\mathrm{ext}}(x, t)+I_{f_{1}}^{\mathrm{ext}}(x, t) \tag{258}
\end{equation*}
$$

where $I_{0}(x, t)$ is defined in (143), $I_{f_{0}}^{\text {ext }}(x, t)$ and $I_{f_{1}}^{\text {ext }}(x, t)$ are defined in (165) and (166), respectively.

- The backward-discretized advection equation on $n>0$ with Dirichlet boundary conditions, see (184). The solution for $n \in \mathbb{Z}$ may be represented by (204).
- The second-order backward-discretized advection equation on $n>0$ with Dirichlet boundary conditions, see (210). The solution for $n \in \mathbb{Z}$ may be represented by (211) for $n>0$, and by (223) for $n<0$.
- The second-order centered-discretized heat equation on $n>0$ with Dirichlet boundary conditions, see (224). The solution for $n \in \mathbb{Z}$ may be represented by (226) for $n>0$, and by (247) for $n<0$.


## Appendix. The second-order centered discretized advection equation

This section demonstrates the discretization of (183) using the standard centered stencil, which is known to be a poor choice [13]. The SDUTM may still be used to solve the discretized system and the analytic continuation method still gives a continuation formula. However, since the method is entirely dispersive, it should not be surprising that undesirable dispersive behavior occurs.

Discretizing (183) using the standard centered stencil gives

$$
\begin{equation*}
\dot{u}_{n}(t)=c \frac{u_{n-1}(t)-u_{n+1}(t)}{2 h} \tag{259}
\end{equation*}
$$

with dispersion relation

$$
\begin{equation*}
W(k)=c \frac{e^{i k h}-e^{-i k h}}{2 h} \tag{260}
\end{equation*}
$$

and nontrivial symmetry $\nu_{1}(k)=-k-\pi / h$. The semi-discrete solution is

$$
\begin{equation*}
u_{n}(T)=\frac{1}{2 \pi} \int_{-\pi / h}^{\pi / h} e^{i k n h} e^{-W T}\left[\hat{u}(k, 0)-\hat{u}\left(\nu_{1}, 0\right)\right] d k+\frac{c}{2 \pi} \int_{-\pi / h}^{\pi / h} e^{i k n h} e^{-W T} \cos (k h) F_{0}(W, T) d k \tag{261}
\end{equation*}
$$

Using this representation, it follows that $u_{-n}(T)=(-1)^{n+1} u_{n}(T)$ for $n \in \mathbb{Z}^{+}$. Repeating similar steps as before, the correct analytic extension satisfies

$$
\begin{equation*}
u_{-n}(T)=(-1)^{n+1} u_{n}(T)+n \sum_{\substack{k=0 \\ n-k \text { even }}}^{n} f_{0}^{(k)}(T)\left(\frac{2 h}{c}\right)^{k} \frac{\Gamma\left(\frac{n+k}{2}\right)}{k!\left(\frac{n-k}{2}\right)!}, \quad n>0 \tag{262}
\end{equation*}
$$

Due to the purely dispersive nature of the discretization (259), if the boundary function $f_{0}(t)$ and the initial condition $\phi(x)$ are not compatible (i.e., $f_{0}(t) \neq \phi(-c t)$ ), this solution exhibits a
dispersive fan, see Figure 22a and 23a. The dispersive fan seen in this figure travels upwind at speed $c$. Its envelope is determined by a combination of the initial and the boundary conditions, see (264). For $c<0$, (183) does not have a boundary condition, but the discrete solution (261) requires one. If we consider the discretized system in its own right and apply an incompatible boundary condition, we observe the dispersive fan for $n>0$, demonstrating the appearance of the dispersive fan is not an analytic continuation issue, but one of the discretization itself. If we choose the compatible boundary condition $f_{0}(t)=\phi(-c t)$ with analytic $\phi(x)$, we do not see the dispersive fan, see Figure 23b. In general, discontinuous initial data $\phi(x)$ gives rise to dispersive effects (as in [14], see Figure 24), further demonstrating that (259) is an inappropriate discretization [13] and that the dispersive fan in Figure 22a and 23a is not caused by discontinuity or the analytic continuation formula (262), but by the incompatibility of the boundary and initial data in conjunction with the dispersive nature of (259).

It is interesting to note that one can "average out" the dispersive fan wave to get the extension formula

$$
\begin{equation*}
u_{-n}(T)=\frac{n}{2} \sum_{k=0}^{n} f_{0}^{(k)}(T)\left(\frac{2 h}{c}\right)^{k} \frac{\Gamma\left(\frac{n+k}{2}\right)}{k!\left(\frac{n-k}{2}\right)!} \tag{263}
\end{equation*}
$$

which is second-order accurate, but does not solve the discretized equation (259). Considering (262) in its continuum limit,

$$
\begin{align*}
u_{-n}(T) & \sim(-1)^{n+1} u_{n}(T)+2 \sum_{\substack{k=0 \\
n-k \text { even }}}^{n} \frac{f_{0}^{(k)}(T)}{k!}\left(-\frac{x}{c}\right)^{k} \\
& \sim(-1)^{n+1} u_{n}(T)+f_{0}\left(T-\frac{x}{c}\right)+(-1)^{n} f_{0}\left(T+\frac{x}{c}\right) \tag{264}
\end{align*}
$$

and since for $x<0$,

$$
u_{n}(T) \rightarrow \begin{cases}f_{0}(T+x / c), & x>-c T  \tag{265}\\ \phi(-x-c t), & x<-c T\end{cases}
$$

which has a limit as $n \rightarrow \infty\left(h \rightarrow 0^{+}\right.$and $\left.n h \rightarrow-x\right)$ if and only if $u_{n}(T)=f_{0}(T+x / c)$, i.e., if the initial and boundary conditions are compatible. Further, these previous two equations give the envelope of the dispersive fan.

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Figure 22: For $c=1$ : (a) The SDUTM solution (261) for incompatible boundary and initial conditions given by (209), shown with the analytic continuation given by (262) and the continuum solution $u_{c}(x, t)$ given by (4). (b) The SDUTM solution, (261), for initial condition (209) and compatible boundary conditions, shown with the analytic continuation given by (262) and the continuum solution $u_{c}(x, t)$ given by (4).
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Figure 23: For $c=-1$ : (a) The SDUTM solution (261) for incompatible boundary and initial conditions given by (209), shown with the analytic continuation given by (262) and the continuum solution $u_{c}(x, t)$ given by (4). (b) The SDUTM solution, (261), for initial condition (209) and compatible boundary conditions, shown with the analytic continuation given by (262) and the continuum solution $u_{c}(x, t)$ given by (4).
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Figure 24: The centered discretization with step function initial data, resulting in Gibbs phenomenon-like behavior for (a) $c=1$, and (b) $c=-1$.


[^0]:    ${ }^{2}$ The calculations for the Neumann problem are similar, and the details are left to the reader. The main difference with the Dirichlet problem is the role played by the discretization of the Neumann conditions.

