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Elliptic solutions of the defocusing NLS equation are stable

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Abstract

The stability of the stationary periodic solutions of the integrable (one-dimensional, cubic) defocusing nonlinear Schrödinger (NLS) equation is reasonably well understood, especially for solutions of small amplitude. In this paper, we exploit the integrability of the NLS equation to establish the spectral stability of all such stationary solutions, this time by explicitly computing the spectrum and the corresponding eigenfunctions associated with their linear stability problem. An additional argument using an appropriate Krein signature allows us to conclude the (nonlinear) orbital stability of all stationary solutions of the defocusing NLS equation with respect to so-called subharmonic perturbations: perturbations that have period equal to an integer multiple of the period of the amplitude of the solution. All results presented here are independent of the size of the amplitude of the solutions and apply equally to solutions with trivial and nontrivial phase profiles.

1. Introduction

The defocusing one-dimensional nonlinear Schrödinger equation with cubic nonlinearity is given by

$$i\Psi_t = -\tfrac{1}{2}\Psi_{xx} + \Psi\,|\Psi|^2\,. \tag{1}$$

Here $\Psi(x,t)$ is a complex-valued function, describing the slow modulation of a carrier wave in a dispersive medium. Due to both its physical relevance and its mathematical properties, (1) is one of the canonical equations of nonlinear dynamics. The equation has been used extensively to model, among other applications, waves in deep water [2, 39], propagation

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in nonlinear optics with normal dispersion [23, 28], Bose–Einstein condensates with repulsive self-interaction [21, 34] and electron plasma waves [12]. Equation (1) is completely integrable [1, 40]. This will be used extensively later on.

The equation has a large class of stationary solutions; these are written as

$$\Psi = \mathrm{e}^{-\imath\omega t}\phi(x), \tag{2}$$

where ω is a real constant. Among this class of solutions are the dark and grey solitons, for which $\phi(x)$ is expressed in terms of hyperbolic functions. These solutions may be regarded as limit cases of the so-called elliptic solutions studied in this paper. The stationary solutions (2) are either periodic or quasi-periodic as functions in x. The amplitude of $\phi(x)$ of the elliptic solutions is expressed in terms of Jacobi elliptic functions. A thorough discussion of the stationary solutions is found in, for instance, [9]. The details relevant to our investigations are presented in section 2.

The stability analysis of the stationary solutions was begun in [39], where the now classical calculation for the modulational stability of the plane-wave solution ($\phi(x)$ constant) is given. The literature discussing the stability of the soliton solutions is extensive, see [29], and references therein. Rowlands [35] may have been the first to consider the stability of the elliptic solutions directly. He studied the spectral stability problem for these solutions using regular perturbation theory with the Floquet parameter as a small expansion parameter. At the origin in the spectral plane, this parameter is zero, thus Rowlands was able to obtain expressions for the different branches of the continuous spectrum near the origin. For the focusing NLS equation these calculations demonstrate that the spectrum lies partially in the right-half plane, which leads to the conclusion of instability. For the defocusing NLS equation (1), the first approximation to these branches lies on the imaginary axis, and Rowlands' method is inconclusive with regards to stability or instability of the elliptic solutions. More recently, the stability of the elliptic solutions has been examined by Gallay and Hărăguş [17, 18]. In [18], they established the spectral stability of small-amplitude solutions of the form (2) of (1), as well as their (nonlinear) orbital stability with respect to perturbations that are of the same period as $|\phi(x)|$. In [17], the restriction on the amplitude for the orbital stability result is removed. Hărăguş and Kapitula [22] put some of these results in a more general framework valid for spectral problems with periodic coefficients originating from Hamiltonian systems. They establish that the small-amplitude elliptic solutions investigated in [18] are not only spectrally but also *linearly* stable. Lastly, we should mention a recent paper by Ivey and Lafortune [26]. They undertake a spectral stability analysis of the cnoidal wave solution of the focusing NLS equation, by exploiting the squared-eigenfunction connection, like we do in [5] for the cnoidal wave solutions of the Korteweg–de Vries equation and here, see below. Their calculations use Floquet theory for the spatial Lax operator to construct an Evans function for the spectral stability problem, whose zeros give the point spectrum corresponding to periodic perturbations. They also obtain a description of the continuous spectrum (which contains this point spectrum) using a Floquet discriminant. Their description of the spectrum is explicit in the sense that no differential equations remain to be solved. By computing level curves of this Floquet discriminant numerically, they obtain a numerical description of the spectrum.

In this paper, we confirm the recent findings on spectral and orbital stability of the elliptic solutions of the defocusing equation and extend their validity to solutions of arbitrary amplitude. In addition, we extend the stability results to the class of so-called subharmonic perturbations, *i.e.* perturbations that are periodic with period equal to an integer multiple of the period of the amplitude $|\phi(x)|$. Further, exploiting the integrability of (1), we are able to provide an explicit analytic description of the spectrum and the eigenfunctions associated with the linear stability problem of *all* elliptic solutions. We follow the same method as in [5], using the algebraic connection between the eigenfunctions of the Lax pair of (1) and those of the spectral stability problem. This explicit characterization of the spectrum, as well as the extension of the spectral stability results to arbitrary amplitude, and the results involving subharmonic perturbations are new. It appears that the methods of Ivey and Lafortune [26] allow for an equally explicit description when applied to the defocusing case. They rely on the general theory of hyperelliptic Riemann surfaces and theta functions, which are restricted to the elliptic case, through a nontrivial reduction process. We never leave the realm of elliptic functions, resulting in a significantly more straightforward approach. The explicit characterization of the spectrum is an obvious starting point for the stability analysis of more general solutions to non-integrable generalizations of the NLS equations, such as the two-dimensional NLS equation [10, 11] or one-dimensional perturbations of the NLS equation which might include such effects as dissipation or external potentials, see *e.g.*, [7, 26]. As in [17, 18, 22], we prove the spectral stability of the elliptic solutions of (1), without imposing a restriction on the amplitude. The results of [22] allow us to prove the completeness of the eigenfunctions of the linear stability problem, resulting in a conclusion of linear stability. Similarly to the last section of [13], we employ an appropriate Krein signature calculation to allow us to invoke the classical results of Grillakis, Shatah and Strauss [20], from which (nonlinear) orbital stability follows.

It should be emphasized that our results are equally valid for elliptic solutions that have trivial phase ($\phi(x)$ real) as for solutions with a non-trivial phase profile ($\phi(x)$ not purely real). Similar calculations to the ones presented here apply to the focusing NLS equation, without the conclusion of stability, of course. That case is more complicated, due to the Lax operator associated with that integrable equation not being self adjoint. It will be presented separately elsewhere.

Before entering the main body of the paper, we wish to apologize to the reader for the use of no less than three different incarnations of the NLS equation, in addition to (1). One is obtained through a scaling transformation with a time-dependent exponential factor, to allow the stationary solutions to appear as equilibrium solutions. The second one is used to facilitate our proof of spectral stability and involves a time- and space-dependent exponential factor. The last NLS form writes the second one in terms of its real and imaginary parts, and is useful for our proof of orbital stability. All forms are introduced because we benefit greatly from their use. None are new to the literature. It does not appear straighforward to avoid the use of any of them without much added complication.

2. Elliptic solutions of the defocusing NLS equation

The results of this section are presented in more detail in [9]. We restrict our considerations to the bare necessities for what follows.

Stationary solutions (2) of (1) satisfy the ordinary differential equation

$$\omega\phi = -\tfrac{1}{2}\phi_{xx} + \phi|\phi|^2. \tag{3}$$

Substituting an amplitude-phase decomposition

$$\phi(x) = R(x)\,\mathrm{e}^{i\theta(x)} \tag{4}$$

in (3), we find ordinary differential equations satisfied by the amplitude $R(x)$ and the phase $\theta(x)$ by separating real and imaginary parts, after factoring out the overall exponential factor. Here we explicitly use that both amplitude and phase are real-valued functions. The equation for the phase $\theta(x)$ is easily solved in terms of the amplitude. One finds

$$\theta(x) = c\int_0^x \frac{1}{R^2(y)}\,\mathrm{d}y. \tag{5}$$

Here c is a constant of integration. Using standard methods for elliptic differential equations (see for instance [8, 30]), one shows that the amplitude $R(x)$ is given by

$$R^2(x) = k^2 \mathrm{sn}^2(x, k) + b, \tag{6}$$

where $\mathrm{sn}(x, k)$ is the Jacobi elliptic sine function, and $k \in [0, 1)$ is the elliptic modulus [8, 30]. The amplitude $R(x)$ is periodic with period $T(k) = 2K(k)$, where $K(k)$ is the complete elliptic integral of the first kind [8, 30]:

$$K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 y}} \mathrm{d}y. \tag{7}$$

The form of the solution (6) leads to

$$\omega = \tfrac{1}{2}(1 + k^2) + \tfrac{3}{2}b, \tag{8}$$

and

$$c^2 = b(b + 1)(b + k^2). \tag{9}$$

Conditions on the reality of the amplitude and phase lead to the constraint $b \in \mathbb{R}^+$ (including zero) on the offset parameter. The class of solutions constructed here is not the most general class of stationary solutions of (1). We did not specify the full class of parameters allowed by the Lie point symmetries of (1), which allow for a scaling in x, multiplying by a unitary constant, etc. The methods introduced in the remainder of this paper apply equally well and with similar results to the full class of stationary elliptic solutions.

If the constant c is zero, the solution is referred to as a trivial-phase solution. Otherwise it is called a nontrivial-phase solution. It is clear from the above that the only trivial-phase solutions are (up to symmetry transformations)

$$\Psi(x, t) = k \mathrm{sn}(x, k)\, \mathrm{e}^{-\frac{\imath}{2}(1+k^2)t}. \tag{10}$$

This one-parameter family of solutions is found from the two-parameter family of stationary solutions by equating $b = 0$. The trivial-phase solutions are periodic in x. Their period is $4K(k)$. In contrast, the nontrivial-phase solutions are typically not periodic in x. The period of their amplitude is $T(k) = 2K(k)$, whereas the period $\tau(k)$ of their phase is determined by $\theta(\tau(k)) = 2\pi$. Unless $\tau(k)$ and $T(k)$ are rationally related, the nontrivial-phase solution is quasi-periodic instead of periodic.

This quasi-periodicity is more immediately obvious using a different form of the elliptic solutions (see [17, 18]), which will prove useful in section 6. We split the integrand of (5) as

$$\frac{c}{R^2(x)} = \kappa(k, b) + \mathcal{K}(x; k, b), \tag{11}$$

where $\kappa(k, b)$ is the average value of $c/R^2(x)$ over an interval of length $T(k)$. Thus the average value of $\mathcal{K}(x; k, b)$ is zero. Then the elliptic solutions may be written as

$$\Psi(x, t) = \mathrm{e}^{-\imath\omega t + \imath\kappa x} \hat{R}(x), \tag{12}$$

where $\hat{R}(x + T(k)) = \hat{R}(x)$ is typically not real. It is clear from this formulation of the elliptic solutions that they are generically quasiperiodic with two incommensurate spatial periods $T(k)$ and $2\pi/\kappa(k, b)$.

3. The linear stability problem

Before we study the orbital stability of the elliptic solutions, we examine their spectral and linear stability. To this end, we transform (1) so that the elliptic solutions are time-independent solutions of this new equation. Let

$$\Psi(x, t) = \mathrm{e}^{-\imath\omega t} \psi(x, t). \tag{13}$$

Then

$$\imath\psi_t = -\omega\psi - \tfrac{1}{2}\psi_{xx} + \psi|\psi|^2. \tag{14}$$

As stated, the elliptic solutions are those solutions for which $\psi_t \equiv 0$. Next, we consider perturbations of such an elliptic solution. Let

$$\psi(x,t) = \mathrm{e}^{\imath\theta(x)}\left(R(x) + \epsilon u(x,t) + \imath\epsilon v(x,t)\right) + \mathcal{O}(\epsilon^2), \tag{15}$$

where ϵ is a small parameter and $u(x, t)$ and $v(x, t)$ are real-valued functions. Since their dependence on both x and t is unrestricted, there is no loss of generality from factoring out the temporal and spatial phase factors. Substituting (15) into (1) and separating real and imaginary parts, the terms of zero order in ϵ vanish, since $R(x)\,\mathrm{e}^{\imath\theta(x)}$ solves (1). Next, we equate terms of order ϵ to zero and separate real and imaginary parts, resulting in

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{L}\begin{pmatrix} u \\ v \end{pmatrix} = J\begin{pmatrix} L_+ & S \\ -S & L_- \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{16}$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{17}$$

and the linear operators L_-, L_+ and S are defined by

$$L_- = -\frac{1}{2}\partial_x^2 + R^2(x) - \omega + \frac{c^2}{2R^4(x)}, \tag{18}$$

$$L_+ = -\frac{1}{2}\partial_x^2 + 3R^2(x) - \omega + \frac{c^2}{2R^4(x)}, \tag{19}$$

$$S = \frac{c}{R^2(x)}\partial_x - \frac{cR'(x)}{R^3(x)} = \frac{c}{R(x)}\partial_x\frac{1}{R(x)}. \tag{20}$$

We wish to show that perturbations u and v that are initially bounded remain so for all times. By ignoring terms of order ϵ^2 and higher we are restricting ourselves to linear stability. The elliptic solution $\phi(x) = R(x)\,\mathrm{e}^{\imath\theta(x)}$ is by definition *linearly stable* if for all $\varepsilon > 0$ there is a $\delta > 0$ such that if $||u(x,0) + \imath v(x,0)|| < \delta$ then $||u(x,t) + \imath v(x,t)|| < \varepsilon$ for all $t > 0$. It should be noted that this definition depends on the choice of the norm $||\cdot||$ of the perturbations. In the next section this norm will be specified. The linear stability problem (16) is written in its standard form to allow for a straightforward comparison with the results of other authors, see for instance [17, 18, 22, 35], and many references where only the soliton case is considered. Some of our calculations are more conveniently done using a different form of the linear stability problem (16) or the spectral stability problem (22, below). These forms will be introduced as necessary.

Since (16) is autonomous in t, we can separate variables and consider solutions of the form

$$\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} = \mathrm{e}^{\lambda t}\begin{pmatrix} U(x,\lambda) \\ V(x,\lambda) \end{pmatrix}, \tag{21}$$

so that the eigenfunction vector $(U(x,\lambda), V(x,\lambda))^T$ satisfies the spectral problem

$$\lambda\begin{pmatrix} U \\ V \end{pmatrix} = \mathcal{L}\begin{pmatrix} U \\ V \end{pmatrix} = J\begin{pmatrix} L_+ & S \\ -S & L_- \end{pmatrix}\begin{pmatrix} U \\ V \end{pmatrix}. \tag{22}$$

Since $-S$ is the Hermitian conjugate of S, this latter form of the spectral problem emphasizes the Hamiltonian structure of the problem. In what follows, we suppress the λ

dependence of U and V. In order to show that the solution $\phi(x) = R(x)\,e^{\imath\theta(x)}$ is *spectrally stable*, we need to verify that the spectrum $\sigma(\mathcal{L})$ does not intersect the open right-half of the complex λ plane. To avoid confusion with other spectra defined below, we refer to $\sigma(\mathcal{L})$ as the *stability spectrum* of the elliptic solution $\phi(x)$. Since the nonlinear Schrödinger equation (1) is Hamiltonian [2], the spectrum of its linearization is symmetric with respect to both the real and the imaginary axis [38], so proving the spectral stability of an elliptic solution is equivalent to proving the inclusion $\sigma(\mathcal{L}) \subset \imath\mathbb{R}$.

Spectral stability of an elliptic solution implies its linear stability if the eigenfunctions corresponding to the stability spectrum $\sigma(\mathcal{L})$ are complete in the space defined by the norm $\|\cdot\|$. In that case all solutions of (16) may be obtained as linear combinations of solutions of (22).

The first goal of this paper is to prove the spectral and linear stability of all solutions (2) by analytically determining the stability spectrum $\sigma(\mathcal{L})$, as well as its associated eigenfunctions. It is already known from [18] and [22] that the inclusion $\sigma(\mathcal{L}) \subset \imath\mathbb{R}$ holds for solutions of small amplitude, or, equivalently, solutions with small elliptic modulus, leading to spectral stability. We strengthen these results by providing a completely explicit description of $\sigma(\mathcal{L})$ and its eigenfunctions, without requiring any restriction on the elliptic modulus. To conclude the completeness of the eigenfunctions associated with $\sigma(\mathcal{L})$, and thus the linear stability of the elliptic solutions, we rely on the SCS lemma, see Hărăguş and Kapitula [22].

4. Numerical results

In the next few sections, we determine the spectrum of (22) analytically. Before we do so, we compute it numerically, using Hill's method [15]. Hill's method is ideally suited to a periodic-coefficient problem such as (22). It should be emphasized that almost none of the elliptic solutions are periodic in x, as discussed in section 2. Nevertheless, since we have factored out the exponential phase factor $e^{\imath\theta(x)}$ and the remaining coefficients are all expressed in terms of $R(x)$, the spectral problem (22) is a problem with periodic coefficients, even for elliptic solutions that are quasi-periodic.

Using Hill's method, we compute all eigenfunctions using the Floquet–Bloch decomposition

$$\begin{pmatrix} U(x) \\ V(x) \end{pmatrix} = e^{i\mu x}\begin{pmatrix} \hat{U}(x) \\ \hat{V}(x) \end{pmatrix}, \quad \hat{U}(x+T(k)) = \hat{U}(x), \quad \hat{V}(x+T(k)) = \hat{V}(x), \tag{23}$$

with $\mu \in [-\pi/2T(k), \pi/2T(k))$. It follows from Floquet's theorem [3] that *all* bounded solutions of (22) are of this form. Here bounded means that $\max_{x\in\mathbb{R}}\{|U(x)|, |V(x)|\}$ is finite. Thus

$$U, V \in C_b^0(\mathbb{R}). \tag{24}$$

By a similar argument as that given at the end of section 2, the typical eigenfunction (23) obtained this way is quasi-periodic, with periodic eigenfunctions ensuing when the two periods $T(k)$ and $2\pi/\mu$ are commensurate. Specifically, our investigations include perturbations of an arbitrary period that is an integer multiple of $T(k)$, i.e., subharmonic perturbations.

Figure 1 shows discrete approximations to the spectrum of (22), computed using SpectrUW 2.0 [14]. The solution parameters for the top two panels (*a*) and (*b*) are $b = 0$ (thus corresponding to a trivial-phase solution (10)) and $k = 0.8$. The numerical parameters (see [14, 15]) are $N = 20$ (41 Fourier modes) and $D = 40$ (39 different Floquet exponents). The right panel (*b*) is a blow-up of the left panel (*a*) around the origin. First, it appears

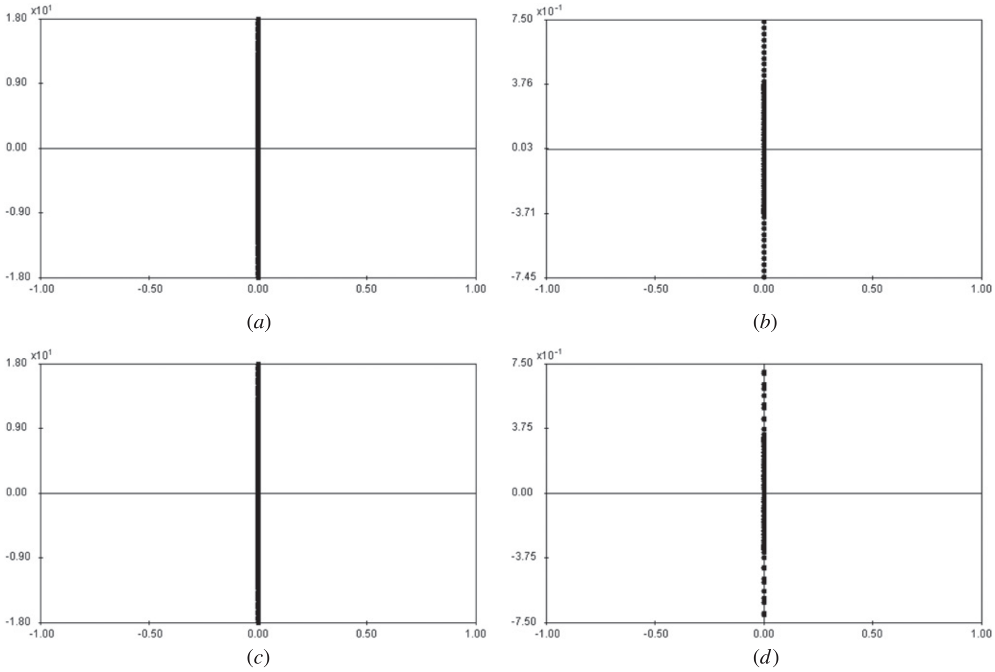

Figure 1. Numerically computed spectra (imaginary part of λ vs. real part of λ) of (22) for different solutions (2), with parameter values given below, using Hill's method with $N = 20$ (41 Fourier modes) and $D = 40$ (39 different Floquet exponents), see [14, 15]. (*a*) A trivial-phase sn-solution with $k = 0.5$. (*b*) A blow-up of (*a*) around the origin, showing a band of higher spectral density. (*c*) A nontrivial-phase solution with $b = 0.2$ and $k = 0.5$. (*d*) A blow-up of (*c*) around the origin, similarly showing a band of higher spectral density.

that the spectrum is on the imaginary axis[4], indicating spectral stability of the snoidal solution (10). Second, the numerics show that a symmetric band around the origin has a higher spectral density than does the rest of the imaginary axis. This is indeed the case, as shown in more detail in figure 2(*a*), where the imaginary parts in $[-1, 1]$ of the computed eigenvalues are displayed as a function of the Floquet parameter μ. This shows that λ values with imaginary parts in $[-0.37, 0.37]$ (approximately) are attained for four different μ values in $[-\pi/2T(k), \pi/2T(k))$. The rest of the imaginary axis is only attained for two different μ values. This picture persists if a larger portion of the imaginary λ axis is examined. These numerical results are in perfect agreement with the theoretical results below.

The bottom two panels (*c*) and (*d*) correspond to a nontrivial-phase solution with $b = 0.2$ and $k = 0.5$. The numerical parameters are identical to those for panels (*a*) and (*b*). Again, the spectrum appears to lie on the imaginary axis, with a higher spectral density around the origin. The clumping of the eigenvalues outside of the higher-density band is a consequence of aliasing. This is an artifact of the numerics and the graphics. A plot of the imaginary parts of the computed eigenvalues as a function of μ is shown in figure 2(*b*). As for the trivial-phase case this shows the quadruple covering of the spectrum of a band around the origin of the imaginary axis, and the double covering of the rest of the imaginary axis. Due to the nontrivial-phase profile, the curves in figure 2(*b*) have lost some symmetry compared to those in figure 2(*a*).

[4] The order of magnitude of the largest real part computed is 10^{-10}.

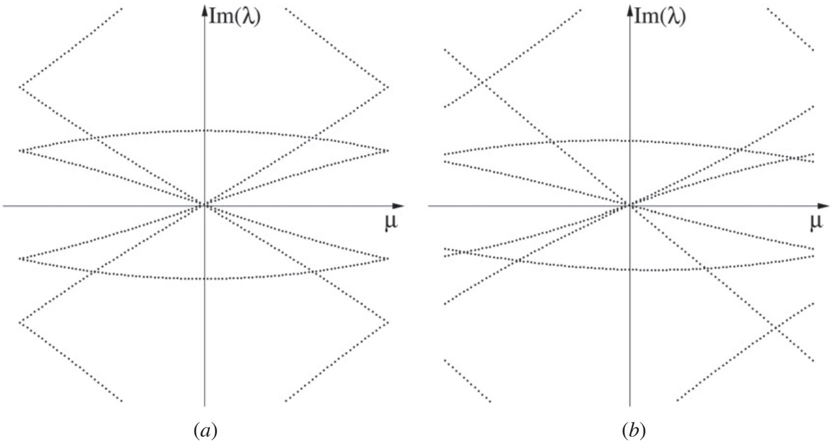

Figure 2. The imaginary part of λ as a function of μ, demonstrating the higher spectral density (four vs. two) corresponding to figure 1(*b*) (left panel) and to figure 1(*d*) (right panel). The parameter values are identical to those of figure 1.

Making the opposite choice for the sign on c in (9) results in the figure being slanted in the other direction.

The above considerations remain true for different values of the offset $b \in \mathbb{R}^+$ and the elliptic modulus $k \in [0, 1)$, although the spectrum does depend on both, as we will prove in the following sections. Thus, for all values of $(b, k) \in \mathbb{R}^+ \times [0, 1)$, the spectrum of the elliptic solutions appears to be confined to the imaginary axis, indicating the spectral stability of these solutions. Similarly, for all these parameter values, the spectrum $\sigma(\mathcal{L})$ covers a symmetric interval around the origin four times, whereas the rest of the imaginary axis is double covered. The edge point on the imaginary axis where the transition from spectral density four to two occurs depends on both b and k and is denoted $\lambda_c(b, k)$. The k-dependence of $\lambda_c(b = 0.2, k)$ is shown in figure 3. Again, both numerical and analytical results (see section 6) are displayed. For these numerical results, Hill's method with $N = 50$ was used.

5. Lax pair representation

Since our analytical stability results originate from the squared-eigenfunction connection between the defocusing NLS linear stability problem (16) and its Lax pair, in this section we examine this Lax pair, restricted to the elliptic solutions of the defocusing NLS.

As for the stability problem, we consider the generalized defocusing NLS (14). This equation is integrable, thus it has a Lax pair representation. Specifically, (14) is equivalent to the compatibility condition $\chi_{xt} = \chi_{tx}$ of the two first-order linear differential equations

$$\chi_x = \begin{pmatrix} -\imath\zeta & \psi \\ \psi^* & \imath\zeta \end{pmatrix} \chi, \qquad \chi_t = \begin{pmatrix} -\imath\zeta^2 - \frac{\imath}{2}|\psi|^2 + \frac{\imath}{2}\omega & \zeta\psi + \frac{\imath}{2}\psi_x \\ \zeta\psi^* - \frac{\imath}{2}\psi_x^* & \imath\zeta^2 + \frac{\imath}{2}|\psi|^2 - \frac{\imath}{2}\omega \end{pmatrix} \chi. \tag{25}$$

Thus (14) is satisfied if and only if both equations for χ of (25) are satisfiable. Written as a spectral problem with parameter ζ, the first equation is seen to be formally self adjoint [27], thus the spectral parameter ζ is confined to the real axis. Restricting to the elliptic solutions gives

$$\chi_x = \begin{pmatrix} -\imath\zeta & \phi \\ \phi^* & \imath\zeta \end{pmatrix} \chi, \qquad \chi_t = \begin{pmatrix} -\imath\zeta^2 - \frac{\imath}{2}|\phi|^2 + \frac{\imath}{2}\omega & \zeta\phi + \frac{\imath}{2}\phi_x \\ \zeta\phi^* - \frac{\imath}{2}\phi_x^* & \imath\zeta^2 + \frac{\imath}{2}|\phi|^2 - \frac{\imath}{2}\omega \end{pmatrix} \chi. \tag{26}$$

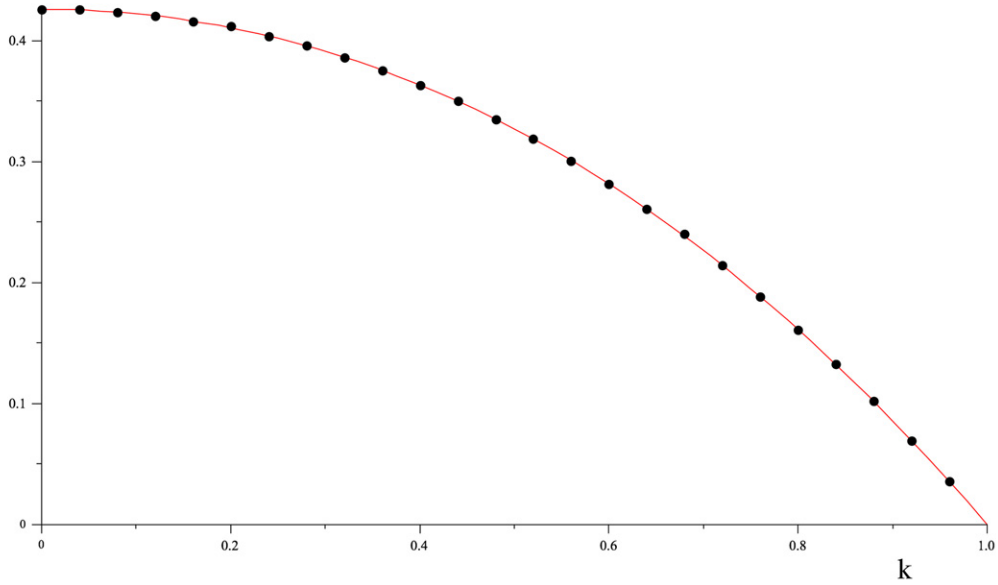

Figure 3. Numerical and analytical results for the imaginary part of the edge point $\lambda_c(b, k)$ of the quadruple-covered region as a function of the elliptic modulus k for $b = 0.2$. The solid curve displays the analytical result, the small circles are obtained numerically.

(This figure is in colour only in the electronic version)

We refer to the spectrum of the first equation of (26) as σ_L. It is the set of all ζ values for which this equation has a solution bounded in x (as in section (4)). As discussed above, $\sigma_L \subset \mathbb{R}$. The main goal of this section is the complete analytic determination of σ_L. For ease of notation, we rewrite the second equation of (26) as

$$\chi_t = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \chi. \tag{27}$$

Since A, B and C are independent of t, we may separate variables. Consider the ansatz

$$\chi(x, t) = \mathrm{e}^{\Omega t}\varphi(x), \tag{28}$$

where Ω is independent of t. We refer to the set of all Ω such that χ is a bounded function of x as the t-spectrum σ_t. Substituting (28) into (27) and canceling the exponential, we find

$$\begin{pmatrix} A - \Omega & B \\ C & -A - \Omega \end{pmatrix} \varphi = \mathbf{0}. \tag{29}$$

This implies that the existence of nontrivial solutions requires

$$\Omega^2 = A^2 + BC = -\zeta^4 + \omega\zeta^2 - c\zeta + \tfrac{1}{16}(4\omega b - 3b^2 - k'^4), \tag{30}$$

where $k'^2 = 1 - k^2$. We have used the explicit form of $\phi(x)$, given in section 2. This demonstrates that Ω is not only independent of t, but also of x. Such a conclusion could also be arrived at by expressing the derivatives of the operators of (26) as matrix commutators, and applying the fact that the trace of a matrix commutator is identically zero [4, 16].

Having determined Ω as a function of ζ for any given elliptic solution of defocusing NLS (i.e., in terms of the parameters b and k), we now wish to do the same for the eigenvector $\varphi(x)$, determined by (29). Immediately,

$$\varphi = \gamma(x)\begin{pmatrix} -B(x) \\ A(x)-\Omega \end{pmatrix}, \tag{31}$$

where $\gamma(x)$ is a scalar function. Indeed, the vector part of (31) ensures that $\chi(x,t)$ satisfies the second equation of (26). Next, we determine $\gamma(x)$ so that $\chi(x,t)$ also satisfies the first equation. Substituting (31) in this first equation results in two homogeneous linear scalar differential equations for $\gamma(x)$ which are linearly dependent. Solving gives

$$\gamma(x) = \gamma_0 \exp\left(-\int \frac{(A-\Omega)\phi + B_x + \imath\zeta B}{B}\,\mathrm{d}x\right). \tag{32}$$

For almost all $\zeta \in \mathbb{C}$, we have explicitly determined two linearly independent solutions of the first equation of (26). Indeed, for all ζ, there should be two such solutions, and two have been constructed for all $\zeta \in \mathbb{C}$ for which $\Omega \neq 0$: the combination of (31) and (32) gives two solutions, corresponding to the different signs for Ω in (30). These solutions are clearly linearly independent. For those values of ζ for which $\Omega = 0$, only one solution is generated. A second one may be found using the method of reduction of order.

To determine the spectrum σ_L, we need to determine the set of all $\zeta \subset \mathbb{R}$ such that (31) is bounded for all x. Clearly, the vector part of (31) is bounded as a function of x. Thus, we need to determine for which ζ the scalar function $\gamma(x)$ is bounded. For this, it is necessary and sufficient that

$$\left\langle \Re\left(\frac{(A-\Omega)\phi + B_x + \imath\zeta B}{B}\right)\right\rangle = 0. \tag{33}$$

Here $\langle\cdot\rangle = \frac{1}{T(k)}\int_0^{T(k)} \cdot\,\mathrm{d}x$ is the average over a period and $\Re$ denotes the real part. The investigation of (33) is significantly simpler for the trivial-phase case $b = 0$ than for the general nontrivial-phase case. We treat these cases separately.

5.1. The trivial-phase case: $b = 0$

With $b = 0$, (30) becomes

$$\Omega^2 = -\zeta^4 + \omega\zeta^2 - \frac{k'^4}{16} = -(\zeta-\zeta_1)(\zeta-\zeta_2)(\zeta-\zeta_3)(\zeta-\zeta_4), \tag{34}$$

with

$$\zeta_1 = -\tfrac{1}{2}(1+k), \qquad \zeta_2 = -\tfrac{1}{2}(1-k), \qquad \zeta_3 = \tfrac{1}{2}(1-k), \qquad \zeta_4 = \tfrac{1}{2}(1+k). \tag{35}$$

The graph for Ω^2 as a function of ζ is shown in figure 4(*a*).

The explicit form of (33) is different depending on whether Ω is real or imaginary. It should be noted that since $\zeta \in \mathbb{R}$, it follows from (34) that these are the only possibilities.

First, we consider Ω being imaginary or zero, requiring $|\zeta| \geqslant (k+1)/2$ or $|\zeta| \leqslant (1-k)/2$. It follows from the definitions of A and B that the integrand in (33) may be written as a rational function of the periodic function $\mathrm{sn}^2(x,k)$, multiplied by its derivative $2\mathrm{sn}(x,k)\mathrm{cn}(x,k)\mathrm{dn}(x,k)$. As a consequence the average of this integrand is zero. Thus, all these values of ζ belong to the Lax spectrum. Extra care should be taken when $\zeta = 0$, in which case the denominator in (33) is singular, and not integrable. This case may be dealt with separately. One finds that the vector part of (31) cancels the singularity in $\gamma(x)$. In fact, the two eigenfunctions of the first equation of (26) are $(-\mathrm{dn}(x,k), k\mathrm{cn}(x,k))^T$ and $(-k\mathrm{cn}(x,k), \mathrm{dn}(x,k))^T$.

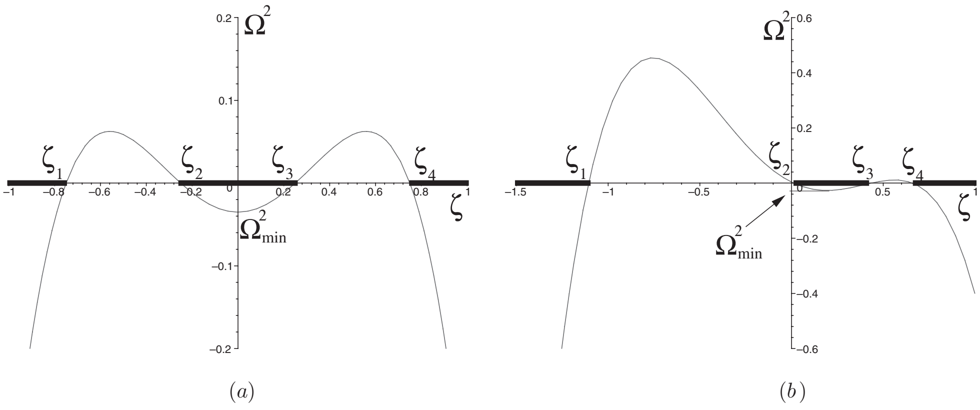

Figure 4. Ω^2 as a function of real ζ, for $k = 0.5$. The union of the bold line segments is the Lax spectrum σ_L. (*a*) shows the symmetric trivial-phase case with $b = 0$. (*b*) illustrates a nontrivial-phase case, with $b = 0.2$.

Next, we consider the case where Ω is real, requiring $(1-k)/2 < |\zeta| < (1+k)/2$. Similar to the above, the integrand contains many terms of the form $R(\mathrm{sn}^2(x,k))(\mathrm{sn}^2(x,k))'$, where R is a rational, nonsingular function. The average of such terms vanishes, leaving a single term $-4\Omega\zeta\,\mathrm{sn}^2(x,k)/(4\zeta^2\mathrm{sn}^2(x,k)+\mathrm{cn}^2(x,k)\mathrm{dn}^2(x,k))$. This term is of fixed sign and never results in zero average. The corresponding values of ζ are not in σ_L.

In summary, we have established that

$$\sigma_L = (-\infty, \zeta_1] \cup [\zeta_2, \zeta_3] \cup [\zeta_4, \infty). \tag{36}$$

This set is indicated in figure 4(*a*) as a bold line. Furthermore, we find that the corresponding values of Ω are imaginary, covering the entire imaginary axis. Thus,

$$\sigma_t = \imath\mathbb{R}. \tag{37}$$

We may be more specific. The segment $\zeta \in (-\infty, \zeta_1]$ gives rise to a complete covering of the imaginary axis, as does $\zeta \in [\zeta_4, \infty)$. Next, the segment $\zeta \in [0, \zeta_3]$ gives rise to $\Omega \in [-i|\Omega_{\min}|, i|\Omega_{\min}|] = [-ik'^2/4, ik'^2/4]$, as does $\zeta \in [\zeta_2, 0]$. Thus, there is an interval on the imaginary axis around the origin that is quadruple covered, while the rest of the imaginary axis is double covered. Thus

$$\sigma_t = (\imath\mathbb{R})^2 \cup \left[-\frac{\imath k'^2}{4}, \frac{\imath k'^2}{4}\right]^2, \tag{38}$$

where the exponents denote multiplicities.

5.2. *The nontrivial-phase case: $b > 0$*

The nontrivial-phase case is more complicated. First, note that the discriminant of (30) is $k^4k'^4 \neq 0$ for $k \neq 0, 1$. This implies that the four roots of the right-hand side of (30) are always real, for all values of $b > 0$. Indeed, complex roots would come about by the collision of real roots, which is not possible since the discriminant is never zero. In fact, the explicit expressions for these roots are quite simple:

$$\begin{aligned}\zeta_1 &= \tfrac{1}{2}(-\sqrt{b} - \sqrt{b+k^2} - \sqrt{b+1}),\\ \zeta_2 &= \tfrac{1}{2}(\sqrt{b} + \sqrt{b+k^2} - \sqrt{b+1}),\\ \zeta_3 &= \tfrac{1}{2}(\sqrt{b} - \sqrt{b+k^2} + \sqrt{b+1}),\\ \zeta_4 &= \tfrac{1}{2}(-\sqrt{b} + \sqrt{b+k^2} + \sqrt{b+1}).\end{aligned} \tag{39}$$

An indicative graph for Ω^2 as a function of ζ is given in figure 4(*b*), using $k = 0.5$ and $b = 0.2$.

As for the trivial-phase case, we split the examination of the real ζ-axis in two parts: those ζ-values for which Ω is pure imaginary, and those for which Ω is real.

If $\zeta \in (-\infty, \zeta_1] \cup [\zeta_2, \zeta_3] \cup [\zeta_4, \infty)$, then Ω is pure imaginary. As before, the integrand of (33) is of the form $\mathcal{R}(\mathrm{sn}^2(x,k))(\mathrm{sn}^2(x,k))'$, where $\mathcal{R}$ is a rational function of its argument, resulting in a zero average. Thus all these values of ζ are in the Lax spectrum. Again one has to consider the case where B might have zeros. It is easy to see that this occurs only when either $\zeta = c/2b$ or when $\zeta = c/2(b+k^2)$. Note that both values are in the specified ζ-range, as the corresponding values for Ω^2 are negative. Although the expressions of the corresponding eigenfunctions are not as compact as for the trivial-phase case, one easily shows that all singularities of $\gamma(x)$ cancel with roots of the vector part of (31). Thus these values are legitimate members of the Lax spectrum.

Next, if Ω is real, up to terms with zero average, the integrand may be written as

$$\frac{2\Omega}{P(\mathrm{sn}^2(x,k))}(k^2\mathrm{sn}^2(x,k)+b)(c - 2\zeta(k^2\mathrm{sn}^2(x,k)+b)), \tag{40}$$

where $P(\mathrm{sn}^2(x,k))$ is a polynomial with no real roots. Unlike the trivial-phase case, the numerator of this expression has roots for $-K(k) < x < K(k)$, and it is not obvious to see that its average is nonzero. We use a more abstract argument. The left-hand side of (33) depends analytically on b and on ζ, at least for $\zeta \in (\zeta_3, \zeta_4)$ and $b > 0$. An identical argument holds for $\zeta \in (\zeta_1, \zeta_2)$. For convenience, we denote this left-hand side as $F(\zeta, b)$. Thus, elements of σ_L are real values of ζ for which $F(\zeta, b) = 0$. It should be noted that ζ_3 and ζ_4 depend on b (see (35)), but since ζ_3 and ζ_4 are always well separated, this is no cause for concern. For a fixed value of b, and using the analytical dependence of $F(\zeta, b)$ on ζ, it follows that $F(\zeta, b)$ is either identically zero, or has isolated zeros. If $F(\zeta, b)$ were to have isolated zeros, these would correspond to isolated points in σ_L. Since σ_L is the spectrum of a period problem, this is not possible [37]. Thus we investigate the possibility that for a fixed value of $b > 0$, $F(\zeta, b)$ is identically zero for all $\zeta \in (\zeta_3, \zeta_4)$. We know this is not true for $b = 0$. Due to the analytic dependence on b, it follows that it is not true for $0 < b \leqslant b_1$, for b_1 sufficiently small. The last possibility to examine is whether there can exist a value of $b > b_1$ for which $F(\zeta, b)$ is identically zero as a function of ζ. If we think of the spectra σ_L parameterized by increasing values of b, this would imply the sudden presence of a continuous subset of σ_L out of a vacuum: i.e., this subset would not emerge from or be connected to other parts of σ_L. Since σ_L depends continuously on its parameters [24], this is not possible. We conclude that $F(\zeta, b)$ has no zeros if Ω is real.

It follows that our conclusions are identical to those for the trivial-phase case. Specifically, we have established (36) and (37). As before, the set σ_L is indicated in figure 4(*b*) as a bold line. Analogously to (38), we may write

$$\sigma_t = (\imath\mathbb{R})^2 \cup \left[-\imath\sqrt{|\Omega^2_{\min}|}, \imath\sqrt{|\Omega^2_{\min}|}\right]^2, \tag{41}$$

where the exponents denote multiplicities, as before. Here $\Omega^2_{\min}$ is the minimal value of Ω^2 as a function of ζ. This value depends on the two parameters b and k. If desired, it can be calculated using Cardano's formulae, but we do not subject the reader to its explicit form. The perfect agreement between the numerics and this analytical result is illustrated in figure 3 for $b = 0.2$, for varying k.

6. Spectral stability

The connection between the eigenfunctions of the Lax pair (25) and the eigenfunctions of the linear stability problem (16) for the defocusing NLS equation (1) is well known [1, 2, 19, 32, 36]. It is convenient to phrase the result using the form (12) of the solutions. Write

$$\Psi(x,t) = \mathrm{e}^{-\imath\omega t+\imath\kappa x}\tilde{R}(x,t), \tag{42}$$

where κ is the average value of $c/R^2(x)$, with $R(x)$ the amplitude of the stationary solution under consideration, as before. We see that the periodic part $\hat{R}(x)$ of the considered elliptic solution (12) is a stationary solution of

$$\imath\tilde{R}_t = -\omega\tilde{R} - \frac{1}{2}\tilde{R}_{xx} - \imath\kappa\tilde{R}_x + \frac{\kappa^2}{2}\tilde{R} + \tilde{R}|\tilde{R}|^2. \tag{43}$$

To linearize around the elliptic solution with $\hat{R}(x) = \hat{R}_1(x) + \imath\hat{R}_2(x)$, we let

$$\tilde{R}(x,t) = \hat{R}(x) + \epsilon(w_1(x,t) + \imath w_2(x,t)) + \mathcal{O}(\epsilon^2), \tag{44}$$

which results in

$$\frac{\partial}{\partial t}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = J\mathcal{L}_\kappa \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \tag{45}$$

with

$$\mathcal{L}_\kappa = \begin{pmatrix} -\frac{1}{2}\partial_x^2 + \frac{1}{2}\kappa^2 + 3\hat{R}_1^2(x) + \hat{R}_2^2(x) - \omega & \kappa\partial_x + 2\hat{R}_1\hat{R}_2 \\ -\kappa\partial_x + 2\hat{R}_1\hat{R}_2 & -\frac{1}{2}\partial_x^2 + \frac{1}{2}\kappa^2 + \hat{R}_1^2(x) + 3\hat{R}_2^2(x) - \omega \end{pmatrix}. \tag{46}$$

It should be noted that although $\hat{R}(x)$ is a periodic solution of (43), it is not necessary for $\tilde{R}(x,t)$ to be periodic. Indeed, we wish to allow for perturbations (44) that are bounded and sufficiently smooth, but otherwise arbitrary. Noting the independence of $J\mathcal{L}_\kappa$ on t, we separate variables as before,

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \mathrm{e}^{\lambda t}\begin{pmatrix} W_1 \\ W_2 \end{pmatrix}, \tag{47}$$

to obtain the spectral problem

$$\lambda\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = J\mathcal{L}_\kappa\begin{pmatrix} W_1 \\ W_2 \end{pmatrix}. \tag{48}$$

We easily prove the following theorem.

Theorem 1. *The vector* $(w_1, w_2)^T = \left(\mathrm{e}^{-\imath\kappa x}\chi_1^2 + \mathrm{e}^{\imath\kappa x}\chi_2^2, -\imath\mathrm{e}^{-\imath\kappa x}\chi_1^2 + \imath\,\mathrm{e}^{\imath\kappa x}\chi_2^2\right)^T$ *satisfies the linear stability problem (45). Here* $\chi = (\chi_1, \chi_2)^T$ *is any solution of (25) with the corresponding elliptic solution* $\phi(x) = R(x)\,\mathrm{e}^{\imath\theta(x)} = \hat{R}(x)\,\mathrm{e}^{\imath\kappa x}$.

Proof. The proof is by direct calculation: calculate $\partial_t(w_1, w_2)^T$ using the product rule and the second equation of (25). Alternatively, calculate $\partial_t(w_1, w_2)^T$ using (45), substituting $(w_1, w_2)^T = \left(\mathrm{e}^{-\imath\kappa x}\chi_1^2 + \mathrm{e}^{\imath\kappa x}\chi_2^2, -\imath\mathrm{e}^{-\imath\kappa x}\chi_1^2 + \imath\mathrm{e}^{\imath\kappa x}\chi_2^2\right)^T$. In both expressions so obtained, eliminate x-derivatives of w_1 and w_2 (up to order 2) using the first equation of (25). The resulting expressions are equal, finishing the proof. □

Remarks.

- It is possible to repeat this proof for any solution $\psi(x,t)$ of (14). It is not necessary that the solution is a stationary elliptic solution.
- Despite the different form of the spectral stability problem (compare 48 with (22)), it is clear that they determine the same spectra, with different but equivalent eigenfunctions. Indeed, if an eigenfunction (W_1, W_2) corresponds to an element of the spectrum λ for (48), then there is a corresponding eigenfunction (U, V) with the same spectral element λ for (22). Thus, there is no confusion when we use (48) to determine the stability spectrum of an elliptic solution of (1).

To establish the spectral stability of the elliptic solutions of the defocusing NLS equation (1), we need to establish that *all* bounded solutions (W_1, W_2) of (48) are obtained through the squared-eigenfunction connection by

$$\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \mathrm{e}^{2\Omega t} \begin{pmatrix} \mathrm{e}^{-\imath\kappa x}\varphi_1^2 + \mathrm{e}^{\imath\kappa x}\varphi_2^2 \\ -\imath\,\mathrm{e}^{-\imath\kappa x}\varphi_1^2 + \imath\,\mathrm{e}^{\imath\kappa x}\varphi_2^2 \end{pmatrix}. \tag{49}$$

If we manage to do so then by comparing with (47) we immediately conclude that

$$\lambda = 2\Omega. \tag{50}$$

Since $\sigma_t = i\mathbb{R}$, we conclude that the stability spectrum is given by

$$\sigma(\mathcal{L}) = \sigma(\mathcal{L}_\kappa) = \imath\mathbb{R}. \tag{51}$$

Here the norm that is necessary to define the spectrum is the supremum norm in $C_b^0(\mathbb{R})$. In order to obtain this conclusion, we need the following theorem.

Theorem 2. *All but six solutions of (48) are obtained through (49), where $\varphi = (\varphi_1, \varphi_2)^T$ solves the first equation of (26) and (29). Specifically, all solutions of (48) bounded on the whole real line are obtained through the squared eigenfunction connection (49), with one exception corresponding to $\lambda = 0$.*

Proof. For any given value of $\lambda \in \mathbb{C}$, (48) can be written as four-dimensional first-order system of ordinary differential equations. Thus, for any value of $\lambda \in \mathbb{C}$, (48) has four linearly independent solutions. On the other hand, we have already shown (theorem 1) that (49) provides solutions of this ordinary differential equation. Let us count how many solutions are obtained this way, for a fixed value of λ. For any value of $\lambda \in \mathbb{C}$, exactly one value of $\Omega \in \mathbb{C}$ is obtained through $\Omega = \lambda/2$. Excluding the six values of λ for which the discriminant of (30) as a function of ζ is zero (these turn out to be only the values of λ for which Ω^2 reaches its maximum or minimum value in figure 4), (30) gives rise to four values of $\zeta \in \mathbb{C}$. It should be noted that we are not restricting ourselves to $\zeta \in \sigma_L$ now, since the boundedness of the solutions is not a concern in this counting argument. Next, for a given pair $(\Omega, \zeta) \in \mathbb{C}^2$, (29) defines a unique solution of the system consisting of the first equation of (26) and (29). Thus, any choice of $\lambda \in \mathbb{C}$ not equal to the six values mentioned above, gives rise to exactly four solutions of (48), through the squared eigenfunction connection of theorem 1 Before we consider the six excluded values, we need to show that the four solutions $(W_1(x), W_2(x))^T$ just obtained are linearly independent. As in [5], there are two parts to this.

(1) If there is an exponential contribution to $(W_1, W_2)^T$ from $\gamma(x)$ then an argument similar to that given in [5] establishes the linear independence of the four solutions.

(2) As in [5], the only possibility for the exponential factor due to $\gamma(x)$ not to contribute is for the integrand in that factor to be proportional to a logarithmic derivative. It is easily checked that this occurs only for $\lambda = 0 = \Omega$. It is a tedious calculation to verify that the four solutions $(W_1(x), W_2(x))^T$ obtained through the squared eigenfunction connection are linearly dependent. In fact, no two of them are linearly independent. Using the invariances of the equation, one can construct four linearly independent solutions, two of which are bounded, while the other two are unbounded. One of the bounded solutions is the one obtained using the squared eigenfunction connection. Unlike for the KdV equation [5], no linear combination of the unbounded solutions is bounded. Thus, in this case, three of the solutions of (48) are not obtained through the squared eigenfunction connection. Of these three solutions, only one is bounded.

For the six excluded values, three linearly independent solutions of (48) are found. The fourth one may be constructed using reduction of order, and introduces algebraic growth. Extra care is required for the trivial-phase case, for which both maxima are equal, but the same conclusion follows. For the two λ values for which Ω^2 reaches its minimum value, the two solutions obtained from (49) are bounded, thus these values of λ are part of the spectrum. The two values of λ for which Ω^2 reaches its maximum value only give rise to unbounded solutions and are not part of the spectrum.

We conclude that *all* but one of the bounded solutions of (48) are obtained through the squared eigenfunction connection. This finishes the proof. □

Remark. It is important to remember that the algebraically growing solutions discussed above (corresponding to $\lambda = 0 = \Omega$) do not lead to solutions of (48) through the squared eigenfunction connection. Indeed, those solutions do not solve the second equation of (26), and therefore theorem 1 does not apply to them. If it did, eight solutions would be obtained corresponding to $\lambda = 0$.

The above considerations are summarized in the following theorem.

Theorem 3 (Spectral Stability)**.** *The elliptic solutions of the NLS equation (1) are spectrally stable. The spectrum of their associated linear stability problem (48) (or (22)) is explicitly given by $\sigma(\mathcal{L}) = \imath\mathbb{R}$, or, accounting for multiple coverings,*

$$\sigma(\mathcal{L}) = (\imath\mathbb{R})^2 \cup \left[-2\imath\sqrt{|\Omega^2_{\min}|}, 2\imath\sqrt{|\Omega^2_{\min}|}\right]^2, \tag{52}$$

where $|\Omega^2_{\min}|$ is as before.

It follows from theorem 3 that the value of $\lambda_c(b, k)$ in figure 3 is given by

$$\lambda_c(b, k) = 2\sqrt{|\Omega^2_{\min}|}, \tag{53}$$

which is the expression for the solid curve in figure 3.

Similar to the calculations in [5], we could obtain parametric representations for the Floquet parameter and the imaginary part of the spectrum as a function of ζ. This would reproduce the curves in figure 2.

Although the results of [22] specific to (1) are valid for small amplitude, no such restriction is imposed for the use of their SCS lemma, as done in section 3 of [22]. Thus, we inherit the conclusion of linear stability from [22]. Since the SCS lemma makes a statement about the eigenfunctions of the spectral problem after Floquet decomposition, the linear stability result holds with respect to what we called subharmonic perturbations in the introduction: perturbations that are periodic with a period that is an integer multiple of the period of the periodic part $\hat{R}(x)$ of the elliptic solution.

Theorem 4 (Linear Stability)**.** *The elliptic solutions of the defocusing NLS equation (1) are linearly stable with respect to square integrable subharmonic perturbations.*

In other words, all solutions of (43) or, equivalently, (16) with sufficiently small square integrable initial conditions remain small for all $t > 0$.

7. Nonlinear stability

Next, we consider the nonlinear stability of the elliptic solutions. It proves convenient to rewrite the NLS equation (1) using real-valued dependent variables with $\Psi(x,t) = \mathrm{e}^{-\imath\omega t}\psi(x,t) = \mathrm{e}^{-\imath\omega t}(r(x,t) + \imath l(x,t))$. We have

$$\frac{\partial}{\partial t}\begin{pmatrix} r \\ l \end{pmatrix} = \begin{pmatrix} -l_{xx}/2 + l(r^2+l^2) - \omega l \\ r_{xx}/2 - r(r^2+l^2) + \omega r \end{pmatrix}. \tag{54}$$

In these variables, we write the elliptic solutions using

$$\phi(x) = \tilde{r}(x) + \imath\tilde{l}(x) = \mathrm{e}^{\imath\theta(x)}R(x). \tag{55}$$

This is almost identical to the reformulation used in section 6, but without the exponential $\exp(\imath\kappa x)$ factored out.

As stated, we wish to allow for subharmonic perturbations: perturbations whose period is a positive integer multiple of that of the minimal period $2T(k)$ of the amplitude of the elliptic solutions. Note that there is no need to consider so-called superharmonic perturbations, which have a period that is equal to this minimal period divided by a positive integer. Such perturbations are also periodic with period equal to the minimal period, and they are *de facto* included in our considerations. Further, in order to properly define the higher-order members of the NLS hierarchy that are necessary for our stability argument below, we require that $r(x, t)$, $l(x, t)$ and their derivatives of up to third order are square integrable. Hence we consider (54) on the function space

$$\mathbb{V} = H^3_{\mathrm{per}}([-NT, NT]) \times H^3_{\mathrm{per}}([-NT, NT]),$$

where N is a fixed positive integer.

To facilitate our approach to the nonlinear stability problem, some tools from the theory of the NLS equation as an integrable system are necessary. We review these here, partially to aid the exposition, but also to reformulate these for our reformulation of the NLS equation (54) using real coordinates. The essence of all these results is found in different classical sources such as [1, 2, 16].

7.1. Hamiltonian structure of the NLS equation and the NLS hierarchy

Using its real form (54), the Hamiltonian structure of the NLS equation is given by

$$\frac{\partial}{\partial t}\begin{pmatrix} r \\ l \end{pmatrix} = JH'(r,l), \tag{56}$$

where J is the skew-symmetric matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{57}$$

the Hamiltonian $H(r, l)$ is the nonlinear functional

$$H(r,l) = \int_{-NT}^{NT} \mathcal{H}(r,l,r_x,l_x)\,\mathrm{d}x = \int_{-NT}^{NT}\left(\frac{1}{4}\left(r_x^2+l_x^2\right) + \frac{1}{4}(r^2+l^2)^2 - \frac{\omega}{2}(r^2+l^2)\right)\mathrm{d}x, \tag{58}$$

and $H'(r,l)$ denotes the variational derivative of H:

$$H'(r,l) = \begin{pmatrix} \mathcal{H}_r - \partial_x \mathcal{H}_{r_x} \\ \mathcal{H}_l - \partial_x \mathcal{H}_{l_x} \end{pmatrix}. \tag{59}$$

By virtue of its integrability, the NLS equation possesses an infinite number of conserved quantities $H_0, H_1, H_2, \ldots$, which are mutually in involution under the (canonical) Poisson bracket specified by (57). Just like $H = H_2 + \omega H_0$, with $H_0 = -\frac{1}{2}\int_{-NT}^{NT}(r^2+l^2)\mathrm{d}x$, defines the NLS flow (54) with dynamical variable t, each H_j defines a dynamical equation with evolution variable τ_j by

$$\frac{\partial}{\partial \tau_j}\begin{pmatrix} r \\ l \end{pmatrix} = J H_j'(r,l). \tag{60}$$

The collection of all these equations corresponding to $j \in \mathbb{N}$ is the NLS hierarchy. Each equation in the NLS hierarchy is integrable, and has the same infinite sequence of conserved quantities $\{H_j, j \in \mathbb{N}\}$ as the NLS equation (1), which is itself a member of the hierarchy, corresponding to $j = 2$. Further, due to the involution property of their Hamiltonians, the flows of all members of the NLS hierarchy mutually commute. This makes it possible for us to think of simultaneous solutions to the entire hierarchy: $(r,l) = (r,l)(\tau_0, \tau_1, \tau_2, \tau_3, \ldots)$.

It is clear that any linear combination of right-hand sides of equations in the NLS hierarchy gives rise to a dynamical equation whose flow commutes with all equations of the hierarchy. We define the n-th NLS equation[5] with evolution variable t_n as

$$\frac{\partial}{\partial t_n}\begin{pmatrix} r \\ l \end{pmatrix} = J \hat{H}_n'(r,l), \tag{61}$$

where

$$\hat{H}_n = H_n + \sum_{j=0}^{n-1} c_{n,j} H_j, n \geqslant 1. \tag{62}$$

Here the coefficients $c_{n,j}$ are constants. At the moment these constants are arbitrary. The crux of our stability proof hangs on the freedom we have obtained from the introduction of these constants. These constants will be fixed as convenient below. As seen above, the NLS equation in the form (54) is the 2nd NLS equation with $\hat{H}_2 = H = H_2 + \omega H_0$.

Every member of the NLS hierarchy has a Lax pair. These Lax pairs share the same first component $\chi_{\tau_1} = T_1(\zeta, r, l)\chi$, while their second component $\chi_{\tau_j} = T_j(\zeta, r, l, r_x, l_x, \ldots)\chi$ is different. The collection of all these Lax equations is called the linear NLS hierarchy. Its first three members, using the (r,l) coordinates, are

$$\chi_{\tau_0} = \begin{pmatrix} \imath/2 & 0 \\ 0 & -\imath/2 \end{pmatrix}\chi, \qquad \chi_{\tau_1} = \begin{pmatrix} -\imath\zeta & r+\imath l \\ r-\imath l & \imath\zeta \end{pmatrix}\chi,$$
$$\chi_{\tau_2} = \begin{pmatrix} -\imath\zeta^2 - \imath(r^2+l^2)/2 & \zeta(r+\imath l) + \imath(r_x + \imath l_x)/2 \\ \zeta(r-\imath l) - \imath(r_x - \imath l_x)/2 & \imath\zeta^2 + \imath(r^2+l^2)/2 \end{pmatrix}\chi. \tag{63}$$

Additional members of the linear NLS hierarchy are easily constructed using the AKNS method [1]. The compatibility condition of the τ_1 equation with the n-th member of the linear hierarchy results in the n-th equation of the NLS hierarchy.

A Lax equation for the n-th NLS equation is constructed by taking a linear combination with the same coefficients of the lower-order Lax equations, as was done for the nonlinear hierarchy. We define the n-th linear NLS equation as

$$\chi_{t_n} = \hat{T}_n \chi, \tag{64}$$

[5] Truly, this is an n-parameter family of equations parameterized by the n parameters $c_{n,j}$. We refer to this family simply as the n-th NLS equation.

where

$$\hat{T}_n = \begin{pmatrix} \hat{A}_n & \hat{B}_n \\ \hat{C}_n & -\hat{A}_n \end{pmatrix} = T_n + \sum_{j=0}^{n-1} c_{n,j} T_j, \qquad \hat{T}_0 = T_0. \tag{65}$$

7.2. Stationary solutions of the NLS hierarchy

The n-stationary solutions of the NLS hierarchy are defined to be functions so that

$$\frac{\partial}{\partial t_n} \begin{pmatrix} r \\ l \end{pmatrix} = 0, \tag{66}$$

for some choice of the constants $c_{n,0}$, $c_{n,1}$, ..., $c_{n,n-1}$. Thus, an n-stationary solution satisfies the ordinary differential equation

$$J \hat{H}_n'(r,l) = 0 \iff \hat{H}_n'(r,l) = 0,$$

since J is invertible.

It is known that the n-stationary solutions can be written in terms of genus $n-1$ Riemann theta functions, or limits of these. We do not need this explicit representation, but we do need some classical properties of these solutions, see e.g. [4].

- Since all flows of the NLS hierarchy commute, the set of n-stationary solutions is invariant under any of the NLS equations. In other words, an n-stationary solution remains an n-stationary solution after evolution under any of the NLS flows.
- Any n-stationary solution is also stationary with respect to all of the higher-order time variables t_m, $m > n$. In such cases, the constants $c_{m,j}$ with $j \geqslant n$ are free parameters. An example is provided below. In what follows, we make use of this fact in our construction of a Lyapunov function.

Returning to the elliptic solutions $\psi(x,t) = \phi(x) = \tilde{r}(x) + \imath \tilde{l}(x)$, we know that they satisfy the ordinary differential equation (3). This is nothing but the second stationary NLS equation in complex coordinates:

$$\psi_{t_2} = 0 \iff H_2' + c_{2,1} H_1' + c_{2,0} H_0' = 0, \tag{67}$$

with $c_{2,1} = 0$ and $c_{2,0} = \omega$. Thus the elliptic solutions are 2-stationary solutions of the NLS hierarchy. As stated above, this implies they are m-stationary solutions for any $m > 2$ as well. As an example, consider the fourth NLS equation

$$\hat{H}_4'(r,l) = 0 = H_4(r,l) + c_{4,3} H_3'(r,l) + c_{4,2} H_2'(r,l) + c_{4,1} H_1'(r,l) + c_{4,0} H_0'(r,l). \tag{68}$$

The elliptic solutions satisfy this equation with

$$c_{4,0} = \omega c_{4,2} + 2\imath c c_{4,3} - \frac{k^4}{2} - 2k^2 - 5bk^2 - \frac{1}{2} - \frac{15b^2}{2} - 5b, \tag{69}$$

$$c_{4,1} = 4\imath c + 2\omega c_{4,3}, \tag{70}$$

for any values of $c_{4,2}$ and $c_{4,3}$.

7.3. Orbital stability of the elliptic solutions

We begin by translating some of the results of the previous sections to the (r, l) coordinates used here. Expanding about the elliptic solution (55),

$$\begin{pmatrix} r(x,t) \\ l(x,t) \end{pmatrix} = \begin{pmatrix} \tilde{r}(x) \\ \tilde{l}(x) \end{pmatrix} + \epsilon \begin{pmatrix} w_1(x,t) \\ w_2(x,t) \end{pmatrix} + \mathcal{O}(\epsilon^2). \tag{71}$$

Note that this $w = (w_1, w_2)^T$ is different from, but obviously related to, that of the previous section. Substituting this in (14) and ignoring terms beyond first order in ϵ, we find

$$w_t = J\mathcal{M}w, \tag{72}$$

where the symmetric differential operator $\mathcal{M} = \hat{H}_2''(\tilde{r}, \tilde{l})$ is the Hessian of $\hat{H}_2(r, l)$ evaluated at the elliptic solution:

$$\hat{H}_2''(\tilde{r}, \tilde{l}) = \begin{pmatrix} -\frac{1}{2}\partial_x^2 + 3\tilde{r}^2 + \tilde{l}^2 - \omega & 2\tilde{r}\tilde{l} \\ 2\tilde{r}\tilde{l} & -\frac{1}{2}\partial_x^2 + \tilde{r}^2 + 3\tilde{l}^2 - \omega \end{pmatrix}. \tag{73}$$

Again, we separate variables. Let $w(x, t) = \mathrm{e}^{\lambda t} W(x)$. This results in the eigenvalue problem

$$\lambda W = J\mathcal{M}W. \tag{74}$$

As in the previous section, the solutions of this problem are related to those of the Lax pair problem (25):

$$\lambda = 2\Omega, \qquad W(x) = \begin{pmatrix} \chi_1^2 + \chi_2^2 \\ -\imath\chi_1^2 + \imath\chi_2^2 \end{pmatrix}. \tag{75}$$

These relations are verified in the same way as before.

Before we turn to nonlinear stability, we briefly review the simplest symmetries of the NLS equation, namely those that leave the class of elliptic solutions invariant. These are the multiplication by a constant phase factor, and the translation in x. These symmetries are represented by the Lie group $G = \mathbb{R} \times S^1$. Let $g = (x_0, \gamma) \in G$. Then elements of G act on $\Psi(x, t)$ according to

$$T(g)\Psi(x,t) = \mathrm{e}^{\imath\gamma}\Psi(x + x_0, t), \tag{76}$$

or, in real coordinates,

$$T(g)\begin{pmatrix} r(x,t) \\ l(x,t) \end{pmatrix} = \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} r(x+x_0,t) \\ l(x+x_0,t) \end{pmatrix}. \tag{77}$$

The orbit of a solution $\Psi(x, t)$ under the group G is the collection $\{T(g)\Psi(x, t), g \in G\}$. Stability of the elliptic solutions is considered modulo these two symmetries: an elliptic solution is considered stable if it never strays far from its orbit when perturbed with sufficiently small perturbations. Concretely, an elliptic solution $\phi(x) = \tilde{r}(x) + \imath\tilde{l}(x)$ is stable if nearby solutions stay near elliptic solutions for all time.

Definition. *The stationary solution $\Psi = \mathrm{e}^{-\imath\omega t}(\tilde{r}(x) + \imath\tilde{l}(x))$ is (nonlinearly) orbitally stable in $\mathbb{V}$ if for any given $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that if $(r(x, 0), l(x, 0))^T \in \mathbb{V}$ then for all $t > 0$*

$$||(r(x,0), l(x,0))^T - (\tilde{r}(x), \tilde{l}(x))^T|| < \delta$$
$$\Rightarrow \inf_{g \in G} ||(r(x,t), l(x,t))^T - T(g)(\tilde{r}(x), \tilde{l}(x))^T|| < \epsilon.$$

To prove nonlinear stability, we start by constructing a Lyapunov functional. For Hamiltonian systems, this is typically a constant of the motion $K(r,l)$ for which $(\tilde{r}, \tilde{l})^T$ is an unconstrained minimizer:

$$\frac{\mathrm{d}}{\mathrm{d}t}K(r,l) = 0, \quad K'(\tilde{r}, \tilde{l}) = 0, \quad \langle v, K''(\tilde{r}, \tilde{l})v\rangle > 0, \quad \text{for all } v \in \mathbb{V}, v \neq 0. \tag{78}$$

Here the notation $K'(r,l)$ denotes the variational derivative, as in (59), but allowing for the possibility of functionals depending on higher than first derivatives. If $K(r,l) = \int_{-NT}^{NT}\mathcal{K}(r,l,r_x,l_x,r_{xx},l_{xx},\ldots)\,\mathrm{d}x$, then

$$K'(r,l) = \sum_j (-1)^j \begin{pmatrix} \partial_x^j(\partial\mathcal{K}/\partial r_{jx}) \\ \partial_x^j(\partial\mathcal{K}/\partial l_{jx}) \end{pmatrix}, \tag{79}$$

where the index jx denotes the jth derivative with respect to x. The above sum is finite, with the number of terms determined by the order of the differential expression $\mathcal{K}(r,l,r_x,l_x,r_{xx},l_{xx},\ldots)$. The above concept of a Lyapunov function will be generalized slightly below, to accommodate orbital stability.

An infinite number of candidates for a Lyapunov functional suggest themselves, since NLS is an integrable Hamiltonian system. Indeed, all conserved quantities of the equation satisfy the first condition. We want to construct one that satisfies the other requirements as well.

Linearizing the n-th NLS equation about the elliptic solution $(\tilde r,\tilde l)^T$ results in

$$w_{t_n} = J\mathcal{L}_n w, \tag{80}$$

where $\mathcal{L}_n$ is the Hessian of $\hat H_n$ evaluated at $(\tilde r,\tilde l)^T$, and w is the same as above, but it is regarded now as a function of all t_n. The squared eigenfunction connection (75) and separation of variables give

$$2\Omega_n W(x) = J\mathcal{L}_n W(x), \tag{81}$$

where Ω_n is obtained through

$$\chi = \mathrm{e}^{\Omega_n t_n}\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}. \tag{82}$$

Note that due to their commutativity, the different equations of the linear NLS hierarchy share the common set of complete eigenfunctions $(\chi_1,\chi_2)^T$. Substituting this in (64) determines a relationship between Ω_n and ζ, similar to that obtained in section 5. In general, this relationship defines a Riemann surface of genus n, see section 4.6 of [4]. However, the algebraic curve determining this Riemann surface is singular for $n > 2$ when evaluated at the elliptic solutions, as detailed by the next theorem.

Theorem 5. *Let $(\tilde r,\tilde l)^T$ be a stationary solution of the second NLS equation. For all $n > 2$ the algebraic relationship between Ω_n and ζ reduces to*

$$\Omega_n^2(\zeta) = p_n^2(\zeta)\Omega^2(\zeta), \tag{83}$$

where $\Omega^2(\zeta)$ is given in (30), and $p_n(\zeta)$ is a polynomial of degree $n-2$ in ζ. Furthermore, the choice of the free parameters $c_{n,j}$, $j > 1$ gives complete control over the roots of $p_n(\zeta)$.

Proof. The proof is analogous to a special case of that given in section 4 of [33] for the finite-genus solutions of the KdV equation. When evaluated at a stationary solution of the n-th NLS equation, the flows of all higher-order NLS equations become linearly dependent. The theorem is a consequence of this linear dependence and the specific functional form (obtained through a standard AKNS calculation [1]) the matrices in the linear NLS hierarchy take as polynomials in ζ. □

With these facts established, we return to proving orbital stability. The construction of a Lyapunov functional establishes so-called *formal stability* [25]. We go from formal stability to nonlinear (orbital) stability using the following theorem, which is a rephrasing of results due to Grillakis, Shatah and Strauss [20] and Maddocks and Sachs [31].

Theorem 6 (Orbital Stability)**.** *Let* $(\tilde r, \tilde l)^T$ *be a linearly stable equilibrium solution of the integrable Hamiltonian problem* $(\tilde r, \tilde l)_t^T = JH'(\tilde r, \tilde l)$ *on the function space* $\mathbb{V}$*. Further, suppose there exist constants* $c_{n,j}$*,* $0 \leqslant j \leqslant n-1$ *such that the Hamiltonian* $\hat H_n$ *for the n-th equation of the hierarchy satisfies:*

(1) The kernel of $\hat H_n(\tilde r, \tilde l)$ *is spanned by the generators of the symmetry group G acting on* $(\tilde r, \tilde l)^T$*.*

(2) For all eigenfunctions W corresponding to nonzero eigenvalues of $\hat H_n$*, we have that* $K_n(W) := \langle W, \hat H_n''(\tilde r, \tilde l) W\rangle > 0$*.*

Then $(\tilde r, \tilde l)^T$ is an orbitally stable solution of $(r, l)_t^T = JH'(r, l)$ in $\mathbb{V}$.

It should be noted that the concept of orbital stability requires well-posedness of the initial-value problem in $\mathbb{V}$, for which the reader is referred to chapter 5, theorem 2.1 of [6].

Let us consider what remains to be done to use this theorem to establish orbital stability of the elliptic solutions of the NLS equation. First, the linear stability of the elliptic solutions was established in the previous section, using the SCS lemma of Hărăguş and Kapitula [22]. Second, the kernel of $\hat H_2 = H$ has geometric multiplicity two, when considered on $\mathbb{V}$ [17]. Using the original complex formulation (1), the infinitesimal generator corresponding to phase invariance is $(\Psi, \Psi^*) \to (\imath\Psi, -\imath\Psi^*)$, while that for translational invariance is $(\Psi, \Psi^*) \to (\Psi_x, \Psi_x^*)$. In terms of the real coordinates, those invariances are generated by $(r, l) \to (-l, r)$ and $(r, l) \to (r_x, l_x)$, respectively. This implies that the two-dimensional null space of $\hat H_2''(\tilde r, \tilde l)$ is spanned by

$$\left\{ \begin{pmatrix} -\tilde l \\ \tilde r \end{pmatrix}, \begin{pmatrix} \tilde r_x \\ \tilde l_x \end{pmatrix} \right\}. \tag{84}$$

Note that it follows from the Riemann surface relations in theorem 5 that the kernel of $\hat H_n''(\tilde r, \tilde l)$, $n > 2$, is the same as that of $\hat H_2''(\tilde r, \tilde l)$, provided $p_n(\zeta)$ has no zeros on the Lax spectrum. Thus, it remains to be seen whether we can satisfy condition 2 of theorem 6: is it possible to find $\hat H_n$ such that this requirement is satisfied? Let us calculate $K_n(W)$. For any n, we have by definition of $\mathcal{L}_n$ that (see (81))

$$J\mathcal{L}_n W = 2\Omega_n W \iff \mathcal{L}_n W = 2\Omega_n J^{-1} W, \tag{85}$$

from which

$$\begin{aligned} K_n(W) &= \int_{-NT}^{NT} W^* \cdot \mathcal{L}_n W \, \mathrm{d}x = 2\Omega_n \int_{-NT}^{NT} W^* \cdot J^{-1} W \, \mathrm{d}x \\ &= \frac{\Omega_n}{\Omega} \int_{-NT}^{NT} W^* \cdot \mathcal{L}_2 W \, \mathrm{d}x, \end{aligned} \tag{86}$$

so that

$$K_n(W(\zeta)) = \frac{\Omega_n(\zeta)}{\Omega(\zeta)} K_2(W(\zeta)) = p_n(\zeta) K_2(W(\zeta)). \tag{87}$$

This shows that any roots of $\Omega(\zeta)$ result in removable singularities. In what follows we write $K_n(\zeta)$ for $K_n(W(\zeta))$. Thus, in order to calculate $K_n(\zeta)$, it suffices to calculate $K_2(\zeta)$ and to know the polynomial $p_n(\zeta)$.

Let us calculate $K_2(\zeta)$. As the notation indicates, we consider all quantities (spectral elements, Ω, eigenfunctions W, etc) parametrized by the Lax spectral parameter ζ. Some care has to be taken to ensure that a consistent branch of $\Omega(\zeta)$ is used. From $\mathcal{L}W = 2\Omega J^{-1} W = 2\Omega(-W_2, W_1)^T$, we get $W^* \cdot \mathcal{L} W = 2\Omega(W_1 W_2^* - W_2 W_1^*)$. Using the squared eigenfunction connection (75), we obtain

$$W^* \cdot \mathcal{L} W = 4\imath\Omega(|\chi_1|^4 - |\chi_2|^4). \tag{88}$$

We know that (see section 5) $\chi_1 = -\gamma(x)B(x)\exp(\Omega t)$, $\chi_2 = \gamma(x)(A(x) - \Omega)\exp(\Omega t)$. The factor $\exp(\Omega t)$ is irrelevant for this calculation since Ω is imaginary on the Lax spectrum, thus the exponential factor has unit magnitude. Next, we calculate the magnitude of $\gamma(x)$. From (32),

$$\gamma = \gamma_0 \exp\left(-\int \frac{(A-\Omega)\phi + B_x + \imath\zeta B}{B}\,\mathrm{d}x\right) = \frac{\gamma_0}{B}\exp(\imath \text{ real})\exp\left(-\int \frac{(A-\Omega)\phi}{B}\,\mathrm{d}x\right), \tag{89}$$

where 'real' denotes a sum of real quantities, so that the resulting exponential has no effect on the magnitude. The constant factor γ_0 has no effect on the sign of $W^* \cdot \mathcal{L}W$, and we equate it to 1. Simplifying the integrand using (30) and $C = B^*$ yields

$$\begin{aligned} -\frac{(A-\Omega)\phi}{B} &= \frac{B^*\phi}{A+\Omega} = \frac{\zeta|\phi|^2 - \frac{\imath}{2}\phi_x^*\phi}{-\imath\zeta^2 - \frac{\imath}{2}|\phi|^2 + \frac{\imath}{2}\omega - \Omega} \\ &= \frac{\zeta|\phi|^2 - \frac{\imath}{2}(\tilde{r}_x - \imath\tilde{l}_x)(\tilde{r} + \imath\tilde{l})}{-\zeta^2 - \frac{1}{2}|\phi|^2 + \frac{\omega}{2} + \imath\Omega} = \text{imag} + \frac{-\frac{1}{2}(\tilde{r}\tilde{r}_x + \tilde{l}\tilde{l}_x)}{-\zeta^2 - \frac{1}{2}|\phi|^2 + \frac{\omega}{2} + \imath\Omega} \\ &= \text{imag} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\ln\left|-\frac{1}{2}(\tilde{r}^2 + \tilde{l}^2) + \frac{\omega}{2} - \zeta^2 + \imath\Omega\right|, \end{aligned} \tag{90}$$

where 'imag' denotes an imaginary quantity, having no effect on the final calculation of $|\gamma|$. Note that $A + \Omega$ is imaginary, so that the argument of the absolute value in (90) is real. It can also be written as $-\imath A - \imath\Omega$. We get

$$|\gamma|^2 = \frac{|-\imath A - \imath\Omega|}{|B|^2} = \frac{|A+\Omega|}{|B|^2} = \frac{1}{|A-\Omega|}, \tag{91}$$

using (30) once more. This results in

$$|\chi_1|^4 = |\gamma|^4|B|^4 = |A+\Omega|^2, \qquad |\chi_2|^4 = |\gamma|^4|A-\Omega|^2 = |A-\Omega|^2, \tag{92}$$

so that

$$\begin{aligned} W^* \cdot \mathcal{L}W &= 4\imath\Omega(|A+\Omega|^2 - |A-\Omega|^2) = 4\imath\Omega(|\imath A + \imath\Omega|^2 - |\imath A - \imath\Omega|^2) \\ &= 4\imath\Omega((\imath A + \imath\Omega)^2 - (\imath A - \imath\Omega)^2) = -16\Omega^2\imath A, \end{aligned} \tag{93}$$

and finally

$$K_2(\zeta) = -16\Omega^2(\zeta)\int_{-NT}^{NT}\left(\zeta^2 - \frac{\omega}{2} + \frac{1}{2}|\phi|^2\right)\mathrm{d}x. \tag{94}$$

Before we calculate this integral explicitly, we note the following: (i) the factor $-16\Omega^2(\zeta)$ is strictly positive or zero on the Lax spectrum $\zeta \in \mathbb{R}$. Those zero values are allowed, since clearly they correspond to the kernel of $\mathcal{L}$. (ii) the integrand $\imath A$ has fixed sign on any component of the Lax spectrum, since $-A^2 = -\Omega^2 + |B|^2$, which is strictly positive on the Lax spectrum. It follows that the value of the integral as a function of ζ can only change sign for values of $\zeta \notin \sigma_L$. In order to determine that such sign changes do in fact occur for all elliptic solutions, we calculate (94) explicitly, using standard tables [8]. Using $|\phi|^2 = k^2\mathrm{sn}^2(x,k) + b$, we obtain

$$K_2(\zeta) = -32\Omega^2(\zeta)NT(k)\left(\zeta^2 - \frac{b}{4} + \frac{1}{4}\left(k'^2 - 2\frac{E(k)}{K(k)}\right)\right), \tag{95}$$

where $E(k)$ denotes the complete elliptic integral of the second kind:

$$E(k) = \int_0^{\pi/2}\sqrt{1 - k^2\sin^2 y}\,\mathrm{d}y. \tag{96}$$

Using this definition and the definition of $K(k)$ (7), it is easily seen that the expression $k'^2 - 2E(k)/K(k)$ is positive for all $k \in [0, 1)$. It follows that the expression inside the outer parenthesis is negative for $\zeta = 0$, thus two sign changes of $K_2(\zeta)$ occur for each elliptic solution. As remarked above, these sign changes occur for $\zeta = \pm\zeta_0 \notin \sigma_L$, but the overall result is that $K_2(\zeta)$ has a different fixed sign on the different components of σ_L. Thus no conclusion about orbital stability can be drawn from $K_2(\zeta)$.

We turn to the other candidates $K_n(\zeta)$. Two sign changes need to be made undone. We wish to introduce two free parameters to do so, hence we turn to $K_4(\zeta) = p_4(\zeta)K_2(\zeta)$. Recall that $p_4(\zeta)$ is a polynomial of degree two, depending on the free constants $c_{4,2}$ and $c_{4,3}$. A direct calculation gives

$$p_4(\zeta) = \frac{\Omega_4(\zeta)}{\Omega(\zeta)} = 4\zeta^2 + 2\imath c_{4,3}\zeta - c_{4,2} + 3b + 1 + k^2. \tag{97}$$

Whole ranges of choices for the constants $c_{4,2}$ and $c_{4,3}$ allow us to achieve $K_4(\zeta) \geqslant 0$ for all $\zeta \in \sigma_L$, with equality attained only for ζ such that $W(\zeta)$ is in the null space of $\mathcal{L}$. Perhaps the most convenient choice is

$$c_{4,2} = 4\zeta_0^2 + 3b + 1 + k^2, c_{4,3} = 0. \tag{98}$$

We summarize our result in the following theorem.

Theorem 7. *There exist constants $c_{4,2}$ and $c_{4,3}$ such that $K_4(\zeta)$ is strictly positive on the Lax spectrum σ_L, with the exception of those values $\zeta \in \sigma_L$ that give rise to elements of the null space of $\mathcal{L}$, for which $K_4(\zeta) = 0$. Therefore, the elliptic solutions of the NLS equation are orbitally stable in $\mathbb{V}$.*

In other words, we have demonstrated that any elliptic solution of the NLS equation is orbitally stable with respect to subharmonic perturbations that are sufficiently smooth.

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