Numerical computation of the finite-genus solutions of the Korteweg-de Vries equation via Riemann–Hilbert problems

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July 30, 2012

Abstract

In this letter we describe how to compute the finite-genus solutions of the Korteweg-de Vries equation using a Riemann-Hilbert problem that is satisfied by the Baker-Akhiezer function corresponding to a Schrödinger operator with finite-gap spectrum. The recovery of the corresponding finite-genus solution is performed using the asymptotics of the Baker–Akhiezer function. This method has the benefit that the space and time dependence of the Baker-Akhiezer function appear in an explicit, linear and computable way. We make use of recent advances in the numerical solution of Riemann-Hilbert problems to produce an efficient and uniformly accurate numerical method for computing all finite-genus solutions of the KdV equation.

1 Introduction

The goal of this letter is to announce the results of [9]. A new and convenient representation of solutions of the Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R} \tag{1.1}$$

is found. In general we obtain not only periodic but also quasi-periodic solutions. This problem has been studied in great detail. Of significant impact are the results of Lax [4] and Novikov [6]. Full reviews of developments are found in Chapter 2 of [5] and Dubrovin's review article [2].

If we make a fundamental assumption on $u_0(x)$ (specifically that it admits a finite-gap Bloch spectrum, see Section 2) then the solution of (1.1) is expressed in terms of the zeros of a specific Baker–Akhiezer (BA) function [5].

This paper focuses on the derivation and numerical solution of a Riemann–Hilbert representation of the BA function. We use a Riemann–Hilbert problem (RHP) that, when solved, is used to find the BA function and extract the associated solution of the KdV equation. The x and t dependence for the solution appears in an explicit way. Furthermore, like its whole line counterpart derived from the inverse scattering transform, the dependence appears linearly in an exponential.

The RHP requires a regularization procedure using a g-function [1] which further simplifies the x and t dependence. The resulting RHP has piecewise constant jumps. Straightforward modifications allow the RHP to be numerically solved effectively using the techniques in [7]. These results extend some ideas of [10, 11] to the periodic and quasi-periodic regime. This results in an approximation of the BA function that is seen to be uniformly valid on its associated Riemann surface. The theory of [8] can be used to explain

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this uniform convergence. This produces a uniform approximation of the associated solution of the KdV equation in the entire (x, t)-plane.

In this paper we discuss the RHP problem for the BA function. Then we present theorems that allow all finite-gap BA functions to be reduced to a RHP with piecewise-constant jumps and fixed square-root singularities. We discuss the extraction of the solution to the KdV equation from the BA function and present some numerical results.

2 The finite-genus solutions of the KdV equation

We begin by considering the scattering problem associated with the KdV equation. The time-independent Schrödinger equation

$$-\Psi_{xx} - u_0(x)\Psi = \lambda\Psi, \tag{2.1}$$

is solved for bounded eigenfunctions $\Psi(x, \lambda)$. We define the Bloch spectrum

$$\mathcal{S}(u_0) = \{\lambda \in \mathbb{C} : \text{ there exists a solution } \Psi(x, \lambda) \text{ such that } \sup_{x\in\mathbb{R}} |\Psi(x, \lambda)| < \infty\}.$$

Assumption 2.1. $\mathcal{S}(u_0)$ *consists of a finite number of intervals. We say that* u_0 *is a finite gap potential.*

Define the Riemann surface Γ by the hyperelliptic algebraic curve

$$F(\lambda, w) = w^2 - P(\lambda) = 0, \quad P(\lambda) = (\lambda - \alpha_{g+1}) \prod_{j=1}^{g} (\lambda - \alpha_j)(\lambda - \beta_j).$$

We divide Γ in two sheets $\Gamma_\pm$. For a function f defined on Γ, we use $f_\pm$ to denote its restriction to $\Gamma_\pm$. It is known [5] that the two, linearly independent solutions of (2.1) can be interpreted as the piecewise definition of the BA function $\Psi_\pm$ on Γ. This function is uniquely determined by a divisor for its poles and the asymptotic behavior [5]. In what follows we assume without loss of generality that $a_1 = 0$, using the symmetries of the KdV equation.

3 From a g-genus Riemann surface to the cut plane

Consider the hyperelliptic Riemann surface Γ from Section 2. Given a point $Q = (\lambda, w) \in \Gamma$, we follow [3] and define the involution $*$ by $Q^* = (\lambda, -w)$. This is an isomorphism from one sheet of the Riemann surface to the other. We find a planar representation of the BA function that satisfies

$$\begin{aligned} \mathbf{\Psi}^+(x, t, \lambda) &= \mathbf{\Psi}^-(x, t, \lambda) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \lambda \in (\alpha_{n+1}, \infty) \cup \bigcup_{j=1}^{g} (\alpha_j, \beta_j), \\ \mathbf{\Psi}(x, t, \lambda) &= \begin{bmatrix} e^{i\lambda^{1/2}x + 4i\lambda^{3/2}} & e^{-i\lambda^{1/2}x - 4i\lambda^{3/2}} \end{bmatrix} (I + \mathcal{O}(\lambda^{-1/2})). \end{aligned}$$

Note that in general $\mathbf{\Psi}$ has poles in every interval $[\beta_j, \alpha_{j+1}]$ on either Γ_+ or Γ_-. We leave a discussion of the poles to later. Define

$$\begin{aligned} \boldsymbol{R}(x, t, \lambda) &= \begin{bmatrix} e^{-\zeta(x,t,\lambda)/2} & 0 \\ 0 & e^{\zeta(x,t,\lambda)/2} \end{bmatrix}, \\ \zeta(x, t, \lambda) &= 2ix\lambda^{1/2} + 8it\lambda^{3/2} \end{aligned}$$

The function $\mathbf{\Phi}(x,t,\lambda) = \Psi(x,t,\lambda)\boldsymbol{R}(x,t,\lambda)$ satisfies

$$\begin{aligned}
\mathbf{\Phi}^+(x,t,\lambda) &= \mathbf{\Phi}^-(x,t,\lambda) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \lambda \in (\alpha_{n+1},\infty) \cup \bigcup_{j=1}^{g} (\alpha_j,\beta_j), \\
\mathbf{\Phi}^+(x,t,\lambda) &= \mathbf{\Phi}^-(x,t,\lambda) \begin{bmatrix} e^{-\zeta(x,t,\lambda)} & 0 \\ 0 & e^{\zeta(x,t,\lambda)} \end{bmatrix}, \ \lambda \in \bigcup_{j=1}^{g} (\beta_j,\alpha_{j+1}), \\
\mathbf{\Phi}(x,t,\lambda) &= \begin{bmatrix} 1 & 1 \end{bmatrix} (I + \mathcal{O}(\lambda^{-1/2})).
\end{aligned} \tag{3.1}$$

The jump relations of the function on each of the (β_j, α_{j+1}) are oscillatory. Define the g-function

$$\mathcal{G}(x,t,\lambda) = \frac{\sqrt{P(\lambda)}}{2\pi i} \sum_{j=1}^{g} \int_{\beta_j}^{\alpha_{j+1}} \frac{-\zeta(x,t,s) + i\Omega_j(x,t)}{\sqrt{P(s)}^+} \frac{ds}{s-\lambda},$$

where the $\Omega_j(x,t)$ are constant in λ.

Lemma 3.1. *There exists a choice of the $\Omega_j(x,t)$ such that:*

- *each $\Omega_j(x,t)$ is real valued, and*
- *$\mathcal{G}(x,t,\lambda) = \mathcal{O}(\lambda^{-1/2})$ as $\lambda \to \infty$.*

Define

$$\boldsymbol{G}(x,t,\lambda) = \begin{bmatrix} e^{-\mathcal{G}(x,t,\lambda)} & 0 \\ 0 & e^{\mathcal{G}(x,t,\lambda)} \end{bmatrix},$$

and consider the function

$$\mathbf{\Sigma}(x,t,\lambda) = \mathbf{\Phi}(x,t,\lambda)\boldsymbol{G}(x,t,\lambda).$$

It can be shown that [9]

$$\begin{aligned}
\mathbf{\Sigma}^+(x,t,\lambda) &= \mathbf{\Sigma}^-(x,t,\lambda) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \lambda \in (\alpha_{g+1},\infty) \cup \bigcup_{j=1}^{g} (\alpha_j,\beta_j), \\
\mathbf{\Sigma}^+(x,t,\lambda) &= \mathbf{\Sigma}^-(x,t,\lambda) \begin{bmatrix} e^{i\Omega_j(x,t)} & 0 \\ 0 & e^{-i\Omega_j(x,t)} \end{bmatrix}, \ \lambda \in \bigcup_{j=1}^{g} (\beta_j,\alpha_{j+1}), \\
\mathbf{\Sigma}(x,t,\lambda) &= \begin{bmatrix} 1 & 1 \end{bmatrix} (1 + \mathcal{O}(\lambda^{-1/2})).
\end{aligned} \tag{3.2}$$

These relations do not uniquely determine a function until we specify where we allow the function to be unbounded. We assume that the poles of the BA function are at $(\beta_j, 0)$ for all j and we reduce the general case to this: Assume the poles of $\Psi_\pm$ are at points $Q_j = (z_j, \sigma_j \sqrt{P(z_j)}^+)$, $\ z_j \in [\beta_j, \alpha_{j+1}]$ for $j = 1, \ldots, g$. Here $\sigma_j = \pm$ is chosen so that the pole is on $\Gamma_\pm$. If we find a BA function $(\Psi_p)_\pm$ with poles at $(\beta_j, 0)$ and zeros at Q_j then the pointwise product

$$(\Psi_r)_\pm = (\Psi_p)_\pm \Psi_\pm,$$

has poles at $(0, \beta_j)$ and zeros in the gaps. We reduced the problem of computing $\Psi_\pm$ to that of computing $(\Psi_r)_\pm$ and $(\Psi_p)_\pm$ both of which have poles at $(\beta_j, 0)$. To compute these functions we must generalize the asymptotic behavior of the BA functions. We replace $\zeta(x,t,\lambda)$ with

$$\kappa(x,t,\lambda) = 2i(x+t_1)\lambda^{1/2} + 2i(4t+t_2)\lambda^{3/2} + 2i \sum_{j=3}^{g} t_j \lambda^{j-1/2}.$$

This alters the definition of $\Omega_j(x,t)$ but they still exist, satisfying the properties in Lemma 3.1.

Theorem 3.1. *The real constants t_j are chosen such that the BA function $(\Psi_p)_\pm$ with poles at $(\beta_j, 0)$ and asymptotic behavior*

$$(\Psi_p)_\pm \sim e^{\pm\kappa(0,0,\lambda)/2},$$

has its zeros at Q_j, $j = 1, \ldots, g$ and

$$(\Psi_r)\pm \sim e^{\pm\kappa(x,t,\lambda)/2}.$$

4 Solving the piecewise-constant Riemann–Hilbert problem

We introduce a slew of local parametrices. Define

$$\boldsymbol{Y}(k; a, b, \alpha, \beta, c) = \begin{bmatrix} -i(k-a)^\alpha (k-b)^\beta / c & i(k-a)^\alpha (k-b)^\beta \\ 1/c & 1 \end{bmatrix}, \quad \alpha, \beta = \pm\frac{1}{2}.$$

We choose the branch cut of $(k-a)^\alpha(k-b)^\beta$ to be along the interval $[a, b]$. Define

$$\begin{aligned}
\boldsymbol{A}_1(x,t,\lambda) &= \boldsymbol{Y}(\lambda; \alpha_1, \beta_1, 1/2, -1/2, 1), \\
\boldsymbol{A}_j(x,t,\lambda) &= \boldsymbol{Y}(\lambda; \alpha_j, \beta_j, 1/2, -1/2, \exp(-i\Omega_{j-1}(x,t))), \quad j = 2, \ldots, g+1, \\
\boldsymbol{B}_j(x,t,\lambda) &= \boldsymbol{Y}(\lambda; \alpha_j, \beta_j, 1/2, -1/2, \exp(-i\Omega_j(x,t))), \quad j = 1, \ldots, g.
\end{aligned}$$

This allows us to enforce boundedness at each α_j with a possible unbounded singularity at β_j. The matrices $\boldsymbol{A}_j$ are used locally at α_j and $\boldsymbol{B}_j$ at β_j. Further, define

$$\boldsymbol{\Delta}(x,t,\lambda) = \begin{bmatrix} \delta(x,t,\lambda) & 0 \\ 0 & 1/\delta(x,t,\lambda) \end{bmatrix}, \quad \delta(x,t,\lambda) = \prod_{j=1}^{g} \left(\frac{\lambda - \alpha_{j+1}}{\lambda - \beta_j} \right)^{\Omega_j(x,t)/(2\pi)}.$$

We take the branch cut for δ to be along the intervals $[\beta_j, \alpha_{j+1}]$ and we assume $\Omega_j(x,t) \in [0, 2\pi)$. Note that $\boldsymbol{\Delta}$ satisfies

$$\boldsymbol{\Delta}^+(x,t,\lambda) = \boldsymbol{\Delta}^-(x,t,\lambda) \begin{bmatrix} e^{i\Omega_j(x,t)} & 0 \\ 0 & e^{-i\Omega_j(x,t)} \end{bmatrix}, \quad \lambda \in (\beta_j, \alpha_{j+1}).$$

The last function needed is

$$\boldsymbol{H}(\lambda) = \frac{1}{2} \begin{bmatrix} 1 & 1 + \sqrt{\lambda - \alpha_{n+1}} \\ 1 & 1 - \sqrt{\lambda - \alpha_{n+1}} \end{bmatrix},$$

where $\sqrt{\lambda - \alpha_{n+1}}$ has its branch cut on $[\alpha_{n+1}, \infty)$. We define $\boldsymbol{K}(x,t,\lambda)$ to be the solution of the RHP shown in Figure 1. Note that this RHP is derived from the deformation of $(\Psi_r)_\pm$. We have the following theorem

Theorem 4.1. *Let $u(x,t)$ be a solution of the KdV equation arising from a finite-gap initial condition such that the poles of the associated BA function are Q_j. Then*

$$u(x,t) = 2i(s_2(x,t) - s_1(x,t)) + 2iE(x,t) \text{ where} \tag{4.1}$$

$$E(x,t) = -\frac{1}{2\pi} \sum_{n=1}^{g} \int_{\beta_n}^{\alpha_{n+1}} \frac{\partial_x \Omega_n(x,t) - 2\lambda^{1/2}}{\sqrt{P(\lambda)_+}} \lambda^g d\lambda,$$

$$\begin{bmatrix} s_1(x,t) & s_2(x,t) \end{bmatrix} = \lim_{\lambda \to \infty} \lambda \partial_x \boldsymbol{K}(x,t,\lambda). \tag{4.2}$$

Remark 4.1. *A consequence of this theorem is that we do not need to find both $(\Psi_p)_\pm$ and $(\Psi_r)_\pm$. Finding $(\Psi_r)_\pm$ allows us to reconstruct the solution of the KdV equation.*

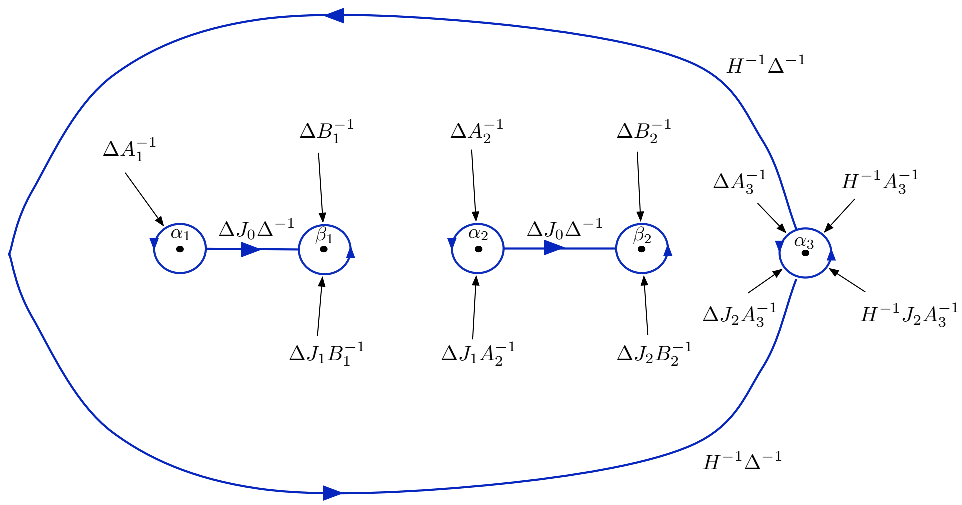

Figure 1: The RHP for $\boldsymbol{K}$. The same deformation works for RHPs which arise from arbitrary genus BA functions by adding additional contours.

5 Numerical Results

We set up systems of equations for $\Omega_j(x,t)$, $j = 1,\ldots,g$ and t_j, $j = 1,\ldots,g$ by evaluating integrals of the form

$$\int_{a_j}^{\lambda} \frac{f(s)}{\sqrt{P(s)}^+} ds \quad \text{and} \quad \int_{b_j}^{\lambda} \frac{f(s)}{\sqrt{P(s)}^+} ds$$

numerically using Chebyshev integration techniques. Once these constants are known we solve the RHP for $\boldsymbol{K}$ using the method in [7]. See Figures 2 and 3 for plots of solutions.

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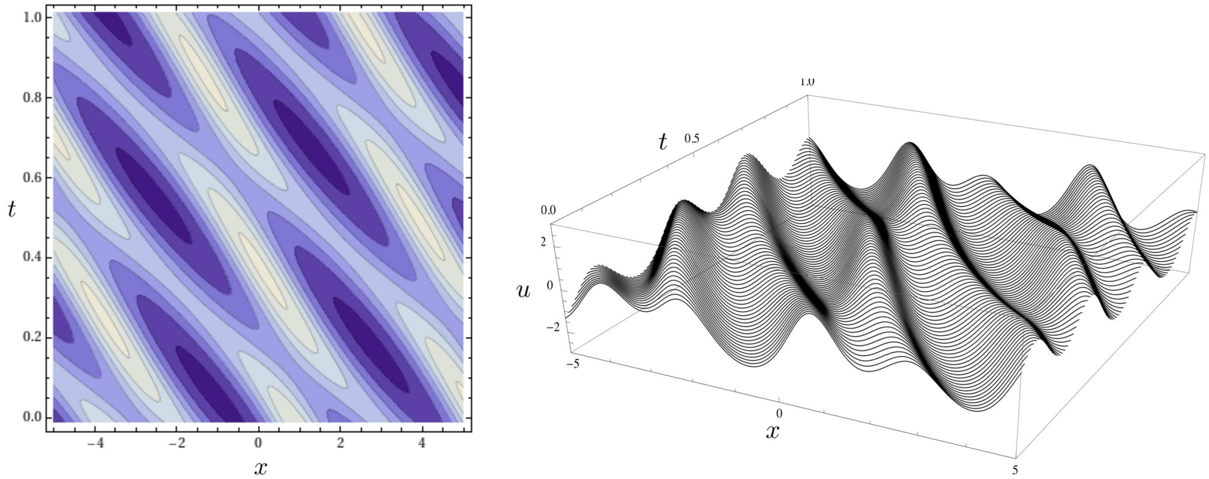
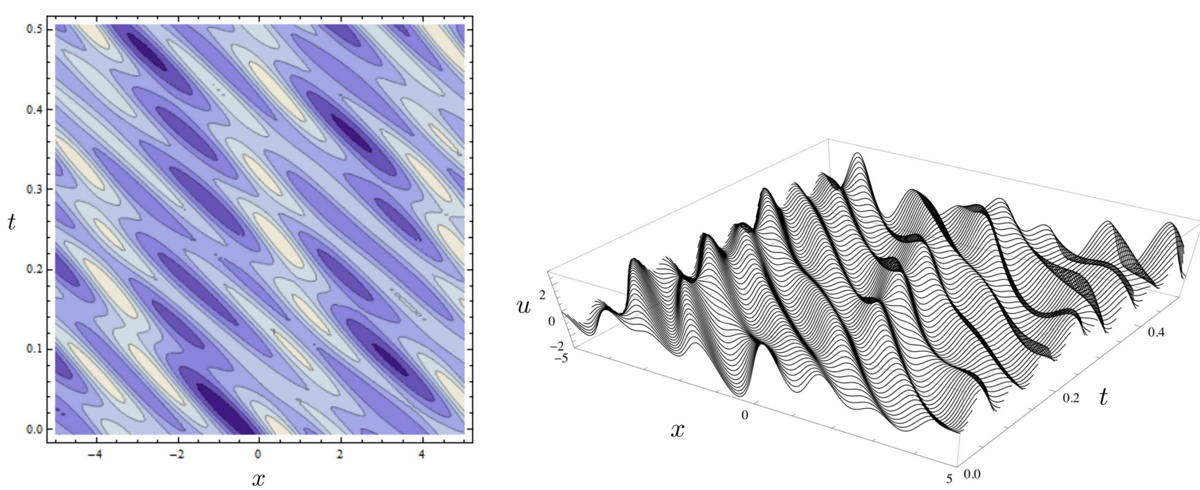


(a) (b)

Figure 2: A genus two solution with $\alpha_1 = 0, \beta_1 = .25, \alpha_2 = 1, \beta_2 = 2$ and $\alpha_3 = 2.25$ with the zeros of the BA function at $(.5, \sqrt{P(.5)}^+)$ and $(2.2, \sqrt{P(2.2)}^+)$ when $t = 0$. (a) A contour plot of the solution. Darker shades represent troughs. (b) A three-dimensional plot of the same solution.



(a) (b)

Figure 3: A genus five solution of the KdV equation with $\alpha_1 = 0, \beta_1 = .25, \alpha_2 = 1, \beta_2 = 2, \alpha_3 = 2.5, \beta_3 = 3, \alpha_4 = 3.3, \beta_4 = 3.5, \alpha_5 = 4, \beta_5 = 5.1$ and $\alpha_6 = 6$ with the zeros of the BA function at $(.5, \sqrt{P(.5)}^+)$, $(2.2, \sqrt{P(2,2)}^+)$, $(3.2, \sqrt{P(3.2)}^+)$, $(3.6, \sqrt{P(3.6)}^+)$ and $(5.3, \sqrt{P(5.3)}^+)$ when $t = 0$. (a) A contour plot of the solution. Darker shades represent troughs. (b) A three-dimensional plot of the same solution.

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