The explicit solution of linear, dissipative, second-order initial-boundary value problems with variable coefficients

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June 3, 2024

Abstract

We derive explicit solution representations for linear, dissipative, second-order Initial-Boundary Value Problems (IBVPs) with coefficients that are spatially varying, with linear, constant-coefficient, two-point boundary conditions. We accomplish this by considering the variable-coefficient problem as the limit of a constant-coefficient interface problem, previously solved using the Unified Transform Method of Fokas. Our method produces an explicit representation of the solution, allowing us to determine properties of the solution directly. As explicit examples, we demonstrate the solution procedure for different IBVPs of variations of the heat equation, and the linearized complex Ginzburg-Landau (CGL) equation (periodic boundary conditions). We can use this to find the eigenvalues of dissipative second-order linear operators (including non-self-adjoint ones) as roots of a transcendental function, and we can write their eigenfunctions explicitly in terms of the eigenvalues.

Contents

1 Introduction

The Unified Transform Method (UTM), or Method of Fokas, is used to solve Initial Value Problems (IVPs) and Initial-Boundary Value Problems (IBVPs) for integrable equations. Its application to linear, constant-coefficient partial differential equations (PDEs) is particularly convenient and straightforward. The UTM leads to many new insights on PDEs and IBVPs, see for instance [1, 7, 10, 11, 12, 13, 14, 29], and references therein. Especially relevant for us, the method has been used to explicitly solve interface problems and problems with piecewise-constant coefficients, see [5, 6, 17, 23, 24, 25, 26]. The purpose of this paper is to generalize the UTM to solve variable-coefficient IBVPs. In [29], Fokas and Treharne use a Lax Pair approach to analyze specific variable-coefficient IBVPs. Their approach reduces the problem from solving a *Partial* Differential Equation to solving an *Ordinary* Differential Equation (ODE) by writing the solution of the PDE as an integral over the solutions to a non-autonomous ODE, but it does not provide an explicit representation of the solution. This approach, like separation of variables, is useful if the associated ODE is a second-order, self-adjoint problem on a finite domain, for which we have regular Sturm-Liouville theory [3], but does not generalize well to problems that are not self adjoint, of higher order, or posed on an unbounded domain.

In our approach to variable-coefficient IBVPs, we divide the domain into N parts and approximate the equation by a constant-coefficient equation on each part. We solve the resulting interface problem using the UTM as shown in [5, 6, 17, 23, 24, 25, 26]. Using Cramer's rule, the solution in each part is found as a ratio of determinants. Through the nontrivial steps of obtaining an explicit expression for the determinants and taking the limit as N goes to infinity, a complicated but explicit solution expression is obtained. As in previous applications of the UTM (*e.g.,* [1, 13] for constant-coefficient problems, [6, 27] for interface problems), one of the benefits of our approach is characterizing which boundary conditions give rise to a well-posed IBVP. In particular, for the finite-interval problem, this work is consistent with Locker's work on Birkhoff regularity, *e.g.,* [16]. Since the UTM is generalizable to large classes of varying boundary conditions, IBVPs of higher order, including non-self-adjoint problems, we expect our method to generalize in these same directions as well.

In this manuscript, expanding on work presented in [9], we construct explicit solution expressions for general, second-order IBVPs with spatially-varying coefficients and with linear, two-point boundary conditions, as integrals over known quantities. We present the solutions for the whole-line problem, the half-line problem, and the finite-interval problem. We choose to demonstrate the solution process to the three problems in all their generality starting with the simplest, so that we start with the whole-line problem in Section 3, followed by the half-line problem in Section 4, finishing with the finite-interval problem in Section 5. In Sections 3.1, 4.1, 5.1, and 5.2, we restrict to specific examples. For the finite-interval problems, our explicit representation characterizes the eigenvalues of the ODE obtained through separation of variables and gives the eigenfunctions explicitly in terms of these eigenvalues, see Section 5.3. This allows for the numerical approximation of the eigenvalues, including for non-self-adjoint problems. Other numerical applications are presented in [9]. In Appendix A, we present a formal derivation of the solution. In this appendix we switch the order of exposition by deriving the solution to the finite-interval problem first, as the solution of the other two problems follows from it. We finish the paper with rigorous proofs in Appendices B–E.

Our formulae may seem complicated; however, they are similar to the solutions found in [21], which have been used to prove a variety of properties of solutions to ODEs and eigenvalue problems. Indeed, our notation is inspired by this book. While our solutions are similar, our methods are entirely different. The reader may also find our expressions reminiscent of path integrals [28], although those are usually used to propagate in time, unlike our spatial "discretization" approach.

2 Assumptions and definitions

Throughout this paper, we consider the linear, second-order evolution equation with spatially variable coefficients:

$$q_t = \alpha(x)\left(\beta(x)q_x\right)_x + \gamma(x)q + f(x,t), \qquad x \in \mathcal{D} \subset \mathbb{R}, \quad t > 0, \tag{2.1a}$$

$$q(x,0) = q_0(x), \qquad x \in \mathcal{D}, \tag{2.1b}$$

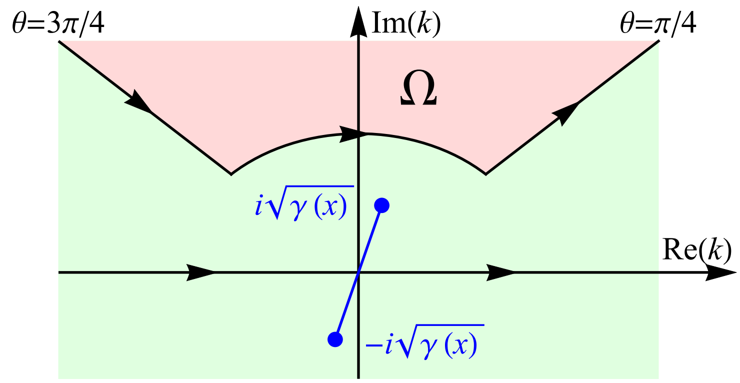

Figure 1: The region Ω with the branch cuts for $\mathfrak{g}(k,x)$.

on different domains $\mathcal{D}$ with (possibly) some functions $f_0(t)$, $f_1(t)$ prescribed at the boundary of $\mathcal{D}$. In all cases, the solution is written as

$$q(x,t) = \frac{1}{2\pi}\int_{\partial\Omega}\frac{\Phi(k,x,t)}{\Delta(k)}e^{-k^2t}\,dk, \tag{2.2}$$

where the functions $\Phi(k,x,t)$ and $\Delta(k)$ depend on $\mathcal{D}$ and the initial and boundary conditions provided. The region $\Omega = \{k\in\mathbb{C} : |k| > r \text{ and } \pi/4 < \arg(k) < 3\pi/4\}$, for some $r > \sqrt{M_\gamma}$, where $M_\gamma = \|\gamma\|_\infty$, as shown in Figure 1. In this section, we establish notation and introduce assumptions on the functions in (2.1) that we use throughout the paper.

We define $\arg(\,\cdot\,)\in[-\pi/2, 3\pi/2)$. We use $\mathcal{D}$ to denote the domain of the problem, so that $\mathcal{D}=\mathbb{R}$, $\mathcal{D}=(x_l,\infty)$, and $\mathcal{D}=(x_l,x_r)$ for the whole-line, half-line, and the finite-interval problems, respectively. The domain $\mathcal{D}$ is given by the open set, and we denote the closure by $\overline{\mathcal{D}}$. We write the L^1–norm over the domain $\mathcal{D}\subseteq\mathbb{R}$ as $\|\cdot\|_{\mathcal{D}}$. When used on a function of multiple variables, we implicitly assume a supremum norm on the other variables, *e.g.,* for a function $f(k,x)$ for $k\in\Omega\subseteq\mathbb{C}$ and $x\in\mathcal{D}$,

$$\|f\|_{\mathcal{D}} = \sup_{k\in\Omega}\int_{\mathcal{D}}|f(k,x)|\,dx.$$

In this way, the norms always represent fixed numbers, never functions. The notation $\mathrm{AC}(\,\cdot\,)$ represents the space of locally absolutely continuous functions on the closure of the domain. We use the 'big-oh' notation $O(\,\cdot\,)$ and the 'little-oh' notation $o(\,\cdot\,)$, as described in [19].

Assumption 1. *We assume the following about the coefficient functions α,β,γ:*

1. $\sup_{x\in\mathcal{D}}|\arg(\alpha(x)\beta(x)| < \pi/2$,
2. $\alpha,\beta\in\mathrm{AC}(\mathcal{D})$,
3. $m_{\alpha\beta} = \inf_{x\in D}|\alpha(x)\beta(x)| > 0$,
4. $\alpha\beta$, $\gamma\in L^\infty(\mathcal{D})$, *and we define* $M_{\alpha\beta} = \|\alpha\beta\|_\infty$ *and* $M_\gamma = \|\gamma\|_\infty$,
5. $(\beta'/\beta - \alpha'/\alpha)$, $\gamma'\in L^1(\mathcal{D})$.

Assumption 2. *We assume the following about the inhomogeneous, initial, and boundary functions f, q_0, f_m:*

1. *For the inhomogeneous function $f(x,t)$, we assume $f(x,\cdot)\in\mathrm{AC}((0,T))$ for each $x\in\overline{\mathcal{D}}$, and*

$$\|f\|_{\mathcal{D}} = \sup_{t\in[0,T]}\int_{\mathcal{D}}|f(x,t)|\,dx < \infty \qquad \text{and} \qquad \|f_t\|_{\mathcal{D}} = \sup_{t\in[0,T]}\int_{\mathcal{D}}|f_t(x,t)|\,dx < \infty.$$

2. *For the initial condition $q_0(x)$, we assume $q_0\in L^1(\mathcal{D})$.*
3. *For the boundary functions $f_m(t)$, $m=0,1$, we assume $f_m\in\mathrm{AC}((0,T))$ and $f_m'\in L^\infty((0,T))$.*

Assumption 3. *The finite-interval IBVP has a number of subcases. One of these, Case 4 (see Definition 8), requires the following assumption, in addition to Assumption 1:*

1. $\beta'/\beta - \alpha'/\alpha \in \mathrm{AC}(\mathcal{D})$.
2. *For the boundary functions* $f_m(t)$, $m = 0, 1$, *we require* $f_m' \in \mathrm{AC}((0,T))$ *and* $f_m'' \in L^\infty((0,T))$.

Remark 4.

- *Assumption 1.1 is equivalent to restricting to problems that we call* fully dissipative. *This is in contrast to problems that we call* partially dissipative *and/or* partially dispersive *where* $\sup_{x\in\mathcal{D}} |\arg(\alpha(x)\beta(x))| = \pi/2$, *but* $\inf_{x\in\mathcal{D}} |\arg(\alpha(x)\beta(x))| < \pi/2$ *or* fully dispersive *where* $|\arg(\alpha(x)\beta(x))| = \pi/2$ *for all* $x \in \mathcal{D}$.
- *Assumption 1.2 may seem odd considering that we derive our results from those for an interface problem in Appendix A. However, in that section, we use the mean value theorem as we let the limit of the number of interfaces* N *approach infinity, and thus we assume continuity of our function. This section can be amended to include piecewise continuous functions, but makes the solution formulas even more complicated. For simplicity, we restrict to continuous functions. Alternatively, we could employ distribution theory to extend the current results to discontinuous functions.*
- *Assumptions 1.3 and 1.4 are physically natural conditions to impose. Assumption 1.5 ensures that our solution is well defined. It may be possible to extend this to other* L^p *spaces or other more general spaces with some more work.*
- *Throughout this paper, we always use Assumptions 1 and 2. We clearly state when Assumption 3 is used, which is only for the 4th Boundary Case of the finite-interval.*

Definition 5. *For* $x \in \mathcal{D}$, *since* $\alpha(x)$ *and* $\beta(x)$ *are continuous by Assumption 1.2, we define arguments of* $\alpha(x)$ *and* $\beta(x)$ *to be* $\theta_\alpha(x)$ *and* $\theta_\beta(x)$, *chosen to be continuous*[1], *so that*

$$\alpha(x) = |\alpha(x)|e^{i\theta_\alpha(x)} \qquad \text{and} \qquad \beta(x) = |\beta(x)|e^{i\theta_\beta(x)}. \tag{2.3}$$

Using this, we define, for $x \in \mathcal{D}$,

$$\mu(x) = \frac{1}{\sqrt{\alpha(x)\beta(x)}} = \frac{1}{\sqrt{|\alpha(x)\beta(x)|}} e^{-\frac{i}{2}(\theta_\alpha(x)+\theta_\beta(x))}, \tag{2.4}$$

and, for $x \in \mathcal{D}$ *and* $k \in \mathbb{C}$,

$$\mathfrak{g}(k,x) = \sqrt{1 + \frac{\gamma(x)}{k^2}} = \sqrt{\left|1 + \frac{\gamma(x)}{k^2}\right|} e^{\frac{i}{2}\arg(1+\gamma(x)/k^2)}. \tag{2.5}$$

We also define $\mathfrak{n}(k,x) = \mu(x)\mathfrak{g}(k,x)$, $(\beta\mu)(x) = \beta(x)\mu(x)$, *and* $(\beta\mathfrak{n})(k,x) = \beta(x)\mathfrak{n}(k,x)$,

$$\sqrt{(\beta\mu)(x)} = \sqrt[4]{\left|\frac{\beta(x)}{\alpha(x)}\right|} e^{\frac{i}{4}(\theta_\beta(x)-\theta_\alpha(x))}, \qquad \sqrt{\mathfrak{g}(k,x)} = \sqrt[4]{\left|1 + \frac{\gamma(x)}{k^2}\right|} e^{\frac{i}{4}\arg(1+\gamma(x)/k^2)}, \tag{2.6}$$

and $\sqrt{(\beta\mathfrak{n})(k,x)} = \sqrt{(\beta\mu)(x)}\sqrt{\mathfrak{g}(k,x)}$. *Finally, we define*

$$\mathfrak{u}(x) = \frac{1}{\mu(x)}\left(\frac{\beta'(x)}{\beta(x)} - \frac{\alpha'(x)}{\alpha(x)}\right), \tag{2.7}$$

and

$$q_\alpha(x) = \frac{q_0(x)}{\alpha(x)}, \qquad f_\alpha(x,t) = \frac{f(x,t)}{\alpha(x)}, \qquad \tilde{f}_\alpha(k^2,x,t) = \int_0^t f_\alpha(x,s)e^{k^2 s}\,ds, \tag{2.8}$$

and $\psi_\alpha(k^2,x,t) = q_\alpha(x) + \tilde{f}_\alpha(k^2,x,t)$.

[1] Note that not necessarily $\theta_\alpha(x) = \arg(\alpha(x))$, given how we defined the range of $\arg(\,\cdot\,)$ above, because of the continuity requirement. For instance, if $\alpha(x) = \exp(ix)$ (and say $\beta(x) = \exp(-ix)$), we can set $\theta_\alpha(x) = x$.

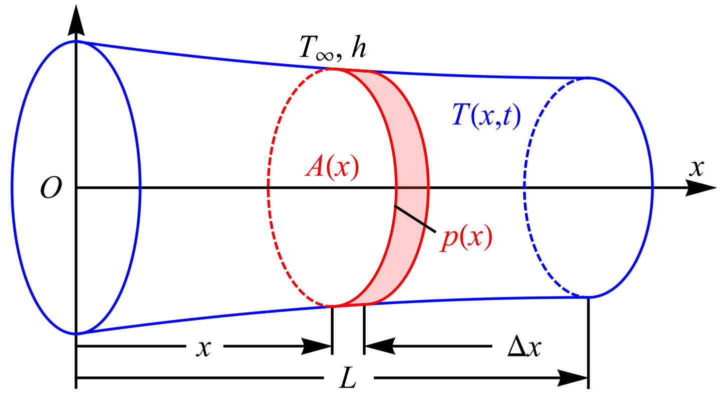

Figure 2: Terminology for the derivation of the partially lumped heat equation [20].

3 The whole-line problem

Consider (2.1) for $x \in \mathbb{R}$ and with decay at infinity,

$$\begin{aligned} q_t &= \alpha(x)\left(\beta(x) q_x\right)_x + \gamma(x) q + f(x,t), && x \in \mathbb{R}, \quad t > 0, && (3.1a)\\ q(x,0) &= q_0(x), && x \in \mathbb{R}, && (3.1b)\\ \lim_{|x|\to\infty} q(x,t) &= 0, && t > 0. && (3.1c) \end{aligned}$$

Theorem 6. *Under Assumptions 1 and 2, the IVP (3.1) has the solution*

$$q(x,t) = \frac{1}{2\pi}\int_{\partial\Omega} \frac{\Phi(k,x,t)}{\Delta(k)} e^{-k^2 t}\, dk, \tag{3.2}$$

where Ω is shown in Figure 1. Here

$$\Phi(k,x,t) = \int_{-\infty}^{\infty} \frac{\Psi(k,x,y)\psi_\alpha(k^2,y,t)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\, dy \qquad \text{and} \qquad \Delta(k) = \sum_{n=0}^{\infty} \mathcal{E}_{2n}^{(-\infty,\infty)}(k), \tag{3.3}$$

with, for $y < x$,

$$\Psi(k,x,y) = \exp\left(ik\int_y^x \mathfrak{n}(k,\xi)\, d\xi\right) \sum_{n=0}^{\infty}\sum_{\ell=0}^{n} (-1)^\ell \tilde{\mathcal{E}}_{n-\ell}^{(-\infty,y)}(k)\mathcal{E}_\ell^{(x,\infty)}(k), \tag{3.4}$$

and for $y > x$, $\Psi(k,x,y) = \Psi(k,y,x)$. Here, $\mathcal{E}_0^{(a,b)}(k) = 1$, $\tilde{\mathcal{E}}_0^{(a,b)}(k) = 1$, and for $n \geq 1$,

$$\mathcal{E}_n^{(a,b)}(k) = \frac{1}{2^n}\int_{a=y_0<y_1<\cdots<y_n<y_{n+1}=b} \left(\prod_{p=1}^{n} \frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right) \exp\left(ik\sum_{p=0}^{n}(1-(-1)^{n-p})\int_{y_p}^{y_{p+1}} \mathfrak{n}(k,\xi) d\xi\right) d\mathbf{y}_n, \tag{3.5a}$$

$$\tilde{\mathcal{E}}_n^{(a,b)}(k) = \frac{1}{2^n}\int_{a=y_0<y_1<\cdots<y_n<y_{n+1}=b} \left(\prod_{p=1}^{n} \frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right) \exp\left(ik\sum_{p=0}^{n}(1-(-1)^{p})\int_{y_p}^{y_{p+1}} \mathfrak{n}(k,\xi) d\xi\right) d\mathbf{y}_n, \tag{3.5b}$$

where $d\mathbf{y}_n = dy_1 \cdots dy_n$ and the prime denotes the derivative with respect to the second variable. The function $\mathcal{E}_n^{(a,b)}(k)$ is defined for $b = \infty$ and if n is even, for $a = -\infty$. The function $\tilde{\mathcal{E}}_n^{(a,b)}(k)$ is defined for $a = -\infty$ and if n is even, for $b = \infty$. The functions $\psi_\alpha(k^2,x,t)$, $\mathfrak{n}(k,x)$, and $(\beta\mathfrak{n})(k,x)$ are given in Definition 5.

Proof. The formal derivation is given in Appendix A. Its validity is proven in Appendices B–E. □

3.1 Example: The partially lumped heat equation

Consider the heat equation with partial lumping analysis [20], describing the temperature $T(x,t)$ in a body with minimal temperature variation in the y and z directions with ambient temperature T_∞, heat transfer coefficient h_0, thermal conductivity k_0, cross-sectional area $A(x)$, and perimeter $p(x)$, see Figure 2. We assume the length L is much greater than the width in the y and z directions. Ignoring temperature deviations in the y and z-directions, this IBVP takes the form

$$\theta_t = \frac{1}{A(x)}\big(A(x)\theta_x\big)_x - C(x)\theta, \qquad x\in\mathbb{R}, \quad t>0, \tag{3.6a}$$

$$\theta(x,0) = \theta_0(x), \qquad x\in\mathbb{R}, \tag{3.6b}$$

$$\lim_{|x|\to\infty}\theta(x,t) = 0, \qquad t>0. \tag{3.6c}$$

Here $\theta(x,t) = T(x,t) - T_\infty$ represents the difference of the temperature in the body $T(x,t)$ and the ambient temperature T_∞, the function $C(x) = h_0 p(x)/(k_0 A(x)) > 0$, and we equate the thermal diffusivity to 1 ($\alpha = 1$). Comparing this to (3.1), we have $\alpha(x) = 1/A(x)$, $\beta(x) = A(x)$, $\gamma(x) = -C(x)$, and $f(x,t)\equiv 0$. We require the absolute continuity of $A(x) > 0$, the boundedness of $C(x)$, and the absolute integrability of $A'(x)/A(x)$ and $C'(x)$, so that Assumption 1 is satisfied. Then we have the solution

$$\theta(x,t) = \frac{1}{2\pi}\int_{\partial\Omega}\frac{\Phi(k,x,t)}{\Delta(k)}e^{-k^2 t}\,dk, \tag{3.7}$$

where Ω is shown in Figure 1,

$$\mathfrak{n}(k,x) = \sqrt{1-\frac{C(x)}{k^2}}, \tag{3.8}$$

and $\psi_\alpha(k^2,x,t) = A(x)\theta_0(x)$. The functions $\Phi(k,x,t)$ and $\Delta(k)$ are given in (3.3).

3.2 A note about the integrability conditions.

A variable coefficient PDE in the form

$$q_t = a(x)q_{xx} + b(x)q_x + c(x)q, \tag{3.9}$$

can always be written in the form of (3.1a) as

$$q_t = a(x)\exp\left(-\int_{x_0}^x \frac{b(y)}{a(y)}\,dy\right)\left[\exp\left(\int_{x_0}^x \frac{b(y)}{a(y)}\,dy\right)q_x\right]_x + c(x)q, \tag{3.10}$$

which gives

$$\alpha(x) = a(x)\exp\left(-\int_{x_0}^x \frac{b(y)}{a(y)}\,dy\right), \qquad \beta(x) = \exp\left(\int_{x_0}^x \frac{b(y)}{a(y)}\,dy\right), \qquad \text{and} \qquad \gamma(x) = c(x). \tag{3.11}$$

From this, we have

$$\frac{(\beta\mathfrak{n})'(k,x)}{(\beta\mathfrak{n})(k,x)} = \frac{1}{2}\left(\frac{\beta'(x)}{\beta(x)} - \frac{\alpha'(x)}{\alpha(x)} + \frac{\gamma'(x)}{k^2+\gamma(x)}\right) = \frac{1}{2}\left(\frac{2b(x)}{a(x)} - \frac{a'(x)}{a(x)} + \frac{c'(x)}{k^2+c(x)}\right), \tag{3.12}$$

which we can see is not integrable (over an infinite or semi-infinite domain) if a, b, c are constants with $ab \neq 0$. This presents a problem for our solution. However, we can make the change of variables,

$$q(x,t) = \exp\left(-\int_{x_0}^x \frac{b(y)}{2a(y)}\,dy\right)u(x,t). \tag{3.13}$$

The PDE becomes

$$u_t = a(x)u_{xx} + \left(\frac{a'(x)b(x) - a(x)b'(x)}{2a(x)} - \frac{b(x)^2}{4a(x)} + c(x)\right)u, \tag{3.14}$$

for which, we have

$$\alpha(x) = a(x), \qquad \beta(x) = 1, \qquad \gamma(x) = \frac{a'(x)b(x) - a(x)b'(x)}{2a(x)} - \frac{b(x)^2}{4a(x)} + c(x), \tag{3.15}$$

and

$$\frac{(\beta\mathfrak{n})'(k,x)}{(\beta\mathfrak{n})(k,x)} = \frac{1}{2}\left(-\frac{a'(x)}{a(x)} + \frac{\gamma'(x)}{k^2+\gamma(x)}\right). \tag{3.16}$$

In the case of constant coefficients, the integrability condition, Assumption 1.5, is satisfied and our solution is well defined.

3.2.1 Example: The constant-coefficient, advected heat equation

Consider the constant-coefficient IBVP

$$q_t = q_{xx} + c q_x, \qquad x \in \mathbb{R}, \quad t > 0, \tag{3.17a}$$

$$q(x,0) = q_0(x), \qquad x \in \mathbb{R}, \tag{3.17b}$$

$$\lim_{|x|\to\infty} q(x,t) = 0, \qquad t > 0. \tag{3.17c}$$

This problem is well posed for $c \in \mathbb{R}$ [13]. The PDE (3.17a) can be written in the form (3.1a) as

$$q_t = e^{-cx}\left(e^{cx} q_x\right)_x, \tag{3.18}$$

with $\alpha(x) = e^{-cx}$, $\beta(x) = e^{cx}$, and $\gamma(x) = 0$. Since $\beta'/\beta - \alpha'/\alpha = 2c$ is not absolutely integrable over the real line, and Assumption 1.5 is not satisfied. With the change of variables $q(x,t) = e^{-cx/2} u(x,t)$, the IBVP (3.17) becomes

$$u_t = u_{xx} - \frac{c^2}{4} u, \qquad x \in \mathbb{R}, \quad t > 0, \tag{3.19a}$$

$$u(x,0) = e^{cx/2} q_0(x), \qquad x \in \mathbb{R}, \tag{3.19b}$$

$$\lim_{|x|\to\infty} u(x,t) = 0, \qquad t > 0. \tag{3.19c}$$

Now $\alpha(x) = 1$, $\beta(x) = 1$, and $\gamma(x) = -c^2/4$, so that $\beta'/\beta - \alpha'/\alpha = 0$ and $\gamma' = 0$, and Assumption 1 is satisfied. This example shows that, although all evolution equations can be written in the form (3.1a), a transformation may be needed before the integrability conditions are met and the solution expression (3.2) applies.

4 The half-line problem

Consider (2.1) on the half line $x > x_l$ with a linear, constant-coefficient boundary condition and decay at infinity,

$$q_t = \alpha(x)\left(\beta(x) q_x\right)_x + \gamma(x) q + f(x,t), \qquad x > x_l, \quad t > 0, \tag{4.1a}$$

$$q(x,0) = q_0(x), \qquad x > x_l, \tag{4.1b}$$

$$f_0(t) = a_0 q(x_l,t) + a_1 q_x(x_l,t), \qquad t > 0, \tag{4.1c}$$

$$\lim_{x\to\infty} q(x,t) = 0, \qquad t > 0, \tag{4.1d}$$

with $(a_0, a_1) \neq (0,0)$.

Theorem 7. *Under Assumptions 1 and 2, the IBVP (4.1) has the solution*

$$q(x,t) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{\Phi(k,x,t)}{\Delta(k)} e^{-k^2 t}\, dk, \tag{4.2}$$

where Ω is shown in Figure 1. Here

$$\Delta(k) = 2 \sum_{n=0}^{\infty} \left(\frac{(-1)^n i a_0}{k \mathfrak{n}(k,x_l)} - a_1 \right) \mathcal{E}_n^{(x_l,\infty)}(k), \tag{4.3}$$

and

$$\Phi(k,x,t) = \mathcal{B}_0(k,x) F_0(k^2,t) + \Phi_\psi(k,x,t), \tag{4.4}$$

where

$$\Phi_\psi(k,x,t) = \int_{x_l}^{\infty} \frac{\Psi(k,x,y)\psi_\alpha(k^2,y,t)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\, dy. \tag{4.5}$$

The functions $\psi_\alpha(k^2,x,t)$ and $\mathfrak{n}(k,x)$ are defined in Definition 5. The boundary term $\mathcal{B}_0(k,x)$ is defined by

$$\mathcal{B}_0(k,x) = \frac{4\beta(x_l) \exp\left(ik \int_{x_l}^{x} \mathfrak{n}(k,\xi)\, d\xi\right)}{\sqrt{(\beta\mathfrak{n})(k,x_l)}\sqrt{(\beta\mathfrak{n})(k,x)}} \sum_{n=0}^{\infty} (-1)^n \mathcal{E}_n^{(x,\infty)}(k), \tag{4.6}$$

and

$$F_m(k^2,t) = \int_0^t e^{k^2 s} f_m(s)\, ds, \qquad m = 0, 1. \tag{4.7}$$

Note that we will use $F_1(k^2,t)$ in the finite-interval problem in Section 5. For $x_l < y < x$,

$$\Psi(k,x,y) = 4\exp\left(ik\int_{x_l}^x \mathfrak{n}(k,\xi)\, d\xi\right) \sum_{n=0}^\infty \sum_{\ell=0}^n (-1)^\ell \left(\frac{a_0}{k\mathfrak{n}(k,x_l)}\mathcal{S}_{n-\ell}^{(x_l,y)}(k) - a_1\mathcal{C}_{n-\ell}^{(x_l,y)}(k)\right)\mathcal{E}_\ell^{(x,\infty)}(k), \tag{4.8}$$

and $\Psi(k,x,y) = \Psi(k,y,x)$ for $x_l < x < y$. $\mathcal{E}_n^{(a,b)}(k)$ is defined in (3.5a), $\mathcal{C}_0^{(a,b)}(k) = 1$, $\mathcal{S}_0^{(a,b)}(k) = 0$, and for $n \geq 1$,

$$\mathcal{C}_n^{(a,b)}(k) = \frac{1}{2^n}\int_{a=y_0<y_1<\cdots<y_n<y_{n+1}=b} \left(\prod_{p=1}^n \frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right) \cos\left(k\sum_{p=0}^n (-1)^p \int_{y_p}^{y_{p+1}} \mathfrak{n}(k,\xi)\, d\xi\right) d\mathbf{y}_n, \tag{4.9a}$$

$$\mathcal{S}_n^{(a,b)}(k) = \frac{1}{2^n}\int_{a=y_0<y_1<\cdots<y_n<y_{n+1}=b} \left(\prod_{p=1}^n \frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right) \sin\left(k\sum_{p=0}^n (-1)^p \int_{y_p}^{y_{p+1}} \mathfrak{n}(k,\xi)\, d\xi\right) d\mathbf{y}_n, \tag{4.9b}$$

where $d\mathbf{y}_n = dy_1\cdots dy_n$ and the prime denotes the derivative with respect to the second variable, as before.

4.1 Example: The advected heat equation

Consider the advected heat equation on the half line with spatially variable thermal conductivity $\sigma^2(x) > 0$ and velocity $c(x)$, without forcing and with homogeneous Dirichlet boundary conditions, i.e.,

$$q_t = \left(\sigma^2(x)q_x\right)_x - c(x)q_x, \qquad x>0, \quad t>0, \tag{4.10a}$$

$$q(x,0) = q_0(x), \qquad x>0, \tag{4.10b}$$

$$q(0,t) = 0, \qquad t>0, \tag{4.10c}$$

$$\lim_{x\to\infty} q(x,t) = 0, \qquad t>0. \tag{4.10d}$$

Here $x_l = 0$, $a_0 = 1$ and $a_1 = 0$. Further,

$$\alpha(x) = \exp\left(\int_0^x \frac{c(\xi)}{\sigma^2(\xi)}\, d\xi\right), \qquad \beta(x) = \sigma^2(x)\exp\left(-\int_0^x \frac{c(\xi)}{\sigma^2(\xi)}\, d\xi\right), \qquad \gamma(x) = 0, \tag{4.11}$$

$f(x,t) = 0$, and $f_0(t) = 0$. We require absolute continuity of $\sigma(x)$, boundedness of $c(x)$, and since

$$\frac{\beta'(x)}{\beta'(x)} - \frac{\alpha'(x)}{\alpha(x)} = \frac{2\sigma'(x)}{\sigma(x)} - \frac{2c(x)}{\sigma^2(x)}, \tag{4.12}$$

we require absolute integrability of $\sigma'(x)/\sigma(x)$ and $c(x)$, so that Assumption 1 is satisfied. Note that if $\sigma(x)$ is absolutely continuous and $\sigma'(x)/\sigma(x)$ is absolutely integrable, then $\sigma(x)$ is bounded above and below. This problem has the solution (4.2), where Ω is shown in Figure 1, $\mathfrak{n}(k,x) = 1/\sigma(x)$,

$$k\mathfrak{n}(k,0)\Delta(k) = 2i\sum_{n=0}^\infty (-1)^n \mathcal{E}_n^{(0,\infty)}(k). \tag{4.13}$$

Since $\mathcal{B}_0(k,x,t) = 0$ and $\psi(k^2,y,t) = q_0(y)$, we have

$$\Phi(k,x,t) = \int_0^\infty \exp\left(\frac{1}{2}\int_y^x \frac{c(\xi)}{\sigma^2(\xi)}\, d\xi\right)\frac{\Psi(k,x,y)q_0(y)}{\sqrt{\sigma(x)\sigma(y)}}\, dy, \tag{4.14}$$

and for $0 < y < x$,

$$k\mathfrak{n}(k,0)\Psi(k,x,y) = 4\exp\left(ik\int_0^x \frac{d\xi}{\sigma(\xi)}\right)\sum_{n=0}^\infty\sum_{\ell=0}^n (-1)^\ell \mathcal{S}_{n-\ell}^{(0,y)}(k)\mathcal{E}_\ell^{(x,\infty)}(k), \tag{4.15}$$

and $\Psi(k,x,y) = \Psi(k,y,x)$ for $0 < x < y$.

5 The finite-interval problem

Consider (2.1) on the finite interval $x_l < x < x_r$ with linear, constant-coefficient boundary conditions,

$$q_t = \alpha(x)\left(\beta(x)q_x\right)_x + \gamma(x)q + f(x,t), \qquad x \in (x_l, x_r), \quad t > 0, \tag{5.1a}$$

$$q(x,0) = q_0(x), \qquad x \in (x_l, x_r), \tag{5.1b}$$

$$f_0(t) = a_{11}q(x_l,t) + a_{12}q_x(x_l,t) + b_{11}q(x_r,t) + b_{12}q_x(x_r,t), \qquad t > 0, \tag{5.1c}$$

$$f_1(t) = a_{21}q(x_l,t) + a_{22}q_x(x_l,t) + b_{21}q(x_r,t) + b_{22}q_x(x_r,t), \qquad t > 0. \tag{5.1d}$$

Considering the concatenated matrix

$$(a:b) = \begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \end{pmatrix}, \tag{5.2}$$

we let $(a:b)_{i,j} = \det((a:b)_{\{1,2\},\{i,j\}})$ denote the determinant of the 2×2 minor with columns at i and j [22]. We require $\operatorname{rank}(a:b) = 2$ and one of the following *Boundary Cases.*

Definition 8. *For $x \in \mathcal{D} = (x_l, x_r)$, we define the constants*

$$m_{\mathfrak{c}_0} = \frac{(a:b)_{1,4}}{\mu(x_l)} - \frac{(a:b)_{2,3}}{\mu(x_r)}, \qquad m_{\mathfrak{c}_1} = \frac{(a:b)_{1,4}}{\mu(x_l)} + \frac{(a:b)_{2,3}}{\mu(x_r)}, \qquad m_{\mathfrak{s}} = \frac{(a:b)_{1,3}}{\mu(x_l)\mu(x_r)}, \tag{5.3}$$

and $\mathfrak{u}_\pm = \mathfrak{u}(x_r) \pm \mathfrak{u}(x_l)$, where $\mu(x)$ and $\mathfrak{u}(x)$ are defined in Definition 5. We define the following Boundary Cases:

1. *$(a:b)_{2,4} \neq 0$,*
2. *$(a:b)_{2,4} = 0$ and $m_{\mathfrak{c}_0} \neq 0$,*
3. *$(a:b)_{2,4} = 0$, $m_{\mathfrak{c}_0} = 0$, $m_{\mathfrak{c}_1} = 0$, and $(a:b)_{1,3} \neq 0$,*
4. *$(a:b)_{2,4} = 0$, $m_{\mathfrak{c}_0} = 0$, $m_{\mathfrak{c}_1} \neq 0$, and $m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}} \neq 0$.*

Please refer to Remark 10 for an interpretation of these different Boundary Cases.

Theorem 9. *Under Assumptions 1 and 2 (and for Boundary Case 4, Assumption 3), the IBVP (5.1) has the solution*

$$q(x,t) = \frac{1}{2\pi}\int_{\partial\Omega} \frac{\Phi(k,x,t)}{\Delta(k)} e^{-k^2 t}\, dk, \tag{5.4}$$

where Ω is shown in Figure 1. We define

$$\Xi(k) = \exp\left(ik\int_{x_l}^{x_r} \mathfrak{n}(k,\xi)\, d\xi\right), \tag{5.5}$$

where $\mathfrak{n}(k,x)$ is defined in Definition 5. Then

$$\Delta(k) = 2i\,\Xi(k)\left(\mathfrak{a}(k) + \sum_{n=0}^{\infty} \mathfrak{c}_n(k)\mathcal{C}_n^{(x_l,x_r)}(k) + \sum_{n=0}^{\infty} \mathfrak{s}_n(k)\mathcal{S}_n^{(x_l,x_r)}(k)\right), \tag{5.6}$$

with

$$\mathfrak{a}(k) = \frac{\beta(x_r)(a:b)_{1,2} + \beta(x_l)(a:b)_{3,4}}{k\sqrt{(\beta\mathfrak{n})(k,x_l)}\sqrt{(\beta\mathfrak{n})(k,x_r)}}, \tag{5.7a}$$

$$\mathfrak{c}_n(k) = (-1)^n\frac{(a:b)_{1,4}}{k\mathfrak{n}(k,x_l)} - \frac{(a:b)_{2,3}}{k\mathfrak{n}(k,x_r)}, \tag{5.7b}$$

$$\mathfrak{s}_n(k) = (-1)^n(a:b)_{2,4} + \frac{(a:b)_{1,3}}{k^2\mathfrak{n}(k,x_l)\mathfrak{n}(k,x_r)}. \tag{5.7c}$$

The numerator of (5.4) is

$$\Phi(k,x,t) = \mathcal{B}_0(k,x)F_0(k^2,t) + \mathcal{B}_1(k,x)F_1(k^2,t) + \Phi_\psi(k,x,t), \tag{5.8a}$$

where

$$\Phi_\psi(k,x,t)=\int_{x_l}^{x_r}\frac{\Psi(k,x,y)\psi_\alpha(k^2,y,t)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\,dy. \tag{5.8b}$$

The function $\psi_\alpha(k^2,x,t)$ *is defined in Definition 5,* $F_m(k^2,t)$ *is defined in (4.7), and the boundary terms* $\mathcal{B}_0(k,x)$ *and* $\mathcal{B}_1(k,x)$ *are given by*

$$\begin{aligned}\mathcal{B}_{2-j}(k,x)=(-1)^j\frac{4\Xi(k)}{\sqrt{(\beta\mathfrak{n})(k,x)}}\Bigg\{&\frac{\beta(x_r)}{\sqrt{(\beta\mathfrak{n})(k,x_r)}}\left[-\frac{a_{j1}}{k\mathfrak{n}(k,x_l)}\sum_{n=0}^\infty\mathcal{S}_n^{(x_l,x)}(k)+a_{j2}\sum_{n=0}^\infty\mathcal{C}_n^{(x_l,x)}(k)\right]\\&+\frac{\beta(x_l)}{\sqrt{(\beta\mathfrak{n})(k,x_l)}}\left[\frac{b_{j1}}{k\mathfrak{n}(k,x_r)}\sum_{n=0}^\infty\mathcal{S}_n^{(x,x_r)}(k)+b_{j2}\sum_{n=0}^\infty(-1)^n\mathcal{C}_n^{(x,x_r)}(k)\right]\Bigg\},\quad j=1,2.\end{aligned} \tag{5.8c}$$

Further, for $x_l<y<x<x_r$*,*

$$\begin{aligned}\Psi(k,x,y)=4\Xi(k)\Bigg\{&-(a:b)_{2,4}\sum_{n=0}^\infty\sum_{\ell=0}^n(-1)^\ell\mathcal{C}_{n-\ell}^{(x_l,y)}(k)\mathcal{C}_\ell^{(x,x_r)}(k)+\frac{(a:b)_{1,3}}{k^2\mathfrak{n}(k,x_l)\mathfrak{n}(k,x_r)}\sum_{n=0}^\infty\sum_{\ell=0}^n\mathcal{S}_{n-\ell}^{(x_l,y)}(k)\mathcal{S}_\ell^{(x,x_r)}(k)\\&+\frac{(a:b)_{1,4}}{k\mathfrak{n}(k,x_l)}\sum_{n=0}^\infty\sum_{\ell=0}^n(-1)^\ell\mathcal{S}_{n-\ell}^{(x_l,y)}(k)\mathcal{C}_\ell^{(x,x_r)}(k)-\frac{(a:b)_{2,3}}{k\mathfrak{n}(k,x_r)}\sum_{n=0}^\infty\sum_{\ell=0}^n\mathcal{C}_{n-\ell}^{(x_l,y)}(k)\mathcal{S}_\ell^{(x,x_r)}(k)\\&-\frac{\beta(x_r)(a:b)_{1,2}}{k\sqrt{(\beta\mathfrak{n})(k,x_l)}\sqrt{(\beta\mathfrak{n})(k,x_r)}}\sum_{n=0}^\infty\mathcal{S}_n^{(y,x)}(k)\Bigg\},\end{aligned} \tag{5.9a}$$

and, for $x_l<x<y<x_r$*,*

$$\begin{aligned}\Psi(k,x,y)=4\Xi(k)\Bigg\{&-(a:b)_{2,4}\sum_{n=0}^\infty\sum_{\ell=0}^n(-1)^\ell\mathcal{C}_{n-\ell}^{(x_l,x)}(k)\mathcal{C}_\ell^{(y,x_r)}(k)+\frac{(a:b)_{1,3}}{k^2\mathfrak{n}(k,x_l)\mathfrak{n}(k,x_r)}\sum_{n=0}^\infty\sum_{\ell=0}^n\mathcal{S}_{n-\ell}^{(x_l,x)}(k)\mathcal{S}_\ell^{(y,x_r)}(k)\\&+\frac{(a:b)_{1,4}}{k\mathfrak{n}(k,x_l)}\sum_{n=0}^\infty\sum_{\ell=0}^n(-1)^\ell\mathcal{S}_{n-\ell}^{(x_l,x)}(k)\mathcal{C}_\ell^{(y,x_r)}(k)-\frac{(a:b)_{2,3}}{k\mathfrak{n}(k,x_r)}\sum_{n=0}^\infty\sum_{\ell=0}^n\mathcal{C}_{n-\ell}^{(x_l,x)}(k)\mathcal{S}_\ell^{(y,x_r)}(k)\\&-\frac{\beta(x_l)(a:b)_{3,4}}{k\sqrt{(\beta\mathfrak{n})(k,x_l)}\sqrt{(\beta\mathfrak{n})(k,x_r)}}\sum_{n=0}^\infty\mathcal{S}_n^{(x,y)}(k)\Bigg\}.\end{aligned} \tag{5.9b}$$

Note that $\Psi(k,x,y)\neq\Psi(k,y,x)$ *unless* $\beta(x_r)(a:b)_{1,2}=\beta(x_l)(a:b)_{3,4}$*. The functions* $\mathcal{C}_n^{(a,b)}(k)$ *and* $\mathcal{S}_n^{(a,b)}(k)$ *are defined in (4.9a) and (4.9b), respectively.*

Remark 10. *We use underline to denote non-zero terms in this remark. Further, we use row reduction and the fact that the order of equations (5.1c) and (5.1d) is irrelevant.*

1. *If* $(a:b)_{2,4}\neq 0$*, the most general form of the matrix* $(a:b)$ *in (5.2) is*

$$(a:b)=\begin{pmatrix}a_{11}&\underline{a_{12}}&b_{11}&0\\a_{21}&0&b_{21}&\underline{b_{22}}\end{pmatrix}.$$

This case includes the classical Neumann and Robin boundary conditions at both boundaries. We refer to these as Robin-type boundary conditions. *In the case of constant coefficients, this is Birkhoff regular [16].*

2. *If* $(a:b)_{2,4}=0$ *and* $m_{\mathfrak{c}_0}\neq 0$*, the most general form of the matrix* $(a:b)$ *in (5.2) are*

$$(a:b)=\begin{pmatrix}a_{11}&\underline{a_{12}}&0&0\\a_{21}&0&\underline{b_{21}}&0\end{pmatrix},\ \begin{pmatrix}0&0&b_{11}&\underline{b_{12}}\\\underline{a_{21}}&0&b_{21}&0\end{pmatrix},\ \begin{pmatrix}\underline{a_{11}}&0&0&0\\0&a_{22}&b_{21}&\underline{b_{22}}\end{pmatrix},\ \begin{pmatrix}0&0&\underline{b_{11}}&0\\a_{21}&\underline{a_{22}}&0&b_{22}\end{pmatrix};$$

or

$$(a:b)=\begin{pmatrix}\underline{a_{11}}&0&\underline{b_{11}}&0\\0&\underline{a_{22}}&b_{21}&\underline{b_{22}}\end{pmatrix},\qquad\text{where}\qquad\frac{\underline{a_{11}}\,\underline{b_{22}}}{\mu(x_l)}+\frac{\underline{a_{22}}\,\underline{b_{11}}}{\mu(x_r)}\neq 0.$$

This case includes a Robin boundary condition on the left (or right) and a Dirichlet boundary condition on the right (or left). It also includes the classical periodic 'boundary conditions'. We refer to these as mixed-type or periodic-type boundary conditions. *In the case of constant coefficients, this is Birkhoff regular [16].*

3. *If $(a:b)_{2,4}=0$, $m_{\mathfrak{c}_0}=0$, $m_{\mathfrak{c}_1}=0$, and $(a:b)_{1,3}\neq 0$, the most general form of the matrix $(a:b)$ in (5.2) are*

$$(a:b)=\begin{pmatrix}\underline{a_{11}} & 0 & 0 & b_{12}\\ 0 & 0 & \underline{b_{21}} & 0\end{pmatrix} \qquad \text{or} \qquad (a:b)=\begin{pmatrix}\underline{a_{11}} & 0 & 0 & 0\\ 0 & a_{22} & \underline{b_{21}} & 0\end{pmatrix}.$$

This case includes the case of the classical Dirichlet boundary conditions. We refer to these as Dirichlet-type boundary conditions. *In the case of constant coefficients, this is Birkhoff regular for the case of Dirichlet boundary conditions* (i.e., *if $a_{22}=0=b_{12}$ or, equivalently, if $(a:b)_{1,2}=0=(a:b)_{3,4}$) and is Birkhoff irregular if $a_{22}\neq 0$ or $b_{12}\neq 0$ (or, equivalently, if $(a:b)_{1,2}\neq 0$ or $(a:b)_{3,4}\neq 0$) [16].*

4. *If $(a:b)_{2,4}=0$, $m_{\mathfrak{c}_0}=0$, $m_{\mathfrak{c}_1}\neq 0$, and $m_{\mathfrak{c}_1}\mathfrak{u}_+-8m_{\mathfrak{s}}\neq 0$, the most general form of the matrix $(a:b)$ is*

$$(a:b)=\begin{pmatrix}\underline{a_{11}} & 0 & \underline{b_{11}} & 0\\ a_{21} & \underline{a_{22}} & 0 & \underline{b_{22}}\end{pmatrix}, \qquad \text{where} \qquad \frac{\underline{a_{11}}\,\underline{b_{22}}}{\mu(x_l)}+\frac{\underline{a_{22}}\,\underline{b_{11}}}{\mu(x_r)}=0 \qquad \text{and} \qquad a_{21}\neq -\frac{1}{4}\mu(x_l)\underline{a_{22}}\mathfrak{u}_+.$$

This case does not include any classical boundary conditions. Instead, it is an interface problem on a circle. In the case of constant coefficients, this is Birkhoff irregular [16].

5.1 Example: The heat equation with homogeneous, Dirichlet boundary conditions

Consider the heat equation on the finite interval with spatially varying thermal conductivity $\sigma^2(x)$ without forcing and with homogeneous Dirichlet boundary conditions, *i.e.,*

$$q_t=\left(\sigma^2(x)q_x\right)_x, \qquad x\in(0,1), \quad t>0, \tag{5.10a}$$

$$q(x,0)=q_0(x), \qquad x\in(0,1), \tag{5.10b}$$

$$q(0,t)=0, \qquad t>0, \tag{5.10c}$$

$$q(1,t)=0, \qquad t>0. \tag{5.10d}$$

We let $x_l=0$, $x_r=1$, $\alpha(x)=1$, $\beta(x)=\sigma^2(x)$, $\gamma(x)=0$, $f(x,t)=0$, $f_m(t)=0$ ($m=0,1$), and

$$(a:b)=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}. \tag{5.11}$$

Since $(a:b)_{2,4}=0$, $m_{\mathfrak{c}_0}=0$, $m_{\mathfrak{c}_1}=0$, and $(a:b)_{1,3}=1\neq 0$, this is an example of Boundary Case 3 and it is *regular.* We require absolute continuity of $\sigma(x)$, integrability of $q_0(x)$, and absolutely integrability of $\sigma'(x)/\sigma(x)$. This has the solution

$$q(x,t)=\frac{1}{2\pi}\int_{\partial\Omega}\frac{\Phi(k,x)}{\Delta(k)}e^{-k^2t}\,dk, \tag{5.12}$$

where Ω is shown in Figure 1. Since $\mathfrak{n}(k,x)=1/\sigma(x)$, $\mathfrak{a}(k)=0$, $\mathfrak{c}_n(k)=0$, and $\mathfrak{s}_n(k)=\sigma(0)\sigma(1)/k^2$, then

$$\frac{k^2\Delta(k)}{2i\sigma(0)\sigma(1)}=\exp\left(ik\int_0^1\frac{d\xi}{\sigma(\xi)}\right)\sum_{n=0}^{\infty}\mathcal{S}_n^{(0,1)}(k), \tag{5.13}$$

and since $\mathcal{B}_0(k,x)=\mathcal{B}_1(k,x)=0$ and $\psi_\alpha(k^2,y,t)=q_0(y)$, we have

$$\Phi(k,x)=\int_0^1\frac{\Psi(k,x,y)q_0(y)}{\sqrt{\sigma(x)\sigma(y)}}\,dy, \tag{5.14}$$

where, for $0<y<x<1$,

$$\frac{k^2\Psi(k,x,y)}{4\sigma(0)\sigma(1)}=\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}\mathcal{S}_{n-\ell}^{(0,y)}(k)\mathcal{S}_{\ell}^{(x,1)}(k), \tag{5.15}$$

and $\Psi(k,x,y)=\Psi(k,y,x)$ for $0<x<y<1$. This is the same solution given in [9]. It reduces to the solution given in [13] for constant $\sigma(x)$.

5.2 Example: The CGL equation with periodic boundary conditions

The complex Ginzburg-Landau (CGL) equation is the nonlinear PDE

$$A_t = (1 + ia(x))A_{xx} + A - (1 + ib(x))|A|^2 A, \tag{5.16}$$

where a, b are real functions of x. In the special case $a(x) = 0 = b(x)$, (5.16) is the real Ginzburg-Landau equation. If $a(x), b(x) \to \infty$, (5.16) becomes the Nonlinear Schrödinger (NLS) equation [2]. Consider the linearized (about $A = 0$), CGL equation with periodic boundary conditions:

$$A_t = (1 + ia(x))A_{xx} + A, \qquad x \in (0, 1), \quad t > 0, \tag{5.17a}$$
$$A(x, 0) = A_0(x), \qquad x \in (0, 1), \tag{5.17b}$$
$$A(0, t) = A(1, t), \qquad t > 0, \tag{5.17c}$$
$$A_x(0, t) = A_x(1, t), \qquad t > 0. \tag{5.17d}$$

Here $x_l = 0$, $x_r = 1$, $\alpha(x) = 1 + ia(x)$, $\beta(x) = 1$, $\gamma(x) = 1$, $f(x, t) = 0$, $f_0(t) = 0 = f_1(t)$, and

$$(a : b) = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}. \tag{5.18}$$

We assume $a(x) \in \mathbb{R}$ and $a \in \mathrm{AC}(\mathcal{D})$, so that Assumption 1 is satisfied. Here,

$$\mu(x) = \frac{e^{-\frac{i}{2}\arctan(a(x))}}{\sqrt[4]{1 + a(x)^2}} \qquad \text{and} \qquad \mathfrak{g}(k) = \sqrt{1 + \frac{1}{k^2}}, \tag{5.19}$$

and $\mathfrak{n}(k, x) = \mu(x)\mathfrak{g}(k)$, where the square root in $\mathfrak{g}(k)$ is defined in (2.5). Since $(a : b)_{2,4} = 0$ and $m_{\mathfrak{c}_0} \neq 0$, this is a Boundary Case 2 example, which is *regular*. For simplicity, we assume that $a(x)$ is periodic, *i.e.,* $a(0) = a(1)$. This problem has the solution

$$q(x, t) = -\frac{1}{2\pi i}\int_{\partial\Omega} \frac{\tilde{\Phi}(k, x)}{\tilde{\Delta}(k)} e^{-k^2 t}\, dk, \tag{5.20}$$

where we define $\tilde{\Delta}(k) = k\mathfrak{n}(k, 0)\Xi(-k)\Delta(k)/(4i)$ and $\tilde{\Phi}(k, x) = -k\mathfrak{n}(k, 0)\Xi(-k)\Phi(k, x)/4$, and where Ω is shown in Figure 1. Here, $\mathfrak{a}(k) = 2/(k\mathfrak{n}(k, 0))$, $\mathfrak{c}_n(k) = -(1+(-1)^n)/(k\mathfrak{n}(k, 0))$, $\mathfrak{s}_n(k) = 0$, and since $\mathcal{B}_0(k, x, t) = 0 = \mathcal{B}_1(k, x, t)$ and $\psi_\alpha(k^2, x, t) = A_0(x)/(1 + ia(x))$,

$$\tilde{\Delta}(k) = 1 - \sum_{n=0}^{\infty} \mathcal{C}_{2n}^{(0,1)}(k) \qquad \text{and} \qquad \tilde{\Phi}(k, x) = \int_0^1 \frac{\tilde{\Psi}(k, x, y)A_0(y)}{(1 + ia(y))\sqrt{\mathfrak{n}(k, x)}\sqrt{\mathfrak{n}(k, y)}}\, dy. \tag{5.21}$$

We define $\tilde{\Psi}(k, x, y) = -k\mathfrak{n}(k, 0)\Xi(-k)\Psi(k, x)/4$,

$$\tilde{\Psi}(k, x, y) = \sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^\ell \mathcal{S}_{n-\ell}^{(0,y)}(k)\mathcal{C}_\ell^{(x,1)}(k) + \sum_{n=0}^{\infty}\sum_{\ell=0}^{n}\mathcal{C}_{n-\ell}^{(0,y)}(k)\mathcal{S}_\ell^{(x,1)}(k) + \sum_{n=0}^{\infty}\mathcal{S}_n^{(y,x)}(k), \tag{5.22}$$

for $0 < y < x < 1$, and $\tilde{\Psi}(k, x, y) = \tilde{\Psi}(k, y, x)$ for $0 < x < y < 1$.

5.3 Sturm-Liouville Problems: Eigenvalues and Eigenfunctions

Theorem 11. *The Sturm–Liouville problem*

$$\alpha(x)\left(\beta(x)y'\right)' + \gamma(x)y = \lambda y, \tag{5.23}$$

with boundary conditions

$$a_{11}y(x_l) + a_{12}y'(x_l) + b_{11}y(x_r) + b_{12}y'(x_r) = 0, \tag{5.24a}$$
$$a_{21}y(x_l) + a_{22}y'(x_l) + b_{21}y(x_r) + b_{22}y'(x_r) = 0, \tag{5.24b}$$

has the eigenfunctions

$$X_m(x) = \frac{C_m}{\sqrt{(\beta\mathfrak{n})(\kappa_m, x)}} \sum_{n=0}^{\infty} \mathcal{C}_n^{(x_l,x)}(\kappa_m) + \frac{S_m}{\sqrt{(\beta\mathfrak{n})(\kappa_m, x)}} \sum_{n=0}^{\infty} \mathcal{S}_n^{(x_l,x)}(\kappa_m), \tag{5.25}$$

corresponding to the eigenvalues $\lambda_m = -\kappa_m^2$*, where* $\{\kappa_m\}_{m=1}^{\infty}$ *are the zeros of* $\Delta(k)$ *(5.6). Here,*

$$C_m = -\frac{a_{12}\kappa_m\mathfrak{n}(\kappa_m, x_l)}{\sqrt{(\beta\mathfrak{n})(\kappa_m, x_l)}} - \frac{b_{11}}{\sqrt{(\beta\mathfrak{n})(\kappa_m, x_r)}} \sum_{n=0}^{\infty} \mathcal{S}_n^{(x_l,x_r)}(\kappa_m) - \frac{b_{12}\kappa_m\mathfrak{n}(\kappa_m, x_r)}{\sqrt{(\beta\mathfrak{n})(\kappa_m, x_r)}} \sum_{n=0}^{\infty} (-1)^n \mathcal{C}_n^{(x_l,x_r)}(\kappa_m), \tag{5.26a}$$

$$S_m = \frac{a_{11}}{\sqrt{(\beta\mathfrak{n})(\kappa_m, x_l)}} + \frac{b_{11}}{\sqrt{(\beta\mathfrak{n})(\kappa_m, x_r)}} \sum_{n=0}^{\infty} \mathcal{C}_n^{(x_l,x_r)}(\kappa_m) - \frac{b_{12}\kappa_m\mathfrak{n}(\kappa_m, x_r)}{\sqrt{(\beta\mathfrak{n})(\kappa_m, x_r)}} \sum_{n=0}^{\infty} (-1)^n \mathcal{S}_n^{(x_l,x_r)}(\kappa_m). \tag{5.26b}$$

Proof. Using (C.1) in (5.25) gives that the eigenfunctions solve the eigenvalue equation (5.23). Inserting (5.25) into the boundary conditions (5.24a), we find

$$a_{11}X_m(x_l) + a_{12}X_m'(x_l) + b_{11}X_m(x_r) + b_{12}X_m'(x_r) = C_mS_m - S_mC_m = 0. \tag{5.27}$$

For (5.24b), we find

$$\begin{aligned} &a_{21}X_m(x_l) + a_{22}X_m'(x_l) + b_{21}X_m(x_r) + b_{22}X_m'(x_r) \\ &= C_m\left[\frac{a_{21}}{\sqrt{(\beta\mathfrak{n})(\kappa_m, x_l)}} + \frac{b_{21}}{\sqrt{(\beta\mathfrak{n})(\kappa_m, x_r)}} \sum_{n=0}^{\infty} \mathcal{C}_n^{(x_l,x_r)}(\kappa_m) - \frac{b_{22}\kappa_m\mathfrak{n}(\kappa_m, x_r)}{\sqrt{(\beta\mathfrak{n})(\kappa_m, x_r)}} \sum_{n=0}^{\infty} (-1)^n \mathcal{S}_n^{(x_l,x_r)}(\kappa_m)\right] \\ &\quad + S_m\left[\frac{a_{22}\kappa_m\mathfrak{n}(\kappa_m, x_l)}{\sqrt{(\beta\mathfrak{n})(\kappa_m, x_l)}} + \frac{b_{21}}{\sqrt{\beta\mathfrak{n})(\kappa_m, x_r)}} \sum_{n=0}^{\infty} \mathcal{S}_n^{(x_l,x_r)}(\kappa_m) + \frac{b_{22}\kappa_m\mathfrak{n}(\kappa_m, x_r)}{\sqrt{(\beta\mathfrak{n})(\kappa_m, x_r)}} \sum_{n=0}^{\infty} (-1)^n \mathcal{C}_n^{(x_l,x_r)}(\kappa_m)\right]. \end{aligned} \tag{5.28}$$

Expanding (5.28), we obtain

$$\begin{aligned} &a_{21}X_m(x_l) + a_{22}X_m'(x_l) + b_{21}X_m(x_r) + b_{22}X_m'(x_r) \\ &= \frac{(a:b)_{1,2}\kappa_m}{\beta(x_l)} + \frac{(a:b)_{3,4}\kappa_m}{\beta(x_r)}\left[\sum_{n=0}^{\infty}(-1)^n\mathcal{C}_n^{(x_l,x_r)}(\kappa_m)\sum_{n=0}^{\infty}\mathcal{C}_n^{(x_l,x_r)}(\kappa_m) + \sum_{n=0}^{\infty}(-1)^n\mathcal{S}_n^{(x_l,x_r)}(\kappa_m)\sum_{n=0}^{\infty}\mathcal{S}_n^{(x_l,x_r)}(\kappa_m)\right] \\ &\quad + \frac{\kappa_m^2\mathfrak{n}(\kappa_m, x_l)\mathfrak{n}(\kappa_m, x_r)}{\sqrt{(\beta\mathfrak{n})(\kappa_m, x_l)}\sqrt{(\beta\mathfrak{n})(\kappa_m, x_r)}}\left[\sum_{n=0}^{\infty}\mathfrak{c}_n(k)\mathcal{C}_n^{(x_l,x_r)}(k) + \sum_{n=0}^{\infty}\mathfrak{s}_n(k)\mathcal{S}_n^{(x_l,x_r)}(k)\right]. \end{aligned} \tag{5.29}$$

Using the identity

$$1 = \sum_{n=0}^{\infty}(-1)^n\mathcal{C}_n^{(x_l,x_r)}(k)\sum_{n=0}^{\infty}\mathcal{C}_n^{(x_l,x_r)}(k) + \sum_{n=0}^{\infty}(-1)^n\mathcal{S}_n^{(x_l,x_r)}(k)\sum_{n=0}^{\infty}\mathcal{S}_n^{(x_l,x_r)}(k), \tag{5.30}$$

in (5.29), this becomes

$$a_{21}X_m(x_l) + a_{22}X_m'(x_l) + b_{21}X_m(x_r) + b_{22}X_m'(x_r) = \frac{\kappa_m^2\mathfrak{n}(\kappa_m, x_l)\mathfrak{n}(\kappa_m, x_r)}{\sqrt{(\beta\mathfrak{n})(\kappa_m, x_l)}\sqrt{(\beta\mathfrak{n})(\kappa_m, x_r)}}\Delta(\kappa_m) = 0, \tag{5.31}$$

and the second boundary condition (5.24b) is satisfied.

To prove (5.30), we define the right-hand side as $\mathfrak{e}_1$ and rewrite it as a Cauchy product, obtaining

$$\mathfrak{e}_1 = \sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^\ell\left[\mathcal{C}_\ell^{(x_l,x_r)}(k)\mathcal{C}_{n-\ell}^{(x_l,x_r)}(k) + \mathcal{S}_\ell^{(x_l,x_r)}(k)\mathcal{S}_{n-\ell}^{(x_l,x_r)}(k)\right]. \tag{5.32}$$

Letting $\ell \to n - \ell$ in the inner sum, we see that

$$\sum_{\ell=0}^{n}(-1)^\ell\left[\mathcal{C}_\ell^{(x_l,x_r)}(k)\mathcal{C}_{n-\ell}^{(x_l,x_r)}(k) + \mathcal{S}_\ell^{(x_l,x_r)}(k)\mathcal{S}_{n-\ell}^{(x_l,x_r)}(k)\right] = 0, \tag{5.33}$$

for odd ℓ. The $n=0$ term is 1. The $n \geq 2$ even terms are 0 and thus gives $\mathfrak{e}_1 = 1$. For $n=2$, we show

$$0 = \int_{x_l}^{x_r} dz_1 \int_{z_1}^{x_r} dz_2 \, \cos\left(\int_{x_l}^{x_r} - \int_{x_l}^{z_1} + \int_{z_1}^{z_2} - \int_{z_2}^{x_r} \nu(k,\xi)\, d\xi\right)$$
$$- \int_{x_l}^{x_r} dy_1 \int_{x_l}^{x_r} dz_1 \, \cos\left(\int_{x_l}^{y_1} - \int_{y_1}^{x_r} - \int_{x_l}^{z_1} + \int_{z_1}^{x_r} \nu(k,\xi)\, d\xi\right)$$
$$+ \int_{x_l}^{x_r} dy_1 \int_{y_1}^{x_r} dy_2 \, \cos\left(\int_{x_l}^{y_1} - \int_{y_1}^{y_2} + \int_{y_2}^{x_r} - \int_{x_l}^{x_r} \nu(k,\xi)\, d\xi\right). \tag{5.34}$$

Let I_j denote the three integrals above, in order. Since the first and the last term are equal and equal to

$$I_1 = I_3 = \int_{x_l}^{x_r} dy_1 \int_{y_1}^{x_r} dy_2 \, \cos\left(2 \int_{y_1}^{y_2} \nu(k,\xi)\, d\xi\right), \tag{5.35}$$

and since the second term is

$$I_2 = -\int_{x_l}^{x_r} dy_1 \int_{x_l}^{x_r} dz_1 \, \cos\left(2 \int_{z_1}^{y_1} \nu(k,\xi)\, d\xi\right)$$
$$= -\int_{x_l}^{x_r} dy_1 \int_{x_l}^{y_1} dz_1 \, \cos\left(2 \int_{z_1}^{y_1} \nu(k,\xi)\, d\xi\right) - \int_{x_l}^{x_r} dy_1 \int_{y_1}^{x_r} dz_1 \, \cos\left(2 \int_{z_1}^{y_1} \nu(k,\xi)\, d\xi\right)$$
$$= -2 \int_{x_l}^{x_r} dy_1 \int_{y_1}^{x_r} dz_1 \, \cos\left(2 \int_{z_1}^{y_1} \nu(k,\xi)\, d\xi\right), \tag{5.36}$$

and so the $n=2$ term is 0. The other n terms are similar. □

Comparing to Pöschel and Trubowitz [21] ($\alpha(x) = \beta(x) = 1$, $\gamma(x) = -q(x)$, $\lambda = k^2$), we find that

$$y_1(x,k^2,q) = \sqrt{\frac{\mathfrak{n}(k,0)}{\mathfrak{n}(k,x)}} \sum_{n=0}^{\infty} \mathcal{C}_n^{(0,x)}(k) \qquad \text{and} \qquad y_2(x,k^2,q) = \frac{1}{k\sqrt{\mathfrak{n}(k,0)}\sqrt{\mathfrak{n}(k,x)}} \sum_{n=0}^{\infty} \mathcal{S}_n^{(0,x)}(k), \tag{5.37}$$

where $\mathfrak{n}(k,x) = \sqrt{1 - q(x)/k^2}$.

5.3.1 Example: Eigenvalues for the CGL equation with periodic boundary conditions

We revisit the complex Ginzburg-Landau equation described in Section 5.2, setting $a(x) = x\sin(2\pi x)$. The associated eigenvalue problem is of the form

$$(1 + ia(x))y'' + y = \lambda y, \qquad y(0) = y(1), \qquad y'(0) = y'(1). \tag{5.38}$$

The eigenvalues $\lambda_m = -\kappa_m^2$ are related to the zeroes κ_m ($m = 0,1,2,\ldots$) of $\tilde{\Delta}(k)$ (5.21). Since $\tilde{\Delta}(k)$ is even in k, if κ_m is a root, so is $-\kappa_m$, and each gives rise to the same eigenvalue. Since $\mathfrak{g}(\pm i) = 0$, and

$$\mathcal{C}_n^{(0,1)}(0) = \frac{1}{2^n} \int_{0<\cdots<1} \left(\prod_{p=1}^{n} \frac{\mu'(y_p)}{\mu(y_p)}\right) d\mathbf{y}_n = \frac{1}{2^n n!} \left(\int_0^1 \frac{\mu'(y)}{\mu(y)}\, dy\right)^n = \frac{1}{2^n n!} \left(\log\left(\frac{\mu(1)}{\mu(0)}\right)\right)^n = 0, \tag{5.39}$$

then $\tilde{\Delta}(\pm i) = 0$, and $\kappa_0 = i$ (and $-i$) is an exact double root of $\tilde{\Delta}(k)$, and $\lambda_0 = -\kappa_0^2 = 1$ is an exact eigenvalue of the problem, which can be confirmed directly (with the constant eigenfunction). We define

$$\mathfrak{m}(x) = \int_0^x \mu(\xi)\, d\xi \qquad \text{and} \qquad \eta(y) = \frac{(\beta\mathfrak{n})'(k,x)}{(\beta\mathfrak{n})(k,x)} = \frac{\mu'(x)}{\mu(x)} = \frac{2\pi x\cos(2\pi x) + \sin(2\pi x)}{2i - 2x\sin(2\pi x)}. \tag{5.40}$$

We truncate (5.21) at order $n = N$ and denote as $\tilde{\Delta}_N(k)$. Denoting $\kappa = k\mathfrak{g}(k)$, the zeroth-order approximations of the roots of $\tilde{\Delta}(k)$ are

$$\tilde{\Delta}_0(k) = 1 - \cos(\mathfrak{m}(1)\kappa) = 0 \qquad \Rightarrow \qquad \kappa_m^{(0)} = \pm\frac{\sqrt{4m^2\pi^2 - \mathfrak{m}(1)^2}}{\mathfrak{m}(1)}, \qquad m = 1,2,3,\ldots. \tag{5.41}$$

Method:	λ_1	λ_2	λ_3	λ_4
chebfun	$-41.585+3.3357i$	$-41.689+7.7171i$	$-170.71+19.919i$	$-170.62+23.463i$
NDEigenvalues	$-41.585+3.3364i$	$-41.689+7.7167i$	$-170.73+19.929i$	$-170.65+23.464i$
Hill's Method	$-41.585+3.3358i$	$-41.689+7.7171i$	$-170.71+19.919i$	$-170.62+23.463i$
FindRoot: $\Delta_0(k)$	$-42.012+5.3928i$	$-42.012+5.3928i$	$-171.05+21.571i$	$-171.05+21.571i$
FindRoot: $\Delta_1(k)$	$-41.595+3.3501i$	$-41.671+7.7097i$	$-170.73+19.949i$	$-170.60+23.434i$
FindRoot: $\Delta_2(k)$	$-41.585+3.3356i$	$-41.689+7.7172i$	$-170.70+19.916i$	$-170.63+23.466i$

Table 1: Eigenvalues of the system (5.23) calculated using MATLAB's chebfun package, Mathematica's NDEigenvalues, Hill's method [4], and a root finding algorithm on $\tilde{\Delta}_N(k)$ for $N=0,1,2$.

As in the case $\kappa_0=\pm i$, these approximations are double roots. However, the actual eigenvalues are simple roots that are near these points. The next-order approximations $\kappa_m^{(1)}$ are the roots of

$$0=\tilde{\Delta}_1(k)=1-\cos(\mathfrak{m}(1)\kappa)-\mathcal{C}_2^{(0,1)}(k). \tag{5.42}$$

In order to compute $\mathcal{C}_2^{(0,1)}(k)$, we use an interpolation function for $\mathfrak{m}(x)$, and rewrite

$$k\sum_{p=0}^{n}(-1)^p\int_{y_p}^{y_{p+1}}\mathfrak{n}(k,\xi)\,d\xi=k\mathfrak{g}(k)\sum_{p=0}^{n}(-1)^p(\mathfrak{m}(y_{p+1})-\mathfrak{m}(y_p))=\kappa\left(\mathfrak{m}(1)-2\sum_{p=0}^{n}(-1)^p\mathfrak{m}(y_p)\right). \tag{5.43}$$

Then we use (4.9a) to compute the $\tilde{\Delta}_1(k)$. We use a root finding algorithm to find the roots, using that

$$\partial_\kappa\mathcal{C}_n^{(0,1)}(k)=\frac{1}{2^n}\int_{0<\cdots<1}\left(\prod_{p=1}^{n}\eta(y_p)\right)\cos\left(\kappa\left(\mathfrak{m}(1)-2\sum_{p=0}^{n}(-1)^p\mathfrak{m}(y_p)\right)\right)\left(\mathfrak{m}(1)-2\sum_{p=0}^{n}(-1)^p\mathfrak{m}(y_p)\right)d\mathbf{y}_n. \tag{5.44}$$

The results are shown in Table 1.

Appendices

A Derivations

In this appendix, we derive the solution expressions for the finite-interval, half-line, and whole-line IBVPs, in that order. The solution for each successive problem is obtained from the preceding one in a straightforward manner.

A.1 The finite-interval problem

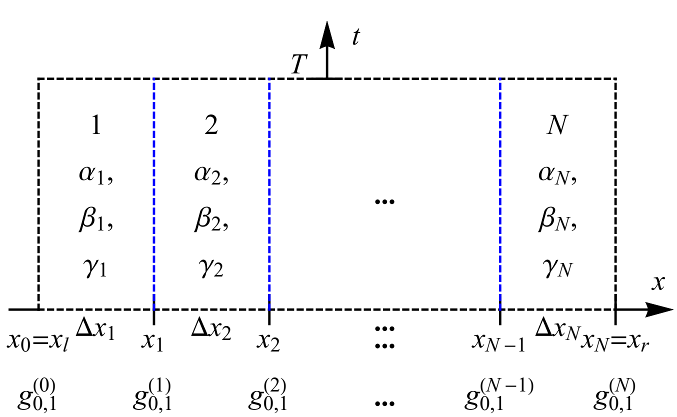

Figure 3: A partition of the finite interval $[x_l,x_r]$.

To consider the finite-interval problem (5.1), we form a partition $\{x_j, j = 0, \ldots, N\}$ of the interval $[x_l, x_r]$, see Figure 3. For simplicity, we assume that the partition is evenly spaced, *i.e.*, $\Delta x_j = \Delta x = (x_r - x_l)/N$ for $j = 1, \ldots, N$, although this assumption may be relaxed easily. On each subinterval, we solve the evolution equation (5.1a) with constant-coefficient approximations α_j, β_j, γ_j, $j = 1, \ldots, N$ for $\alpha(x)$, $\beta(x)$, and $\gamma(x)$ (such that $\alpha_j \to \alpha(x_j)$, *etc.*, in the limit as $N \to \infty$), with the initial condition restricted to the subinterval. At each interface x_j, $j = 1, \ldots, N-1$, we impose continuity of the solution and a jump discontinuity on the derivative, corresponding to the evolution equation, *i.e.*, we solve the following interface problem:

$$q_t^{(j)} = \alpha_j \beta_j q_{xx}^{(j)} + \gamma_j q^{(j)} + f(x,t), \qquad x \in (x_{j-1}, x_j), \quad t > 0, \quad j = 1, \ldots, N, \tag{A.1a}$$

$$q^{(j)}(x,0) = q_0(x), \qquad x \in (x_{j-1}, x_j), \quad t > 0, \quad j = 1, \ldots, N, \tag{A.1b}$$

$$q^{(j)}(x_j,t) = q^{(j+1)}(x_j,t), \qquad t > 0, \quad j = 1, \ldots, N-1, \tag{A.1c}$$

$$\beta_j q_x^{(j)}(x_j,t) = \beta_{j+1} q_x^{(j+1)}(x_j,t), \qquad t > 0, \quad j = 1, \ldots, N-1, \tag{A.1d}$$

with the boundary conditions

$$a_{11} q^{(1)}(x_l,t) + a_{12} q_x^{(1)}(x_l,t) + b_{11} q^{(N)}(x_r,t) + b_{12} q_x^{(N)}(x_r,t) = f_0(t), \qquad t > 0, \tag{A.2a}$$

$$a_{21} q^{(1)}(x_l,t) + a_{22} q_x^{(1)}(x_l,t) + b_{21} q^{(N)}(x_r,t) + b_{22} q_x^{(N)}(x_r,t) = f_1(t), \qquad t > 0. \tag{A.2b}$$

The jump discontinuity in the derivative (A.1d) can be derived by dividing the PDE (5.1a) by $\alpha(x)$ and integrating over a small interval containing x_j. Following [5, 6, 23, 24, 25, 26], we find the *local relations*

$$\left(e^{-i\kappa x + w_j t} q^{(j)}(x,t)\right)_t = \alpha_j \beta_j \left(e^{-i\kappa x + w_j t}\left(q_x^{(j)}(x,t) + i\kappa q^{(j)}(x,t)\right)\right)_x + e^{-i\kappa x + w_j t} f(x,t), \tag{A.3}$$

for $x \in (x_{j-1}, x_j)$, $1 \le j \le N$, and $w_j(\kappa) = \alpha_j \beta_j \kappa^2 - \gamma_j$. We define the "transforms"

$$\hat{q}_0^{(j)}(k) = \frac{1}{\alpha_j} \int_{x_{j-1}}^{x_j} e^{-iky} q_0(y)\, dy, \qquad j = 1, \ldots, N, \tag{A.4a}$$

$$\hat{q}^{(j)}(k,t) = \frac{1}{\alpha_j} \int_{x_{j-1}}^{x_j} e^{-iky} q^{(j)}(y,t)\, dy, \qquad j = 1, \ldots, N, \tag{A.4b}$$

$$\tilde{f}_j(k,t) = \frac{1}{\alpha_j} \int_0^t ds \int_{x_{j-1}}^{x_j} e^{-iky + Ws} f(y,s)\, dy, \qquad j = 1, \ldots, N, \tag{A.4c}$$

$$F_m(W,t) = \int_0^t e^{Ws} f_m(s)\, ds, \qquad m = 0, 1, \tag{A.4d}$$

$$g_m^{(j)}(W,t) = \int_0^t e^{Ws} q_{mx}^{(j)}(x_j,s)\, ds, \qquad j = 0, \ldots, N, \quad m = 0, 1, \tag{A.4e}$$

with $q_{mx}^{(0)}(x_l,t) = q_{mx}^{(1)}(x_l,t)$, for consistency at $j = 0$. Using the interface conditions (A.1c) and (A.1d), we have

$$g_0^{(j)}(W,t) = \int_0^t e^{Ws} q^{(j+1)}(x_j,s)\, ds, \qquad g_1^{(j)}(W,t) = \frac{\beta_{j+1}}{\beta_j} \int_0^t e^{Ws} q_x^{(j+1)}(x_j,s)\, ds, \qquad j = 0, \ldots, N-1, \tag{A.5}$$

where we define $\beta_0 = \beta_1$, again for consistency. From the boundary conditions (A.2),

$$a_{11} g_0^{(0)}(k^2,t) + a_{12} g_1^{(0)}(k^2,t) + b_{11} g_0^{(N)}(k^2,t) + b_{12} g_1^{(N)}(k^2,t) = F_0(k^2,t), \tag{A.6a}$$

$$a_{21} g_0^{(0)}(k^2,t) + a_{22} g_1^{(0)}(k^2,t) + b_{21} g_0^{(N)}(k^2,t) + b_{22} g_1^{(N)}(k^2,t) = F_1(k^2,t). \tag{A.6b}$$

Integrating the local relations (A.3) over $D_j = (x_{j-1}, x_j) \times (0, T)$, we find

$$\alpha_j \tilde{f}_j(\kappa, T) = \int_0^T dt \int_{x_{j-1}}^{x_j} dx \left[\left(e^{-i\kappa x + w_j t} q^{(j)}\right)_t - \alpha_j \beta_j \left(e^{-i\kappa x + w_j t}\left(q_x^{(j)} + i\kappa q^{(j)}\right)\right)_x\right], \qquad j = 1, \ldots, N. \tag{A.7}$$

Using Green's theorem,

$$\begin{aligned}\alpha_j \tilde{f}_j(\kappa,T) &= \int_{x_{j-1}}^{x_j} e^{-i\kappa x} q_0(x)\,dx - e^{w_j T}\int_{x_{j-1}}^{x_j} e^{-i\kappa x} q^{(j)}(x,T)\,dx \\ &\quad + \alpha_j e^{-i\kappa x_j}\int_0^T e^{w_j t}\left(\beta_j q_x^{(j)}(x_j,t) + i\beta_j\kappa q^{(j)}(x_j,t)\right)dt \\ &\quad - \alpha_j e^{-i\kappa x_{j-1}}\int_0^T e^{w_j t}\left(\beta_j q_x^{(j)}(x_{j-1},t) + i\beta_j\kappa q^{(j)}(x_{j-1},t)\right)dt, \qquad j=1,\ldots,N, \end{aligned} \tag{A.8}$$

which are rewritten as *global relations* using (A.4),

$$\begin{aligned} e^{w_j t}\hat{q}^{(j)}(\kappa,t) &= \hat{q}_0^{(j)}(\kappa) - \tilde{f}_j(\kappa,t) + e^{-i\kappa x_j}\left(\beta_j g_1^{(j)}(w_j,t) + i\beta_j\kappa g_0^{(j)}(w_j,t)\right) \\ &\quad - e^{-i\kappa x_{j-1}}\left(\beta_{j-1} g_1^{(j-1)}(w_j,t) + i\beta_j\kappa g_0^{(j-1)}(w_j,t)\right), \qquad j=1,\ldots,N. \end{aligned} \tag{A.9}$$

As in [17, 24, 26], it is convenient for the first arguments of $g_m^{(j)}(w_j,t)$ to be identical. We transform the independent variable κ in the jth equation as

$$\kappa = \nu_j(k) = \frac{k}{\sqrt{\alpha_j\beta_j}}\sqrt{1+\frac{\gamma_j}{k^2}}, \qquad j=1,\ldots,N. \tag{A.10}$$

We do not worry about the branch cuts here. The resulting branch cuts in the solution are defined in Section 2 and proven to be correct in Appendices B–E. However, since we assume that $\gamma(x)$ is bounded, there are no branch cuts for $|k| > \sqrt{M_\gamma}$ for $k \in \Omega$ where $M_\gamma > 0$ is defined in Assumption 1.4. Until we take the limit, we suppress the k dependence of $\nu_j(k)$. Our global relations (A.9) become

$$\begin{aligned} e^{k^2 t}\hat{q}^{(j)}(\nu_j,t) &= \hat{q}_0^{(j)}(\nu_j) - \tilde{f}_j(\nu_j,t) + e^{-i\nu_j x_j}\left(\beta_j g_1^{(j)}(k^2,t) + i\beta_j\nu_j g_0^{(j)}(k^2,t)\right) \\ &\quad - e^{-i\nu_j x_{j-1}}\left(\beta_{j-1} g_1^{(j-1)}(k^2,t) + i\beta_j\nu_j g_0^{(j-1)}(k^2,t)\right), \qquad j=1,\ldots,N. \end{aligned} \tag{A.11}$$

These relations are valid for $k \in \mathbb{C}$, since the domains are bounded. Letting $k \mapsto -k$, (and $\nu_j \mapsto -\nu_j$), gives $2N$ equations, along with (A.6) for $2N+2$ unknowns. We write this linear system of equations in matrix form as

$$\mathcal{A}_N(k) X_N(k^2,t) = Y_N(k,t) - e^{k^2 t}\mathcal{Y}_N(k,t), \tag{A.12}$$

where

$$X_N(k^2,t) = \left(g_0^{(0)}(k^2,t),\ldots,g_0^{(N)}(k^2,t),\beta_0 g_1^{(0)}(k^2,t),\ldots,\beta_N g_1^{(N)}(k^2,t)\right)^\top, \tag{A.13a}$$

$$\begin{aligned} Y_N(k,t) &= \left(0,\hat{q}_0^{(1)}(\nu_1),\ldots,\hat{q}_0^{(N)}(\nu_N),\hat{q}_0^{(1)}(-\nu_1),\ldots,\hat{q}_0^{(N)}(-\nu_N),0\right)^\top \\ &\quad - \left(-F_0(k^2,t),\ \tilde{f}_1(\nu_1,t),\ldots,\tilde{f}_N(\nu_N,t),\tilde{f}_1(-\nu_1,t),\ldots,\tilde{f}_N(-\nu_N,t),\ -F_1(k^2,t)\right)^\top, \end{aligned} \tag{A.13b}$$

$$\mathcal{Y}_N(k,t) = \left(0,\hat{q}^{(1)}(\nu_1,t),\ldots,\hat{q}^{(N)}(\nu_N,t),\hat{q}^{(1)}(-\nu_1,t),\ldots,\hat{q}^{(N)}(-\nu_N,t),0\right)^\top, \tag{A.13c}$$

and

$$\mathcal{A}_N(k) = \left(\begin{array}{ccccc:ccccc} a_{11} & 0 & \cdots & 0 & b_{11} & a_{12}/\beta_0 & 0 & \cdots & 0 & b_{12}/\beta_N \\ \hdashline & & A_{N,N+1}(k) & & & & & B_{N,N+1}(k) & & \\ \hdashline & & A_{N,N+1}(-k) & & & & & B_{N,N+1}(-k) & & \\ \hdashline a_{21} & 0 & \cdots & 0 & b_{21} & a_{22}/\beta_0 & 0 & \cdots & 0 & b_{22}/\beta_N \end{array}\right), \tag{A.14}$$

where

$$A_{N,N+1}(k) = \begin{pmatrix} i\beta_1\nu_1 e^{-i\nu_1 x_0} & -i\beta_1\nu_1 e^{-i\nu_1 x_1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & i\beta_N\nu_N e^{-i\nu_N x_{N-1}} & -i\beta_N\nu_N e^{-i\nu_N x_N} \end{pmatrix}, \tag{A.15}$$

and

$$B_{N,N+1}(k) = \begin{pmatrix} e^{-i\nu_1 x_0} & -e^{-i\nu_1 x_1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & e^{-i\nu_N x_{N-1}} & -e^{-i\nu_N x_N} \end{pmatrix}. \tag{A.16}$$

Here $A_{N,N+1}(k)$ and $B_{N,N+1}(k)$ are $N \times (N+1)$-dimensional matrices. Since the contribution involving $\mathcal{Y}_N$ along the contour $\partial\Omega$ (see below) is zero [26], it suffices to solve $\mathcal{A}_N X_N = Y_N$ for the unknown functions $g_m^{(j)}$. This is further justified in Appendices B–E. Using Cramer's rule,

$$X_N^{(j)}(k^2,t) = g_0^{(j-1)}(k^2,t) = \frac{\det\left(\mathcal{A}_N^{(j)}(k)\right)}{\det\left(\mathcal{A}_N(k)\right)}, \qquad j = 1, \dots, N+1, \tag{A.17}$$

where the matrix $\mathcal{A}_N^{(j)}(k)$ is $\mathcal{A}_N(k)$ with the jth column replaced by Y_N. If we multiply this equation by $ke^{-k^2 t}$ and integrate over $\partial\Omega$, where $\Omega = \{k \in \mathbb{C} : |k| > r \text{ and } \pi/4 < \arg(k) < 3\pi/4\}$ for some $r > \sqrt{M_\gamma}$, see Figure 1, we can invert the time "transform" $g_0^{(j-1)}(k^2,t)$ [13], to find

$$q^{(j)}(x_{j-1},t) = \frac{1}{i\pi}\int_{\partial\Omega} \frac{\det(\mathcal{A}_N^{(j)}(k))}{\det(\mathcal{A}_N(k))} ke^{-k^2 t}\, dk, \qquad j = 1, \dots, N+1. \tag{A.18}$$

This gives the solution at the interface boundary points, which is all that is needed to consider the limit $q(x,t) = \lim_{N\to\infty} q^{(j)}(x_{j-1},t)$, where the N dependence of $q^{(j)}(x_{j-1},t)$ is implicit. Alternatively, we could compute the full solution of the interface problem as in [5, 24, 25, 26] and calculate that limit. This gives the same result.

Define

$$D_N(k) = i^{N+1}\frac{\det(\mathcal{A}_N(k))}{\nu_1\nu_N}\left(\prod_{p=1}^{N-1}\frac{1}{\Lambda_p^+}\right), \tag{A.19}$$

with

$$\Lambda_p^{\boldsymbol{\ell}} = (\beta\nu)_{p+1} + (-1)^{\ell_p+\ell_{p+1}}(\beta\nu)_p \qquad \text{and} \qquad \Lambda_p^{\pm} = (\beta\nu)_{p+1} \pm (\beta\nu)_p, \tag{A.20}$$

where $\boldsymbol{\ell} \in \{0,1\}^N$, so that $\ell_p, \ell_{p+1} \in \{0,1\}$ and $(\beta\nu)_j = \beta_j\nu_j$. Note that $\Lambda_p^{\boldsymbol{\ell}} = \Lambda_p^+$ when $\ell_p = \ell_{p+1}$ and $\Lambda_p^{\boldsymbol{\ell}} = \Lambda_p^-$ when $\ell_p \neq \ell_{p+1}$. For $N \le 8$, we explicitly verify using Mathematica that

$$\begin{aligned} D_N(k) = 2i\Bigg\{&\frac{\beta_N(a:b)_{1,2} + \beta_1(a:b)_{3,4}}{\sqrt{(\beta\nu)_1}\sqrt{(\beta\nu)_N}}\frac{\sqrt{(\beta\nu)_N}}{\sqrt{(\beta\nu)_1}}\left(\prod_{p=1}^{N-1}\frac{2(\beta\nu)_p}{\Lambda_p^+}\right) + (a:b)_{2,4}\mathfrak{S}_{N,1}^{(1,N)}(k) + \frac{(a:b)_{1,3}}{\nu_1\nu_N}\mathfrak{S}_{N,0}^{(1,N)}(k) \\ &+ \frac{(a:b)_{1,4}}{\nu_1}\mathfrak{C}_{N,1}^{(1,N)}(k) - \frac{(a:b)_{2,3}}{\nu_N}\mathfrak{C}_{N,0}^{(1,N)}(k)\Bigg\}, \end{aligned} \tag{A.21}$$

where $(a:b)_{i,j} = \det((a:b)_{\{1,2\},\{i,j\}})$ is the determinant of the minor of maximal size with columns at i and j [22] of the concatenated matrix $(a:b)$. We define

$$\mathfrak{C}_{N,\lambda}^{(q,s)}(k) = \sum_{\substack{\boldsymbol{\ell}\in\{0,1\}^{s-q+1} \\ \ell_q=0}} (-1)^{\lambda\ell_s}\left(\prod_{p=q}^{s-1}\frac{\Lambda_p^{\boldsymbol{\ell}}}{\Lambda_p^+}\right)\cos\left(\sum_{p=q}^{s}(-1)^{\ell_p}\nu_p\Delta x\right), \tag{A.22a}$$

$$\mathfrak{S}_{N,\lambda}^{(q,s)}(k) = \sum_{\substack{\boldsymbol{\ell}\in\{0,1\}^{s-q+1} \\ \ell_q=0}} (-1)^{\lambda\ell_s}\left(\prod_{p=q}^{s-1}\frac{\Lambda_p^{\boldsymbol{\ell}}}{\Lambda_p^+}\right)\sin\left(\sum_{p=q}^{s}(-1)^{\ell_p}\nu_p\Delta x\right), \tag{A.22b}$$

where $\lambda = 0, 1$. We do not prove this result for general N. Its justification follows indirectly from the proofs in Appendices B–E. We can show that

$$\prod_{p=1}^{N-1} \frac{2(\beta\nu)_p}{\Lambda_p^+} = \exp\left(\sum_{p=1}^{N-1} \left(\ln(2(\beta\nu)_p) - \ln(\Lambda_p^+)\right)\right) = \frac{\sqrt{(\beta\nu)_1}}{\sqrt{(\beta\nu)_N}} + O(\Delta x), \tag{A.23}$$

as $N \to \infty$ and $\Delta x \to 0^+$. Similarly,

$$\frac{\Lambda_p^-}{\Lambda_p^+} = \frac{1}{2}\frac{(\beta\mathfrak{n})'(k, x_p)}{(\beta\mathfrak{n})(k, x_p)}\Delta x + O((\Delta x)^2) \qquad \text{and} \qquad \prod_{p=\ell}^{m-1} \frac{2(\beta\nu)_p}{\Lambda_p^+} = \frac{\sqrt{(\beta\mathfrak{n})(k, x_\ell)}}{\sqrt{(\beta\mathfrak{n})(k, x_m)}} + O(\Delta x), \tag{A.24}$$

as $\ell, m, N \to \infty$, $\Delta x \to 0^+$, and where the prime denotes the derivative with respect to the spatial variable, and $\mathfrak{n}(k, x)$, $(\beta\mathfrak{n})(k, x)$ are defined in Definition 5. Note that to use (A.24), we assume $(\beta\mathfrak{n})(k, x)$ is a smooth function of x. If this function has a countable number of discontinuities, it is possible to proceed, but we have to account for the jumps.

We wish to consider (A.21) as $N \to \infty$ (*i.e.*, $\Delta x \to 0^+$). To this end, we break up the sum in (A.22a) by the number of times n the entries of the vector $\boldsymbol{\ell} = (\ell_q, \ldots, \ell_s)$ switch from 0 to 1 or from 1 to 0, *e.g.*, $(0, \ldots, 0, 1, \ldots, 1)$ switches once, so $n = 1$. We sum over where the possible switches of each order n can occur *i.e.*, $q-1 < y_1 < y_2 < \cdots < y_n < s$. At the location of each switch, $\Lambda_p^{\boldsymbol{\ell}}/\Lambda_p^+ = \Lambda_p^-/\Lambda_p^+$, whereas $\Lambda_p^{\boldsymbol{\ell}}/\Lambda_p^+ = 1$ otherwise. Defining $y_0 = q-1$ and $y_{n+1} = s$, this gives

$$\mathfrak{C}_{N,\lambda}^{(q,s)}(k) = \sum_{n=0}^{s-q} \sum_{y_0 < y_1 < \cdots < y_n < y_{n+1}} (-1)^{\lambda n} \left(\prod_{p=1}^{n} \frac{\Lambda_{y_p}^-}{\Lambda_{y_p}^+}\right) \cos\left(\sum_{p=0}^{n} (-1)^p \sum_{r=y_p+1}^{y_{p+1}} \nu_p \Delta x\right). \tag{A.25}$$

Using (A.24), we arrive at a sum of n-dimensional Riemann sums which limit to n-dimensional integrals, giving

$$\mathfrak{C}_{N,\lambda}^{(q,s)}(k) = \sum_{n=0}^{\infty} (-1)^{\lambda n} \mathcal{C}_n^{(x_q, x_s)}(k) + O(\Delta x), \tag{A.26}$$

where $\mathcal{C}_n^{(a,b)}(k)$ is defined in (4.9a). The limit of (A.22b) is

$$\mathfrak{S}_{N,\lambda}^{(q,s)}(k) = \sum_{n=0}^{\infty} (-1)^{\lambda n} \mathcal{S}_n^{(x_q, x_s)}(k) + O(\Delta x), \tag{A.27}$$

obtained the same way, with $\mathcal{S}_n^{(a,b)}(k)$ defined in (4.9b). No more rigor is required at this point, as we prove in Appendices B–E that our result is a solution under less restrictive assumptions needed to justify these steps.

Using (A.23), (A.26), and (A.27) in (A.21), we have that

$$\Delta(k) = \lim_{N \to \infty} \Xi(k) D_N(k), \tag{A.28}$$

gives (5.6). For the numerator, similar to $D_N(k)$ in (A.19), we define

$$E_N(k, j, t) = i^N \frac{2\det(\mathcal{A}_N^{(j)}(k))}{\nu_1 \nu_N} \left(\prod_{p=1}^{N-1} \frac{1}{\Lambda_p^+}\right), \tag{A.29}$$

and use a cofactor expansion along the jth column of $\mathcal{A}_N^{(j)}$, so that

$$E_N(k, j, t) = \sum_{m=1}^{N+1} Y_N^{(m)} M_N^{(m,j)}(k) + \sum_{m=1}^{N+1} Y_N^{(m+N+1)} M_N^{(m+N+1,j)}(k), \tag{A.30}$$

where $M_N^{(m,j)}(k)$ are cofactors of the matrix $\mathcal{A}_N^{(j)}$, scaled by the same factor as in (A.29). With $x = x_j = j\Delta x = j/N$, which is fixed, we let

$$\mathcal{B}_0(k, x) = k\Xi(k) \lim_{N \to \infty} M_N^{(1,j)}(k) \qquad \text{and} \qquad \mathcal{B}_1(k, x) = k\Xi(k) \lim_{N \to \infty} M_N^{(2N+2,j)}(k). \tag{A.31}$$

Since, for $m = 1, \ldots, 2N$,

$$Y_N^{(m+1)} = \hat{q}_0^{(m)}(\nu_m) - \tilde{f}_m(\nu_m, t) = \frac{e^{-i\nu_m x_m}}{\alpha_m} \left(q_0(x_m) - \int_0^t f(x_m, s) e^{k^2 s} \, ds \right) \Delta x + O((\Delta x)^2), \tag{A.32}$$

so that, for $m = 1, \ldots, N$,

$$Y_N^{(m+1)} = \frac{e^{-i\nu_m x_m} \psi_N^{(m)}(k^2, t)}{\alpha_m} \Delta x + O((\Delta x)^2) \quad \text{and} \quad Y_N^{(m+N+1)} = \frac{e^{i\nu_m x_m} \psi_N^{(m)}(k^2, t)}{\alpha_m} \Delta x + O((\Delta x)^2), \tag{A.33}$$

which defines $\psi_N^{(m)}(k^2, t)$. Then

$$\Phi(k, x, t) = \lim_{N \to \infty} k \Xi(k) E_N(k, j, t), \tag{A.34}$$

which gives (5.8a). Here

$$\Phi_\psi(k, x, t) = \lim_{N \to \infty} k \Xi(k) \sum_{m=1}^{N} \frac{\psi_N^{(m)}(k^2, t)}{\alpha_m} \left(e^{-i\nu_m x_m} M_N^{(m+1,j)}(k) + e^{i\nu_m x_m} M_N^{(m+N+1,j)}(k) \right) \Delta x, \tag{A.35}$$

where we let $y = x_m = m\Delta x = m/N$, which is kept fixed. This gives (5.8b), after defining

$$\Psi(k, x, y) = \lim_{N \to \infty} \Psi_N^{(j,m)}(k) = \lim_{N \to \infty} \Xi(k) \sqrt{(\beta\nu)_m} \sqrt{(\beta\nu)_j} \left(e^{-i\nu_m x_m} M_N^{(m+1,j)}(k) + e^{i\nu_m x_m} M_N^{(m+N+1,j)}(k) \right), \tag{A.36a}$$

$$\psi_\alpha(k^2, y, t) = \lim_{N \to \infty} \frac{\psi_N^{(m)}(k^2, t)}{\alpha_m} = \frac{q_0(y)}{\alpha(y)} - \int_0^t \frac{f(y, s)}{\alpha(y)} e^{k^2 s} \, ds, \tag{A.36b}$$

which defines $\Psi_N^{(j,m)}$.

For the boundary term $\mathcal{B}_0(k, x)$, similar to (A.21), we explicitly verify using Mathematica for $N \le 8$ and for $1 = m < j \le N$,

$$\begin{aligned} M_N^{(1,j)}(k) = \frac{4}{\sqrt{(\beta\nu)_{j-1}}} \Bigg\{ & \frac{\beta_N}{\sqrt{(\beta\nu)_{j-1}}} \left(\prod_{p=j-1}^{N-1} \frac{2(\beta\nu)_p}{\Lambda_p^+} \right) \left[-\frac{a_{21}}{\nu_1} \mathfrak{S}_{N,0}^{(1,j-1)}(k) + a_{22} \mathfrak{C}_{N,0}^{(1,j-1)}(k) \right] \\ & + \frac{\sqrt{(\beta\nu)_{j-1}}}{\nu_1} \left(\prod_{p=1}^{j-1} \frac{2(\beta\nu)_p}{\Lambda_p^+} \right) \left[\frac{b_{21}}{\nu_N} \mathfrak{S}_{N,0}^{(j,N)}(k) + b_{22} \mathfrak{C}_{N,1}^{(j,N)}(k) \right] \Bigg\}, \end{aligned} \tag{A.37}$$

and using (A.24), (A.26), (A.27), and (A.31), we find (5.8c) for $j = 2$. Similarly, for the other boundary term $\mathcal{B}_1(k, x)$, we find (5.8c) for $j = 1$.

For the remaining terms, for $1 \le m < j \le N$, we derive

$$\begin{aligned} \Psi_N^{(j,m)}(k) = 4\Xi(k) \frac{\sqrt{(\beta\nu)_j}}{\sqrt{(\beta\nu)_m}} \left(\prod_{p=m}^{j-1} \frac{2(\beta\nu)_p}{\Lambda_p^+} \right) \Bigg[& -(a:b)_{2,4} \mathfrak{C}_{N,0}^{(1,m)}(k) \mathfrak{C}_{N,1}^{(j,N)}(k) + \frac{(a:b)_{1,3}}{\nu_1 \nu_N} \mathfrak{S}_{N,0}^{(1,m)}(k) \mathfrak{S}_{N,0}^{(j,N)}(k) \\ & + \frac{(a:b)_{1,4}}{\nu_1} \mathfrak{S}_{N,0}^{(1,m)}(k) \mathfrak{C}_{N,1}^{(j,N)}(k) - \frac{(a:b)_{2,3}}{\nu_N} \mathfrak{C}_{N,0}^{(1,m)}(k) \mathfrak{S}_{N,0}^{(j,N)}(k) \Bigg] \\ - \frac{4(a:b)_{1,2}\beta_N}{\sqrt{(\beta\nu)_1}\sqrt{(\beta\nu)_N}} \frac{\sqrt{(\beta\nu)_j}}{\sqrt{(\beta\nu)_{j-1}}} \frac{\sqrt{(\beta\nu)_m}}{\sqrt{(\beta\nu)_1}} & \left(\prod_{p=1}^{m} \frac{(\beta\nu)_p}{\Lambda_p^+} \right) \frac{\sqrt{(\beta\nu)_N}}{\sqrt{(\beta\nu)_{j-1}}} \left(\prod_{p=j-1}^{N-1} \frac{(\beta\nu)_p}{\Lambda_p^+} \right) \mathfrak{S}_{N,0}^{(m+1,j-1)}(k). \end{aligned} \tag{A.38}$$

Taking the limit using (A.24), (A.26), and (A.27), as before letting $x_j \to x$ and $x_m \to y$, we find for $x_r \le y < x \le x_\ell$,

$$\Psi(k,x,y) = 4\Xi(k)\left\{-(a:b)_{2,4}\left(\sum_{n=0}^{\infty}\mathcal{C}_n^{(x_l,y)}\right)\left(\sum_{n=0}^{\infty}(-1)^n\mathcal{C}_n^{(x,x_r)}\right) + \frac{(a:b)_{1,3}}{k^2\mathfrak{n}(k,x_l)\mathfrak{n}(k,x_r)}\left(\sum_{n=0}^{\infty}\mathcal{S}_n^{(x_l,y)}\right)\left(\sum_{n=0}^{\infty}\mathcal{S}_n^{(x,x_r)}\right)\right.$$
$$+\frac{(a:b)_{1,4}}{k\mathfrak{n}(k,x_l)}\left(\sum_{n=0}^{\infty}\mathcal{S}_n^{(x_l,y)}\right)\left(\sum_{n=0}^{\infty}(-1)^n\mathcal{C}_n^{(x,x_r)}\right) - \frac{(a:b)_{2,3}}{k\mathfrak{n}(k,x_r)}\left(\sum_{n=0}^{\infty}\mathcal{C}_n^{(x_l,y)}\right)\left(\sum_{n=0}^{\infty}\mathcal{S}_n^{(x,x_r)}\right)$$
$$\left.-\frac{(a:b)_{1,2}\beta(x_r)}{k\sqrt{(\beta\mathfrak{n})(k,x_l)}\sqrt{(\beta\mathfrak{n})(k,x_r)}}\sum_{n=0}^{\infty}\mathcal{S}_n^{(y,x)}\right\}, \tag{A.39}$$

which may be rewritten as (5.9a). Similarly, for $x_l \le x < y \le x_r$, we find (5.9b). Finally, we have

$$q(x,t) = \lim_{N\to\infty}\frac{1}{i\pi}\int_{\partial\Omega}\frac{\det(\mathcal{A}_N^{(j)}(k))}{\det(\mathcal{A}_N(k))}ke^{-k^2t}\,dk = \lim_{N\to\infty}\frac{1}{2\pi}\int_{\partial\Omega}\frac{k\Xi(k)E_N(k,j,t)}{\Xi(k)D_N(k)}e^{-k^2t}\,dk, \tag{A.40}$$

which gives (5.4).

A.2 The half-line problem

We obtain the solution of the half-line problem by taking the limit as $x_r \to \infty$ of the solution of the finite-interval problem (5.1) with $f_1(t) = 0$ and

$$(a:b) = \begin{pmatrix} a_0 & a_1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}. \tag{A.41}$$

In this limit, (5.4) becomes (4.2) with the same Ω, shown in Figure 1. This process is detailed below.

Using (A.41), we find the coefficients $\mathfrak{a}(k) = 0$, $\mathfrak{c}_n(k) = -a_1/(k\mathfrak{n}(k,x_r))$, and $\mathfrak{s}_n(k) = a_0/(k^2\mathfrak{n}(k,x_l)\mathfrak{n}(k,x_r))$. We define $\tilde\Delta(k) = k\mathfrak{n}(k,x_r)\Delta(k)$, and (5.6) becomes

$$\tilde\Delta(k) = 2i\Xi(k)\left\{\frac{a_0}{k\mathfrak{n}(k,x_l)}\sum_{n=0}^{\infty}\mathcal{S}_n^{(x_l,x_r)}(k) - a_1\sum_{n=0}^{\infty}\mathcal{C}_n^{(x_l,x_r)}(k)\right\}. \tag{A.42}$$

We have $\mathcal{B}_1(k,x) = 0$, and (5.8c) for $j = 2$ becomes

$$\tilde{\mathcal{B}}_0(k,x) = k\mathfrak{n}(k,x_r)\mathcal{B}_0(k,x) = \frac{4\beta(x_l)\Xi(k)}{\sqrt{(\beta\mathfrak{n})(k,x_l)}\sqrt{(\beta\mathfrak{n})(k,x)}}\sum_{n=0}^{\infty}\mathcal{S}_n^{(x,x_r)}(k). \tag{A.43}$$

For $x_l \le y < x \le x_r$, (5.9a) becomes

$$\tilde\Psi(k,x,y) = k\mathfrak{n}(k,x_r)\Psi(k,x,y) = 4\Xi(k)\left\{\frac{a_0}{k\mathfrak{n}(k,x_l)}\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}\mathcal{S}_\ell^{(x,x_r)}(k)\mathcal{S}_{n-\ell}^{(x_l,y)}(k) - a_1\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}\mathcal{S}_\ell^{(x,x_r)}(k)\mathcal{C}_{n-\ell}^{(x_l,y)}(k)\right\}, \tag{A.44}$$

and, for $x_l < x < y < x_r$, $\tilde\Psi(k,x,y) = \tilde\Psi(k,y,x)$. From (5.8a),

$$\tilde\Phi(k,x,t) = k\mathfrak{n}(k,x_r)\Phi(k,x,t) = \tilde{\mathcal{B}}_0(k,x) + \tilde\Phi_\psi(k,x,t), \tag{A.45}$$

where

$$\tilde\Phi_\psi(k,x,t) = k\mathfrak{n}(k,x_r)\Phi_\psi(k,x,t) = \int_{x_l}^{x_r}\frac{\tilde\Psi(k,x,y)\psi_\alpha(k^2,y,t)}{\sqrt{(\beta\mathfrak{n})(k,x)(\beta\mathfrak{n})(k,y)}}\,dy, \tag{A.46}$$

and $\psi_\alpha(k^2,y,t)$ is defined in Definition 5.

To take the limit as $x_r \to \infty$, using (4.9b), we write

$$\exp\left(\int_a^{x_r} ik\mathfrak{n}(k,\xi)\,d\xi\right)\mathcal{S}_n^{(a,x_r)}(k) = \frac{1}{2i\cdot 2^n}\int_{a<\cdots<x_r}\left(\prod_{p=1}^{n}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\left[\exp\left(\sum_{p=0}^{n}(1+(-1)^p)\int_{y_p}^{y_{p+1}} ik\mathfrak{n}(k,\xi)\,d\xi\right)\right.$$
$$\left.-\exp\left(\sum_{p=0}^{n}(1-(-1)^p)\int_{y_p}^{y_{p+1}} ik\mathfrak{n}(k,\xi)\,d\xi\right)\right]dy_1\cdots dy_n. \tag{A.47}$$

Since $\mathrm{Re}(ik\mathfrak{n}(k,x)) < 0$ for all $k \in \Omega$ and all $x > x_l$, see Lemma 15 in Section B, it follows that

$$\exp\left(\int_{y_n}^{x_r} ik\mathfrak{n}(k,\xi)\,d\xi\right) \to 0, \tag{A.48}$$

as $x_r \to \infty$. Thus the term in (A.47) which does not contain the $p = n$ term survives and considering even and odd n separately, we conclude

$$\exp\left(\int_{a}^{x_r} ik\mathfrak{n}(k,\xi)\,d\xi\right) \mathcal{S}_n^{(a,x_r)}(k) \to -\frac{(-1)^n}{2i}\mathcal{E}_n^{(a,\infty)}(k), \quad \text{as} \quad x_r \to \infty, \tag{A.49}$$

where $\mathcal{E}_n^{(a,b)}(k)$ is defined in (3.5b). Similarly,

$$\exp\left(\int_{a}^{x_r} ik\mathfrak{n}(k,\xi)\,d\xi\right) \mathcal{C}_n^{(a,x_r)}(k) \to \frac{1}{2}\mathcal{E}_n^{(a,\infty)}(k), \quad \text{as} \quad x_r \to \infty. \tag{A.50}$$

Therefore, we have as $x_r \to \infty$,

$$-2i\tilde{\Delta}(k) \to 2\sum_{n=0}^{\infty}\left(\frac{(-1)^n i a_0}{k\mathfrak{n}(k,x_l)} - a_1\right)\mathcal{E}_n^{(x_l,\infty)}(k), \tag{A.51}$$

and

$$-2i\tilde{\mathcal{B}}_0(k,x) \to \frac{4\beta(x_l)\exp\left(\int_{x_l}^{x} ik\mathfrak{n}(k,\xi)\,d\xi\right)}{\sqrt{(\beta\mathfrak{n})(k,x_l)}\sqrt{(\beta\mathfrak{n})(k,x)}}\sum_{n=0}^{\infty}(-1)^n\mathcal{E}_n^{(x,\infty)}(k), \tag{A.52}$$

and, for $x_l \le y < x$,

$$-2i\tilde{\Psi}(k,x,y) \to 4\exp\left(\int_{x_l}^{x} i\nu(k,\xi)\,d\xi\right)\sum_{n=0}^{\infty}\sum_{n=0}^{n}(-1)^{\ell}\left(\frac{a_0}{k\mathfrak{n}(k,x_l)}\mathcal{S}_{n-\ell}^{(x_l,y)}(k) - a_1\mathcal{C}_{n-\ell}^{(x_l,y)}(k)\right)\mathcal{E}_{\ell}^{(x,\infty)}(k), \tag{A.53}$$

and similarly for $x_l \le x < y$. These final results combine to give (4.2).

A.3 The whole-line problem

We repeat the process from the previous section, now letting $x_l \to -\infty$. Starting from the half-line solution (4.2) with $f_0(t) = 0$, $a_0 = 1$, and $a_1 = 0$. The denominator in (4.2) is determined by

$$k\mathfrak{n}(k,x_l)\Delta(k) = 2i\sum_{n=0}^{\infty}(-1)^n\mathcal{E}_n^{(x_l,\infty)}(k). \tag{A.54}$$

Since $\mathcal{B}_0(k,x,t) = 0$, we also have from (4.4)

$$\Phi(k,x,t) = \int_{-\infty}^{\infty}\frac{\Psi(k,x,y)\psi_\alpha(k^2,y,t)}{\sqrt{(\beta\mathfrak{n})(k,x)(\beta\mathfrak{n})(k,y)}}\,dy, \tag{A.55}$$

where $\psi_\alpha(k^2,y,t)$ is defined in Definition 5. For $x_l < y < x$, (4.8) becomes

$$k\mathfrak{n}(k,x_l)\Psi(k,x,y) = 4\exp\left(\int_{x_l}^{x} ik\mathfrak{n}(k,\xi)\,d\xi\right)\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^{\ell}\mathcal{S}_{n-\ell}^{(x_l,y)}(k)\mathcal{E}_{\ell}^{(x,\infty)}(k), \tag{A.56}$$

and $\Psi(k,x,y) = \Psi(k,y,x)$ for $x_l < x < y$. Since $\mathcal{E}_n^{(x_l,\infty)}(k) \to 0$ if n is odd, and from (A.47), we have

$$\exp\left(\int_{x_l}^{b} ik\mathfrak{n}(k,\xi)\,d\xi\right)\mathcal{S}_n^{(x_l,b)}(k) \to -\frac{1}{2i}\tilde{\mathcal{E}}_n^{(-\infty,b)}(k), \tag{A.57}$$

as $x_l \to -\infty$. Therefore,

$$k\mathfrak{n}(k,x_l)\Delta(k) \to 2i\sum_{\substack{n=0\\ n \text{ even}}}^{\infty}\mathcal{E}_n^{(-\infty,\infty)}(k). \tag{A.58}$$

For $x_l < y < x$,

$$k\mathfrak{n}(k,x_l)\Psi(k,x,y) = 2i\exp\left(\int_y^x ik\mathfrak{n}(k,\xi)\,d\xi\right)\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^\ell\tilde{\mathcal{E}}_{n-\ell}^{(-\infty,y)}(k)\mathcal{E}_\ell^{(x,\infty)}(k), \tag{A.59}$$

and $\Psi(k,x,y) = \Psi(k,y,x)$ for $x_l < x < y$. Combining these results gives (3.2).

B Proofs: the solution expressions are well defined

Prior to proving that the solution expression (2.2) solves the evolution equation 2.1a and satisfies the initial and boundary conditions for the problem considered, we show in this appendix that this expression is well defined for all problems considered. We refer to the whole-line, half-line, and *regular* finite-interval problems as *regular problems*, and the *irregular* finite-interval problems as *irregular problems*. Throughout, we need Assumptions 1 and 2 from Section 2. For Boundary Case 4 of the finite-interval problem, we also require Assumption 3. Note that Assumption 3.1 is not needed for all *irregular problems*, only for Boundary Case 4. Therefore, we will be explicit as to when Assumption 3 is required.

In this appendix, we denote the r dependence of Ω explicitly as $\Omega(r) = \{k \in \mathbb{C} : |k| > r \text{ and } \pi/4 < \arg(k) < 3\pi/4\}$. We define $\arg(\,\cdot\,) \in [-\pi/2, 3\pi/2)$ with $\theta = \arg(k)$. The following lemma characterizes some properties of the coefficient functions α, β that follow from the assumptions.

Lemma 12. *If $\alpha(x)\beta(x)$ is not identically zero, the following are equivalent:*

a. $\alpha\beta \in L^\infty(\mathcal{D})$,

b. $\alpha \in L^\infty(\mathcal{D})$,

c. $\beta \in L^\infty(\mathcal{D})$,

as are the following:

i. $m_{\alpha\beta} = \inf_{x\in\mathcal{D}} |\alpha(x)\beta(x)| > 0$,

ii. $m_\alpha = \inf_{x\in\mathcal{D}} |\alpha(x)| > 0$,

iii. $m_\beta = \inf_{x\in\mathcal{D}} |\beta(x)| > 0$.

Proof. Under Assumption 1.3, there exists an $x_0 \in \mathcal{D}$ such that $0 < |\alpha(x_0)\beta(x_0)| < \infty$. From Assumptions 1.2 and 1.5,

$$\left|\frac{\beta(x)}{\beta(x_0)}\right| = \left|\frac{\alpha(x)}{\alpha(x_0)}\exp\left(\int_{x_0}^x \frac{\beta'(y)}{\beta(y)} - \frac{\alpha'(y)}{\alpha(y)}\,dy\right)\right| \le \left|\frac{\alpha(x)}{\alpha(x_0)}\right|\exp\left(\left\|\frac{\beta'}{\beta} - \frac{\alpha'}{\alpha}\right\|_{\mathcal{D}}\right) = E\left|\frac{\alpha(x)}{\alpha(x_0)}\right|, \tag{B.1a}$$

with $E = \exp(\|\beta'/\beta - \alpha'/\alpha\|_{\mathcal{D}})$. We conclude that $b \Rightarrow c$, $b \Rightarrow a$, $a \Rightarrow c$, $iii \Rightarrow ii$, $iii \Rightarrow i$, and $i \Rightarrow ii$. Similarly,

$$\left|\frac{\alpha(x)}{\alpha(x_0)}\right| \le E\left|\frac{\beta(x)}{\beta(x_0)}\right|, \tag{B.1b}$$

so that $c \Rightarrow b$, $c \Rightarrow a$, $a \Rightarrow b$, $ii \Rightarrow iii$, $ii \Rightarrow i$, and $i \Rightarrow iii$. □

Next, Lemmas 13–15 present some properties of the functions $\mathfrak{n}(k,x)$ and $(\beta\mathfrak{n})(k,x)$.

Lemma 13. *For $|k| \ge r > \sqrt{M_\gamma}$, where M_γ is defined in Assumption 1.4, we have*

$$m_\mathfrak{n} = \frac{1}{\sqrt{M_{\alpha\beta}}}\sqrt{1 - \frac{M_\gamma}{r^2}} \le |\mathfrak{n}(k,x)| \le \frac{1}{\sqrt{m_{\alpha\beta}}}\sqrt{1 + \frac{M_\gamma}{r^2}} = M_\mathfrak{n}, \tag{B.2}$$

which defines $m_\mathfrak{n}, M_\mathfrak{n} > 0$. From this, we also have $m_\mathfrak{n} \le |\mu(x)| \le M_\mathfrak{n}$.

Proof. The proof is trivial from the definition of $\mathfrak{n}(k,x)$ in Definition 5 using Assumptions 1.3 and 1.4. □

Lemma 14. *For $|k| \ge r > \sqrt{M_\gamma}$, $(\beta\mathfrak{n})'/(\beta\mathfrak{n}) \in L^1(\mathcal{D})$, and under Assumption 3, $\mathfrak{u} \in \mathrm{AC}(\mathcal{D})$.*

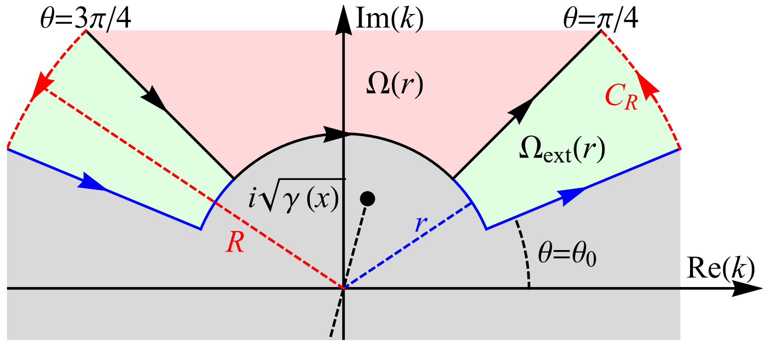

Figure 4: The region $\Omega_{\text{ext}}(r) = \{k \in \mathbb{C} : |k| > r \text{ and } \theta_0 < \arg(k) < \pi - \theta_0\}$ described in Lemma 15 ($\Omega(r) \cup$ the green regions) and the contour $C_R = \{k \in \mathbb{C} : |k| = R \text{ and } \theta_0 < \arg(k) < \pi/4 \text{ or } 3\pi/4 < \theta < \pi - \theta_0\}$.

Proof. The function

$$\frac{(\beta\mathfrak{n})'(k,x)}{(\beta\mathfrak{n})(k,x)} = \frac{1}{2}\left(\frac{\beta'(x)}{\beta(x)} - \frac{\alpha'(x)}{\alpha(x)} + \frac{\gamma'(x)}{k^2+\gamma(x)}\right) \in L^1(\mathcal{D}), \tag{B.3}$$

for $|k| \geq r > \sqrt{M_\gamma}$, by Assumption 1.5. On the finite interval, by Assumptions 1.2 and 1.3, $\mu \in \text{AC}(\mathcal{D})$ and, from Assumption 3.1, $\mathfrak{u} \in \text{AC}(\mathcal{D})$ and thus $\mathfrak{u} \in L^\infty(\mathcal{D})$ and $\mathfrak{u}' \in L^1(\mathcal{D})$ [15]. □

Lemma 15. *There exists an $r > \sqrt{M_\gamma}$, $m_{\mathfrak{in}} > 0$, and $0 < \theta_0 < \pi/4$ such that*

$$\text{Re}(ik\mathfrak{n}(k,x)) \leq -m_{\mathfrak{in}}|k|, \tag{B.4}$$

for $k \in \Omega_{\text{ext}}(r)$, where $\Omega_{\text{ext}}(r) = \{k \in \mathbb{C} : |k| > r$ and $\theta_0 < \arg(k) < \pi - \theta_0\}$, see Figure 4.

Proof. With $\phi = \arg(k\mathfrak{n}(k,x))$, $\Theta = \sup_{x\in\mathcal{D}} |\arg(\alpha(x)\beta(x))|$, $\psi = \arg(1+\gamma(x)/k^2)$, (and $\theta = \arg(k)$), we have from the definition of $\mathfrak{n}(k,x)$ in Definition 5,

$$\theta - \frac{1}{2}(\Theta - \psi) \leq \phi \leq \theta + \frac{1}{2}(\Theta + \psi), \tag{B.5}$$

see Figure 5a. Using Assumption 1.4 and Figure 5b,

$$|\psi| \leq \arcsin\left(\left|\frac{\gamma(x)}{k^2}\right|\right) \leq \arcsin\left(\frac{M_\gamma}{r^2}\right) \leq \frac{2M_\gamma}{r^2}. \tag{B.6}$$

Since $0 \leq \Theta < \pi/2$, we can choose $r > \sqrt{M_\gamma}$ large enough so that

$$|\psi| \leq \frac{1}{2}\left(\frac{\pi}{2} - \Theta\right), \tag{B.7}$$

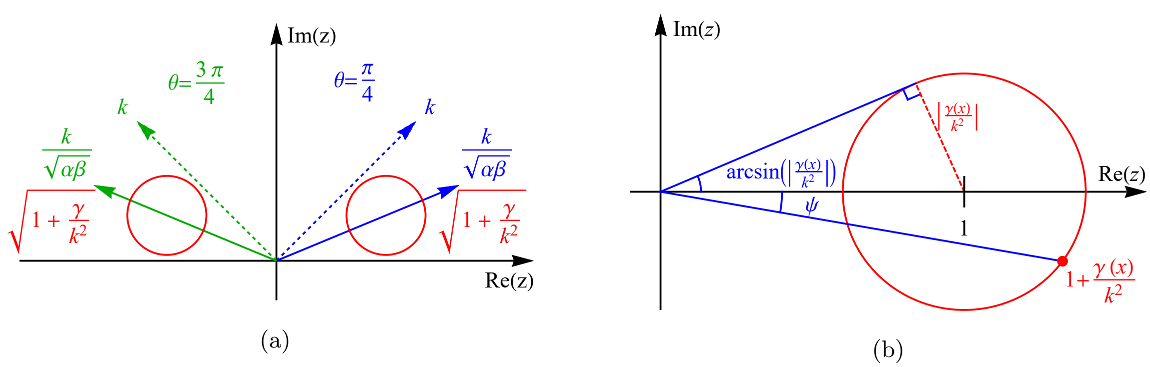

Figure 5: (a) The arguments of $k\mathfrak{n}(k,x)$ and its components, (b) $\psi = \arg\left(1+\gamma(x)/k^2\right)$.

which gives, from (B.5),

$$\theta-\theta_1=\theta-\frac{1}{4}\left(\Theta+\frac{\pi}{2}\right)\leq\theta-\frac{1}{2}\left(\Theta+|\psi|\right)\leq\phi\leq\theta+\frac{1}{2}\left(\Theta+|\psi|\right)\leq\theta+\frac{1}{4}\left(\Theta+\frac{\pi}{2}\right)=\theta+\theta_1, \tag{B.8}$$

which defines $0<\theta_1<\pi/4$. For $|k|>r$ and $\theta_1\leq\arg(k)\leq\pi-\theta_1$, we have that $0\leq\phi\leq\pi$, so that $\operatorname{Re}(ik\mathfrak{n}(k,x))\leq 0$. In particular, $\operatorname{Re}(ik\mathfrak{n}(k,x))<0$ for $k\in\Omega_{\text{ext}}(r)$. More specifically, using that $\sin(\phi)\geq\phi(\pi-\phi)/\pi$ for $0\leq\phi\leq\pi$, then for $\theta_1\leq\theta\leq\pi-\theta_1$, we have

$$\operatorname{Re}(ik\mathfrak{n}(k,x))=-|k\mathfrak{n}(k,x)|\sin(\phi)\leq-\frac{1}{\pi}m_{\mathfrak{n}}|k|\phi(\pi-\phi)\leq-\frac{1}{\pi}m_{\mathfrak{n}}|k|(\theta-\theta_1)(\pi-\theta_1-\theta). \tag{B.9}$$

Finally, (B.4) follows from choosing θ_0 such that $0\leq\theta_1<\theta_0<\pi/4$ and letting $m_{i\mathfrak{n}}=m_{\mathfrak{n}}(\theta_0-\theta_1)/4$. □

Having established some properties of the coefficient functions α,β and the *dispersion functions* $\mathfrak{n}$, $(\beta\mathfrak{n})$, we define a generalization $\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)$ of the *accumulation functions* $\mathcal{E}_n^{(a,b)}(k)$, $\tilde{\mathcal{E}}_n^{(a,b)}(k)$, $\mathcal{C}_n^{(a,b)}(k)$, and $\mathcal{S}_n^{(a,b)}(k)$, and we show some relations between these functions. Further, we show these functions are bounded and well defined, and we find their large-k asymptotics.

Definition 16. *With r from Lemma 15, $(a,b)\subseteq\mathcal{D}$ and $k\in\Omega_{\text{ext}}(r)$. For integer $n>0$, we define the function*

$$\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)=\frac{1}{2^n}\int_{a=y_0<y_1<\cdots<y_n<y_{n+1}=b}\left(\prod_{p=1}^{n}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\exp\left(\sum_{p=0}^{n}\sigma_{p,n}\int_{y_p}^{y_{p+1}}ik\mathfrak{n}(k,\xi)\,d\xi\right)d\mathbf{y}_n, \tag{B.10a}$$

where $\sigma_{p,n}$ is a non-negative integer-valued function of n and $p=0,1,\ldots,n$. Here we require for any p that $\sigma_{p,n}\neq\sigma_{p+1,n}$. For $n<0$, we define $\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)=0$, and for $n=0$, we define

$$\mathcal{J}_0^{(a,b)}[\sigma_{0,0}](k)=\exp\left(\sigma_{0,0}\int_a^b ik\mathfrak{n}(k,\xi)\,d\xi\right). \tag{B.10b}$$

The function $\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)$ is defined as $a\to-\infty$ if $\sigma_{0,n}=0$, and as $b\to\infty$ if $\sigma_{n,n}=0$ (if $\mathcal{D}$ is unbounded).
Finally, we define

$$\mathfrak{C}_n^{(a,b)}(k)=\exp\left(\int_a^b ik\mathfrak{n}(k,\xi)\,d\xi\right)\mathcal{C}_n^{(a,b)}(k)\qquad\text{and}\qquad\mathfrak{S}_n^{(a,b)}(k)=\exp\left(\int_a^b ik\mathfrak{n}(k,\xi)\,d\xi\right)\mathcal{S}_n^{(a,b)}(k), \tag{B.11}$$

where $\mathcal{C}_n^{(a,b)}(k)$ and $\mathcal{S}_n^{(a,b)}(k)$ are defined in (4.9).

Lemma 17. *With $\mathcal{E}_n^{(a,b)}(k)$ and $\tilde{\mathcal{E}}_n^{(a,b)}(k)$ defined in (3.5), and $\mathfrak{C}_n^{(a,b)}(k)$ and $\mathfrak{S}_n^{(a,b)}(k)$ defined in (B.11), we have the following relations:*

$$\mathcal{E}_n^{(a,b)}(k)=\mathcal{J}_n^{(a,b)}[1-(-1)^{n-p}](k), \tag{B.12a}$$

$$\tilde{\mathcal{E}}_n^{(a,b)}(k)=\mathcal{J}_n^{(a,b)}[1-(-1)^{p}](k), \tag{B.12b}$$

$$\mathfrak{C}_n^{(a,b)}(k)=\frac{1}{2}\left[\mathcal{J}_n^{(a,b)}[1+(-1)^p](k)+\mathcal{J}_n^{(a,b)}[1-(-1)^p](k)\right], \tag{B.12c}$$

$$\mathfrak{S}_n^{(a,b)}(k)=\frac{1}{2i}\left[\mathcal{J}_n^{(a,b)}[1+(-1)^p](k)-\mathcal{J}_n^{(a,b)}[1-(-1)^p](k)\right]. \tag{B.12d}$$

Proof. The proofs follow immediately from the definitions in (3.5) and (4.9). □

The next two lemmas give bounds and asymptotics for the function $\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)$.

Lemma 18. *For $(x,y)\subseteq(a,b)\subseteq\mathcal{D}$, $k\in\Omega_{\text{ext}}(r)$, and r from Lemma 15,*

$$\left|\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)\right|\leq\frac{1}{2^n n!}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{(a,b)}^n\qquad\text{and}\qquad\left|\sum_{\ell=0}^{n}(-1)^{\lambda\ell}\mathcal{J}_{n-\ell}^{(a,x)}[\sigma_{p,n-\ell}](k)\mathcal{J}_{\ell}^{(y,b)}[\tilde{\sigma}_{p,\ell}](k)\right|\leq\frac{1}{2^n n!}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{(a,b)}^n, \tag{B.13}$$

where $\lambda=0,1$. These inequalities hold as $a\to-\infty$ and $b\to\infty$, provided the functions are defined. Thus, $\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)$ is well defined. The same bounds hold for $\mathcal{E}_n^{(a,b)}(k)$, $\tilde{\mathcal{E}}_n^{(a,b)}(k)$, $\mathfrak{C}_n^{(a,b)}(k)$, and $\mathfrak{S}_n^{(a,b)}(k)$.

Proof. By Lemma 15, the exponentials in (B.10) are bounded by 1 for $k \in \Omega_{\text{ext}}(r)$. Using Lemma 14,

$$\left|\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)\right| \le \frac{1}{2^n}\int_{a<y_1<\cdots<y_n<b}\left|\prod_{p=1}^{n}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right| dy_1\cdots dy_n = \frac{1}{2^n n!}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{(a,b)}^n, \tag{B.14a}$$

so that

$$\begin{aligned}\left|\sum_{\ell=0}^{n}(-1)^{\lambda\ell}\mathcal{J}_{n-\ell}^{(a,x)}[\sigma_{p,n-\ell}](k)\mathcal{J}_{\ell}^{(y,b)}[\sigma_{p,\ell}](k)\right| &\le \sum_{\ell=0}^{n}\frac{1}{2^n(n-\ell)!\ell!}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{(a,x)}^{n-\ell}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{(y,b)}^{\ell}\\ &= \frac{1}{2^n n!}\left(\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{(a,x)}+\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{(y,b)}\right)^n \le \frac{1}{2^n n!}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{(a,b)}^n. \end{aligned}\tag{B.14b}$$

For $\mathcal{E}_n^{(a,b)}(k)$, $\tilde{\mathcal{E}}_n^{(a,b)}(k)$, $\mathcal{C}_n^{(a,b)}(k)$, and $\mathcal{S}_n^{(a,b)}(k)$, the result follows from (B.12). These bounds hold as $a \to -\infty$ or $b \to \infty$, provided the functions are defined. ∎

Lemma 19. *There exists $r > \sqrt{M_\gamma}$, such that for any $(a,b) \subseteq \mathcal{D}$ and $n \ge 1$,*

$$\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k) \to 0, \qquad \text{as } |k| \to \infty, \qquad k \in \Omega_{\text{ext}}(r). \tag{B.15a}$$

This result holds as $a \to -\infty$ and $b \to \infty$, provided the functions are defined. The result extends to $\mathcal{E}_n^{(a,b)}(k)$, $\tilde{\mathcal{E}}_n^{(a,b)}(k)$, $\mathcal{C}_n^{(a,b)}(k)$, and $\mathcal{S}_n^{(a,b)}(k)$.

Next, we define $\lambda_{p,n} = \sigma_{p-1,n} - \sigma_{p,n}$. Using Assumption 3 and since $\lambda_{p,n} \ne 0$ (see Definition 16), we have

$$\mathcal{J}_1^{(a,b)}[\sigma_{p,n}](k) = \frac{1}{4\lambda_{1,1}ik}\left[\mathfrak{u}(b)\exp\left(\sigma_{0,1}\int_a^b ik\mu(\xi)\,d\xi\right) - \mathfrak{u}(a)\exp\left(\sigma_{1,1}\int_a^b ik\mu(\xi)\,d\xi\right)\right] + o(k^{-1}). \tag{B.15b}$$

There exists $r > \sqrt{M_\gamma}$ and $C > 1$ such that, for any $(a,b) \subseteq \mathcal{D}$ and $k \in \Omega_{\text{ext}}(r)$,

$$\left|\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)\right| \le \frac{C^n}{|k|^{\lfloor\frac{n+1}{2}\rfloor}}, \tag{B.15c}$$

where $\lfloor\cdot\rfloor$ is the floor function.

Proof. From Lemma 15, for any $r > \sqrt{M_\gamma}$ and for all $k \in \Omega_{\text{ext}}(r)$, we have

$$\begin{aligned}\left|\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)\right| &= \frac{1}{2^n}\left|\int_{a=y_0<y_1<\cdots<y_n<y_{n+1}=b}\left(\prod_{p=1}^{n}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\exp\left(\sum_{p=0}^{n}\sigma_{p,n}\int_{y_p}^{y_{p+1}} ik\mathfrak{n}(k,\xi)\,d\xi\right)dy_1\cdots dy_n\right|\\ &\le \frac{1}{2^n}\int_{a=y_0<\cdots<y_{n+1}=b}\left(\prod_{p=1}^{n}\left|\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right|\right)\exp\left(-m_{\mathfrak{in}}|k|\sum_{p=0}^{n}\sigma_{p,n}(y_{p+1}-y_p)\right)dy_1\cdots dy_n, \end{aligned}\tag{B.16}$$

and since $\sigma_{p,n} \ge 0$ and $\sigma_{p,n} \ne \sigma_{p+1,n}$ for any p, the argument of the exponential is strictly negative. Thus, by Lemma 18 and the Dominated Convergence Theorem (DCT), we have (B.15a). For $\mathcal{E}_n^{(a,b)}(k)$, $\tilde{\mathcal{E}}_n^{(a,b)}(k)$, $\mathcal{C}_n^{(a,b)}(k)$, and $\mathcal{S}_n^{(a,b)}(k)$, the result follows from (B.12).

Using the definition of $\mathfrak{n}(k,x)$ in Definition 5, it is straightforward to show that

$$\exp\left(ik\int_y^x \mathfrak{n}(k,\xi)\,d\xi\right) = \exp\left(ik\int_y^x \mu(\xi)\,d\xi\right)\left(1 + O(|x-y|k^{-1})\right), \tag{B.17}$$

for which, on a finite-interval, we may omit $|x-y|$. Using Assumption 1.5, (2.7), (B.3), and (B.17) in (B.10) for $n = 1$, we have

$$\mathcal{J}_1^{(a,b)}[\sigma_{p,n}](k) = \frac{1+O(k^{-1})}{4}\int_a^b \mathfrak{u}(y)\mu(y)\exp\left(\sigma_{0,1}\int_a^y + \sigma_{1,1}\int_y^b ik\mu(\xi)\,d\xi\right)dy + O(k^{-2}). \tag{B.18}$$

By Lemma 14, $\mathfrak{u} \in \mathrm{AC}(\mathcal{D})$ and integration by parts gives

$$\mathcal{J}_1^{(a,b)}[\sigma_{p,n}](k) = \frac{1}{4\lambda_{1,1} ik}\left(\mathfrak{u}(b)\exp\left(\sigma_{0,1}\int_a^b ik\mu(\xi)\,d\xi\right) - \mathfrak{u}(a)\exp\left(\sigma_{1,1}\int_a^b ik\mu(\xi)\,d\xi\right)\right)$$
$$- \frac{1}{4\lambda_{1,1} ik}\int_a^b \mathfrak{u}'(y)\exp\left(\sigma_{0,1}\int_a^y + \sigma_{1,1}\int_y^b ik\mu(\xi)\,d\xi\right)dy + O(k^{-2}). \tag{B.19}$$

By Lemma 14 and the DCT, we obtain (B.15b).

Inequality (B.15c) for $n = 0$ and $n = 1$ follows from (B.10b) and (B.15b), respectively. Using (2.7), (B.3), and (B.17) in (B.10) for $n \geq 2$, we have

$$\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k) = \frac{1+O(k^{-1})}{2^{n+1}}\int_{a<\cdots<b}\left(\prod_{p=1}^{n-1}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\mathfrak{u}(y_n)\mu(y_n)\exp\left(\sum_{p=0}^{n}\sigma_{p,n}\int_{y_p}^{y_{p+1}} ik\mu(\xi)\,d\xi\right)d\mathbf{y}_n$$
$$+ \frac{1+O(k^{-1})}{2^{n+1}}\int_{a<\cdots<b}\left(\prod_{p=1}^{n-1}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\left(\frac{\gamma'(y_n)}{k^2+\gamma(y_n)}\right)\exp\left(\sum_{p=0}^{n}\sigma_{p,n}\int_{y_p}^{y_{p+1}} ik\mu(\xi)\,d\xi\right)d\mathbf{y}_n. \tag{B.20}$$

Let $\mathcal{I}_n^{(a,b)}(k)$ denote the integral in the first line of (B.20):

$$\mathcal{I}_n^{(a,b)}(k) = \int_{a<\cdots<b}\left(\prod_{p=1}^{n-1}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\mathfrak{u}(y_n)\mu(y_n)\exp\left(\sum_{p=0}^{n}\sigma_{p,n}\int_{y_p}^{y_{p+1}} ik\mu(\xi)\,d\xi\right)d\mathbf{y}_n. \tag{B.21}$$

Integration by parts with respect to $y_n \in (y_{n-1}, b)$ gives

$$\mathcal{I}_n^{(a,b)}(k) = \frac{\mathfrak{u}(b)}{ik\lambda_{n,n}}\int_{a=y_0<\cdots<y_n=b}\left(\prod_{p=1}^{n-1}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\exp\left(\sum_{p=0}^{n-1}\sigma_{p,n}\int_{y_p}^{y_{p+1}} ik\mu(\xi)\,d\xi\right)d\mathbf{y}_{n-1}$$
$$- \int_{a<\cdots<b}\left(\prod_{p=1}^{n-1}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\frac{\mathfrak{u}(y_{n-1})}{ik\lambda_{n,n}}\exp\left(\sigma_{n,n}\int_{y_{n-1}}^{b} ik\mu(\xi)\,d\xi + \sum_{p=0}^{n-2}\sigma_{p,n}\int_{y_p}^{y_{p+1}} ik\mu(\xi)\,d\xi\right)d\mathbf{y}_{n-1}$$
$$- \int_{a<\cdots<b}\left(\prod_{p=1}^{n-1}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\frac{\mathfrak{u}'(y_n)}{ik\lambda_{n,n}}\exp\left(\sum_{p=0}^{n-1}\sigma_{p,n}\int_{y_p}^{y_{p+1}} ik\mu(\xi)\,d\xi\right)d\mathbf{y}_n. \tag{B.22}$$

In the second line of (B.22), we integrate over $y_{n-1} \in (a,b)$ last, leaving the remaining integral over $a = y_0 < y_1 < \cdots < y_{n-2} < y_{n-1}$ to be done first. Similarly, in the third line of (B.22), we integrate over $y_n \in (a,b)$ last and leave the remaining integral over $a = y_0 < y_1 < \cdots < y_{n-1} < y_n$ to be done first. Returning to (B.20) yields

$$\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k) = \frac{1+O(k^{-1})}{2^{n+1}}\frac{\mathfrak{u}(b)}{\lambda_{n,n} ik}\int_{a=y_0<\cdots<y_n=b}\left(\prod_{p=1}^{n-1}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\exp\left(\sum_{p=0}^{n-1}\sigma_{p,n}\int_{y_p}^{y_{p+1}} ik\mu(\xi)\,d\xi\right)d\mathbf{y}_{n-1}$$
$$- \frac{1+O(k^{-1})}{2^{n+1}}\int_a^b dy_{n-1}\,\frac{\mathfrak{u}(y_{n-1})}{\lambda_{n,n} ik}\frac{(\beta\mathfrak{n})'(k,y_{n-1})}{(\beta\mathfrak{n})(k,y_{n-1})}\exp\left(\sigma_{n,n}\int_{y_{n-1}}^{b} ik\mu(\xi)\,d\xi\right)\times$$
$$\times\int_{a=y_0<\cdots<y_{n-2}<y_{n-1}}\left(\prod_{p=1}^{n-2}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\exp\left(\sum_{p=0}^{n-2}\sigma_{p,n}\int_{y_p}^{y_{p+1}} ik\mu(\xi)\,d\xi\right)d\mathbf{y}_{n-2}$$
$$- \frac{1+O(k^{-1})}{2^{n+1}}\int_a^b dy_n\left(\frac{\mathfrak{u}'(y_n)}{\lambda_{n,n} ik} + \frac{\gamma'(y_n)}{k^2+\gamma(y_n)}\right)\exp\left(\sigma_{n,n}\int_{y_n}^{b} ik\mu(\xi)\,d\xi\right)\times$$
$$\times\int_{a=y_0<\cdots<y_{n-1}<y_n}\left(\prod_{p=1}^{n-1}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\exp\left(\sum_{p=0}^{n-1}\sigma_{p,n}\int_{y_p}^{y_{p+1}} ik\mu(\xi)\,d\xi\right)d\mathbf{y}_{n-1}, \tag{B.23}$$

which gives the asymptotic recurrence relation

$$\begin{aligned}\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k) &= \frac{1+O(k^{-1})}{4}\frac{\mathfrak{u}(b)}{\lambda_{n,n}ik}\mathcal{J}_{n-1}^{(a,b)}[\sigma_{p,n}](k) \\ &\quad - \frac{1+O(k^{-1})}{8}\int_a^b \frac{\mathfrak{u}(y_{n-1})}{\lambda_{n,n}ik}\frac{(\beta\mathfrak{n})'(k,y_{n-1})}{(\beta\mathfrak{n})(k,y_{n-1})}\exp\left(\sigma_{n,n}\int_{y_{n-1}}^b ik\mu(\xi)\,d\xi\right)\mathcal{J}_{n-2}^{(a,y_{n-1})}[\sigma_{p,n}](k)\,dy_{n-1} \\ &\quad - \frac{1+O(k^{-1})}{4}\int_a^b \left(\frac{\mathfrak{u}'(y_n)}{\lambda_{n,n}ik} + \frac{\gamma'(y_n)}{k^2+\gamma(y_n)}\right)\exp\left(\sigma_{n,n}\int_{y_n}^b ik\mu(\xi)\,d\xi\right)\mathcal{J}_{n-1}^{(a,y_n)}[\sigma_{p,n}](k)\,dy_n. \end{aligned} \tag{B.24}$$

Assuming (B.15c) holds for $n = 0, 1, \ldots, m-1$, and using that $|\lambda_{p,n}| \geq 1$, we find

$$\left|\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)\right| \leq \frac{1+O(k^{-1})}{4}\left[\frac{\|\mathfrak{u}\|_\infty}{|k|}\frac{C^{n-1}}{|k|^{\lfloor\frac{n}{2}\rfloor}} + \frac{\|\mathfrak{u}\|_\infty}{2|k|}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathcal{D}}\frac{C^{n-2}}{|k|^{\lfloor\frac{n-1}{2}\rfloor}} + \left(\frac{\|\mathfrak{u}'\|_{\mathcal{D}}}{|k|} + \frac{\|\gamma'\|_{\mathcal{D}}}{|k|^2 - M_\gamma}\right)\frac{C^{n-1}}{|k|^{\lfloor\frac{n}{2}\rfloor}}\right], \tag{B.25}$$

which, using Lemma 14, gives (B.15c) for $n \geq 0$. □

Having defined the function $\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)$ and established some of its properties, we prove that the function $\Delta(k)$ is bounded and well defined, and that the "transforms" $\Phi_0(k,x)$, $\Phi_f(k,x,t)$ and $\mathcal{B}_m(k,x)$, of the initial condition $q_0(x)$, the inhomogeneous function $f(x,t)$, and the boundary functions $f_m(t)$, respectively, are bounded and well defined.

Definition 20. *We define*

$$\Phi_0(k,x) = \int_{\mathcal{D}} \frac{\Psi(k,x,y)q_\alpha(y)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\,dy, \tag{B.26a}$$

$$\Phi_f(k,x,t) = \int_{\mathcal{D}} \frac{\Psi(k,x,y)\tilde{f}_\alpha(k^2,y,t)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\,dy, \tag{B.26b}$$

so that $\Phi_\psi(k,x,t) = \Phi_0(k,x) + \Phi_f(k,x,t)$. We define the corresponding parts of the solution as

$$q_0(x,t) = \frac{1}{2\pi}\int_{\partial\Omega(r)} \frac{\Phi_0(k,x)}{\Delta(k)}e^{-k^2t}\,dk, \tag{B.27a}$$

$$q_f(x,t) = \frac{1}{2\pi}\int_{\partial\Omega(r)} \frac{\Phi_f(k,x,t)}{\Delta(k)}e^{-k^2t}\,dk, \tag{B.27b}$$

$$q_{\mathcal{B}_m}(x,t) = \frac{1}{2\pi}\int_{\partial\Omega(r)} \frac{\mathcal{B}_m(k,x)}{\Delta(k)}F_m(k^2,t)e^{-k^2t}\,dk, \qquad m = 0, 1, \tag{B.27c}$$

where we define $\mathcal{B}_m(k,x) = 0$ $(m = 0, 1)$ for the whole-line problem and $\mathcal{B}_1(k,x) = 0$ for the half-line problem. Thus $q(x,t) = q_0(x,t) + q_f(x,t) + q_{\mathcal{B}_0}(x,t) + q_{\mathcal{B}_1}(x,t)$ for the finite-interval, half-line and whole-line problems.

Lemma 21. *For all three problems, there exists $r > \sqrt{M_\gamma}$ and $M_\Delta > 0$, so that for all $k \in \Omega_{\text{ext}}(r)$,*

$$\Delta(k) = \mathfrak{b}_0(k)(1+\varepsilon(k)) \qquad \text{and} \qquad \frac{1}{2}|\mathfrak{b}_0(k)| \leq |\Delta(k)| \leq M_\Delta, \tag{B.28a}$$

where $|\varepsilon(k)| < 1/2$. For the whole-line problem,

$$\mathfrak{b}_0(k) = 1; \tag{B.28b}$$

for the half-line problem,

$$\mathfrak{b}_0(k) = 2\left(\frac{ia_0}{k\mathfrak{n}(k,x_l)} - a_1\right); \tag{B.28c}$$

and, for the finite-interval problem,

1. *if* $(a:b)_{2,4} \neq 0$, *then*

$$\mathfrak{b}_0(k) = -(a:b)_{2,4}; \tag{B.28d}$$

2. *if* $(a:b)_{2,4} = 0$ *and* $m_{\mathfrak{c}_0} \neq 0$, *then*

$$\mathfrak{b}_0(k) = \frac{i m_{\mathfrak{c}_0}}{k}; \tag{B.28e}$$

3. *if* $(a:b)_{2,4} = 0$, $m_{\mathfrak{c}_0} = 0$, $m_{\mathfrak{c}_1} = 0$, *and* $(a:b)_{1,3} \neq 0$, *then*

$$\mathfrak{b}_0(k) = -\frac{m_{\mathfrak{s}}}{k^2}; \tag{B.28f}$$

4. *with Assumption 3, if* $(a:b)_{2,4} = 0$, $m_{\mathfrak{c}_0} = 0$, $m_{\mathfrak{c}_1} \neq 0$, *and* $m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}} \neq 0$, *then*

$$\mathfrak{b}_0(k) = \frac{1}{8k^2}\left(m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}}\right). \tag{B.28g}$$

Proof. For the whole-line problem,

$$\Delta(k) = 1 + \sum_{n=1}^{\infty} \mathcal{E}_{2n}^{(-\infty,\infty)}(k) = 1 + \varepsilon(k). \tag{B.29}$$

By Lemmas 18 and 19 and the DCT,

$$\varepsilon(k) = \sum_{n=1}^{\infty} \mathcal{E}_{2n}^{(-\infty,\infty)}(k) \to 0, \tag{B.30}$$

as $|k| \to \infty$. Thus, we can choose r sufficiently large so that for $k \in \Omega_{\text{ext}}(r)$, $|\varepsilon(k)| < 1/2$, and

$$\frac{1}{2} \leq 1 - |\varepsilon(k)| \leq |\Delta(k)| \leq 1 + |\varepsilon(k)| < \frac{3}{2}. \tag{B.31}$$

For the half-line problem, we write

$$\Delta(k) = 2\sum_{n=0}^{\infty}\left(\frac{(-1)^n i a_0}{k\mathfrak{n}(k,x_l)} - a_1\right)\mathcal{E}_n^{(x_l,\infty)}(k) = 2\left(\frac{i a_0}{k\mathfrak{n}(k,x_l)} - a_1\right)\left[1 + \sum_{n=1}^{\infty}\frac{\frac{(-1)^n i a_0}{k\mathfrak{n}(k,x_l)} - a_1}{\frac{i a_0}{k\mathfrak{n}(k,x_l)} - a_1}\mathcal{E}_n^{(x_l,\infty)}(k)\right]. \tag{B.32}$$

Recall that we require $(a_0, a_1) \neq (0,0)$. If $a_1 \neq 0$, we choose $r > \sqrt{M_\gamma}$ sufficiently large so that $|a_1| > |a_0|/(m_{\mathfrak{n}} r)$ for $k \in \Omega_{\text{ext}}(r)$. Using this, for either case, $a_0 \neq 0$ or $a_1 \neq 0$, we have

$$\left|\frac{\frac{(-1)^n i a_0}{k\mathfrak{n}(k,x_l)} - a_1}{\frac{i a_0}{k\mathfrak{n}(k,x_l)} - a_1}\right| \leq \frac{|a_1| + \frac{|a_0|}{m_{\mathfrak{n}} r}}{\left||a_1| - \frac{|a_0|}{m_{\mathfrak{n}} r}\right|} = A < \infty, \tag{B.33}$$

which defines $A \geq 1$. We have

$$|\varepsilon(k)| = \left|\sum_{n=1}^{\infty}\frac{\frac{(-1)^n a_0}{k\mathfrak{n}(k,x_l)} + i a_1}{\frac{a_0}{k\mathfrak{n}(k,x_l)} + i a_1}\mathcal{E}_n^{(x_l,\infty)}(k)\right| \leq A\sum_{n=1}^{\infty}\left|\mathcal{E}_n^{(x_l,\infty)}(k)\right| \to 0, \tag{B.34}$$

by the DCT. We choose $r > \sqrt{M_\gamma}$ large enough such that $|\varepsilon(k)| < 1/2$ for $k \in \Omega_{\text{ext}}(r)$. Then

$$0 < \left||a_1| - \left|\frac{a_0}{k\mathfrak{n}(k,x_l)}\right|\right| \leq |\Delta(k)| \leq 3\left(|a_1| + \frac{|a_0|}{m_{\mathfrak{n}} r}\right) < \infty. \tag{B.35}$$

For the finite-interval problem, since

$$\mathfrak{C}_0^{(x_l,x_r)}(k) = \Xi(k)\mathcal{C}_0^{(x_l,x_r)}(k) = \frac{1}{2}\left(\Xi(k)^2 + 1\right) \qquad \text{and} \qquad \mathfrak{S}_0^{(x_l,x_r)}(k) = \Xi(k)\mathcal{S}_0^{(x_l,x_r)}(k) = \frac{1}{2i}\left(\Xi(k)^2 - 1\right), \tag{B.36}$$

where $\Xi(k)$ is defined in (5.5) and $\mathcal{C}_n^{(a,b)}(k)$ and $\mathcal{S}_n^{(a,b)}(k)$ are defined in (B.11), we factor out the $n = 0$ term in (5.6) and write

$$\Delta(k) = 2i\mathfrak{a}(k)\Xi(k) + i\mathfrak{c}_0(k)\left(\Xi(k)^2+1\right) + \mathfrak{s}_0(k)\left(\Xi(k)^2-1\right) + 2i\sum_{n=1}^{\infty}\Big(\mathfrak{c}_n(k)\mathcal{C}_n^{(x_l,x_r)}(k) + \mathfrak{s}_n(k)\mathcal{S}_n^{(x_l,x_r)}(k)\Big). \quad \text{(B.37)}$$

Since $\Xi(k) \to 0$ exponentially fast, we have

$$\Delta(k) = i\mathfrak{c}_0(k) - \mathfrak{s}_0(k) + 2i\sum_{n=1}^{\infty}\Big(\mathfrak{c}_n(k)\mathcal{C}_n^{(x_l,x_r)}(k) + \mathfrak{s}_n(k)\mathcal{S}_n^{(x_l,x_r)}(k)\Big) + o(k^{-2}). \quad \text{(B.38)}$$

1. If $(a:b)_{2,4} \neq 0$, then we can write (B.28a) with $\mathfrak{b}_0(k)$ defined in (B.28d), where

$$\varepsilon(k) = \frac{-1}{(a:b)_{2,4}}\left[i\mathfrak{c}_0(k) - \mathfrak{s}_0(k) + (a:b)_{2,4} + 2i\sum_{n=1}^{\infty}\Big(\mathfrak{c}_n(k)\mathcal{C}_n^{(x_l,x_r)}(k) + \mathfrak{s}_n(k)\mathcal{S}_n^{(x_l,x_r)}(k)\Big)\right] + o(k^{-2}). \quad \text{(B.39)}$$

   Since $\mathfrak{c}_n(k) = O(k^{-1})$, $\mathfrak{s}_0(k) = (a:b)_{2,4} + O(k^{-2})$, $\mathfrak{s}_n(k) = O(k^0)$, and because both $\mathcal{C}_n^{(x_l,x_r)}(k) \to 0$ and $\mathcal{S}_n^{(x_l,x_r)}(k) \to 0$ by Lemma 19 and both are bounded (see Lemma 18), we can choose $r > \sqrt{M_\gamma}$ sufficiently large so that $|\varepsilon(k)| < 1/2$ for $k \in \Omega_{\text{ext}}(r)$, by the DCT. We have

$$\frac{1}{2}|(a:b)_{2,4}| \leq |\Delta(k)| \leq \frac{3}{2}|(a:b)_{2,4}|. \quad \text{(B.40)}$$

2. If $(a:b)_{2,4} = 0$ and $m_{\mathfrak{c}_0} \neq 0$, then we can write (B.28a) with $\mathfrak{b}_0(k)$ defined in (B.28e), where

$$\varepsilon(k) = \frac{k}{im_{\mathfrak{c}_0}}\left[i\mathfrak{c}_0(k) - \frac{im_{\mathfrak{c}_0}}{k} - \mathfrak{s}_0(k) + 2i\sum_{n=1}^{\infty}\Big(\mathfrak{c}_n(k)\mathcal{C}_n^{(x_l,x_r)}(k) + \mathfrak{s}_n(k)\mathcal{S}_n^{(x_l,x_r)}(k)\Big)\right] + o(k^{-1}). \quad \text{(B.41)}$$

   Since $\mathfrak{c}_0(k) = m_{\mathfrak{c}_0}/k + O(k^{-3})$, $\mathfrak{s}_n(k) = O(k^{-2})$, $\mathfrak{c}_n(k) = O(k^{-1})$, and since $\mathcal{C}_n^{(x_l,x_r)}(k) \to 0$ and $\mathcal{S}_n^{(x_l,x_r)}(k) \to 0$ and both are bounded (see Lemma 18), we can choose $r > \sqrt{M_\gamma}$ large enough such that $|\varepsilon(k)| < 1/2$ for $k \in \Omega_{\text{ext}}(r)$, by the DCT. We have

$$\frac{|m_{\mathfrak{c}_0}|}{2|k|} \leq |\Delta(k)| \leq \frac{3|m_{\mathfrak{c}_0}|}{2r}. \quad \text{(B.42)}$$

3. If $(a:b)_{2,4} = 0$, $m_{\mathfrak{c}_0} = 0$, $m_{\mathfrak{c}_1} = 0$, and $(a:b)_{1,3} \neq 0$, then $\mathfrak{c}_0(k) = \mathfrak{c}_1(k) = 0$ and we can write (B.28a) with $\mathfrak{b}_0(k)$ defined in (B.28f), where

$$\varepsilon(k) = -\frac{k^2}{m_{\mathfrak{s}}}\left[-\mathfrak{s}_0(k) + \frac{m_{\mathfrak{s}}}{k^2} + 2i\sum_{n=1}^{\infty}\mathfrak{s}_n(k)\mathcal{S}_n^{(x_l,x_r)}(k)\right] + o(k^0). \quad \text{(B.43)}$$

   Since $\mathfrak{s}_n(k) = m_{\mathfrak{s}}/k^2 + O(k^{-4})$, and since $\mathcal{S}_n^{(x_l,x_r)}(k) \to 0$ and is bounded (see Lemma 18), we can choose $r > \sqrt{M_\gamma}$ sufficiently large so that $|\varepsilon(k)| < 1/2$ for $k \in \Omega_{\text{ext}}(r)$, by the DCT. We have

$$\frac{|m_{\mathfrak{s}}|}{2|k|^2} \leq |\Delta(k)| \leq \frac{3|m_{\mathfrak{s}}|}{2r^2}. \quad \text{(B.44)}$$

4. If $(a:b)_{2,4} = 0$, $m_{\mathfrak{c}_0} = 0$, $m_{\mathfrak{c}_1} \neq 0$, and $m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}} \neq 0$, we can write (B.28a) with $\mathfrak{b}_0(k)$ defined in (B.28g), where

$$\begin{aligned}\varepsilon(k) = \frac{8k^2}{m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}}}\Bigg[& i\mathfrak{c}_0(k) + 2i\mathfrak{c}_0(k)\mathcal{C}_2^{(x_l,x_r)}(k) - \mathfrak{s}_0(k) + 2i\mathfrak{c}_1(k)\mathcal{C}_1^{(x_l,x_r)}(k) - \frac{m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}}}{8k^2} \\ & + 2i\sum_{n=3}^{\infty}\mathfrak{c}_n(k)\mathcal{C}_n^{(x_l,x_r)}(k) + 2i\Xi(k)\sum_{n=1}^{\infty}\mathfrak{s}_n(k)\mathcal{S}_n^{(x_l,x_r)}(k)\Bigg] + o(k^0). \quad \text{(B.45)}\end{aligned}$$

By Lemmas 17 and 19, we have

$$\begin{aligned}\mathcal{C}_1^{(x_l,x_r)}(k) &= \frac{1}{2}\left[\mathcal{J}_1^{(x_l,x_r)}[1+(-1)^p](k)+\mathcal{J}_1^{(x_l,x_r)}[1-(-1)^p](k)\right]\\ &= -\frac{1}{16ik}\left(\mathfrak{u}(x_r)+\mathfrak{u}(x_l)\right)\left(1-\exp\left(\int_{x_l}^{x_r}2ik\mu(\xi)\,d\xi\right)\right)+o(k^{-1}) = -\frac{1}{16ik}\mathfrak{u}_+ + o(k^{-1}).\end{aligned} \tag{B.46}$$

For $n>2$, from Lemma 19, we have

$$\sum_{n=3}^{\infty}\left|k\mathcal{C}_n^{(x_l,x_r)}(k)\right| \le \sum_{n=3}^{\infty}\frac{|k|C^n}{|k|^{\lfloor\frac{n+1}{2}\rfloor}} = \frac{C^4+C^3}{k-C^2} = O(k^{-1}). \tag{B.47}$$

Since $\mathfrak{c}_0(k)=O(k^{-3})$, $\mathfrak{c}_1(k)=-m_{\mathfrak{c}_1}/k+O(k^{-3})$, $\mathfrak{s}_n(k)=m_{\mathfrak{s}}/k^2+O(k^{-4})$, and since $\mathcal{S}_n^{(x_l,x_r)}(k)\to 0$ and $\mathcal{S}_n^{(x_l,x_r)}(k)$ is bounded (see Lemma 18), we can choose $r>\sqrt{M_\gamma}$ sufficiently large so that $|\varepsilon(k)|<1/2$ for $k\in\Omega_{\text{ext}}(r)$, by the DCT. We have

$$\frac{1}{16|k|^2}\left|m_{\mathfrak{c}_1}\mathfrak{u}_+-8m_{\mathfrak{s}}\right| \le |\Delta(k)| \le \frac{3}{16r^2}\left|m_{\mathfrak{c}_1}\mathfrak{u}_+-8m_{\mathfrak{s}}\right|. \tag{B.48}$$

□

Remark 22. *Note that for constant-coefficient IBVPs (α, β, γ constant), the denominator $\Delta(k)$ reduces to*

$$\Delta(k) = 2i\mathfrak{a}(k)\Xi(k)+i\mathfrak{c}_0(k)\left(\Xi(k)^2+1\right)+\mathfrak{s}_0(k)\left(\Xi(k)^2-1\right). \tag{B.49}$$

If $(a:b)_{2,4}=0$ and $m_{\mathfrak{c}_0}=0$ (i.e., *$\mathfrak{c}_0(k)=0$ and $\mathfrak{s}_0(k)=(a:b)_{1,3}$), then we require $(a:b)_{1,3}\neq 0$, so that $\Delta(k)\not\to 0$ exponentially fast (or is not identically zero). Thus, Boundary Cases 1–4 are the only allowable cases giving rise to a well-defined solution for constant-coefficient problems. If the coefficients are not constant, it may be possible to go out to higher order in the asymptotics of Lemma 21,* e.g., *$(a:b)_{2,4}=0$, $m_{\mathfrak{c}_0}=0$, $m_{\mathfrak{c}_1}\neq 0$, and $m_{\mathfrak{c}_1}\mathfrak{u}_+-8m_{\mathfrak{s}}=0$, and additional allowable boundary conditions may be identified. This requires further investigation.*

Lemma 23. *Consider the finite-interval, half-line, and whole-line problems. For all three, there exists an $r>\sqrt{M_\gamma}$ and $M_\Psi>0$ such that, for $k\in\Omega_{\text{ext}}(r)$, $x\in\overline{\mathcal{D}}$, and $y\in\overline{\mathcal{D}}$,*

$$|\Psi(k,x,y)|\le M_\Psi. \tag{B.50a}$$

For the regular problems*,*

$$\left|\frac{\Psi(k,x,y)}{\Delta(k)}\right|\le M_\Psi, \tag{B.50b}$$

and for the irregular problems*,*

$$\left|\frac{\Psi(k,x,y)}{\Delta(k)}\right|\le M_\Psi\left(1+|k|\left(e^{-m_{\mathfrak{i}\mathfrak{n}}|k|(x-x_l)}+e^{-m_{\mathfrak{i}\mathfrak{n}}|k|(x_r-x)}\right)\right)\le 3M_\Psi|k|. \tag{B.50c}$$

Thus $\Psi(k,x,y)$ and $\Psi(k,x,y)/\Delta(k)$ are well-defined functions.

Proof. For the whole-line problem, from (3.4) and Lemma 15,

$$|\Psi(k,x,y)|\le e^{-m_{\mathfrak{i}\mathfrak{n}}|k||x-y|}\sum_{n=0}^{\infty}\frac{1}{2^n n!}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathbb{R}}^n \le \exp\left(\frac{1}{2}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathcal{D}}\right), \tag{B.51}$$

and (B.50a) follows. From Lemma 21, (B.50b) follows. For the half-line problem, with $x_l<y<x$, from (4.8)

$$\begin{aligned}|\Psi(k,x,y)| &\le 4\left|\exp\left(\int_y^x ik\mathfrak{n}(k,\xi)\,d\xi\right)\right|\left|\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}\right|\left(\frac{a_0}{k\mathfrak{n}(k,x_l)}\mathcal{S}_{n-\ell}^{(x_l,y)}(k)-a_1\mathcal{C}_{n-\ell}^{(x_l,y)}(k)\right)\mathcal{E}_\ell^{(x,\infty)}(k)\Bigg|\\ &\le 4e^{-m_{\mathfrak{i}\mathfrak{n}}|k||x-y|}\left(\frac{|a_0|}{m_{\mathfrak{n}}r}+|a_1|\right)\exp\left(\frac{1}{2}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathcal{D}}\right),\end{aligned} \tag{B.52}$$

and similarly for $x_l < x < y$. Therefore (B.50a) follows. From Lemma 21, we have

$$\left|\frac{\Psi(k,x,y)}{\Delta(k)}\right| \le \frac{4\left(|a_1| + \frac{|a_0|}{|k\mathfrak{n}(k,x_l)|}\right)}{\left||a_1| - \left|\frac{a_0}{k\mathfrak{n}(k,x_l)}\right|\right|} \exp\left(\frac{1}{2}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathcal{D}}\right) \le 4A\exp\left(\frac{1}{2}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathcal{D}}\right), \tag{B.53}$$

where A is defined in (B.33). This gives (B.50b).

For the finite-interval problem:

1. if $(a:b)_{2,4} \neq 0$, from (5.9a), we find for $x_l < y < x < r_r$,

$$\begin{aligned}|\Psi(k,x,y)| \le 4&\left(|(a:b)_{2,4}| + \frac{|(a:b)_{1,3}|}{m_{\mathfrak{n}}^2 r^2} + \frac{|(a:b)_{1,4}| + |(a:b)_{2,3}|}{m_{\mathfrak{n}} r}\right)\exp\left(\frac{1}{2}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathcal{D}}\right) \\ &+ \frac{4M_\beta|(a:b)_{1,2}|}{m_\beta m_{\mathfrak{n}}|k|}\exp\left(\frac{1}{2}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathcal{D}}\right) e^{-m_{i\mathfrak{n}}|k|(x_r - x_l - |x-y|)},\end{aligned} \tag{B.54}$$

   and similarly for $x_l < x < y < x_r$. Thus (B.50a) follows. From Lemma 21, (B.50b) follows.

2. If $(a:b)_{2,4} = 0$ and $m_{\mathfrak{c}_0} \neq 0$, from (5.9a), we find for $x_l < y < x < r_r$,

$$\begin{aligned}|\Psi(k,x,y)| \le \frac{4}{|k|}&\left(\frac{|(a:b)_{1,4}| + |(a:b)_{2,3}|}{m_{\mathfrak{n}}} + \frac{|(a:b)_{1,3}|}{m_{\mathfrak{n}}^2 r}\right)\exp\left(\frac{1}{2}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathcal{D}}\right) \\ &+ \frac{4M_\beta|(a:b)_{1,2}|}{m_\beta m_{\mathfrak{n}}|k|}\exp\left(\frac{1}{2}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathcal{D}}\right) e^{-m_{i\mathfrak{n}}|k|(x_r - x_l - |x-y|)},\end{aligned} \tag{B.55}$$

   and similarly for $x_l < x < y < x_r$. This gives (B.50a). From Lemma 21, (B.50b) follows.

3. If $(a:b)_{2,4} = 0$, $m_{\mathfrak{c}_0} = 0$, $m_{\mathfrak{c}_1} = 0$, and $(a:b)_{1,3} \neq 0$, then for $x_l < y < x < x_r$,

$$|\Psi(k,x,y)| \le \frac{4}{|k|^2}\left(\frac{|(a:b)_{1,3}|}{m_{\mathfrak{n}}^2} + \frac{4M_\beta|(a:b)_{1,2}||k|}{m_\beta m_{\mathfrak{n}}} e^{-m_{i\mathfrak{n}}|k|(x_r - x_l - |x-y|)}\right)\exp\left(\frac{1}{2}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathcal{D}}\right), \tag{B.56}$$

   and similarly for $x_l < x < y < x_r$. This gives (B.50a). This Boundary Case is regular if both $(a:b)_{1,2} = 0$ and $(a:b)_{3,4} = 0$ and irregular if either $(a:b)_{1,2} \neq 0$ or $(a:b)_{3,4} \neq 0$, see Remark 10. Lemma 21 gives (B.50b) or (B.50c).

4. If $(a:b)_{2,4} = 0$, $m_{\mathfrak{c}_0} = 0$, $m_{\mathfrak{c}_1} \neq 0$, and $m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}} \neq 0$, then,

$$\frac{(a:b)_{1,4}}{\mu(x_l)} = \frac{(a:b)_{2,3}}{\mu(x_r)} = \frac{m_{\mathfrak{c}_1}}{2}. \tag{B.57}$$

   From this $(a:b)_{1,4}/\mathfrak{n}(k,x_l) = m_{\mathfrak{c}_1}/2 + O(k^{-2})$ and $(a:b)_{2,3}/\mathfrak{n}(k,x_r) = m_{\mathfrak{c}_1}/2 + O(k^{-2})$. Using Lemma 19, there exists an $r > C^2$ such that

$$\left|\sum_{n=1}^{\infty}\sum_{\ell=0}^{n}(-1)^{\lambda\ell}\mathcal{J}_{n-\ell}^{(x_l,y)}[\sigma_{p,n-\ell}](k)\mathcal{J}_{\ell}^{(x,x_r)}[\overline{\sigma}_{p,\ell}](k)\right| \le \sum_{n=1}^{\infty}\sum_{\ell=0}^{n}\frac{C^{n-\ell}}{|k|^{\lfloor\frac{n-\ell+1}{2}\rfloor}}\frac{C^{\ell}}{|k|^{\lfloor\frac{\ell+1}{2}\rfloor}} = \frac{(k+C)^2}{(k-C^2)^2} - 1 = O(k^{-1}), \tag{B.58}$$

   for $k \in \Omega_{\text{ext}}(r)$. For $x_l < y < x < x_r$, the $n = 0$ terms involving $(a:b)_{1,4}$ and $(a:b)_{2,3}$ combine to give

$$\begin{aligned}&\frac{(a:b)_{1,4}}{k\mathfrak{n}(k,x_l)}\mathcal{S}_0^{(x_l,y)}(k)\mathcal{C}_0^{(x,x_r)}(k) - \frac{(a:b)_{2,3}}{k\mathfrak{n}(k,x_r)}\mathcal{C}_0^{(x_l,y)}(k)\mathcal{S}_0^{(x,x_r)}(k) \\ &\quad= \frac{m_{\mathfrak{c}_1}}{2k}\left(\mathcal{S}_0^{(x_l,y)}(k)\mathcal{C}_0^{(x,x_r)}(k) - \mathcal{C}_0^{(x_l,y)}(k)\mathcal{S}_0^{(x,x_r)}(k)\right) + O(k^{-3}) \\ &\quad= \frac{m_{\mathfrak{c}_1}}{2k}\sin\left(\int_{x_l}^{y} - \int_{x}^{x_r} k\mathfrak{n}(k,\xi)\,d\xi\right) + O(k^{-3}),\end{aligned} \tag{B.59}$$

   so that, for $x_l < y < x < x_r$,

$$|\Psi(k,x,y)| \le 4\left\{\frac{|m_{\mathfrak{c}_1}|}{4|k|}\left(e^{-m_{i\mathfrak{n}}|k|(x-x_l)} + e^{-m_{i\mathfrak{n}}|k|(x_r-x)}\right) + O(k^{-2})\right\} + \frac{4M_\beta|(a:b)_{1,2}|}{m_\beta m_{\mathfrak{n}}|k|}e^{-m_{i\mathfrak{n}}|k|(x_r-x)}. \tag{B.60}$$

   This gives (B.50a). Using Lemma 21, we arrive at (B.50c). The same can be shown for $x_l < x < y < x_r$.

□

Lemma 24. *Consider the finite-interval and half-line problems. There exists an $r > \sqrt{M_\gamma}$ and $M_{\mathcal{B}} > 0$ such that for $k \in \Omega_{\text{ext}}(r)$ and $x \in \overline{\mathcal{D}}$, for both the half-line ($m = 0$) and the finite-interval problem ($m = 0, 1$),*

$$|\mathcal{B}_m(k, x)| \leq M_{\mathcal{B}}. \tag{B.61a}$$

Further, for the half-line problem ($m = 0$),

$$\left|\frac{\mathcal{B}_0(k, x)}{\Delta(k)}\right| \leq M_{\mathcal{B}}|k|e^{-m_{i\mathfrak{n}}|k|(x-x_l)}, \tag{B.61b}$$

and for the finite-interval problem ($m = 0, 1$),

$$\left|\frac{\mathcal{B}_m(k, x)}{\Delta(k)}\right| \leq M_{\mathcal{B}}|k|^b\big(e^{-m_{i\mathfrak{n}}|k|(x_r-x)} + e^{-m_{i\mathfrak{n}}|k|(x-x_l)}\big). \tag{B.61c}$$

Here, $b = 1$ for regular *boundary conditions, and $b = 2$ for* irregular *boundary conditions. It follows that the functions $\mathcal{B}_m(k, x)$ and $\mathcal{B}_m(k, x)/\Delta(k)$ are well defined for the half-line and finite-interval problems.*

Proof. For the half-line problem, using Lemmas 15 and 18 in (4.6), we have

$$|\mathcal{B}_0(k, x)| \leq \frac{4M_\beta}{m_\beta m_{\mathfrak{n}}} \exp\left(\frac{1}{2}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathcal{D}}\right) e^{-m_{i\mathfrak{n}}|k|(x-x_l)}, \tag{B.62}$$

which gives (B.61a). Lemma 21 gives (B.61b). Similarly, for the finite-interval problem, using Lemmas 15 and 18 in (5.8c), we have

$$|\mathcal{B}_{2-j}(k, x)| \leq \frac{4M_\beta}{m_\beta m_{\mathfrak{n}}} \exp\left(\frac{1}{2}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathcal{D}}\right) \left\{\left(\frac{|a_{j1}|}{m_{\mathfrak{n}}|k|} + |a_{j2}|\right) e^{-m_{i\mathfrak{n}}|k|(x_r-x)} + \left(\frac{|b_{j1}|}{m_{\mathfrak{n}}|k|} + |b_{j2}|\right) e^{-m_{i\mathfrak{n}}|k|(x-x_l)}\right\}, \tag{B.63}$$

for $j = 1, 2$, which gives (B.61a). For the finite-interval problem with Boundary Case 1 or 2 and for the irregular boundary conditions, (B.61c) follows from the above and Lemma 21. For the regular version of Boundary Case 3, we have $a_{ij} = 0$ for all $i, j = 1, 2$, except for a_{11} and b_{21}, see Remark 10. Thus,

$$|\mathcal{B}_{2-j}(k, x)| \leq \frac{4M_\beta}{m_\beta m_{\mathfrak{n}}^2|k|} \exp\left(\frac{1}{2}\left\|\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right\|_{\mathcal{D}}\right) \left(|a_{j1}|e^{-m_{i\mathfrak{n}}|k|(x_r-x)} + |b_{j1}|e^{-m_{i\mathfrak{n}}|k|(x-x_l)}\right), \tag{B.64}$$

from which (B.61c) follows, using Lemma 21. □

Lemma 25. *Consider the finite-interval, half-line, and whole-line problems. For all three, there exists an $r > \sqrt{M_\gamma}$ and $M_\Phi > 0$ such that for $k \in \Omega_{\text{ext}}(r)$ and $x \in \overline{\mathcal{D}}$,*

$$|\Phi_0(k, x)| \leq M_\Phi\|q_0\|_{\mathcal{D}}. \tag{B.65a}$$

For the regular *problems,*

$$\left|\frac{\Phi_0(k, x)}{\Delta(k)}\right| \leq M_\Phi\|q_0\|_{\mathcal{D}}, \tag{B.65b}$$

and for the irregular *problems,*

$$\left|\frac{\Phi_0(k, x)}{\Delta(k)}\right| = M_\Phi\|q_0\|_{\mathcal{D}}\left(1 + |k|\left(e^{-m_{i\mathfrak{n}}|k|(x-x_l)} + e^{-m_{i\mathfrak{n}}|k|(x_r-x)}\right)\right) \leq 3M_\Phi|k|\|q_0\|_{\mathcal{D}}. \tag{B.65c}$$

It follows that $\Phi_0(k, x)$ and $\Phi_0(k, x)/\Delta(k)$ are well-defined functions.

Proof. The inequalities (B.26a) follow directly from Lemma 23. □

Lemma 26. *Consider the finite-interval, half-line, and whole-line problems. For all three, there exists an* $r > \sqrt{M_\gamma}$ *and* $M_f > 0$ *such that for* $k \in \Omega_{\text{ext}}(r)\backslash\Omega(r)$ *(the green region of Figure 4), for* $x \in \overline{\mathcal{D}}$*, and for* $t \in [0, T]$*,*

$$\left|\Phi_f(k, x, t)e^{-k^2 t}\right| \leq M_f \|f\|_{\mathcal{D}}. \tag{B.66a}$$

Further, for the regular problems*,*

$$\left|\frac{\Phi_f(k, x, t)e^{-k^2 t}}{\Delta(k)}\right| = M_f \|f\|_{\mathcal{D}}, \tag{B.66b}$$

and for the irregular problems*,*

$$\left|\frac{\Phi_f(k, x, t)e^{-k^2 t}}{\Delta(k)}\right| = M_f \|f\|_{\mathcal{D}} \left(1 + |k| \left(e^{-m_{\mathfrak{i}\mathfrak{n}}|k|(x-x_l)} + e^{-m_{\mathfrak{i}\mathfrak{n}}|k|(x_r-x)}\right)\right) \leq 3M_f|k|\|f\|_{\mathcal{D}}. \tag{B.66c}$$

Thus, $\Phi_f(k, x, t)$ *and* $\Phi_f(k, x, t)/\Delta(k)$ *are well-defined functions.*

Proof. For $k \in \Omega_{\text{ext}}(r)\backslash\Omega(r)$, $|e^{-k^2(t-s)}| < 1$. It follows from (2.8) and Assumption 2.1 that

$$\int_{\mathcal{D}} \left|\tilde{f}_\alpha(k^2, x, t)e^{-k^2 t}\right| dx \leq \int_{\mathcal{D}} \int_0^t |f_\alpha(x, s)|\, dsdy \leq \frac{T\|f\|_{\mathcal{D}}}{m_\alpha}. \tag{B.67}$$

Using this and (B.50) in (B.26b), we obtain (B.66) for any $x \in \mathcal{D}$ and for $t \in [0, T]$. □

Lemma 27. *There exists an* $r > \sqrt{M_\gamma}$ *so that for* $x \in \mathcal{D}$ *and* $t \in [0, T]$, $\mathcal{J}_n^{(a,b)}(k)$, $\Delta(k)$, $\Psi(k, x, y)$*, and* $\Phi_0(k, x)$ *are analytic in* k*, for* $k \in \Omega_{\text{ext}}(r)$*. The functions* $\Phi_f(k, x, t)e^{-k^2 t}$ *and* $\mathcal{B}_m(k, x)e^{-k^2 t}$ *are analytic in* k *for* $k \in \Omega_{\text{ext}}(r)\backslash\Omega(r)$*.*

Proof. Consider a closed contour $\Gamma \in \Omega_{\text{ext}}(r)$. Then

$$\oint_\Gamma \mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)\, dk = \frac{1}{2^n} \int_{a<\cdots<b} d\mathbf{y}_n \oint_\Gamma dk \left(\prod_{p=1}^n \frac{(\beta\mathfrak{n})'(k, y_p)}{(\beta\mathfrak{n})(k, y_p)}\right) \exp\left(\sum_{p=0}^n \sigma_{p,n} \int_{y_p}^{y_{p+1}} ik\mathfrak{n}(k, \xi)\, d\xi\right) = 0, \tag{B.68}$$

by Cauchy's theorem. We can switch the order of integration by Fubini's theorem and Lemma 18. Therefore, by Morera's theorem, $\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)$ is analytic for $k \in \Omega_{\text{ext}}(r)$. For all three types of IBVPs considered, the same argument applies for the $\Delta(k)$, $\Psi(k, x, y)$, and the $\Phi_0(k, x)$ functions by Lemmas 21, 23, and 25, and for the $\mathcal{B}_m(k, x)$ and $\Phi_f(k, x, t)$ functions by Lemma 24 and 26. □

The following lemmas prove that the different parts of the solution are well defined.

Lemma 28. *For the half-line problem* $(m = 0)$ *and the finite-interval problem* $(m = 0, 1)$*, there exists an* $r > \sqrt{M_\gamma}$ *such that, for any* $x \in \overline{\mathcal{D}}$ *and* $t \in (0, T)$*, the function* $q_{\mathcal{B}_m}(x, t)$ (B.27c) *can be written as*

$$q_{\mathcal{B}_m}(x, t) = \frac{1}{2\pi} \int_{\partial\Omega_{\text{ext}}(r)} \frac{\mathcal{B}_m(k, x)}{\Delta(k)} \mathfrak{F}_m(k^2, t)e^{-k^2 t}\, dk, \tag{B.69a}$$

where

$$\mathfrak{F}_m(k^2, t) = -\frac{f_m(0)}{k^2} - \frac{1}{k^2} \int_0^t e^{k^2 s} f_m'(s)\, ds, \tag{B.69b}$$

with the bound

$$\left|\mathfrak{F}_m(k^2, t)e^{-k^2 t}\right| \leq \frac{\|f_m\|_\infty e^{-|k|^2 \cos(2\theta)t}}{|k|^2} + \frac{\|f_m'\|_\infty \left(1 - e^{-|k|^2 \cos(2\theta)t}\right)}{|k|^4 \cos(2\theta)}. \tag{B.70}$$

The function $q_{\mathcal{B}_m}(x, t)$ *is well defined.*

Proof. From (4.7) and Assumption 2.3, for $k \in \Omega_{\text{ext}}(r) \backslash \Omega(r)$,

$$\left|F_m(k^2,t)e^{-k^2t}\right| \le \left|\int_0^t e^{-k^2(t-s)} f_m(s)\,ds\right| \le T\|f_m\|_\infty. \tag{B.71}$$

Therefore, for $x \in \mathcal{D}$, we have exponential decay of the integrand of $q_{\mathcal{B}_m}(x,t)$ from Lemma 24. Using Lemma 27, we can deform the contour of (B.27c) from $\Omega(r)$ to $\Omega_{\text{ext}}(r)$. Assumption 2.3 allows us to integrate (4.7) by parts so that

$$F_m(k^2,t)e^{-k^2t} = \frac{f_m(t)}{k^2} - \frac{f_m(0)e^{-k^2t}}{k^2} - \frac{1}{k^2}\int_0^t e^{-k^2(t-s)} f_m'(s)\,ds, \tag{B.72}$$

which gives (B.69), after using Cauchy's theorem on the $f_m(t)$ term. Equation (B.70) follows from (B.69b) and Assumption 2.3. From Lemma 24, for the half-line problem,

$$|q_{\mathcal{B}_m}(x,t)| \le \frac{M_{\mathcal{B}}}{2\pi}\int_{\partial\Omega_{\text{ext}}(r)} |k|e^{-m_{\text{in}}|k|(x-x_l)}\left|\mathfrak{F}_m(k^2,t)e^{-k^2t}\right|dk, \tag{B.73a}$$

and for the finite-interval problem,

$$|q_{\mathcal{B}_m}(x,t)| \le \frac{M_{\mathcal{B}}}{2\pi}\int_{\partial\Omega_{\text{ext}}(r)} |k|^b\left(e^{-m_{\text{in}}|k|(x_r-x)} + e^{-m_{\text{in}}|k|(x-x_l)}\right)\left|\mathfrak{F}_m(k^2,t)e^{-k^2t}\right|dk. \tag{B.73b}$$

From (B.73), we see that $q_{\mathcal{B}_m}(x,t)$ is well defined for $x \in \overline{\mathcal{D}}$ and for $t \in (0,T)$. □

Lemma 29. *Consider the finite-interval, half-line, and whole-line problems. There exists an* $r > \sqrt{M_\gamma}$ *so that for* $x \in \overline{\mathcal{D}}$ *and* $t \in (0,T)$*,* $q_0(x,t)$ (B.27a) *can be written as*

$$q_0(x,t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}(r)} \frac{\Phi_0(k,x)}{\Delta(k)}e^{-k^2t}\,dk, \tag{B.74}$$

which is well defined.

Proof. By Lemmas 25 and 27, $\Phi_0(k,x)/\Delta(k)$ is bounded, well defined, and analytic for $k \in \Omega_{\text{ext}}(r)$. Let $C_R = \{k \in \mathbb{C} : |k| = R \text{ and } \theta_0 < \theta < \pi/4 \text{ or } 3\pi/4 < \theta < \pi - \theta_0\}$, see Figure 4. For the *regular problems*, using symmetry,

$$\left|\int_{C_R} \frac{\Phi_0(k,x)}{\Delta(k)}e^{-k^2t}\,dk\right| \le 2M_\Phi\|q_0\|_\infty \int_{\theta_0}^{\frac{\pi}{4}} e^{-R^2\cos(2\theta)t}R\,d\theta \le \frac{\pi M_\Phi\|q_0\|_{\mathcal{D}}\left(1-e^{-R^2t}\right)}{2Rt} \to 0, \tag{B.75}$$

as $R \to \infty$. Thus we can deform the contour by Cauchy's theorem to conclude (B.74). For the *irregular problems*, the above holds for the integral over the first term of (B.65c) and for $x \in \mathcal{D}$, the second term is exponentially decaying, and we again conclude (B.74). It follows that for all three problems

$$|q_0(x,t)| = \frac{M_\Phi\|q_0\|_{\mathcal{D}}}{2\pi}\int_{\partial\Omega_{\text{ext}}} \left|ke^{-k^2t}\right|\,|dk| < \infty. \tag{B.76}$$

□

Lemma 30. *Consider the finite-interval, half-line, and whole-line problems. There exists an* $r > \sqrt{M_\gamma}$ *so that for* $x \in \overline{\mathcal{D}}$ *and* $t \in (0,T)$*,* $q_f(x,t)$ (B.27b) *can be written as*

$$q_f(x,t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}(r)} \frac{\Phi_{\mathfrak{f}}(k,x,t)e^{-k^2t}}{\Delta(k)}dk, \tag{B.77a}$$

where

$$\Phi_{\mathfrak{f}}(k,x,t) = \int_{\mathcal{D}} \frac{\Psi(k,x,y)\mathfrak{f}_\alpha(k^2,y,t)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\,dy, \tag{B.77b}$$

and

$$\mathfrak{f}_\alpha(k^2,y,t) = -\frac{f_\alpha(y,0)}{k^2} - \frac{1}{k^2}\int_0^t f_{\alpha,s}(y,s)e^{k^2 s}\,ds. \tag{B.77c}$$

Further, we have the bound

$$\int_{\mathcal{D}} \left|\mathfrak{f}_\alpha(k^2,y,t)e^{-k^2t}\right| dy \le \frac{\|f\|_{\mathcal{D}} e^{-|k|^2\cos(2\theta)t}}{m_\alpha |k|^2} + \frac{\|f_t\|_{\mathcal{D}}\left(1-e^{-|k|^2\cos(2\theta)t}\right)}{m_\alpha|k|^4\cos(2\theta)}. \tag{B.78}$$

For all three problems, there exists an $M_f>0$ such that

$$\left|\Phi_{\mathfrak{f}}(k,x,t)e^{-k^2t}\right| \le M_f \int_{\mathcal{D}} \left|\mathfrak{f}_\alpha(k^2,y,t)e^{-k^2t}\right| dy. \tag{B.79a}$$

For the regular problems

$$\left|\frac{\Phi_{\mathfrak{f}}(k,x,t)e^{-k^2t}}{\Delta(k)}\right| \le M_f \int_{\mathcal{D}} \left|\mathfrak{f}_\alpha(k^2,y,t)e^{-k^2t}\right| dy, \tag{B.79b}$$

and for the irregular problems*,*

$$\left|\frac{\Phi_{\mathfrak{f}}(k,x,t)e^{-k^2t}}{\Delta(k)}\right| \le M_f \left(1+|k|\left(e^{-m_{in}|k|(x-x_l)}+e^{-m_{in}|k|(x_r-x)}\right)\right)\int_{\mathcal{D}} \left|\mathfrak{f}_\alpha(k^2,y,t)e^{-k^2t}\right| dy. \tag{B.79c}$$

It follows that $q_f(x,t)$ is well defined for all three problems.

Proof. By Lemmas 26 and 27, $\Phi_f(k,x,t)e^{-k^2t}/\Delta(k)$ is bounded, well defined, and analytic for $k\in\Omega_{\text{ext}}(r)/\Omega(r)$. Let C_R be defined as in the proof of Lemma 29, see Figure 4. Then, for the *regular problems*, using symmetry,

$$\left|\int_{C_R}\frac{\Phi_f(k,x,t)e^{-k^2t}}{\Delta(k)}\,dk\right| \le 2M_f\|f\|_{\mathcal{D}}\int_0^{\frac{\pi}{4}} Re^{-R^2\cos(2\theta)t}\,d\theta \to 0, \tag{B.80}$$

as $R\to\infty$. Thus, we can deform the integral in (B.27b) from $\Omega(r)$ to $\Omega_{\text{ext}}(r)$. For the *irregular problems*, the above holds for the integral over the first term of (B.66c) and for $x\in\mathcal{D}$ the second term is exponentially decaying. Thus, we can still deform from $\Omega(r)$ to $\Omega_{\text{ext}}(r)$. Using Assumption 2.1, we can integrate (2.8) by parts, to obtain

$$\tilde{f}_\alpha(k^2,x,t) = \frac{f_\alpha(x,t)e^{k^2t}-f_\alpha(x,0)}{k^2} - \frac{1}{k^2}\int_0^t f_{\alpha,s}(x,s)e^{k^2s}\,ds, \tag{B.81}$$

which gives (B.77), after using Cauchy's theorem on the $f_\alpha(x,t)$ term. Equation (B.78) follows directly from (B.77c) and Assumption 2.1, and equation (B.79) follows from Lemma 23. From (B.79), we see that the integrand in $q_f(x,t)$ is absolutely integrable and is therefore well defined for all $x\in\overline{\mathcal{D}}$ and any $t\in(0,T)$ (or for any $x\in\mathcal{D}$ and for all $t\in[0,T]$). ☐

Finally, we combine all the results obtained.

Theorem 31. *There exists an $r>\sqrt{M_\gamma}$ such that the functions (3.2), (4.2), and (5.4) are well defined for all $x\in\overline{\mathcal{D}}$ and for any $t\in(0,T)$.*

Proof. Combining Lemmas 28, 29, and 30, we obtain our result. ☐

C Proofs: the solution expressions solve the evolution equation

In this appendix, we prove that the solution expressions (3.2), (4.2), and (5.4) for the whole-line, half-line, and finite-interval problems, respectively, solve the evolution equation (2.1) in their respective domains. Naturally, we are in need of lemmas on the derivatives of various quantities defining the solution expressions. The following lemma deals with derivatives with respect to the spatial variable.

Lemma 32. *For $n \geq 0$, the derivatives of $\mathcal{E}_n^{(x,\infty)}(k)$ and $\tilde{\mathcal{E}}_n^{(-\infty,x)}(k)$ are given by*

$$\partial_x \mathcal{E}_n^{(x,\infty)}(k) = -\frac{1}{2}\frac{(\beta\mathfrak{n})'(k,x)}{(\beta\mathfrak{n})(k,x)}\mathcal{E}_{n-1}^{(x,\infty)}(k) - (1-(-1)^n)ik\mathfrak{n}(k,x)\mathcal{E}_n^{(x,\infty)}(k), \tag{C.1a}$$

$$\partial_x \tilde{\mathcal{E}}_n^{(-\infty,x)}(k) = \frac{1}{2}\frac{(\beta\mathfrak{n})'(k,x)}{(\beta\mathfrak{n})(k,x)}\tilde{\mathcal{E}}_{n-1}^{(-\infty,x)}(k) + (1-(-1)^n)ik\mathfrak{n}(k,x)\tilde{\mathcal{E}}_n^{(-\infty,x)}(k), \tag{C.1b}$$

and those of $\mathcal{C}_n^{(a,b)}(k)$ and $\mathcal{S}_n^{(a,b)}(k)$ are

$$\partial_x \mathcal{C}_n^{(x_l,x)}(k) = \frac{1}{2}\frac{(\beta\mathfrak{n})'(k,x)}{(\beta\mathfrak{n})(k,x)}\mathcal{C}_{n-1}^{(x_l,x)}(k) - (-1)^n k\mathfrak{n}(k,x)\mathcal{S}_n^{(x_l,x)}(k), \tag{C.1c}$$

$$\partial_x \mathcal{C}_n^{(x,x_r)}(k) = -\frac{1}{2}\frac{(\beta\mathfrak{n})'(k,x)}{(\beta\mathfrak{n})(k,x)}\mathcal{C}_{n-1}^{(x,x_r)}(k) + k\mathfrak{n}(k,x)\mathcal{S}_n^{(x,x_r)}(k), \tag{C.1d}$$

$$\partial_x \mathcal{S}_n^{(x_l,x)}(k) = \frac{1}{2}\frac{(\beta\mathfrak{n})'(k,x)}{(\beta\mathfrak{n})(k,x)}\mathcal{S}_{n-1}^{(x_l,x)}(k) + (-1)^n k\mathfrak{n}(k,x)\mathcal{C}_n^{(x_l,x)}(k), \tag{C.1e}$$

$$\partial_x \mathcal{S}_n^{(x,x_r)}(k) = \frac{1}{2}\frac{(\beta\mathfrak{n})'(k,x)}{(\beta\mathfrak{n})(k,x)}\mathcal{S}_{n-1}^{(x,x_r)}(k) - k\mathfrak{n}(k,x)\mathcal{C}_n^{(x,x_r)}(k). \tag{C.1f}$$

Proof. Since $(\beta\mathfrak{n})'/(\beta\mathfrak{n}) \in L^1(\mathcal{D})$ by Lemma 14, the proof is by direct calculation of the derivatives of (3.5) and (4.9) [18]. We show one such calculation. From (3.5b),

$$\partial_x \tilde{\mathcal{E}}_n^{(a,x)}(k) = \frac{\partial_x}{2^n}\int_a^x dy_1 \int_{y_1}^x dy_2 \cdots \int_{y_{n-2}}^x dy_{n-1}\int_{y_{n-1}}^x dy_n \left(\prod_{p=1}^n \frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right)\exp\left(\sum_{p=0}^n (1-(-1)^p)\int_{y_p}^{y_{p+1}} ik\mathfrak{n}(k,\xi)\,d\xi\right), \tag{C.2}$$

so that

$$\begin{aligned}\partial_x \tilde{\mathcal{E}}_n^{(a,x)}(k) = {} & \frac{(\beta\mathfrak{n})'(k,x)}{(\beta\mathfrak{n})(k,x)}\frac{1}{2^n}\int_{a=y_0<\cdots<y_n=x}\left(\prod_{p=1}^{n-1}\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right)\exp\left(\sum_{p=0}^{n-1}(1-(-1)^p)\int_{y_p}^{y_{p+1}} ik\mathfrak{n}(k,\xi)\,d\xi\right)d\mathbf{y}_{n-1} \\ & + (1-(-1)^n)\frac{ik\mathfrak{n}(k,x)}{2^n}\int_{a=y_0<\cdots<y_{n+1}=x}\left(\prod_{p=1}^{n}\frac{(\beta\mathfrak{n})'}{(\beta\mathfrak{n})}\right)\exp\left(\sum_{p=0}^{n}(1-(-1)^p)\int_{y_p}^{y_{p+1}} i\nu(k,\xi)\,d\xi\right)d\mathbf{y}_{n},\end{aligned} \tag{C.3}$$

which gives (C.1b). ∎

In Lemma 33, we prove a general summation identity for the generalized accumulation functions $\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)$. This identity is used to prove the problem-specific identities in Lemma 34. In turn, these are used to prove the relation between $\chi(k,x)$ (C.18b) and $\Delta(k)$ in Lemma 38.

Lemma 33. *Let $\overline{\sigma}_{p,n-\ell}$ and $\tilde{\sigma}_{p,\ell}$ be two non-negative integer-valued functions as described in Definition 16. Denote*

$$\sigma_{p,n} = \begin{cases}\overline{\sigma}_{p,n-\ell}, & \text{if } 0 \leq p \leq n-\ell, \\ \tilde{\sigma}_{p-(n-\ell),\ell}, & \text{if } n-\ell < p \leq n.\end{cases} \tag{C.4a}$$

If $\overline{\sigma}_{n-\ell,n-\ell} = \tilde{\sigma}_{0,\ell}$ and $\sigma_{p,n}$ is independent of ℓ, then

$$\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k) = \sum_{\ell=0}^n \mathcal{J}_{n-\ell}^{(a,x)}[\overline{\sigma}_{p,n-\ell}](k)\mathcal{J}_\ell^{(x,b)}[\tilde{\sigma}_{p,\ell}](k). \tag{C.4b}$$

Proof. Define

$$\mathfrak{j}_n^{(a,b)}[\sigma_{p,n}](k) = \sum_{\ell=0}^n \mathcal{J}_{n-\ell}^{(a,x)}[\overline{\sigma}_{p,n-\ell}](k)\mathcal{J}_\ell^{(x,b)}[\tilde{\sigma}_{p,\ell}](k). \tag{C.5}$$

By the definition of $\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)$ (B.10),

$$\mathfrak{j}_n^{(a,b)}[\sigma_{p,n}](k) = \frac{1}{2^n}\sum_{\ell=0}^{n}\int_{a<\cdots<x}\left(\prod_{p=1}^{n-\ell}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\exp\left(\sum_{p=0}^{n-\ell}\overline{\sigma}_{p,n-\ell}\int_{y_p}^{y_{p+1}} ik\mathfrak{n}(k,\xi)\,d\xi\right)dy_1\cdots dy_{n-\ell}\times$$
$$\times\int_{x<\cdots<b}\left(\prod_{p=n-\ell+1}^{n}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\exp\left(\sum_{p=n-\ell}^{n}\tilde{\sigma}_{p-(n-\ell),\ell}\int_{y_p}^{y_{p+1}} ik\mathfrak{n}(k,\xi)\,d\xi\right)dy_{n-\ell+1}\cdots dy_n. \tag{C.6}$$

In the exponential of the first integral, for the $p = n-\ell$ term, $y_{n-\ell+1}$ is defined as x. In the exponential in the second integral, for the $p = n-\ell$ term, $y_{n-\ell} = x$. Since $\overline{\sigma}_{n-\ell,n-\ell} = \tilde{\sigma}_{0,\ell} = \sigma_{n-\ell,n}$, multiplying the exponentials and adding these terms together, we have

$$\overline{\sigma}_{n-\ell,n-\ell}\int_{y_{n-\ell}}^{x} ik\mathfrak{n}(k,\xi)\,d\xi + \tilde{\sigma}_{0,\ell}\int_{x}^{y_{n-\ell+1}} ik\mathfrak{n}(k,\xi)\,d\xi = \sigma_{n-\ell,n}\int_{y_{n-\ell}}^{y_{n-\ell+1}} ik\mathfrak{n}(k,\xi)\,d\xi, \tag{C.7}$$

and the two integrals are combined as

$$\mathfrak{j}_n^{(a,b)}[\sigma_{p,n}](k) = \frac{1}{2^n}\sum_{\ell=0}^{n}\int_{a<\cdots<y_{n-\ell}<x<y_{n-\ell+1}<\cdots<b}\left(\prod_{p=1}^{n}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\exp\left(\sum_{p=0}^{n}\sigma_{p,n}\int_{y_p}^{y_{p+1}} ik\mathfrak{n}(k,\xi)\,d\xi\right)dy_1\cdots dy_n. \tag{C.8}$$

Summing over ℓ is equivalent to adding up all possibilities of x lying between one of the $y_1,\ldots,y_n$. Since $\sigma_{p,n}$ is independent of ℓ by assumption, the integrand is independent of ℓ, and

$$\mathfrak{j}_n^{(a,b)}[\sigma_{p,n}](k) = \frac{1}{2^n}\int_{a=y_0<\cdots<y_{n+1}<b}\left(\prod_{p=1}^{n}\frac{(\beta\mathfrak{n})'(k,y_p)}{(\beta\mathfrak{n})(k,y_p)}\right)\exp\left(\sum_{p=0}^{n}\sigma_{p,n}\int_{y_p}^{y_{p+1}} ik\mathfrak{n}(k,\xi)\,d\xi\right)dy_1\cdots dy_n, \tag{C.9}$$

which is (C.4). ☐

From the identity in Lemma 33, we can prove the following more specific forms of (C.4b).

Lemma 34. *For the whole-line problem, if n is even,*

$$\mathcal{E}_n^{(-\infty,\infty)}(k) = \sum_{\ell=0}^{n}\tilde{\mathcal{E}}_{n-\ell}^{(-\infty,x)}(k)\mathcal{E}_\ell^{(x,\infty)}(k). \tag{C.10a}$$

For the half-line problem, for any n,

$$\mathcal{E}_n^{(x_l,\infty)}(k) = \sum_{\ell=0}^{n}\left(\mathcal{C}_{n-\ell}^{(x_l,x)}(k) - (-1)^n i\mathcal{S}_{n-\ell}^{(x_l,x)}(k)\right)\mathcal{E}_\ell^{(x,\infty)}(k). \tag{C.10b}$$

Finally, for the finite-interval problem, for any n,

$$\mathcal{C}_n^{(x_l,x_r)}(k) = \sum_{\ell=0}^{n}\left(\mathcal{C}_{n-\ell}^{(x_l,x)}(k)\mathcal{C}_\ell^{(x,x_r)}(k) - (-1)^{n-\ell}\mathcal{S}_{n-\ell}^{(x_l,x)}(k)\mathcal{S}_\ell^{(x,x_r)}(k)\right), \tag{C.10c}$$

$$\mathcal{S}_n^{(x_l,x_r)}(k) = \sum_{\ell=0}^{n}\left(\mathcal{S}_{n-\ell}^{(x_l,x)}(k)\mathcal{C}_\ell^{(x,x_r)}(k) + (-1)^{n-\ell}\mathcal{C}_{n-\ell}^{(x_l,x)}(k)\mathcal{S}_\ell^{(x,x_r)}(k)\right). \tag{C.10d}$$

Proof. For the whole-line problem, we define $\mathfrak{e}_{\mathrm{wl}}$ as the right-hand side of (C.10a). Using (B.12), we write $\mathfrak{e}_{\mathrm{wl}}$ as

$$\mathfrak{e}_{\mathrm{wl}} = \sum_{\ell=0}^{n}\mathcal{J}_{n-\ell}^{(-\infty,x)}[1-(-1)^p](k)\mathcal{J}_\ell^{(x,\infty)}[1-(-1)^{\ell-p}](k). \tag{C.11}$$

From Lemma 33, if n is even, $\overline{\sigma}_{p,n-\ell} = 1-(-1)^p$ and $\tilde{\sigma}_{p,\ell} = 1-(-1)^{\ell-p}$ so that $\overline{\sigma}_{n-\ell,n-\ell} = \tilde{\sigma}_{0,\ell}$ and

$$\sigma_{p,n} = \begin{cases} \overline{\sigma}_{p,n-\ell}, & \text{if } 0 \le p \le n-\ell, \\ \tilde{\sigma}_{p-(n-\ell),\ell}, & \text{if } n-\ell < p \le n, \end{cases} = \begin{cases} 1-(-1)^p, & \text{if } 0 \le p \le n-\ell, \\ 1-(-1)^{\ell-(p-(n-\ell))}, & \text{if } n-\ell < p \le n, \end{cases} = 1-(-1)^{n-p}, \tag{C.12}$$

is independent of ℓ so that (C.10a) follows.

For the half-line problem, we define $\mathfrak{e}_{\mathrm{hl}}$ as the right-hand side of (C.10b). Using (B.12),

$$\mathfrak{e}_{\mathrm{hl}} = \frac{1}{2}\sum_{\ell=0}^{n} \Big((1-(-1)^n)\mathcal{J}_{n-\ell}^{(x_l,x)}[1+(-1)^p](k) + (1+(-1)^n)\mathcal{J}_{n-\ell}^{(x_l,x)}[1-(-1)^p](k) \Big) \mathcal{J}_{\ell}^{(x,\infty)}[1-(-1)^{\ell-p}](k), \tag{C.13}$$

which is simplified to

$$\mathfrak{e}_{\mathrm{hl}} = \sum_{\ell=0}^{n} \mathcal{J}_{n-\ell}^{(x_l,x)}[1-(-1)^{n-p}](k)\mathcal{J}_{\ell}^{(x,\infty)}[1-(-1)^{\ell-p}](k). \tag{C.14}$$

From Lemma 33, $\overline{\sigma}_{p,n-\ell} = 1-(-1)^{n-p}$ and $\tilde{\sigma}_{p,\ell} = 1-(-1)^{\ell-p}$ so that $\overline{\sigma}_{n-\ell,n-\ell} = \tilde{\sigma}_{0,\ell}$ and $\sigma_{p,n} = 1-(-1)^{n-p}$. Equation (C.10b) follows.

For the finite-interval problem, we define $\mathfrak{e}_c$ and $\mathfrak{e}_s$ as the right-hand side of (C.10c) and (C.10d), respectively. Using (B.12), we write these in terms of $\mathcal{J}_n^{(a,b)}[\sigma_{p,n}](k)$, obtaining

$$\begin{aligned} \mathfrak{e}_c = {} & \frac{1}{4}\sum_{\ell=0}^{n}(1+(-1)^{n-\ell}) \Big(\mathcal{J}_{n-\ell}^{(x_l,x)}[1+(-1)^p](k)\mathcal{J}_{\ell}^{(x,x_r)}[1+(-1)^p](k) \\ & \qquad + \mathcal{J}_{n-\ell}^{(x_l,x)}[1-(-1)^p](k)\mathcal{J}_{\ell}^{(x,x_r)}[1-(-1)^p](k) \Big) \\ & + \frac{1}{4}\sum_{\ell=0}^{n}(1-(-1)^{n-\ell}) \Big(\mathcal{J}_{n-\ell}^{(x_l,x)}[1+(-1)^p](k)\mathcal{J}_{\ell}^{(x,x_r)}[1-(-1)^p](k) \\ & \qquad + \mathcal{J}_{n-\ell}^{(x_l,x)}[1-(-1)^p](k)\mathcal{J}_{\ell}^{(x,x_r)}[1+(-1)^p](k) \Big), \end{aligned} \tag{C.15a}$$

$$\begin{aligned} \mathfrak{e}_s = {} & \frac{1}{4i}\sum_{\ell=0}^{n}(1+(-1)^{n-\ell}) \Big(\mathcal{J}_{n-\ell}^{(x_l,x)}[1+(-1)^p](k)\mathcal{J}_{\ell}^{(x,x_r)}[1+(-1)^p](k) \\ & \qquad - \mathcal{J}_{n-\ell}^{(x_l,x)}[1-(-1)^p](k)\mathcal{J}_{\ell}^{(x,x_r)}[1-(-1)^p](k) \Big) \\ & + \frac{1}{4i}\sum_{\ell=0}^{n}(1-(-1)^{n-\ell}) \Big(\mathcal{J}_{n-\ell}^{(x_l,x)}[1+(-1)^p](k)\mathcal{J}_{\ell}^{(x,x_r)}[1-(-1)^p](k) \\ & \qquad - \mathcal{J}_{n-\ell}^{(x_l,x)}[1-(-1)^p](k)\mathcal{J}_{\ell}^{(x,x_r)}[1+(-1)^p](k) \Big), \end{aligned} \tag{C.15b}$$

which simplify to

$$\mathfrak{e}_c = \frac{1}{2}\sum_{\ell=0}^{n} \Big(\mathcal{J}_{n-\ell}^{(x_l,x)}[1+(-1)^p](k)\mathcal{J}_{\ell}^{(x,x_r)}[1+(-1)^{n-\ell+p}](k) + \mathcal{J}_{n-\ell}^{(x_l,x)}[1-(-1)^p](k)\mathcal{J}_{\ell}^{(x,x_r)}[1-(-1)^{n-\ell+p}](k) \Big), \tag{C.16a}$$

$$\mathfrak{e}_s = \frac{1}{2i}\sum_{\ell=0}^{n} \Big(\mathcal{J}_{n-\ell}^{(x_l,x)}[1+(-1)^p](k)\mathcal{J}_{\ell}^{(x,x_r)}[1+(-1)^{n-\ell+p}](k) - \mathcal{J}_{n-\ell}^{(x_l,x)}[1-(-1)^p](k)\mathcal{J}_{\ell}^{(x,x_r)}[1-(-1)^{n-\ell+p}](k) \Big). \tag{C.16b}$$

For the first terms of $\mathfrak{e}_c$ and $\mathfrak{e}_s$, $\overline{\sigma}_{p,n-\ell} = 1+(-1)^p$, and $\tilde{\sigma}_{p,\ell} = 1+(-1)^{n-\ell+p}$ so that $\overline{\sigma}_{n-\ell,n-\ell} = \tilde{\sigma}_{0,\ell}$ and $\sigma_{p,n} = 1+(-1)^p$. For the second terms of $\mathfrak{e}_c$ and $\mathfrak{e}_s$, $\overline{\sigma}_{p,n-\ell} = 1-(-1)^p$, and $\tilde{\sigma}_{p,\ell} = 1-(-1)^{n-\ell+p}$ so that $\overline{\sigma}_{n-\ell,n-\ell} = \tilde{\sigma}_{0,\ell}$ and $\sigma_{p,n} = 1-(-1)^p$. Equations (C.10c) and (C.10d) follow. □

Now, we begin taking derivatives of the solution expressions. In Definition 35, we introduce some functions that appear in the derivatives of the solution expressions. In Lemmas 36–38, we prove some properties of these functions.

Definition 35. *We define*

$$\overline{\Psi}(k,x,y) = \sqrt{(\beta\mathfrak{n})(k,x)}\frac{\partial}{\partial x}\left(\frac{\Psi(k,x,y)}{\sqrt{(\beta\mathfrak{n})(k,x)}}\right) = \Psi_x(k,x,y) - \frac{1}{2}\frac{(\beta\mathfrak{n})'(k,x)}{(\beta\mathfrak{n})(k,x)}\Psi(k,x,y), \quad \text{and} \tag{C.17a}$$

$$\tilde{\Psi}(k,x,y) = \sqrt{(\beta\mathfrak{n})(k,x)}\frac{\partial}{\partial x}\left(\frac{(\beta\overline{\Psi})(k,x,y)}{\sqrt{(\beta\mathfrak{n})(k,x)}}\right) = (\beta\overline{\Psi})_x(k,x,y) - \frac{1}{2}\frac{(\beta\mathfrak{n})'(k,x)}{(\beta\mathfrak{n})(k,x)}\left(\beta\overline{\Psi}\right)(k,x,y), \tag{C.17b}$$

where we use the notation $(\beta\overline{\Psi})(k,x,y) = \beta(x)\overline{\Psi}(k,x,y)$. *We also define*

$$\Psi(k,x,x^\pm) = \lim_{y\to x^\pm}\Psi(k,x,y), \qquad \overline{\Psi}(k,x,x^\pm) = \lim_{y\to x^\pm}\overline{\Psi}(k,x,y), \tag{C.18a}$$

and

$$\chi(k,x) = (\beta\overline{\Psi})(k,x,x^-) - (\beta\overline{\Psi})(k,x,x^+). \tag{C.18b}$$

Lemma 36. *For the whole-line problem, for* $y < x$,

$$\overline{\Psi}(k,x,y) = ik\mathfrak{n}(k,x)\exp\left(\int_y^x ik\mathfrak{n}(k,\xi)\,d\xi\right)\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}\tilde{\mathcal{E}}_{n-\ell}^{(-\infty,y)}(k)\mathcal{E}_{\ell}^{(x,\infty)}(k), \tag{C.19a}$$

and for $x < y$,

$$\overline{\Psi}(k,x,y) = -ik\mathfrak{n}(k,x)\exp\left(\int_x^y ik\mathfrak{n}(k,\xi)\,d\xi\right)\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^n\tilde{\mathcal{E}}_{n-\ell}^{(-\infty,x)}(k)\mathcal{E}_{\ell}^{(y,\infty)}(k). \tag{C.19b}$$

For the half-line problem, for $x_l < y < x$,

$$\overline{\Psi}(k,x,y) = 4ik\mathfrak{n}(k,x)\exp\left(\int_{x_l}^x ik\mathfrak{n}(k,\xi)\,d\xi\right)\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}\left(\frac{a_0}{k\mathfrak{n}(k,x_l)}\mathcal{S}_{n-\ell}^{(x_l,y)}(k) - a_1\mathcal{C}_{n-\ell}^{(x_l,y)}(k)\right)\mathcal{E}_{\ell}^{(x,\infty)}(k), \tag{C.20a}$$

and for $x_l < x < y$,

$$\overline{\Psi}(k,x,y) = 4k\mathfrak{n}(k,x)\exp\left(\int_{x_l}^y ik\mathfrak{n}(k,\xi)\,d\xi\right)\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^n\left(\frac{a_0}{k\mathfrak{n}(k,x_l)}\mathcal{C}_{n-\ell}^{(x_l,x)}(k) + a_1\mathcal{S}_{n-\ell}^{(x_l,x)}(k)\right)\mathcal{E}_{\ell}^{(y,\infty)}(k). \tag{C.20b}$$

For the finite-interval problem, for $x_l < y < x < x_r$,

$$\begin{aligned}\overline{\Psi}(k,x,y) = 4k\mathfrak{n}(k,x)\Xi(k)\Bigg\{&-\frac{\beta(x_r)(a:b)_{1,2}}{k\sqrt{(\beta\mathfrak{n})(k,x_l)}\sqrt{(\beta\mathfrak{n})(k,x_r)}}\sum_{n=0}^{\infty}(-1)^n\mathcal{C}_n^{(y,x)}(k)\\
&- (a:b)_{2,4}\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^\ell\mathcal{C}_{n-\ell}^{(x_l,y)}(k)\mathcal{S}_{\ell}^{(x,x_r)}(k) - \frac{(a:b)_{1,3}}{k^2\mathfrak{n}(k,x_l)\mathfrak{n}(k,x_r)}\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}\mathcal{S}_{n-\ell}^{(x_l,y)}(k)\mathcal{C}_{\ell}^{(x,x_r)}(k)\\
&+\frac{(a:b)_{1,4}}{k\mathfrak{n}(k,x_l)}\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^\ell\mathcal{S}_{n-\ell}^{(x_l,y)}(k)\mathcal{S}_{\ell}^{(x,x_r)}(k) + \frac{(a:b)_{2,3}}{k\mathfrak{n}(k,x_r)}\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}\mathcal{C}_{n-\ell}^{(x_l,y)}(k)\mathcal{C}_{\ell}^{(x,x_r)}(k)\Bigg\},\end{aligned} \tag{C.21a}$$

and for $x_l < x < y < x_r$,

$$\begin{aligned}\overline{\Psi}(k,x,y) = 4k\mathfrak{n}(k,x)\Xi(k)\Bigg\{&\frac{\beta(x_l)(a:b)_{3,4}}{k\sqrt{(\beta\mathfrak{n})(k,x_l)}\sqrt{(\beta\mathfrak{n})(k,x_r)}}\sum_{n=0}^{\infty}\mathcal{C}_n^{(x,y)}(k)\\
&+ (a:b)_{2,4}\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^n\mathcal{S}_{n-\ell}^{(x_l,x)}(k)\mathcal{C}_{\ell}^{(y,x_r)}(k) + \frac{(a:b)_{1,3}}{k^2\mathfrak{n}(k,x_l)\mathfrak{n}(k,x_r)}\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^{n-\ell}\mathcal{C}_{n-\ell}^{(x_l,x)}(k)\mathcal{S}_{\ell}^{(y,x_r)}(k)\\
&+\frac{(a:b)_{1,4}}{k\mathfrak{n}(k,x_l)}\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^n\mathcal{C}_{n-\ell}^{(x_l,x)}(k)\mathcal{C}_{\ell}^{(y,x_r)}(k) + \frac{(a:b)_{2,3}}{k\mathfrak{n}(k,x_r)}\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}(-1)^{n-\ell}\mathcal{S}_{n-\ell}^{(x_l,x)}(k)\mathcal{S}_{\ell}^{(y,x_r)}(k)\Bigg\}.\end{aligned} \tag{C.21b}$$

Proof. Using (C.1) in (3.4), (4.8), and (5.9), we find (C.19), (C.20), and (C.21) for the whole-line, half-line, and finite-interval problems, respectively. ☐

Lemma 37. *Consider the finite-interval, half-line, and whole-line problems. There exists an* $r > \sqrt{M_\gamma}$ *and* $M_\Psi > 0$ *so that for* $k \in \Omega_{\text{ext}}(r)$*, for* $x \in \overline{\mathcal{D}}$*, and for* $y \in \overline{\mathcal{D}}$

$$|\overline{\Psi}(k,x,y)| \leq M_\Psi |k|. \tag{C.22a}$$

For the regular problems

$$\left|\frac{\overline{\Psi}(k,x,y)}{\Delta(k)}\right| \leq M_\Psi |k|, \tag{C.22b}$$

and for the irregular problems

$$\left|\frac{\overline{\Psi}(k,x,y)}{\Delta(k)}\right| \leq M_\Psi |k| \left(1 + |k| \left(e^{-m_{in}|k|(x-x_l)} + e^{-m_{in}|k|(x_r-x)}\right)\right). \tag{C.22c}$$

Therefore, $\overline{\Psi}(k,x,y)$ *and* $\overline{\Psi}(k,x,y)/\Delta(k)$ *are well-defined functions.*

Proof. The proof is identical to that of Lemma 23 in Appendix B. Note that M_Ψ here and from Lemma 13 are identical up to a factor of $M_{\mathfrak{n}}$. Without loss of generality, we take them to be the same. ☐

Lemma 38. *For the finite-interval, half-line, and whole-line problems,*

$$\chi(k,x) = 2ik(\beta\mathfrak{n})(k,x)\Delta(k), \tag{C.23}$$

where $\chi(k,x)$ *is defined in* (C.18b).

Proof. For the whole-line problem, using Lemma 36 in (C.18b),

$$\chi(k,x) = ik(\beta\mathfrak{n})(k,x) \sum_{n=0}^{\infty} (1 + (-1)^n) \sum_{\ell=0}^{n} \tilde{\mathcal{E}}_{n-\ell}^{(-\infty,x)}(k) \mathcal{E}_\ell^{(x,\infty)}(k), \tag{C.24}$$

which gives (C.23), using Lemma 34. Similarly, for the half-line problem,

$$\chi(k,x) = -4k(\beta\mathfrak{n})(k,x) \exp\left(\int_{x_l}^{x} ik\mathfrak{n}(k,\xi)\, d\xi\right) \sum_{n=0}^{\infty} \left(\frac{(-1)^n a_0}{k\mathfrak{n}(k,x_l)} + ia_1\right) \sum_{\ell=0}^{n} \left(\mathcal{C}_{n-\ell}^{(x_l,x)}(k) - (-1)^n i \mathcal{S}_{n-\ell}^{(x_l,x)}(k)\right) \mathcal{E}_\ell^{(x,\infty)}(k). \tag{C.25}$$

Using Lemma 34,

$$\chi(k,x) = -4k(\beta\mathfrak{n})(k,x) \sum_{n=0}^{\infty} \left(\frac{(-1)^n a_0}{k\mathfrak{n}(k,x_l)} + ia_1\right) \mathcal{E}_n^{(x_l,\infty)}(k) = 2ik(\beta\mathfrak{n})(k,x)\Delta(k). \tag{C.26}$$

Finally, for the finite-interval problem, since $\mathcal{C}_n^{(x,x)}(k) = \delta_{0n}$ and $\mathcal{S}_n^{(x,x)}(k) = 0$,

$$\begin{aligned}\chi(k,x) = -4k(\beta\mathfrak{n})(k,x)\Xi(k) \Bigg\{ \mathfrak{a}(k) &+ \sum_{n=0}^{\infty} \mathfrak{c}_n(k) \sum_{\ell=0}^{n} \left(\mathcal{C}_{n-\ell}^{(x_l,x)}(k)\mathcal{C}_\ell^{(x,x_r)}(k) - (-1)^{n-\ell}\mathcal{S}_{n-\ell}^{(x_l,x)}(k)\mathcal{S}_\ell^{(x,x_r)}(k)\right) \\ &+ \sum_{n=0}^{\infty} \mathfrak{s}_n(k) \sum_{\ell=0}^{n} \left(\mathcal{S}_{n-\ell}^{(x_l,x)}(k)\mathcal{C}_\ell^{(x,x_r)}(k) + (-1)^{n-\ell}\mathcal{C}_{n-\ell}^{(x_l,x)}(k)\mathcal{S}_\ell^{(x,x_r)}(k)\right) \Bigg\},\end{aligned} \tag{C.27}$$

which gives (C.23), using Lemma 34. ☐

Lemma 39. *For the half-line problem,*

$$\mathcal{B}_{0,x}(k,x) = \frac{4\beta(x_l) ik\mathfrak{n}(k,x)\exp\left(\int_{x_l}^x ik\mathfrak{n}(k,\xi)\,d\xi\right)}{\sqrt{(\beta\mathfrak{n})(k,x_l)}\sqrt{(\beta\mathfrak{n})(k,x)}} \sum_{n=0}^{\infty} \mathcal{E}_n^{(x,\infty)}(k), \tag{C.28a}$$

and there exists an $r > \sqrt{M_\gamma}$ and $M_{\mathcal{B}} > 0$ so that for $k \in \Omega_{\text{ext}}(r)$ and $x \in \overline{\mathcal{D}}$,

$$|\mathcal{B}_{0,x}(k,x)| \leq M_{\mathcal{B}}|k| \qquad \text{and} \qquad \left|\frac{\mathcal{B}_{0,x}(k,x)}{\Delta(k)}\right| \leq M_{\mathcal{B}}|k|^2 e^{-m_{i\mathfrak{n}}|k|(x-x_l)}. \tag{C.28b}$$

For the finite-interval problem, we have for $j = 1, 2$,

$$\begin{aligned}\mathcal{B}_{2-j,x}(k,x) = -(-1)^j \frac{4k\mathfrak{n}(k,x)\Xi(k)}{\sqrt{(\beta\mathfrak{n})(k,x)}} \Bigg\{ &\frac{\beta(x_r)}{\sqrt{(\beta\mathfrak{n})(k,x_r)}} \left[\frac{a_{j1}}{k\mathfrak{n}(k,x_l)} \sum_{n=0}^{\infty} (-1)^n \mathcal{C}_n^{(x_l,x)}(k) + a_{j2} \sum_{n=0}^{\infty} (-1)^n \mathcal{S}_n^{(x_l,x)}(k)\right] \\ &+ \frac{\beta(x_l)}{\sqrt{(\beta\mathfrak{n})(k,x_l)}} \left[\frac{b_{j1}}{k\mathfrak{n}(k,x_r)} \sum_{n=0}^{\infty} \mathcal{C}_n^{(x,x_r)}(k) - b_{j2} \sum_{n=0}^{\infty} (-1)^n \mathcal{S}_n^{(x,x_r)}(k)\right]\Bigg\},\end{aligned} \tag{C.29a}$$

and there exists an $r > \sqrt{M_\gamma}$ and $M_{\mathcal{B}} > 0$ so that for $k \in \Omega_{\text{ext}}(r)$ and $x \in \overline{\mathcal{D}}$,

$$|\mathcal{B}_{m,x}(k,x)| \leq M_{\mathcal{B}}|k| \qquad \text{and} \qquad \left|\frac{\mathcal{B}_{m,x}(k,x)}{\Delta(k)}\right| \leq M_{\mathcal{B}}|k|^{b+1}\left(e^{-m_{i\mathfrak{n}}|k|(x_r-x)} + e^{-m_{i\mathfrak{n}}|k|(x-x_l)}\right). \tag{C.29b}$$

For regular *boundary conditions $b = 1$, and for* irregular *boundary conditions $b = 2$. Therefore, the functions $\mathcal{B}_{m,x}(x,t)$ and $\mathcal{B}_{m,x}(x,t)/\Delta(k)$ are well defined for the half-line and finite-interval problems.*

Proof. Lemma 32 and a direct calculation gives (C.28a) and (C.29a). The proofs for (C.28b) and (C.29b) are identical to the proof of Lemma 24. Note that, as in Lemma 37, the $M_{\mathcal{B}}$'s differ only by a factor of $M_{\mathfrak{n}}$ (see Lemma 13). Without loss of generality, we may take them to be identical. □

Lemma 40. *Consider the finite-interval, half-line, and whole-line problems. We have*

$$\Phi_{0,x}(k,x) = \int_{\mathcal{D}} \frac{\overline{\Psi}(k,x,y) q_\alpha(y)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\,dy, \tag{C.30}$$

where $\Phi_0(k,x)$ is defined in (B.26a). There exists an $M_\Phi > 0$ so that

$$|\Phi_{0,x}(k,x)| \leq M_\Phi|k|\|q_0\|_{\mathcal{D}} \qquad \text{and} \qquad \left|\frac{\Phi_{0,x}(k,x)}{\Delta(k)}\right| \leq M_\Phi|k|^2\|q_0\|_{\mathcal{D}}. \tag{C.31}$$

Thus $\Phi_{0,x}(k,x)$ and $\Phi_{0,x}(k,x)/\Delta(k)$ are well defined for all three problems.

Proof. Breaking up the integral over $\mathcal{D}$ in (B.26a) into two integrals over the regions $y < x$ and $y > x$ and using the Leibniz integral rule, we obtain

$$\Phi_{0,x}(k,x) = \frac{(\Psi(k,x,x^-) - \Psi(k,x,x^+))q_\alpha(x)}{(\beta\mathfrak{n})(k,x)} + \int_{\mathcal{D}} \frac{\overline{\Psi}(k,x,y) q_\alpha(y)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\,dy. \tag{C.32}$$

Since $\Psi(k,x,x^-) = \Psi(k,x,x^+)$, we find (C.30). We obtain (C.31) from Lemma 37. Since the integrand in (C.30) is absolutely integrable, differentiation under the integral is allowed. □

Lemma 41. *Consider the finite-interval, half-line, and whole-line problems. For $k \in \Omega_{\text{ext}}$, $x \in \overline{\mathcal{D}}$, and $t \in (0,T)$,*

$$\Phi_{\mathfrak{f},x}(k,x,t) = \int_{\mathcal{D}} \frac{\overline{\Psi}(k,x,y)\mathfrak{f}_\alpha(k^2,y,t)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\,dy. \tag{C.33}$$

Further, there exists an $M_f > 0$ so that

$$\left|\Phi_{\mathfrak{f},x}(k,x,t)e^{-k^2t}\right| \leq M_f|k| \int_{\mathcal{D}} \left|\mathfrak{f}_\alpha(k^2,y,t)e^{-k^2t}\right| dy. \tag{C.34a}$$

For the regular problems*,*

$$\left|\frac{\Phi_{\mathfrak{f},x}(k,x,t)e^{-k^2t}}{\Delta(k)}\right| \leq M_f|k|\int_{\mathcal{D}} \left|\mathfrak{f}_\alpha(k^2,y,t)e^{-k^2t}\right| dy, \tag{C.34b}$$

and for the irregular problems*,*

$$\left|\frac{\Phi_{\mathfrak{f}}(k,x,t)e^{-k^2t}}{\Delta(k)}\right| \leq M_f|k|\Big(1+|k|\big(e^{-m_{\rm in}|k|(x-x_l)}+e^{-m_{\rm in}|k|(x_r-x)}\big)\Big)\int_{\mathcal{D}} \left|\mathfrak{f}_\alpha(k^2,y,t)e^{-k^2t}\right| dy. \tag{C.34c}$$

where $\Phi_{\mathfrak{f}}(k,x,t)$ *and* $\mathfrak{f}_\alpha(k^2,y,t)$ *are defined in* (B.77b) *and* (B.77c)*, respectively.*

Proof. Breaking up the integral over $\mathcal{D}$ in (B.26a) into two integrals over the regions $y < x$ and $y > x$ and using the Leibniz integral rule, we obtain

$$\Phi_{\mathfrak{f},x}(k,x,t) = \frac{(\Psi(k,x,x^-)-\Psi(k,x,x^+))\,\mathfrak{f}_\alpha(k^2,x,t)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,x)}} + \int_{\mathcal{D}} \frac{\overline{\Psi}(k,x,y)\mathfrak{f}_\alpha(k^2,y,t)}{\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\,dy. \tag{C.35}$$

Since $\Psi(k,x,x^-) = \Psi(k,x,x^+)$ for all three problems, we obtain (C.33). Equation (C.34) follows from (C.22a). Since the integrand (C.33) is absolutely integrable, differentiation under the integral is allowed. □

Lemma 42. *Consider the finite-interval, half-line, and whole-line problems. For* $x \in \mathcal{D}$ *and* $t \in (0,T)$*,*

$$q_{0,x}(x,t) = \frac{1}{2\pi}\int_{\partial\Omega_{\rm ext}} \frac{\Phi_{0,x}(k,x)}{\Delta(k)}e^{-k^2t}\,dk, \tag{C.36a}$$

$$q_{f,x}(x,t) = \frac{1}{2\pi}\int_{\partial\Omega_{\rm ext}} \frac{\Phi_{\mathfrak{f},x}(k,x,t)e^{-k^2t}}{\Delta(k)}dy, \tag{C.36b}$$

$$q_{\mathcal{B}_m,x}(x,t) = \frac{1}{2\pi}\int_{\partial\Omega_{\rm ext}} \frac{\mathcal{B}_{m,x}(k,x)}{\Delta(k)}\mathfrak{F}_m(k^2,t)e^{-k^2t}\,dk, \tag{C.36c}$$

are well defined, i.e., *we can differentiate under the integral sign. Furthermore,* $q_{0,x}(x,t)$ *and* $q_{f,x}(x,t)$ *are well defined for* $x \in \overline{\mathcal{D}}$*. For the* regular problems*,* $q_{\mathcal{B}_m,x}(x,t)$ *is well defined for* $x \in \overline{\mathcal{D}}$*.*

Proof. The integrand in $q_{0,x}(x,t)$ is exponentially decaying for $t \in (0,T)$, and therefore is well defined for $x \in \overline{\mathcal{D}}$. From (C.34) and (B.78), we see that, for any $t \in (0,T)$, $q_{f,x}(x,t)$ is also well defined for $x \in \overline{\mathcal{D}}$. For $t \in (0,T)$, from (C.28b), (C.29b), and (B.70), we see that for $x \in \mathcal{D}$, $q_{\mathcal{B}_m,x}(x,t)$ has exponential decay and is well defined. For the *regular problems*, $q_{\mathcal{B}_m,x}(x,t)$ is absolutely integrable for $x \in \overline{\mathcal{D}}$ and is well defined. □

Remark. For the *irregular problems*, $q_{\mathcal{B}_m,x}(x,t)$ may be ill defined at the boundaries, but the boundary conditions (5.1c) and (5.1d) are well defined and satisfied, see Section D.

Lemma 43. *Consider the finite-interval, half-line, and whole-line problems. For* $x, y \in \mathcal{D}$ *and* $k \in \Omega_{\rm ext}$*,*

$$\tilde{\Psi}(k,x,y) = -\frac{k^2+\gamma(x)}{\alpha(x)}\Psi(k,x,y). \tag{C.37a}$$

For the half-line $(m = 0)$ *and the finite-interval problems* $(m = 0, 1)$*,*

$$(\beta\mathcal{B}_{m,x})_x(k,x) = -\frac{k^2+\gamma(x)}{\alpha(x)}\mathcal{B}_m(k,x), \tag{C.37b}$$

for $x \in \mathcal{D}$*,* $t \in (0,T)$*, and* $k \in \Omega$*.*

Proof. For the whole-line problem, a direct calculation using Lemma 32 gives (C.37a) from (C.19a) for $y < x$ and from (C.19b) for $y > x$. Similarly, for the half-line problem, we obtain (C.37a) from (C.20a) for $x_l < y < x$ and from (C.20b) for $x_l < x < y$. Equation (C.37b) follows from (C.28a). For the finite-interval problem, we obtain (C.37a) from (C.21a) for $x_l < y < x < x_r$ and from (C.21b) for $x_l < x < y < x_r$. Finally, (C.37b) follows from (C.29a). □

Lemma 44. *Consider the finite-interval, half-line, and whole-line problems. With* $\mathfrak{f}(k^2,x,t)=\alpha(x)\mathfrak{f}_\alpha(k^2,x,t)$,

$$\alpha(x)(\beta\Phi_{0,x})_x(k,x)=2ik\Delta(k)q_0(x)-(k^2+\gamma(x))\Phi_0(k,x), \tag{C.38a}$$

$$\alpha(x)(\beta\Phi_{\mathfrak{f},x})_x(k,x,t)=2ik\Delta(k)\mathfrak{f}(k^2,x,t)-(k^2+\gamma(x))\Phi_{\mathfrak{f}}(k,x,t), \tag{C.38b}$$

Proof. Using Lemmas 40 and 41, we split $\mathcal{D}$ into the two parts $y<x$ and $y>x$, and the Leibniz integral rule gives

$$(\beta\Phi_{0,x})_x(k,x)=\frac{\chi(k,x)q_0(x)}{\alpha(x)(\beta\mathfrak{n})(k,x)}+\int_{\mathcal{D}}\frac{\tilde\Psi(k,x,y)q_0(y)}{\alpha(y)\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\,dy, \tag{C.39a}$$

$$(\beta\Phi_{\mathfrak{f},x})_x(k,x,t)=\frac{\chi(k,x)\mathfrak{f}(k^2,x,t)}{\alpha(x)(\beta\mathfrak{n})(k,x)}+\int_{\mathcal{D}}\frac{\tilde\Psi(k,x,y)\mathfrak{f}(k^2,y,t)}{\alpha(y)\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\,dy, \tag{C.39b}$$

where $\chi(k,x)$ and $\tilde\Psi(k,x,y)$ are defined in Definition 35. Using Lemmas 38 and 43 gives (C.38). ☐

Lemma 45. *Consider the finite-interval, half-line, and whole-line problem. For $x\in\mathcal{D}$ and $t\in(0,T)$, the t-derivatives of $q_0(x,t)$, $q_f(x,t)$ and $q_{\mathcal{B}_m}(x,t)$ are*

$$q_{0,t}(x,t)=-\frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}}\frac{k^2\Phi_0(k,x)}{\Delta(k)}e^{-k^2t}\,dk, \tag{C.40a}$$

$$q_{f,t}(x,t)=-\frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}(r)}\frac{k^2\Phi_{\mathfrak{f}}(k,x,t)e^{-k^2t}}{\Delta(k)}dk, \tag{C.40b}$$

$$q_{\mathcal{B}_m,t}(x,t)=-\frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}(r)}\frac{k^2\mathcal{B}_m(k,x)}{\Delta(k)}\mathfrak{F}_m(k^2,t)e^{-k^2t}\,dk,\qquad m=0,1. \tag{C.40c}$$

These functions are well defined.

Proof. Differentiating (B.74) with respect to t gives (C.40a), since the integrand is absolutely integrable. From (B.77c), $\mathfrak{f}_{\alpha,t}(k^2,x,t)e^{-k^2t}=-f_{\alpha,t}(x,t)/k^2$, and differentiating (B.77b) with respect to t yields

$$\Phi_{\mathfrak{f},t}(k,x,t)e^{-k^2t}=-\int_{\mathcal{D}}\frac{\Psi(k,x,y)f_{\alpha,t}(y,t)}{k^2\sqrt{(\beta\mathfrak{n})(k,x)}\sqrt{(\beta\mathfrak{n})(k,y)}}\,dy, \tag{C.41}$$

so that, using Lemma 23,

$$\left|\frac{\Phi_{\mathfrak{f},t}(k,x,t)e^{-k^2t}}{\Delta(k)}\right|\le\frac{M_f}{|k|^2}\Big(1+|k|\big(e^{-m_{in}|k|(x-x_l)}+e^{-m_{in}|k|(x_r-x)}\big)\Big)\|f_{\alpha,t}\|_{\mathcal{D}}. \tag{C.42}$$

Differentiating (B.77a) with respect to t, we obtain

$$q_{f,t}(x,t)=\frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}(r)}\frac{\Phi_{\mathfrak{f},t}(k,x,t)e^{-k^2t}}{\Delta(k)}dk-\frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}(r)}\frac{k^2\Phi_{\mathfrak{f}}(k,x,t)e^{-k^2t}}{\Delta(k)}dk. \tag{C.43}$$

From (C.42), it follows that the first contour integral can be closed in the upper half plane, implying it is zero by Cauchy's theorem, resulting in (C.40b). From (B.79), $q_{f,t}(x,t)$ is well defined, for $x\in\mathcal{D}$. Since $\mathfrak{F}_{m,t}(k^2,t)e^{-k^2t}=-f'_m(t)/k^2$, differentiating (B.69a) with respect to t,

$$q_{\mathcal{B}_m,t}(x,t)=-\frac{f'_m(t)}{2\pi}\int_{\partial\Omega_{\text{ext}}(r)}\frac{\mathcal{B}_m(k,x)}{k^2\Delta(k)}\,dk-\frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}(r)}\frac{k^2\mathcal{B}_m(k,x)}{\Delta(k)}\mathfrak{F}_m(k^2,t)e^{-k^2t}\,dk. \tag{C.44}$$

As above, (B.61) allows us to close the contour of the first integral in the upper half plane, showing the first term is zero by Cauchy's theorem, obtaining (C.40c). From (B.61) and (B.70), $q_{\mathcal{B}_m,t}(x,t)$ is well defined for $x\in\mathcal{D}$. ☐

Lemma 46. *For $x\in\mathcal{D}$ and $t\in(0,T)$, the derivatives*

$$\alpha(x)(\beta q_{0,x})_x(x,t)+\gamma(x)q_0(x,t)=q_{0,t}(x,t), \tag{C.45a}$$

$$\alpha(x)(\beta q_{f,x})_x(x,t)+\gamma(x)q_f(x,t)+f(x,t)=q_{f,t}(x,t), \tag{C.45b}$$

$$\alpha(x)(\beta q_{\mathcal{B}_m,x})_x(x,t)+\gamma(x)q_{\mathcal{B}_m}(x,t)=q_{\mathcal{B}_m,t}(x,t),\qquad m=0,1. \tag{C.45c}$$

are well defined, i.e., *differentiation under the integral sign is allowed.*

Proof. Direct differentiation of the results in Lemma 42 yields

$$(\beta q_{0,x})_x(x,t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{(\beta\Phi_{0,x})_x(k,x)}{\Delta(k)} e^{-k^2t}\,dk, \tag{C.46a}$$

$$(\beta q_{f,x})_x(x,t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{(\beta\Phi_{\mathfrak{f},x})_x(k,x,t)e^{-k^2t}}{\Delta(k)} dk, \tag{C.46b}$$

$$(\beta q_{\mathcal{B}_m,x})_x(x,t) = \frac{1}{2\pi}\int_{\partial\Omega} \frac{(\beta\mathcal{B}_{m,x})_x(k,x)}{\Delta(k)} \mathfrak{F}_m(k^2,t)e^{-k^2t}\,dk, \qquad m=0,1. \tag{C.46c}$$

Using Lemmas 43, 44, and 45,

$$\alpha(x)(\beta q_{0,x})_x(x,t) + \gamma(x)q_0(x,t) = q_{0,t}(x,t) - \frac{q_0(x)}{i\pi}\int_{\partial\Omega_{\text{ext}}} ke^{-k^2t}\,dk, \tag{C.47a}$$

$$\alpha(x)(\beta q_{f,x})_x(x,t) + \gamma(x)q_f(x,t) = q_{f,t}(x,t) - \frac{1}{i\pi}\int_{\partial\Omega_{\text{ext}}} k\mathfrak{f}(k^2,x,t)e^{-k^2t}dk, \tag{C.47b}$$

$$\alpha(x)(\beta q_{\mathcal{B}_m,x})_x(x,t) + \gamma(x)q_{\mathcal{B}_m}(x,t) = q_{\mathcal{B}_m,t}(x,t), \qquad m=0,1. \tag{C.47c}$$

Since the integrands in (C.47) are absolutely integrable, the differentiation inside the integral is justified. The path for the remaining integral in (C.47a) can be deformed down to the real line showing it is zero. Using (B.77c), the remaining integral in (C.47b) is evaluated as

$$-\int_{\partial\Omega_{\text{ext}}} k\mathfrak{f}(k^2,x,t)e^{-k^2t}dk = \int_{\partial\Omega_{\text{ext}}} \left(\frac{f(y,0)}{k} + \frac{1}{k}\int_0^t f_s(y,s)e^{k^2s}\,ds\right) e^{-k^2t}dk, \tag{C.48}$$

which may also be deformed to an indented contour on the real line. The principal-value part integral is zero, while the indentation integral evaluates to

$$\frac{1}{i\pi}\int_{\partial\Omega_{\text{ext}}} k\mathfrak{f}(k^2,x,t)e^{-k^2t}dk = \operatorname{Res}\left(\left(\frac{f(y,0)}{k} + \frac{1}{k}\int_0^t f_s(y,s)e^{k^2s}\,ds\right)e^{-k^2t};\, k=0\right) = f(y,t). \tag{C.49}$$

Equation (C.47) yields (C.45). ☐

Theorem 47. *The solution expressions* (3.2), (4.2), *and* (5.4) *each solve the evolution equation* (2.1a).

Proof. Since $q(x,t) = q_0(x,t) + q_f(x,t) + q_{\mathcal{B}_0}(x,t) + q_{\mathcal{B}_1}(x,t)$, (C.45) gives the result. ☐

D Proofs: the solution expressions satisfy the boundary values

Definition 48. *In this appendix, $\ell = 0$ corresponds to the half-line problem, while $\ell = 1,2$ correspond to the finite-interval problem. We define, for $k \in \Omega_{\text{ext}}$ and $y \in \mathcal{D}$,*

$$\mathfrak{P}^{(0)}(k,y) = \frac{a_0\Psi(k,x_l,y) + a_1\overline{\Psi}(k,x_l,y)}{\sqrt{(\beta\mathfrak{n})(k,x_l)}}, \tag{D.1a}$$

$$\mathfrak{P}^{(\ell)}(k,y) = \frac{a_{\ell1}\Psi(k,x_l,y) + a_{\ell2}\overline{\Psi}(k,x_l,y)}{\sqrt{(\beta\mathfrak{n})(k,x_l)}} + \frac{b_{\ell1}\Psi(k,x_r,y) + b_{\ell2}\overline{\Psi}(k,x_r,y)}{\sqrt{(\beta\mathfrak{n})(k,x_r)}}, \qquad \ell=1,2. \tag{D.1b}$$

For $k \in \Omega_{\text{ext}}$,

$$\mathfrak{B}_0^{(0)}(k) = a_0\mathcal{B}_0(k,x_l) + a_1\mathcal{B}_{0,x}(k,x_l), \tag{D.2a}$$

$$\mathfrak{B}_m^{(\ell)}(k) = a_{\ell1}\mathcal{B}_m(k,x_l) + a_{\ell2}\mathcal{B}_{m,x}(k,x_l) + b_{\ell1}\mathcal{B}_m(k,x_r) + b_{\ell2}\mathcal{B}_{m,x}(k,x_r), \qquad \ell=1,2, \quad m=0,1, \tag{D.2b}$$

and

$$\mathcal{P}_0^{(0)}(k) = a_0\Phi_0(k,x_l) + a_1\Phi_{0,x}(k,x_l), \tag{D.3a}$$

$$\mathcal{P}_0^{(\ell)}(k) = a_{\ell1}\Phi_0(k,x_l) + a_{\ell2}\Phi_{0,x}(k,x_l) + b_{\ell1}\Phi_0(k,x_r) + b_{\ell2}\Phi_{0,x}(k,x_r), \qquad \ell=1,2. \tag{D.3b}$$

For $k \in \Omega_{\text{ext}}$ and $t \in (0,T)$,

$$\mathcal{P}_f^{(0)}(k,t) = a_0 \Phi_{\mathfrak{f}}(k,x_l,t) + a_1 \Phi_{\mathfrak{f},x}(k,x_l,t), \tag{D.4a}$$

$$\mathcal{P}_f^{(\ell)}(k,t) = a_{\ell 1} \Phi_{\mathfrak{f}}(k,x_l,t) + a_{\ell 2} \Phi_{\mathfrak{f},x}(k,x_l,t) + b_{\ell 1} \Phi_{\mathfrak{f}}(k,x_r,t) + b_{\ell 2} \Phi_{\mathfrak{f},x}(k,x_r,t), \qquad \ell = 1,2. \tag{D.4b}$$

Finally, for $t \in (0,T)$,

$$\mathcal{Q}_{\mathcal{B}_m}^{(0)}(t) = a_0 q_{\mathcal{B}_m}(x_l,t) + a_1 q_{\mathcal{B}_m,x}(x_l,t), \tag{D.5a}$$

$$\mathcal{Q}_{\mathcal{B}_m}^{(\ell)}(t) = a_{\ell 1} q_{\mathcal{B}_m}(x_l,t) + a_{\ell 2} q_{\mathcal{B}_m,x}(x_l,t) + b_{\ell 1} q_{\mathcal{B}_m}(x_r,t) + b_{\ell 2} q_{\mathcal{B}_m,x}(x_r,t), \qquad \ell = 1,2, \tag{D.5b}$$

and

$$\mathcal{Q}_0^{(0)}(t) = a_0 q_0(x_l,t) + a_1 q_{0,x}(x_l,t), \tag{D.6a}$$

$$\mathcal{Q}_0^{(\ell)}(t) = a_{\ell 1} q_0(x_l,t) + a_{\ell 2} q_{0,x}(x_l,t) + b_{\ell 1} q_0(x_r,t) + b_{\ell 2} q_{0,x}(x_r,t), \qquad \ell = 1,2, \tag{D.6b}$$

and

$$\mathcal{Q}_f^{(0)}(t) = a_0 q_f(x_l,t) + a_1 q_{f,x}(x_l,t), \tag{D.7a}$$

$$\mathcal{Q}_f^{(\ell)}(t) = a_{\ell 1} q_f(x_l,t) + a_{\ell 2} q_{f,x}(x_l,t) + b_{\ell 1} q_f(x_r,t) + b_{\ell 2} q_{f,x}(x_r,t), \qquad \ell = 1,2. \tag{D.7b}$$

Lemma 49. *For both the half-line problem and the finite-interval problem, for $k \in \Omega_{\text{ext}}$ and $y \in \overline{\mathcal{D}}$,*

$$\mathfrak{P}^{(\ell)}(k,y) = 0, \quad \ell = 0,1,2. \tag{D.8}$$

Proof. For the half-line, using (4.8) (with $x_l = x < y < x_r$) and (C.20b) in (D.1a), gives (D.8).

$$\Psi(k,x_l,y) = -4 \exp\left(\int_{x_l}^{y} ik\mathfrak{n}(k,\xi)\, d\xi\right) \sum_{n=0}^{\infty} (-1)^n a_1 \mathcal{E}_n^{(y,\infty)}(k),$$

$$\overline{\Psi}(k,x_l,y) = 4 \exp\left(\int_{x_l}^{y} ik\mathfrak{n}(k,\xi)\, d\xi\right) \sum_{n=0}^{\infty} (-1)^n a_0 \mathcal{E}_n^{(y,\infty)}(k),$$

$$\Rightarrow \qquad \mathfrak{P}^{(0)}(k,y) = \frac{a_0 \Psi(k,x_l,y) + a_1 \overline{\Psi}(k,x_l,y)}{\sqrt{(\beta\mathfrak{n})(k,x_l)}} = 0. \tag{D.9}$$

Using (5.9) and (C.21) in (D.1b), the calculations for the finite-interval case are equally straightforward albeit more tedious. □

Lemma 50. *For the half-line problem ($m = 0$), and for the finite-interval problem ($m = 0,1$), for $k \in \Omega_{\text{ext}}$,*

$$\mathfrak{B}_m^{(\ell)}(k) = -2ik\Delta(k)\tilde{\delta}_{\ell-1,m}, \quad \ell = 0,1,2. \tag{D.10a}$$

Here

$$\tilde{\delta}_{\ell-1,m} = \begin{cases} 1, & \ell = 0,\ m = 0, \\ 1, & \ell \neq 0,\ m = \ell - 1, \\ 0, & \ell \neq 0,\ m \neq \ell - 1. \end{cases} \tag{D.10b}$$

Proof. For the half-line problem, using (4.6) and (C.28a) in (D.2a), we find (D.10):

$$\mathcal{B}_0(k,x_l) = \frac{4}{\mathfrak{n}(k,x_l)} \sum_{n=0}^{\infty} (-1)^n \mathcal{E}_n^{(x_l,\infty)}(k),$$

$$\mathcal{B}_{0,x}(k,x_l) = \frac{4ik\mathfrak{n}(k,x_l)}{\mathfrak{n}(k,x_l)} \sum_{n=0}^{\infty} \mathcal{E}_n^{(x_l,\infty)}(k),$$

$$\Rightarrow \qquad \mathfrak{B}_0^{(0)}(k) = a_0 \mathcal{B}_0(k,x_l) + a_1 \mathcal{B}_{0,x}(k,x_l) = 4 \sum_{n=0}^{\infty} \left(\frac{(-1)^n a_0}{\mathfrak{n}(k,x_l)} + a_1 ik\right) \mathcal{E}_n^{(x_l,\infty)}(k) = -2ik\Delta(k). \tag{D.11}$$

The finite-interval case (using (5.8c) and (C.29a) in (D.2b)) is similar but more tedious. Its details are omitted. □

Lemma 51. *For both the half-line problem and the finite-interval problem, for $k \in \Omega_{\text{ext}}$ and $t \in [0,T]$,*

$$\mathcal{P}_0^{(\ell)}(k) = 0, \tag{D.12a}$$

$$\mathcal{P}_f^{(\ell)}(k,t) = 0. \tag{D.12b}$$

Proof. Using (B.26a) and (C.30) in (D.3a) and (D.3b), we find

$$\mathcal{P}_0^{(\ell)}(k) = \int_{\mathcal{D}} \frac{\mathfrak{P}^{(\ell)}(k,y)q_\alpha(y)}{\sqrt{(\beta\mathfrak{n})(k,y)}}\, dy, \tag{D.13a}$$

which gives (D.12a), using Lemma 49. Similarly, using (B.77b) and (C.33) in (D.4a) and (D.4b), we find

$$\mathcal{P}_f^{(\ell)}(k,t) = \int_{\mathcal{D}} \frac{\mathfrak{P}^{(\ell)}(k,y)\mathfrak{f}_\alpha(k^2,y,t)}{\sqrt{(\beta\mathfrak{n})(k,y)}}\, dy, \tag{D.13b}$$

which gives (D.12b), using Lemma 49. □

Lemma 52. *For the half-line ($m = 0$) and the finite-interval problem ($m = 0, 1$), for $k \in \Omega_{\text{ext}}$ and $t \in [0,T]$,*

$$\mathcal{Q}_0^{(\ell)}(t) = 0, \tag{D.14a}$$

$$\mathcal{Q}_f^{(\ell)}(t) = 0, \tag{D.14b}$$

$$\mathcal{Q}_{\mathcal{B}_m}^{(\ell)}(t) = f_m(t)\tilde{\delta}_{\ell-1,m}, \tag{D.14c}$$

where $\tilde{\delta}_{\ell-1,m}$ is defined in (D.10b).

Proof. From Lemmas 29, 30, and 42, $\mathcal{Q}_0^{(\ell)}(t)$ (D.6) and $\mathcal{Q}_f^{(\ell)}(t)$ (D.7) are well-defined functions. Similarly, for the *regular problems*, $Q_{\mathcal{B}_m}^{(\ell)}(t)$ (D.5) is a well-defined function from Lemmas 28 and 42. For the *irregular problems*, for Boundary Case 3, $q_{\mathcal{B}_m,x}(x,t)$ may be undefined at the boundary, but the linear combination of boundary terms $\mathcal{Q}_{\mathcal{B}_m}^{(\ell)}(t)$ (D.5b) is well defined. For Boundary Case 4, using Assumption 3, $q_{\mathcal{B}_m,x}(x,t)$ is well defined at the boundary and therefore $Q_{\mathcal{B}_m}^{(\ell)}(t)$ is well defined.

For the irregular Boundary Case 3, see Remark 10.3, from (C.29a) for $x \approx x_l$,

$$\mathcal{B}_{2-j,x}(k,x) = (-1)^j \frac{4k\mathfrak{n}(k,x)\Xi(k)}{\sqrt{(\beta\mathfrak{n})(k,x)}} \left\{ \frac{\beta(x_l)b_{j2}}{\sqrt{(\beta\mathfrak{n})(k,x_l)}} \sum_{n=0}^{\infty}(-1)^n \mathcal{S}_n^{(x,x_r)}(k) \right\} + O(k^0). \tag{D.15}$$

We can prove that either (i) $b_{12} = 0 = b_{22}$, in which case $\mathcal{B}_{m,x}(k,x_l) = O(k^0)$, $\mathcal{B}_{m,x}(k,x_l)/\Delta(k) = O(k^{-2})$, and $q_{\mathcal{B}_m,x}(x_l,t)$ is well defined, see Lemma 42; or (ii) if $(b_{12}, b_{22}) \neq (0,0)$, then $a_{12} = 0 = a_{22}$, in which case $q_{\mathcal{B}_m,x}(x,t)$ does not appear in $\mathcal{Q}_{\mathcal{B}_m}^{(\ell)}(t)$. The same holds for $x \approx x_r$. It follows that $\mathcal{Q}_{\mathcal{B}_m}^{(\ell)}(t)$ is well defined.

For Boundary Case 4, with Assumption 3, we integrate $\mathfrak{F}_m(k^2,t)$ (B.69b) by parts to obtain

$$\mathfrak{F}_m(k^2,t) = -\frac{f_m(0)}{k^2} - \frac{e^{k^2t}f_m'(t) - f_m'(0)}{k^4} + \frac{1}{k^4}\int_0^t e^{k^2s}f_m''(s)\, ds, \tag{D.16}$$

so that we may write $q_{\mathcal{B}_m,x}(x,t)$ (C.36c) as

$$q_{\mathcal{B}_m,x}(x,t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{\mathcal{B}_{m,x}(k,x)}{\Delta(k)}\tilde{\mathfrak{F}}_m(k^2,t)e^{-k^2t}\, dk, \tag{D.17}$$

where

$$\tilde{\mathfrak{F}}_m(k^2,t) = -\frac{f_m(0)}{k^2} + \frac{f_m'(0)}{k^4} + \frac{1}{k^4}\int_0^t e^{k^2s}f_m''(s)\, ds, \tag{D.18}$$

and where the integral of the $f_m'(t)$ term is zero by Cauchy's theorem (before the x-differentiation). The first two terms of $\tilde{\mathfrak{F}}_m(k^2,t)e^{-k^2t}$ are exponentially decaying for $t \in (0,T)$ and the last term is $O(k^{-4})$, by Assumption 3.2. Therefore $q_{\mathcal{B}_m,x}(x,t)$ is well defined for $x \in \overline{\mathcal{D}}$ and $t \in (0,T)$. Consequentially, $\mathcal{Q}_{\mathcal{B}_m}^{(\ell)}(t)$ is well defined.

Using (B.74) and (C.36a) in (D.6), we find

$$\mathcal{Q}_0^{(\ell)}(t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{\mathcal{P}_0^{(\ell)}(k)}{\Delta(k)} e^{-k^2 t}\, dk, \tag{D.19a}$$

which gives (D.14a), using Lemma 51. Similarly, using (B.77) and (C.36b) in (D.7), we find

$$\mathcal{Q}_f^{(\ell)}(t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{\mathcal{P}_f^{(\ell)}(k,t) e^{-k^2 t}}{\Delta(k)} dk. \tag{D.19b}$$

Using Lemma 51, this gives (D.14b). Using (B.69) and (C.36c) in (D.5),

$$\mathcal{Q}_{\mathcal{B}_m}^{(\ell)}(t) = \frac{1}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{\mathfrak{B}_m^{(\ell)}(k)}{\Delta(k)} \mathfrak{F}_m(k^2,t) e^{-k^2 t}\, dk. \tag{D.19c}$$

Finally, using Lemma 50 and (B.69b), we obtain

$$\mathcal{Q}_{\mathcal{B}_m}^{(\ell)}(t) = -\frac{\tilde{\delta}_{\ell-1,m}}{i\pi}\int_{\partial\Omega_{\text{ext}}} \left(\frac{f_m(0)e^{-k^2 t}}{k} + \frac{1}{k}\int_0^t e^{-k^2(t-s)} f_m'(s)\, ds \right) dk. \tag{D.19d}$$

Since the integrand is $O(k^{-3})$, we can deform the path of integration to the real axis. Using the oddness of the integrand, the principal value integral vanishes and only the residue contribution at the origin needs to be calculated:

$$\mathcal{Q}_{\mathcal{B}_m}^{(\ell)}(t) = \tilde{\delta}_{\ell-1,m} \text{Res} \left(\frac{f_m(0)e^{-k^2 t}}{k} + \frac{1}{k}\int_0^t e^{-k^2(t-s)} f_m'(s)\, ds;\, k=0 \right) = f_m(t)\tilde{\delta}_{\ell-1,m}. \tag{D.19e}$$

□

Lemma 53. *Consider any $t \in (0,T)$, fixed. Then*

$$\lim_{|x|\to\infty} q(x,t) = 0 \qquad \text{and} \qquad \lim_{x\to\infty} q(x,t) = 0, \tag{D.20}$$

for the whole-line and half-line problems, respectively.

Proof. For any fixed $t \in (0,T)$, we have absolute integrability in (B.69), (B.74), and (B.77a). Therefore, we may switch the limit and integrals. Since, from (B.51) and (B.52),

$$\lim_{|x|\to\infty} \left| \frac{\Psi(k,x,y)}{\Delta(k)} \right| \le M_\Psi e^{-m_{i\nu}|k||x-y|} = 0, \qquad \lim_{x\to\infty} \left| \frac{\Psi(k,x,y)}{\Delta(k)} \right| \le M_\Psi e^{-m_{i\nu}|k||x-y|} = 0, \tag{D.21}$$

for the whole-line problem and the half-line problem, respectively, (D.20) follows. □

Remark 54. *Since we have absolute integrability in (C.36a), (C.36b), and in (C.36c), we conclude that also*

$$\lim_{|x|\to\infty} q_x(x,t) = 0 \qquad \text{and} \qquad \lim_{x\to\infty} q_x(x,t) = 0, \tag{D.22}$$

for the whole-line and half-line problems, respectively.

Theorem 55. *Consider the finite-interval, the half-line, and the whole-line problems. For all three problems, the solution expression (2.2) satisfies the appropriate boundary conditions.*

Proof. Lemma 53 shows the boundary conditions for the whole-line problem and the right boundary condition for the half-line problem are satisfied. From Lemma 52,

$$a_0 q(x_l,t) + a_1 q(x_l,t) = \mathcal{Q}_0^{(0)}(t) + \mathcal{Q}_f^{(0)}(t) + \mathcal{Q}_{\mathcal{B}_m}^{(0)}(t) = f_0(t). \tag{D.23a}$$

Similarly, for the finite-interval problem,

$$\begin{aligned} a_{\ell 1} q(x_l,t) + a_{\ell 2} q_x(x_l,t) + b_{\ell 1} q(x_r,t) + b_{\ell 2} q_x(x_r,t) &= \mathcal{Q}_0^{(\ell)}(t) + \mathcal{Q}_f^{(\ell)}(t) + \mathcal{Q}_{\mathcal{B}_0}^{(\ell)}(t) + \mathcal{Q}_{\mathcal{B}_1}^{(\ell)}(t) \\ &= f_0(t)\tilde{\delta}_{\ell-1,0} + f_1(t)\tilde{\delta}_{\ell-1,1}. \end{aligned} \tag{D.23b}$$

□

E Proofs: the solution expressions satisfy the initial condition

Theorem 56. *Consider the finite-interval, half-line, and whole-line problms. For $x \in \mathcal{D}$, fixed,*

$$\lim_{t\to 0^+} q_f(x,t) = 0, \tag{E.1a}$$

$$\lim_{t\to 0^+} q_{\mathcal{B}_m}(x,t) = 0. \tag{E.1b}$$

Proof. Since the integral in (B.77) is absolutely convergent, we can pass the limit $t \to 0^+$ inside the integral to obtain (E.1a) by using Cauchy's theorem. Similarly, we move the limit in the integral in (B.69) to obtain (E.1b). □

Lemma 57. *For fixed $x \in \mathcal{D}$, for $y \in \overline{\mathcal{D}}$, for the finite-interval, half-line, and whole-line problems,*

$$\frac{\Psi(k,x,y)}{\Delta(k)} = \exp\left(\operatorname{sgn}(x-y) ik \int_y^x \mathfrak{n}(k,\xi)\, d\xi\right)\left(1 + o(k^0)\right) + o(k^{-1}), \tag{E.2}$$

as $|k| \to \infty$ for $k \in \Omega_{\text{ext}}$.

Proof. For the whole-line problem, from (3.4), for $y < x$,

$$\Psi(k,x,y) = \exp\left(\operatorname{sgn}(x-y) ik \int_y^x \mathfrak{n}(k,\xi)\, d\xi\right)\left(1 + \sum_{n=1}^{\infty}\sum_{\ell=0}^{n} (-1)^\ell \tilde{\mathcal{E}}_{n-\ell}^{(-\infty,y)}(k)\mathcal{E}_\ell^{(x,\infty)}(k)\right). \tag{E.3}$$

By Lemma 19 and the DCT,

$$\sum_{n=1}^{\infty}\sum_{\ell=0}^{n} (-1)^\ell \tilde{\mathcal{E}}_{n-\ell}^{(-\infty,y)}(k)\mathcal{E}_\ell^{(x,\infty)}(k) = o(k^0). \tag{E.4}$$

Dividing (E.3) by $\Delta(k)$ and using Lemma 21, we obtain (E.2). The proof for $x < y$ is identical.

For the half-line problem, for $x_l < y < x$, we write (4.8) as

$$\Psi(k,x,y) = 4\exp\left(ik\int_y^x \mathfrak{n}(k,\xi)\, d\xi\right)\left[\left(\frac{a_0}{k\mathfrak{n}(k,x_l)}\mathcal{S}_0^{(x_l,y)}(k) - a_1 \mathcal{C}_0^{(x_l,y)}(k)\right) + \left(\frac{|a_0|}{m_{\mathfrak{n}}|k|} + |a_1|\right)o(k^0)\right]. \tag{E.5}$$

Using

$$\mathcal{C}_0^{(a,b)}(k) = \frac{1}{2}\left(\exp\left(2ik\int_a^b \mathfrak{n}(k,\xi)\, d\xi\right) + 1\right), \tag{E.6a}$$

$$\mathcal{S}_0^{(a,b)}(k) = \frac{1}{2i}\left(\exp\left(2ik\int_a^b \mathfrak{n}(k,\xi)\, d\xi\right) - 1\right), \tag{E.6b}$$

in (E.5), we find

$$\Psi(k,x,y) = 2\exp\left(ik\int_y^x \mathfrak{n}(k,\xi)\, d\xi\right)\left[\frac{ia_0}{k\mathfrak{n}(k,x_l)} - a_1 + \left(\frac{|a_0|}{m_{\mathfrak{n}}|k|} + |a_1|\right)o(k^0)\right] + O\big(e^{-m_{\mathfrak{in}}|k|(x-x_l)}\big), \tag{E.7}$$

which, from (B.28a) with (B.28c), gives (E.2). The proof is identical for $x_l < x < y$.

For the finite-interval problem we consider the 4 different cases.

1. If $(a:b)_{2,4} \neq 0$, then for $x_l < y < x < x_r$, using (B.28a) with (B.28d), we write (5.9a) as

$$\begin{aligned}\frac{\Psi(k,x,y)}{\mathfrak{d}_0(k)} = {} & \frac{-4}{(a:b)_{2,4}}\exp\left(ik\int_y^x \mathfrak{n}(k,\xi)\, d\xi\right)\left\{-(a:b)_{2,4}\mathcal{C}_0^{(x_l,y)}(k)\mathcal{C}_0^{(x,x_r)}(k) + o(k^0)\right\} \\ & + \frac{4\beta(x_r)(a:b)_{1,2}}{(a:b)_{2,4}k\sqrt{(\beta\mathfrak{n})(k,x_l)}\sqrt{(\beta\mathfrak{n})(k,x_r)}}\sum_{n=0}^{\infty}\Xi(k)\mathcal{S}_n^{(y,x)}(k).\end{aligned} \tag{E.8}$$

   Using (E.6a) and dividing by $1+\varepsilon(k)$, we arrive at (E.2). The proof for $x_l < x < y < x_r$ is identical.

2. If $(a:b)_{2,4}=0$ and $m_{\mathfrak{c}_0}\neq 0$, then for $x_l<y<x<x_r$, using (E.6) and (B.28a) with (B.28e), we write (5.9a) as

$$\frac{\Psi(k,x,y)}{\mathfrak{b}_0(k)}=\frac{4k}{im_{\mathfrak{c}_0}}\exp\left(ik\int_y^x \mathfrak{n}(k,\xi)\,d\xi\right)\left\{-\frac{1}{4i}\left(\frac{(a:b)_{1,4}}{k\mathfrak{n}(k,x_l)}+\frac{(a:b)_{2,3}}{k\mathfrak{n}(k,x_r)}\right)+o(k^{-1})\right\}+o(k^{-1}). \tag{E.9}$$

Using $1/\mathfrak{n}(k,x)=1/\mu(x)+O(k^{-2})$ and dividing by $1+\varepsilon(k)$, we obtain (E.2). The proof for $x_l<x<y<x_r$ is identical.

3. If $(a:b)_{2,4}=0$, $m_{\mathfrak{c}_0}=0$, $m_{\mathfrak{c}_1}=0$, and $(a:b)_{1,3}\neq 0$, then for $x_l<y<x<x_r$, using (B.28a) with (B.28f), we write (5.9a) as

$$\frac{\Psi(k,x,y)}{\mathfrak{b}_0(k)}=-\frac{4k^2}{m_{\mathfrak{s}}}\exp\left(ik\int_y^x \mathfrak{n}(k,\xi)\,d\xi\right)\left\{-\frac{1}{4}\frac{(a:b)_{1,3}}{k^2\mathfrak{n}(k,x_l)\mathfrak{n}(k,x_r)}+o(k^{-2})\right\}+o(k^{-1}). \tag{E.10}$$

Using the asymptotics for $1/\mathfrak{n}(k,x)$ and dividing by $1+\varepsilon(k)$ gives (E.2). The proof is identical for $x_l<x<y<x_r$.

4. If $(a:b)_{2,4}=0$, $m_{\mathfrak{c}_0}=0$, $m_{\mathfrak{c}_1}\neq 0$, and $m_{\mathfrak{c}_1}\mathfrak{u}_+-8m_{\mathfrak{s}}\neq 0$, then for $x_l<y<x<x_r$, using that

$$\sum_{n=3}^{\infty}\left|k\mathcal{C}_n^{(a,b)}(k)\right|=O(k^{-1}), \tag{E.11}$$

using the asymptotics of $1/\mathfrak{n}(k,x)$, the fact that $(a:b)_{1,4}/\mu(x_r)=(a:b)_{2,3}/\mu(x_l)=m_{\mathfrak{c}_1}/2$, and (B.28a) with (B.28g), we write (5.9a) as

$$\begin{aligned}\frac{\Psi(k,x,y)}{\mathfrak{b}_0(k)}=\frac{32k^2}{m_{\mathfrak{c}_1}\mathfrak{u}_+-8m_{\mathfrak{s}}}\exp\left(ik\int_y^x \mathfrak{n}(k,\xi)\,d\xi\right)&\left\{\frac{m_{\mathfrak{c}_1}}{2k}\sum_{n=1}^{2}\sum_{\ell=0}^{n}(-1)^\ell\mathcal{S}_{n-\ell}^{(x_l,y)}(k)\mathcal{C}_\ell^{(x,x_r)}(k)\right.\\ &\left.-\frac{m_{\mathfrak{c}_1}}{2k}\sum_{n=1}^{2}\sum_{\ell=0}^{n}\mathcal{C}_{n-\ell}^{(x_l,y)}(k)\mathcal{S}_\ell^{(x,x_r)}(k)+o(k^{-2})-\frac{m_{\mathfrak{s}}}{4k^2}\right\}+o(k^{-1}).\end{aligned} \tag{E.12}$$

Using integration by parts as in Lemma 19, we derive

$$\mathcal{C}_1^{(a,b)}(k)=\frac{1}{16ik}\mathfrak{u}_+(a,b)\left(\exp\left(2ik\int_a^b \mathfrak{n}(k,\xi)\,d\xi\right)-1\right)+O(k^{-2}), \tag{E.13a}$$

$$\mathcal{S}_1^{(a,b)}(k)=-\frac{1}{16k}\mathfrak{u}_-(a,b)\left(\exp\left(2ik\int_a^b \mathfrak{n}(k,\xi)\,d\xi\right)+1\right)+O(k^{-2}), \tag{E.13b}$$

$$\mathcal{C}_2^{(a,b)}(k)=\frac{1}{16ik}m_{\text{int}}(a,b)\left(\exp\left(2ik\int_a^b \mathfrak{n}(k,\xi)\,d\xi\right)-1\right)+O(k^{-2}), \tag{E.13c}$$

$$\mathcal{S}_2^{(a,b)}(k)=-\frac{1}{16k}m_{\text{int}}(a,b)\left(\exp\left(2ik\int_a^b \mathfrak{n}(k,\xi)\,d\xi\right)+1\right)+O(k^{-2}), \tag{E.13d}$$

where $\mathfrak{u}_\pm(a,b)=\mathfrak{u}(b)\pm\mathfrak{u}(a)$, and

$$m_{\text{int}}(a,b)=\int_a^b\frac{1}{\mu(y)}\left(\frac{(\beta\mu)'(y)}{(\beta\mu)(y)}\right)^2 dy, \tag{E.14}$$

with $\mathfrak{u}(x)$ defined in (2.7). We find

$$\begin{aligned}\frac{\Psi(k,x,y)}{\mathfrak{b}_0(k)}=\frac{32k^2}{m_{\mathfrak{c}_1}\mathfrak{u}_+-8m_{\mathfrak{s}}}\exp\left(ik\int_y^x \mathfrak{n}(k,\xi)\,d\xi\right)&\left\{-\frac{m_{\mathfrak{s}}}{4k^2}+\frac{m_{\mathfrak{c}_1}}{64k^2}\left(\mathfrak{u}_+(x_l,y)-\mathfrak{u}_-(x_l,y)\right)\right.\\ &\left.+\frac{m_{\mathfrak{c}_1}}{64k^2}\left(\mathfrak{u}_+(x,x_r)+\mathfrak{u}_-(x,x_r)\right)+o(k^{-2})\right\}+o(k^{-1}).\end{aligned} \tag{E.15}$$

Combining terms,

$$\frac{\Psi(k,x,y)}{\mathfrak{b}_0(k)} = \frac{32k^2}{m_{\mathfrak{c}_1}\mathfrak{u}_+ - 8m_{\mathfrak{s}}}\exp\left(ik\int_y^x \mathfrak{n}(k,\xi)\,d\xi\right)\left\{-\frac{m_{\mathfrak{s}}}{4k^2} + \frac{m_{\mathfrak{c}_1}}{32k^2}\mathfrak{u}_+ + o(k^{-2})\right\} + o(k^{-1}), \tag{E.16}$$

which, after dividing by $1+\varepsilon(k)$, gives (E.2). The proof is identical for $x_l < x < y < x_r$.

□

Theorem 58. *Consider the finite-interval, half-line, and whole-line problems. If $q_0 \in L^1(\mathcal{D})$, then for almost every $x \in \mathcal{D}$,*

$$\lim_{t\to 0^+} q_0(x,t) = q_0(x). \tag{E.17}$$

Proof. Using the change of variables $k = \lambda z$ with $\lambda = 1/\sqrt{t}$ in (B.74),

$$q_0(x,t) = \frac{\lambda}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{\Phi_0(\lambda z,x)}{\Delta(\lambda z)}e^{-z^2}\,dz = \frac{\lambda}{2\pi}\int_{\partial\Omega_{\text{ext}}} \frac{e^{-z^2}}{\Delta(\lambda z)}\int_{\mathcal{D}} \frac{\Psi(\lambda z,x,y)q_\alpha(y)}{\sqrt{(\beta\mathfrak{n})(\lambda z,x)}\sqrt{(\beta\mathfrak{n})(\lambda z,y)}}\,dy\,dz. \tag{E.18}$$

By Lemma 23, we can use the Fubini-Tonelli theorem to write this as

$$q_0(x,t) = \frac{\lambda}{2\pi}\int_{\mathcal{D}} q_\alpha(y)\int_{\partial\Omega_{\text{ext}}} \frac{\Psi(\lambda z,x,y)}{\Delta(\lambda z)}\frac{e^{-z^2}}{\sqrt{(\beta\mathfrak{n})(\lambda z,x)}\sqrt{(\beta\mathfrak{n})(\lambda z,y)}}\,dz\,dy. \tag{E.19}$$

Using (E.2),

$$q_0(x,t) = \frac{\lambda\big(1+o(\lambda^0)\big)}{2\pi}\int_{\mathcal{D}} \frac{q_\alpha(y)}{\sqrt{(\beta\mu)(x)}\sqrt{(\beta\mu)(y)}}\int_{\partial\Omega_{\text{ext}}} \exp\left(\operatorname{sgn}(x-y)i\lambda z\int_y^x \mathfrak{n}(\lambda z,\xi)\,d\xi\right)e^{-z^2}\,dz\,dy + o(\lambda^0). \tag{E.20}$$

Since

$$\left|\lambda\exp\left(\operatorname{sgn}(x-y)i\lambda z\int_y^x \mu(\xi)\,d\xi\right)O(|x-y|\lambda^{-1})e^{-z^2}\right| \le O(|x-y|\lambda^0)\left|e^{-z^2}\right|, \tag{E.21}$$

is absolutely integrable, we may use the DCT on the remainder term from (B.17). Substituting this result in (E.20), we obtain

$$q_0(x,t) = \frac{\lambda}{2\pi}\int_{\mathcal{D}} \frac{q_\alpha(y)}{\sqrt{(\beta\mu)(x)}\sqrt{(\beta\mu)(y)}}\int_{\partial\Omega_{\text{ext}}} \exp\left(\operatorname{sgn}(x-y)i\lambda z\int_y^x \mu(\xi)\,d\xi\right)e^{-z^2}\,dz\,dy + o(\lambda^0), \tag{E.22}$$

as $\lambda\to\infty$. Define $M_x(y) = \int_x^y \mu(\xi)\,d\xi$. Deforming $\partial\Omega_{\text{ext}}$ down to the real axis and integrating the z-integral gives

$$q_0(x,t) = \frac{\lambda}{2\sqrt{\pi}}\int_{\mathcal{D}} \frac{q_\alpha(y)e^{-\frac{1}{4}\lambda^2 M_x^2(y)}}{\sqrt{(\beta\mu)(x)}\sqrt{(\beta\mu)(y)}}\,dy + o(\lambda^0). \tag{E.23}$$

For a fixed $x\in\mathcal{D}$, if $q_0(x)$ is finite, using that $q_\alpha(y)/(\mu(y)\sqrt{(\beta\mu)(y)}) \in L^1(\mathcal{D})$, it follows that for any $\epsilon > 0$ and for each λ, there exists $\varphi \in \mathrm{AC}(\mathcal{D})\cap C_0(\mathcal{D})$ [15], so that

$$\int_{\mathcal{D}}\left|\frac{q_0(y)}{\mu(y)\sqrt{(\beta\mu)(y)}} - \varphi(y)\right|dy \le \lambda^{-2}, \qquad \text{and} \qquad \left|\varphi(x) - \frac{q_\alpha(x)}{\mu(x)\sqrt{(\beta\mu)(x)}}\right| \le \frac{\epsilon}{2}. \tag{E.24}$$

Using this,

$$q_0(x,t) = \frac{\lambda}{2\sqrt{\pi}\sqrt{(\beta\mu)(x)}}\left[\int_{\mathcal{D}}\left(\frac{q_\alpha(y)}{\mu(y)\sqrt{(\beta\mu)(y)}} - \varphi(y)\right)\mu(y)e^{-\frac{1}{4}\lambda^2 M_x^2(y)}\,dy + \int_{\mathcal{D}}\varphi(y)\mu(y)e^{-\frac{1}{4}\lambda^2 M_x^2(y)}\,dy\right] + o(\lambda^0). \tag{E.25}$$

For the first integral and any $\epsilon > 0$, we can find λ sufficiently large, so that

$$\left| \frac{\lambda}{2\sqrt{\pi}\sqrt{(\beta\mu)(x)}} \int_{\mathcal{D}} \left(\frac{q_\alpha(y)}{\mu(y)\sqrt{(\beta\mu)(y)}} - \varphi(y) \right) \mu(y) e^{-\frac{1}{4}\lambda^2 M_x^2(y)}\, dy \right| \leq \frac{M_{\mathfrak{n}}}{2\sqrt{\pi m_\beta m_{\mathfrak{n}}}\lambda} \leq \frac{\epsilon}{2}. \tag{E.26}$$

Since $\varphi \in \mathrm{AC}(\mathcal{D})$, we may integrate the second integral of (E.25) by parts to obtain

$$\frac{\lambda}{2\sqrt{\pi}} \int_{\mathcal{D}} \varphi(y)\mu(y) e^{-\frac{1}{4}\lambda^2 M_x^2(y)}\, dy = -\frac{1}{2} \int_{\mathcal{D}} \varphi'(y) \mathrm{erf}\left(\frac{\lambda M_x(y)}{2} \right) dy. \tag{E.27}$$

At this point, we may take the limit as $\lambda \to \infty$ using the DCT. Since $\arg(\mu) \in (-\pi/4, \pi/4)$, if $y > x$, then $\arg(M_x(y)) \in (-\pi/4, \pi/4)$ and the error function limits to 1 as $\lambda \to \infty$ [8]. If $y < x$, then $\arg(M_x(y)) \in (3\pi/4, 5\pi/4)$ and the error function limits to -1. It follows that

$$\lim_{\lambda\to\infty} \frac{\lambda}{2\sqrt{\pi}} \int_{\mathcal{D}} \varphi(y)\mu(y) e^{-\frac{1}{4}\lambda^2 M_x^2(y)}\, dy = \varphi(x), \tag{E.28}$$

and we have

$$q_0(x,t) \to \frac{\varphi(x)}{\sqrt{\beta\mu(x)}} \to q_0(x), \tag{E.29}$$

as $t \to 0^+$ and $\epsilon \to 0^+$. Since $q_0 \in L^1(\mathcal{D})$, $q_0(x)$ is finite for almost every $x \in \mathcal{D}$, concluding the proof. □

Acknowledgements

The authors thank David Smith and Vishal Vasan for useful conversations.

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