Initial-to-Interface Maps for the Heat Equation on Composite Domains

Natalie Sheils and Bernard Deconinck
Department of Applied Mathematics
University of Washington
Seattle, WA 98195-2420
nsheils@amath.washington.edu, bernard@amath.washington.edu

June 28, 2015

This paper is dedicated to Mark Ablowitz on the occasion of his 70th birthday, in recognition of his many important contributions to nonlinear science.

Abstract

A map from the initial conditions to the values of the function and its first spatial derivative evaluated at the interface is constructed for the heat equation on finite and infinite domains with n interfaces. The existence of this map allows changing the problem at hand from an interface problem to a boundary value problem which allows for an alternative to the approach of finding a closed-form solution to the interface problem.

1 Introduction

Interface problems for partial differential equations (PDEs) are initial boundary value problems for which the solution of an equation in one domain prescribes boundary conditions for the equations in adjacent domains. In applications, conditions at the interface follow from conservations laws. Few interface problems allow for an explicit closed-form solution using classical solution methods. Using the Fokas method [7, 8] such solutions may be constructed [2, 3, 4, 11, 12, 13, 14, 15].

The construction of a Dirichlet-to-Neumann map, that is, determining the boundary values that are not prescribed in terms of the initial and boundary conditions, is important in the study of PDEs and particularly inverse problems [6, 16]. In what follows we construct a similar map between the initial values of the PDE and the function (and some number of spatial derivatives) evaluated at the interface. This map allows for an alternative to the approach of finding solutions to interface problems as presented in earlier papers using the Fokas Method by the authors and others. Given the initial conditions one could find the value of the function and its derivatives at the interface(s). This changes the problem at hand from an interface problem to a boundary value problem (BVP). At this point, the BVP could be solved using any number of methods appropriate for the given problem.

2 The heat equation on an infinite domain with n interfaces

Consider

$$u_t = \sigma(x) u_{xx}, \tag{1}$$

together with the initial condition $u_0(x) = u(x,0)$ and the asymptotic conditions $\lim_{|x|\to\infty} u(x,t) = 0$, where $-\infty < x < \infty$, $0 < t < T$, and

$$\sigma(x) = \begin{cases} \sigma_1^2, & x < x_1, \\ \sigma_2^2, & x_1 < x < x_2, \\ \vdots & \\ \sigma_n^2, & x_{n-1} < x < x_n, \\ \sigma_{n+1}^2, & x > x_n. \end{cases}$$

We can rewrite (1) as the set of equations

$$u_t^{(j)} = \sigma_j^2 u_{xx}^{(j)}, \qquad x_{j-1} < x < x_j, \;\; 0 < t < T, \tag{2}$$

for $1 \le j \le n+1$ where $x_0 = -\infty$ and $x_{n+1} = \infty$. We impose the continuity interface conditions [9, 3]

$$\begin{aligned} u^{(j)}(x_j,t) &= u^{(j+1)}(x_j,t), && t > 0, \\ \sigma_j^2 u_x^{(j)}(x_j,t) &= \sigma_{j+1}^2 u_x^{(j+1)}(x_j,t), && t > 0, \end{aligned}$$

for $1 \le j \le n$. Since $u^{(j)}(x,t)$ is defined on the open interval $x_{j-1} < x < x_j$, when we write $u^{(j)}(x_j,t)$ we mean $\lim_{x\to x_j^-} u^{(j)}(x,t)$. Similarly, we denote $\lim_{x\to x_j^+} u^{(j+1)}(x,t)$ by $u^{(j+1)}(x_j,t)$. Without loss of generality we shift the problem so $x_1 = 0$. Using the usual steps of the Fokas method [7, 8, 5] we have the local relations

$$(e^{-ikx+\omega_j t} u^{(j)}(x,t))_t = (\sigma_j^2 e^{-ikx+\omega_j(k)t}(u_x^{(j)}(x,t) + iku^{(j)}(x,t)))_x, \tag{3}$$

where $\omega_j(k) = (\sigma_j k)^2$. These relations are a one-parameter family obtained by rewriting (2).

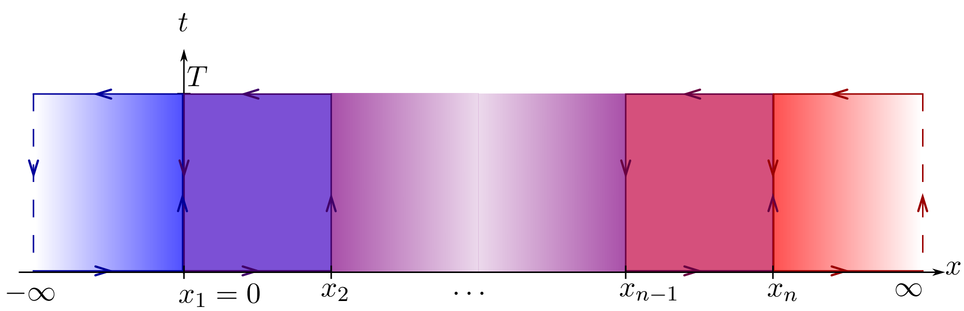

Figure 1: Domains for the application of Green's Theorem in the case of an infinite domain with n interfaces.

Integrating over the appropriate cells of the domain (see Figure 1) and applying Green's Theorem we find the global relations

$$
\begin{aligned}
0 = &\int_{x_{j-1}}^{x_j} e^{-ikx} u_0^{(j)}(x)\,\mathrm{d}x - \int_{x_{j-1}}^{x_j} e^{-ikx+\omega_j(k)T} u^{(j)}(x,T)\,\mathrm{d}x \\
&+ \int_0^T \sigma_j^2 e^{-ikx_j+\omega_j(k)s}(u_x^{(j)}(x_j,s) + iku^{(j)}(x_j,s))\,\mathrm{d}s \\
&- \int_0^T \sigma_j^2 e^{-ikx_{j-1}+\omega_j(k)s}(u_x^{(j)}(x_{j-1},s) + iku^{(j)}(x_{j-1},s))\,\mathrm{d}s,
\end{aligned} \tag{4}
$$

for $1 \le j \le n+1$. Define $D = \{k \in \mathbb{C} : \mathrm{Re}(\omega_j(k)) < 0\}$, $D_R = \{k \in D : |k| < R\}$ and $D_R^+ = \{k \in D_R : \mathrm{Im}(k) > 0\}$ as in Figure 2a where $R > 0$ is an arbitrary finite constant. When $j = 1$ (4) is valid for $k \in \mathbb{C}^+ \setminus D$. Similarly, for $j = n+1$, (4) is valid for $k \in \mathbb{C}^- \setminus D$. For $2 \le j \le n$, (4) is valid for $k \in \mathbb{C} \setminus D$. The dispersion relation $\omega_j(k) = (\sigma_j k)^2$ is invariant under the symmetry $k \to -k$. We supplement the $n+1$ global relations above with their evaluation at $-k$, namely,

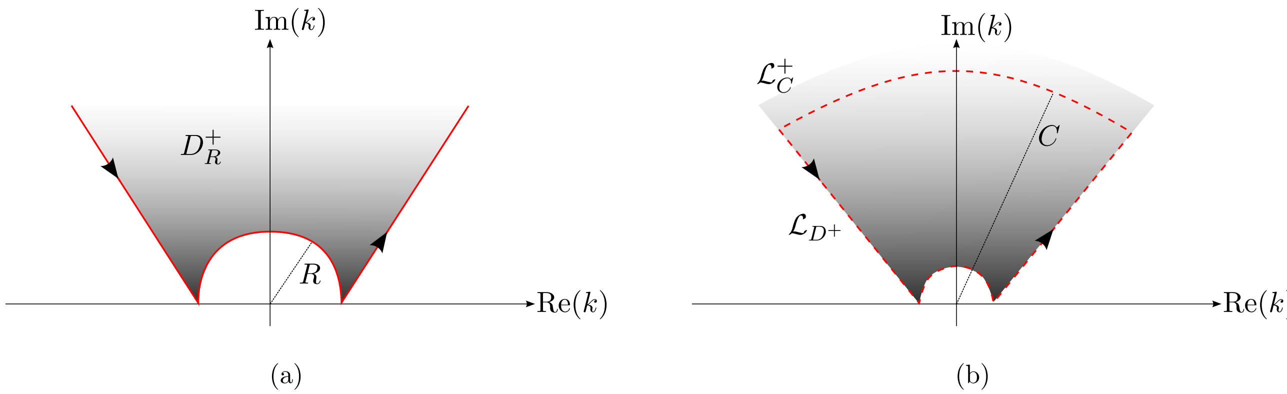

Figure 2: (a) The domain D_R^+ for the heat equation. (b) The contour $\mathcal{L}^+$ is shown as a red dashed line. An application of Cauchy's Integral Theorem using this contour allows elimination of the contribution of terms involving the Fourier transform of the solution.

$$
\begin{aligned}
0 = &\int_{x_{j-1}}^{x_j} e^{ikx} u_0^{(j)}(x)\,\mathrm{d}x - \int_{x_{j-1}}^{x_j} e^{ikx+\omega_j(k)T} u^{(j)}(x,T)\,\mathrm{d}x \\
&+ \int_0^T \sigma_j^2 e^{ikx_j+\omega_j(k)s}(u_x^{(j)}(x_j,s) - iku^{(j)}(x_j,s))\,\mathrm{d}s \\
&- \int_0^T \sigma_j^2 e^{ikx_{j-1}+\omega_j(k)s}(u_x^{(j)}(x_{j-1},s) - iku^{(j)}(x_{j-1},s))\,\mathrm{d}s,
\end{aligned} \tag{5}
$$

for $1 \le j \le n+1$. When $j = 1$, (5) is valid for $k \in \mathbb{C}^- \setminus D$. Similarly, for $j = n+1$, (5) is valid for $k \in \mathbb{C}^+ \setminus D$. For $2 \le j \le n$, (5) is valid for all $k \in \mathbb{C} \setminus D$. Without loss of generality we choose to work with the equations valid in the upper half plane. Define

$$
\begin{aligned}
g_0^{(j)}(\omega,t) &= \int_0^t e^{\omega s} u^{(j)}(x_j,s)\,\mathrm{d}s = \int_0^t e^{\omega s} u^{(j+1)}(x_j,s)\,\mathrm{d}s,\\
g_1^{(j)}(\omega,t) &= \int_0^t e^{\omega s} u_x^{(j)}(x_j,s)\,\mathrm{d}s = \frac{\sigma_2^2}{\sigma_1^2}\int_0^t e^{\omega s} u_x^{(j+1)}(x_j,s)\,\mathrm{d}s,\\
\hat{u}^{(j)}(k,t) &= \int_{x_{j-1}}^{x_j} e^{-ikx} u^{(j)}(x,t)\,\mathrm{d}x,\\
\hat{u}_0^{(j)}(k) &= \int_{x_{j-1}}^{x_j} e^{-ikx} u_0^{(j)}(x)\,\mathrm{d}x,
\end{aligned}
$$

for $1 \le j \le n$. Using the change of variables $k = \kappa/\sigma_j$ on the j^{th} equation, the global relations valid in the upper-half plane are

$$
e^{\kappa^2 t}\hat{u}^{(1)}\left(\frac{\kappa}{\sigma_1},T\right) - \hat{u}_0^{(1)}\left(\frac{\kappa}{\sigma_1}\right) = e^{-i\kappa x_1/\sigma_1}\left(\frac{i\kappa}{\sigma_1} g_0^{(1)}(\kappa^2,T) + g_1^{(1)}(\kappa^2,T)\right), \tag{6a}
$$

$$
\begin{aligned}
e^{\kappa^2 t}\hat{u}^{(j)}\left(\frac{\kappa}{\sigma_j},T\right) - \hat{u}_0^{(j)}\left(\frac{\kappa}{\sigma_j}\right) =& e^{\frac{-i\kappa x_j}{\sigma_j}}\left(\frac{i\kappa}{\sigma_j} g_0^{(j)}(\kappa^2,T) + g_1^{(j)}(\kappa^2,T)\right)\\
&- e^{\frac{-i\kappa x_{j-1}}{\sigma_j}}\left(\frac{i\kappa}{\sigma_j} g_0^{(j-1)}(\kappa^2,T) + \frac{\sigma_{j-1}^2}{\sigma_j^2} g_1^{(j-1)}(\kappa^2,T)\right),
\end{aligned} \tag{6b}
$$

$$
\begin{aligned}
e^{\kappa^2 t}\hat{u}^{(j)}\left(\frac{-\kappa}{\sigma_j},T\right) - \hat{u}_0^{(j)}\left(\frac{-\kappa}{\sigma_j}\right) =& e^{\frac{i\kappa x_j}{\sigma_j}}\left(\frac{-i\kappa}{\sigma_j} g_0^{(j)}(\kappa^2,T) + g_1^{(j)}(\kappa^2,T)\right)\\
&+ e^{\frac{i\kappa x_{j-1}}{\sigma_j}}\left(\frac{i\kappa}{\sigma_j} g_0^{(j-1)}(\kappa^2,T) - \frac{\sigma_{j-1}^2}{\sigma_j^2} g_1^{(j-1)}(\kappa^2,T)\right),
\end{aligned} \tag{6c}
$$

$$
e^{\kappa^2 t}\hat{u}^{(n+1)}\left(\frac{-\kappa}{\sigma_{n+1}},T\right) - \hat{u}_0^{(n+1)}\left(\frac{-\kappa}{\sigma_{n+1}}\right) = e^{\frac{i\kappa x_n}{\sigma_{n+1}}}\left(\frac{i\kappa}{\sigma_{n+1}} g_0^{(n)}(\kappa^2,T) - \frac{\sigma_n^2}{\sigma_{n+1}^2} g_1^{(n)}(\kappa^2,T)\right), \tag{6d}
$$

for $2 \le j \le n$. Equation (6) can be written as a linear system for the interface values:

$$
\mathcal{A}(\kappa)X(\kappa^2,T) = Y(\kappa) + \mathcal{Y}(\kappa,T),
$$

where

$$
X(\kappa^2,T) = \left(g_0^{(1)}, g_0^{(2)}, \dots, g_0^{(n)}, g_1^{(1)}, g_1^{(2)}, \dots, g_1^{(n)}\right)^\top,
$$

$$
Y(\kappa) = -\left(\hat{u}_0^{(1)}\left(\frac{\kappa}{\sigma_1}\right), \dots, \hat{u}_0^{(n)}\left(\frac{\kappa}{\sigma_n}\right), \hat{u}_0^{(2)}\left(\frac{-\kappa}{\sigma_2}\right), \dots, \hat{u}_0^{(n+1)}\left(\frac{-\kappa}{\sigma_{n+1}}\right)\right)^\top,
$$

$$
\mathcal{Y}(\kappa,T) = e^{\kappa^2 T}\left(\hat{u}^{(1)}\left(\frac{\kappa}{\sigma_1},T\right), \dots, \hat{u}^{(n)}\left(\frac{\kappa}{\sigma_n},T\right), \hat{u}^{(2)}\left(\frac{-\kappa}{\sigma_2},T\right), \dots, \hat{u}^{(n+1)}\left(\frac{-\kappa}{\sigma_{n+1}},T\right)\right)^\top,
$$

and

$$
\mathcal{A}(\kappa) =
\left(\begin{array}{ccc|ccc}
\frac{i\kappa}{\sigma_1} e^{-i\frac{\kappa x_1}{\sigma_1}} & & & e^{-i\frac{\kappa x_1}{\sigma_1}} & & \\
\frac{-i\kappa}{\sigma_2} e^{-i\frac{\kappa x_1}{\sigma_2}} & \frac{i\kappa}{\sigma_2} e^{-i\frac{\kappa x_2}{\sigma_2}} & & \frac{-\sigma_1^2}{\sigma_2^2} e^{-i\frac{\kappa x_1}{\sigma_2}} & e^{-i\frac{\kappa x_2}{\sigma_2}} & \\
& \ddots & \ddots & & \ddots & \ddots \\
& \frac{-i\kappa}{\sigma_n} e^{-i\frac{\kappa x_{n-1}}{\sigma_n}} & \frac{i\kappa}{\sigma_n} e^{-i\frac{\kappa x_n}{\sigma_n}} & & \frac{-\sigma_{n-1}^2}{\sigma_n^2} e^{-i\frac{\kappa x_{n-1}}{\sigma_n}} & e^{-i\frac{\kappa x_n}{\sigma_n}} \\
\hline
\frac{i\kappa}{\sigma_2} e^{i\frac{\kappa}{\sigma_2} x_1} & \frac{-i\kappa}{\sigma_2} e^{i\frac{\kappa x_2}{\sigma_2}} & & \frac{-\sigma_1^2}{\sigma_2^2} e^{i\frac{\kappa x_1}{\sigma_2}} & e^{i\frac{\kappa x_2}{\sigma_2}} & \\
& \ddots & \ddots & & \ddots & \ddots \\
& \frac{i\kappa}{\sigma_{n-1}} e^{i\frac{\kappa x_{n-1}}{\sigma_n}} & \frac{-i\kappa}{\sigma_n} e^{i\frac{\kappa x_n}{\sigma_n}} & & \frac{-\sigma_{n-1}^2}{\sigma_n^2} e^{i\frac{\kappa x_{n-1}}{\sigma_n}} & e^{i\frac{\kappa x_n}{\sigma_n}} \\
& & \frac{i\kappa}{\sigma_{n+1}} e^{i\frac{\kappa x_n}{\sigma_{n+1}}} & & & \frac{-\sigma_n^2}{\sigma_{n+1}^2} e^{i\frac{\kappa x_n}{\sigma_{n+1}}}
\end{array}\right).
$$

The matrix $\mathcal{A}(\kappa)$ consists of four $n \times n$ blocks as indicated by the dashed lines. The two blocks in the upper half of $\mathcal{A}(\kappa)$ are zero except for entries on the main and -1 diagonals. The lower two blocks of $\mathcal{A}(\kappa)$ only have nonzero entries on the main and $+1$ diagonals. The matrix $\mathcal{A}(\kappa)$ is singular for isolated values of κ. Asymptotically, for large $|\kappa|$, the zeros of $\det(\mathcal{A}(\kappa))$ are on the real line [10]. Since asymptotically there are no zeros in D_R^+, a sufficiently large R may be chosen such that $\mathcal{A}(\kappa)$ is nonsingular for every $\kappa \in D_R^+$ and $\det(\mathcal{A}(\kappa)) \neq 0$.

Remark. We have been unable to construct physical examples where the zeros of $\det(\mathcal{A}(\kappa))$ are in D^+ and are different from 0. However, if nonphysical values of the parameters are chosen (*e.g.*, σ_j imaginary), then $\det(\mathcal{A}(\kappa))$ has zeros in D^+.

Using Cramer's Rule to solve this system, we have

$$
g_0^{(j)}(\kappa^2, T) = \frac{\det(\mathcal{A}_j(\kappa, T))}{\det(\mathcal{A}(\kappa))}, \tag{7a}
$$

$$
g_1^{(j)}(\kappa^2, T) = \frac{\det(\mathcal{A}_{j+n}(\kappa, T))}{\det(\mathcal{A}(\kappa))}, \tag{7b}
$$

where $1 \le j \le n$ and $\mathcal{A}_j(\kappa, T)$ is the matrix $\mathcal{A}(\kappa)$ with the j^{th} column replaced by $Y + \mathcal{Y}$. This does not give an effective initial-to-interface map because (7) depends on the solutions $\hat{u}^{(j)}(\cdot, T)$. To eliminate this dependence we multiply (7) by $\kappa e^{-\kappa^2 t}$ and integrate around D_R^+, as is typical in the construction of Dirichlet-to-Neumann maps [7]. Switching the order of integration we have

$$
\int_0^T u^{(j)}(x_j, s) \int_{\partial D_R^+} \kappa e^{\kappa^2(s-t)} \, \mathrm{d}\kappa \, \mathrm{d}s = \int_{\partial D_R^+} e^{-\kappa^2 t} \frac{\kappa \det(\mathcal{A}_j(\kappa, T))}{\det(\mathcal{A}(\kappa))} \, \mathrm{d}\kappa, \tag{8a}
$$

$$
\int_0^T u_x^{(j)}(x_j, s) \int_{\partial D_R^+} \kappa e^{\kappa^2(s-t)} \, \mathrm{d}\kappa \, \mathrm{d}s = \int_{\partial D_R^+} e^{-\kappa^2 t} \frac{\kappa \det(\mathcal{A}_{j+n}(\kappa, T))}{\det(\mathcal{A}(\kappa))} \, \mathrm{d}\kappa. \tag{8b}
$$

Using the change of variables $i\ell = \kappa^2$ and the classical Fourier transform formula for the delta function we have

$$u^{(j)}(x_j,t) = \frac{1}{i\pi}\int_{\partial D_R^+} e^{-\kappa^2 t}\frac{\kappa \det(\mathcal{A}_j(\kappa,T))}{\det(\mathcal{A}(\kappa))}\,\mathrm{d}\kappa, \tag{9a}$$

$$u_x^{(j)}(x_j,t) = \frac{1}{i\pi}\int_{\partial D_R^+} e^{-\kappa^2 t}\frac{\kappa \det(\mathcal{A}_{j+n}(\kappa,T))}{\det(\mathcal{A}(\kappa))}\,\mathrm{d}\kappa. \tag{9b}$$

To examine the right-hand-side of (9) we factor the matrix $\mathcal{A}(\kappa)$ as $\mathcal{A}^L(\kappa)\mathcal{A}^M(\kappa)$ where

$$\mathcal{A}^L(\kappa) = \left(\begin{array}{cccc|cccc} e^{-i\frac{\kappa}{\sigma_1}x_1} & & & & & & & \\ & e^{-i\frac{\kappa}{\sigma_2}x_2} & & & & & & \\ & & \ddots & & & & & \\ & & & e^{-i\frac{\kappa}{\sigma_n}x_n} & & & & \\ \hline & & & & e^{i\frac{\kappa}{\sigma_2}x_1} & & & \\ & & & & & e^{i\frac{\kappa}{\sigma_3}x_2} & & \\ & & & & & & \ddots & \\ & & & & & & & e^{i\frac{\kappa}{\sigma_{n+1}}x_n} \end{array}\right)$$

is a diagonal matrix. The elements of $\mathcal{A}^M(\kappa)$ are either 0, $\mathcal{O}(\kappa)$, or decaying exponentially fast for $\kappa \in D_R^+$. Hence,

$$\det(\mathcal{A}^M(\kappa)) = c(\kappa) = \mathcal{O}(\kappa^n),$$

for large κ in D_R^+. Now, $\det(\mathcal{A}(\kappa)) = c(\kappa)\det(\mathcal{A}^L(\kappa))$ as $\kappa \to \infty$ for $\kappa \in D_R^+$. Similarly, factor $\mathcal{A}_j(\kappa,T) = \mathcal{A}^L(\kappa)\mathcal{A}_j^M(\kappa,T)\mathcal{A}_j^R(\kappa,T)$ where $\mathcal{A}_j^R(\kappa,T)$ is the $2n\times 2n$ identity matrix with the (j,j) component replaced by $e^{\kappa^2 T}$. Then $\det(\mathcal{A}_j(\kappa,T)) = e^{\kappa^2 T}\det(\mathcal{A}^L(\kappa))\det(\mathcal{A}_j^M(\kappa,T))$. Thus, the integrand we are considering in (9) is

$$\int_{\partial D_R^+} e^{-\kappa^2 t}\frac{\kappa\det(\mathcal{A}_j(\kappa,T))}{\det(\mathcal{A})}\,\mathrm{d}\kappa = \int_{\partial D_R^+} e^{\kappa^2(T-t)}\frac{\kappa\det(\mathcal{A}_j^M(\kappa,T))}{c(\kappa)}\,\mathrm{d}\kappa.$$

The elements of $\mathcal{A}_j^M(\kappa,T)$ are the same as those in $\mathcal{A}^M(\kappa)$ except in the j^{th} column. Expanding the determinant of $\mathcal{A}_j^M(\kappa,T)$ along the j^{th} column we see that

$$\begin{aligned} e^{\kappa^2(T-t)}\frac{\kappa\det(\mathcal{A}_j^M(\kappa,T))}{c(\kappa)} = \sum_{\ell=1}^{n}\Bigg(c_\ell(\kappa)\left(e^{\frac{i\kappa x_\ell}{\sigma_\ell}+\kappa^2(T-t)}\hat{u}^{(\ell)}\left(\frac{\kappa}{\sigma_\ell},T\right) - e^{-\kappa^2 t+\frac{i\kappa x_\ell}{\sigma_\ell}}\hat{u}_0^{(\ell)}\left(\frac{\kappa}{\sigma_\ell}\right)\right) \\ + c_{\ell+n}(\kappa)\left(e^{\frac{-i\kappa x_\ell}{\sigma_{\ell+1}}+\kappa^2(T-t)}\hat{u}^{(\ell+1)}\left(\frac{-\kappa}{\sigma_{\ell+1}},T\right) - e^{-\kappa^2 t-\frac{i\kappa x_\ell}{\sigma_{\ell+1}}}\hat{u}_0^{(\ell+1)}\left(\frac{-\kappa}{\sigma_{\ell+1}}\right)\right)\Bigg), \end{aligned} \tag{10}$$

where $c_\ell = \mathcal{O}(\kappa^0)$ and $c_{\ell+n} = \mathcal{O}(\kappa)$ for $1 \le \ell \le n$. The terms involving $\hat{u}^{(\ell)}(\cdot,T)$, the solutions of our equation, are decaying exponentially for $\kappa \in D_R^+$. Thus, by Jordan's Lemma [1], the integral of this term along a closed, bounded curve in $\mathbb{C}^+$ vanishes. In particular we consider the closed curve $\mathcal{L}^+ = \mathcal{L}_{D_R^+} \cup \mathcal{L}_C^+$ where $\mathcal{L}_{D_R^+} = \partial D_R^+ \cap \{k : |k| < C\}$ and $\mathcal{L}_C^+ = \{k \in D_R^+ : |k| = C\}$, see Figure 2b. Since the integral along $\mathcal{L}_C^+$ vanishes for large C, (10) must vanish since the contour $\mathcal{L}_{D_R^+}$ becomes ∂D_R^+ as $C \to \infty$.

Since the terms involving the elements of $\mathcal{Y}(\kappa, T)$ evaluate to zero in the solution expression we have the solution

$$
\begin{aligned}
u^{(j)}(x_j,t) =& \frac{1}{i\pi}\int_{\partial D_R^+} e^{-\kappa^2 t}\frac{\kappa \det(A_j(\kappa))}{\det(\mathcal{A}(\kappa))}\,\mathrm{d}\kappa, && \text{(11a)}\\
u_x^{(j)}(x_j,t) =& \frac{1}{i\pi}\int_{\partial D_R^+} e^{-\kappa^2 t}\frac{\kappa \det(A_{j+n}(\kappa))}{\det(\mathcal{A}(\kappa))}\,\mathrm{d}\kappa, && \text{(11b)}
\end{aligned}
$$

where $A_j(\kappa)$ is the matrix $\mathcal{A}(\kappa)$ with the j^{th} column replaced by $Y(\kappa)$. Equation 11 is an effective map between the values of the function at the interface and the given initial conditions.

Remark. Note that since the problem is linear, one could have assumed the initial condition was zero for x outside the region $x_{\ell-1} < x < x_\ell$. Then, the map would be in terms of just $u_0^{(\ell)}$. Summing over $1 \le \ell \le n+1$ would give the complete map for a general initial condition.

As an example of a specific initial-to-interface map we consider the equation (1) with $n = 1$. Using (11) we have

$$
\begin{aligned}
\sigma_1^2 u_x^{(1)}(0,t) =& \frac{i\sigma_1\sigma_2}{\pi(\sigma_1+\sigma_2)}\int_{\partial D_R^+} \kappa e^{-\kappa^2 t}\left(\sigma_1 \hat{u}_0^{(1)}\left(\frac{\kappa}{\sigma_1}\right) - \sigma_2 \hat{u}_0^{(2)}\left(\frac{-\kappa}{\sigma_2}\right)\right)\mathrm{d}\kappa,\\
u^{(1)}(0,t) =& \frac{1}{\pi(\sigma_1+\sigma_2)}\int_{\partial D_R^+} e^{-\kappa^2 t}\left(\sigma_1^2 \hat{u}_0^{(1)}\left(\frac{\kappa}{\sigma_1}\right) + \sigma_2^2 \hat{u}_0^{(2)}\left(\frac{-\kappa}{\sigma_2}\right)\right)\mathrm{d}\kappa.
\end{aligned}
$$

In this case we can deform D_R^+ back to the real line easily. For general n this is not the case. Switching the order of integration and evaluating the κ integral we have

$$
\begin{aligned}
\sigma_1^2 u_x^{(1)}(0,t) =& \frac{\sigma_1\sigma_2}{2t^{3/2}\sqrt{\pi}(\sigma_1+\sigma_2)}\left(\int_{-\infty}^{0} y e^{-y^2/(4t\sigma_1^2)} u_0^{(1)}(y)\,\mathrm{d}y + \int_0^\infty y e^{-y^2/(4t\sigma_2^2)} u_0^{(2)}(y)\,\mathrm{d}y\right), && \text{(12a)}\\
u^{(1)}(0,t) =& \frac{1}{\sqrt{\pi t}(\sigma_1+\sigma_2)}\left(\sigma_1^2\int_{-\infty}^{0} e^{-y^2/(4t\sigma_1^2)} u_0^{(1)}(y)\,\mathrm{d}y + \sigma_2^2\int_0^\infty e^{-y^2/(4t\sigma_2^2)} u_0^{(2)}(y)\,\mathrm{d}y\right), && \text{(12b)}
\end{aligned}
$$

which is an explicit map from the initial data to the value of the temperature and its associated flux at the interface, $x = 0$. If one allows $\sigma_1 = \sigma_2$ the problem is simply that of the heat equation on the whole line. Equation (12) with $\sigma_1 = \sigma_2$ is exactly the Green's Function solution of the whole line problem evaluated at $x = 0$ [9].

3 The heat equation on a finite domain with n interfaces

Consider (1) on a finite domain, $x_0 \le x \le x_{n+1}$, with the boundary conditions

$$
\begin{aligned}
\beta_1 u^{(1)}(x_0,t) + \beta_2 u_x^{(1)}(x_0,t) =& f_1(t), && t > 0, && \text{(13a)}\\
\beta_3 u^{(n+1)}(x_{n+1},t) + \beta_4 u_x^{(n+1)}(x_{n+1},t) =& f_2(t), && t > 0. && \text{(13b)}
\end{aligned}
$$

As before, we rewrite (1) as the set of equations

$$
u_t^{(j)} = \sigma_j^2 u_{xx}^{(j)}, \qquad x_{j-1} < x < x_j, \;\; 0 < t < T,
$$

for $1 \le j \le n+1$, subject to the continuity interface conditions

$$\begin{aligned} u^{(j)}(x_j,t) &= u^{(j+1)}(x_j,t), && t>0, \\ \sigma_j^2 u_x^{(j)}(x_j,t) &= \sigma_{j+1}^2 u_x^{(j+1)}(x_j,t), && t>0, \end{aligned}$$

for $1 \le j \le n$. Without loss of generality we shift the problem so that $x_0 = 0$.

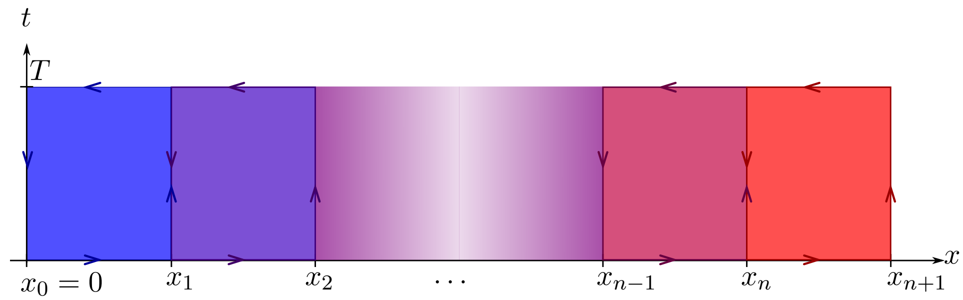

Figure 3: Domains for the application of Green's Theorem in the case of a finite domain with n interfaces.

The following steps are very similar to those presented in the previous section. In what follows we give a brief outline of the changes needed to solve on a finite domain.

Integrating the local relations (3) around the appropriate domain (see Figure 1) and applying Green's Theorem we find the global relations (4) and their evaluation at $-k$ (5). In contrast to Section 2, these $2n+2$ global relations are all valid for $k \in \mathbb{C} \setminus D$. Without loss of generality we choose to work with the equations valid in the upper-half plane. In addition to the definitions in Section 2 we define

$$\begin{aligned} g_0^{(0)}(\omega,t) &= \int_0^t e^{\omega s} u^{(1)}(x_0,s)\,\mathrm{d}s, \\ g_0^{(n+1)}(\omega,t) &= \int_0^t e^{\omega s} u^{(n+1)}(x_{n+1},s)\,\mathrm{d}s, \\ g_1^{(0)}(\omega,t) &= \int_0^t e^{\omega s} u_x^{(1)}(x_0,s)\,\mathrm{d}s, \\ g_1^{(n+1)}(\omega,t) &= \int_0^t e^{\omega s} u_x^{(n+1)}(x_{n+1},s)\,\mathrm{d}s, \\ \tilde{f}_m(\omega,t) &= \int_0^t e^{\omega s} f_m(s)\,\mathrm{d}s, \end{aligned}$$

for $m = 1,2$. Using the change of variables $k = \kappa/\sigma_j$, the global relations valid in the upper-half plane are

$$\begin{aligned} e^{\kappa^2 t}\hat{u}^{(j)}\left(\frac{\kappa}{\sigma_j},T\right) - \hat{u}_0^{(j)}\left(\frac{\kappa}{\sigma_j}\right) &= e^{\frac{-i\kappa x_j}{\sigma_j}}\left(\frac{i\kappa}{\sigma_j}g_0^{(j)}(\kappa^2,T) + g_1^{(j)}(\kappa^2,T)\right) \\ &\quad - e^{\frac{-i\kappa x_{j-1}}{\sigma_j}}\left(\frac{i\kappa}{\sigma_j}g_0^{(j-1)}(\kappa^2,T) + \frac{\sigma_{j-1}^2}{\sigma_j^2}g_1^{(j-1)}(\kappa^2,T)\right), \end{aligned} \tag{14a}$$

$$
\begin{aligned}
e^{\kappa^2 t}\hat{u}^{(j)}\left(\frac{-\kappa}{\sigma_j}, T\right) - \hat{u}_0^{(j)}\left(\frac{-\kappa}{\sigma_j}\right) &= e^{\frac{i\kappa x_j}{\sigma_j}}\left(\frac{-i\kappa}{\sigma_j} g_0^{(j)}(\kappa^2, T) + g_1^{(j)}(\kappa^2, T)\right) \\
&\quad + e^{\frac{i\kappa x_{j-1}}{\sigma_j}}\left(\frac{i\kappa}{\sigma_j} g_0^{(j-1)}(\kappa^2, T) - \frac{\sigma_{j-1}^2}{\sigma_j^2} g_1^{(j-1)}(\kappa^2, T)\right),
\end{aligned} \tag{14b}
$$

for $1 \le j \le n+1$ where we define $\sigma_0 = \sigma_1$ for convenience. These equations, together with the boundary values (13), can be written as a linear system for the interface values

$$\mathcal{A}^F X^F = Y^F + \mathcal{Y}^F,$$

where

$$X^F(\kappa^2, T) = \left(g_0^{(0)}, g_0^{(1)}, \dots, g_0^{(n+1)}, g_1^{(0)}, g_1^{(1)}, \dots, g_1^{(n+1)}\right)^\top,$$

$$Y^F(\kappa, T) = -\left(-\tilde{f}_1(i\kappa^2, T), \hat{u}_0^{(1)}\left(\frac{\kappa}{\sigma_1}\right), \dots, \hat{u}_0^{(n+1)}\left(\frac{\kappa}{\sigma_n}\right), \hat{u}_0^{(1)}\left(\frac{-\kappa}{\sigma_1}\right), \dots, \hat{u}_0^{(n+1)}\left(\frac{-\kappa}{\sigma_{n+1}}\right), -\tilde{f}_2(i\kappa^2, T)\right)^\top,$$

$$\mathcal{Y}^F(\kappa, T) = e^{\kappa^2 T}\left(0, \hat{u}^{(1)}\left(\frac{\kappa}{\sigma_1}, T\right), \dots, \hat{u}^{(n+1)}\left(\frac{\kappa}{\sigma_n}, T\right), \hat{u}^{(1)}\left(\frac{-\kappa}{\sigma_1}, T\right), \dots, \hat{u}^{(n+1)}\left(\frac{-\kappa}{\sigma_{n+1}}, T\right), 0\right)^\top,$$

and

$$
\mathcal{A}^F(\kappa) =
\begin{array}{c}
\begin{array}{cc} \beta_1 \qquad\qquad\qquad\qquad\qquad & \beta_2 \end{array} \\
\left(
\begin{array}{ccc|ccc}
\frac{-i\kappa}{\sigma_1} e^{-i\frac{\kappa x_0}{\sigma_1}} & \frac{i\kappa}{\sigma_1} e^{-i\frac{\kappa x_1}{\sigma_1}} & & -\frac{\sigma_0^2}{\sigma_1^2} e^{-i\frac{\kappa x_0}{\sigma_1}} & e^{-i\frac{\kappa x_1}{\sigma_1}} & \\
\ddots & \ddots & & \ddots & \ddots & \\
& \frac{-i\kappa}{\sigma_{n+1}} e^{-i\frac{\kappa x_n}{\sigma_{n+1}}} & \frac{i\kappa}{\sigma_{n+1}} e^{-i\frac{\kappa x_{n+1}}{\sigma_{n+1}}} & & \frac{-\sigma_n^2}{\sigma_{n+1}^2} e^{-i\frac{\kappa x_n}{\sigma_{n+1}}} & e^{-i\frac{\kappa x_{n+1}}{\sigma_{n+1}}} \\
\hline
\frac{i\kappa}{\sigma_1} e^{i\frac{\kappa x_0}{\sigma_1}} & \frac{-i\kappa}{\sigma_1} e^{i\frac{\kappa x_1}{\sigma_1}} & & \frac{-\sigma_0^2}{\sigma_1^2} e^{i\frac{\kappa x_0}{\sigma_1}} & e^{i\frac{\kappa x_1}{\sigma_1}} & \\
\ddots & \ddots & & \ddots & \ddots & \\
& \frac{i\kappa}{\sigma_{n+1}} e^{i\frac{\kappa x_{n+1}}{\sigma_{n+1}}} & \frac{-i\kappa}{\sigma_{n+1}} e^{i\frac{\kappa x_{n+1}}{\sigma_{n+1}}} & & \frac{-\sigma_n^2}{\sigma_{n+1}^2} e^{i\frac{\kappa x_n}{\sigma_{n+1}}} & e^{i\frac{\kappa x_{n+1}}{\sigma_{n+1}}}
\end{array}
\right) \\
\begin{array}{cc} \beta_3 \qquad\qquad\qquad\qquad\qquad & \beta_4 \end{array}
\end{array}.
$$

The matrix $\mathcal{A}^F(\kappa)$ is made up of four $(n+2)\times(n+2)$ blocks as indicated by the dashed lines. The two blocks in the upper half of $\mathcal{A}^F(\kappa)$ are zero except for entries on the main and -1 diagonals. The lower two blocks of $\mathcal{A}^F(\kappa)$ only have entries on the main and $+1$ diagonals.

As before we use Cramer's Rule to solve this system. After multiplying the solutions by $\kappa e^{-\kappa^2 t}$, integrating around D_R^+, and simplifying as in the previous section we follow a similar process to show the terms from $\mathcal{Y}^F(\kappa, T)$ do not contribute to our solution formula using Jordan's Lemma and Cauchy's Theorem. One can show that $A_j^F(\kappa, T)$ can be replaced by $A_j^F(\kappa, t)$ by writing $\int_0^T \cdot \, \mathrm{d}s$ as $\int_0^t \cdot \, \mathrm{d}s + \int_t^T \cdot \, \mathrm{d}s$ and noticing where the function in analytic and decaying. If the boundary conditions (13) are time-independent then so is A_j^F.

In general, the initial-to-interface map for the heat equation on a finite domain with n interfaces is given by

$$u^{(j)}(x_j, t) = \int_{\partial D_R^+} e^{-\kappa^2 t} \frac{\kappa \det(A_j^F(\kappa, t))}{i\pi \det(\mathcal{A}^F(\kappa))} \, \mathrm{d}\kappa, \tag{16a}$$

$$u_x^{(j)}(x_j, t) = \int_{\partial D_R^+} e^{-\kappa^2 t} \frac{\kappa \det(A_{j+n}^F(\kappa, t))}{i\pi \det(\mathcal{A}^F(\kappa))} \, \mathrm{d}\kappa. \tag{16b}$$

where $A_j^F(\kappa, t)$ is the matrix $\mathcal{A}^F(\kappa, t)$ with the j^{th} column replaced by $Y^F(\kappa, t)$.

Acknowledgements

N.E.S. acknowledges support from the National Science Foundation under grant number NSF-DGE-0718124. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the funding sources.

References

[1] M.J. Ablowitz and A.S. Fokas. *Complex variables: Introduction and Applications*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, second edition, 2003.

[2] M. Asvestas, A.G. Sifalakis, E.P. Papadopoulou, and Y.G. Saridakis. Fokas method for a multi-domain linear reaction-diffusion equation with discontinuous diffusivity. *Journal of Physics: Conference Series*, 490(1):012143, 2014.

[3] B. Deconinck, B. Pelloni, and N.E. Sheils. Non-steady state heat conduction in composite walls. *Proc. R. Soc. A*, 470(2165):22, March 2014.

[4] B. Deconinck, N.E. Sheils, and D.A. Smith. Linear KdV with an interface. *In preparation*, 2015.

[5] B. Deconinck, T. Trogdon, and V. Vasan. The method of Fokas for solving linear partial differential equations. *SIAM Rev.*, 56(1):159–186, 2014.

[6] A.S. Fokas. The generalized Dirichlet-to-Neumann map for certain nonlinear evolution PDEs. *Comm. Pure Appl. Math.*, 58(5):639–670, 2005.

[7] A.S. Fokas. *A unified approach to boundary value problems*, volume 78 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008.

[8] A.S. Fokas and B. Pelloni. A transform method for linear evolution PDEs on a finite interval. *IMA J. Appl. Math.*, 70(4):564–587, 2005.

[9] J. Kevorkian. *Partial differential equations*, volume 35 of *Texts in Applied Mathematics*. Springer-Verlag, New York, second edition, 2000.

[10] E. Langer. The zeros of exponential sums and integrals. *Bull. Amer. Math. Soc.*, 37:213–239, 1931.

[11] D. Mantzavinos, M.G. Papadomanolaki, Y.G. Saridakis, and A.G. Sifalakis. Fokas transform method for a brain tumor invasion model with heterogeneous diffusion in dimensions. *Applied Numerical Mathematics*, 2014.

[12] N.E. Sheils and B. Deconinck. Heat conduction on the ring: Interface problems with periodic boundary conditions. *Appl. Math. Lett.*, 37:107–111, 2014.

[13] N.E. Sheils and B. Deconinck. Interface problems for dispersive equations. *Studies in Applied Mathematics*, 134(3):253–275, 2015.

[14] N.E. Sheils and B. Deconinck. The time-dependent Schrödinger equation with piecewise constant potentials. *In preparation*, 2015.

[15] N.E. Sheils and D.A. Smith. Heat equation on a network using the Fokas Method. *submitted for publication*, 2015.

[16] J. Sylvester and G. Uhlmann. The Dirichlet to Neumann map and applications. In *Inverse problems in partial differential equations*, Proceedings in Applied Mathematics Series, pages 101–139. SIAM, Philadelphia, PA, 1990.