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Global existence for a coupled system of Schrödinger equations with power-type nonlinearities

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In this manuscript, we consider the Cauchy problem for a Schrödinger system with power-type nonlinearities $\begin{cases} i\frac{\partial}{\partial t}u_j + \triangle u_j + \sum_{k=1}^m a_{jk}|u_k|^p|u_j|^{p-2}u_j = 0, \\ u_j(x,0) = \psi_{j0}(x), \end{cases}$ where $u_j : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$, $\psi_{j0} : \mathbb{R}^N \to \mathbb{C}$ for $j = 1, 2, \ldots, m$ and $a_{jk} = a_{kj}$ are positive real numbers. Global existence for the Cauchy problem is established for a certain range of p. A sharp form of a vector-valued Gagliardo-Nirenberg inequality is deduced, which yields the minimal embedding constant for the inequality. Using this minimal embedding constant, global existence for small initial data is shown for the critical case $p = 1 + 2/N$. Finite-time blow-up, as well as stability of solutions in the critical case, is discussed. © *2013 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4774149]

I. INTRODUCTION

The nonlinear Schrödinger (NLS) equation

$$iu_t + \triangle u \pm |u|^2 u = 0, \tag{1.1}$$

where u is a function of $(x, t) \in \mathbb{R}^N \times \mathbb{R}$, arises in many situations. Equation (1.1) describes the evolution of small-amplitude, slowly varying wave packets in a nonlinear medium.[5] Indeed, it has been derived in such diverse fields as waves in deep water,[28] plasma physics,[29] nonlinear fiber optics,[12,13] magneto-static spin waves,[30] and many other settings. Similarly, the m-component coupled nonlinear Schrödinger (CNLS) system with power-type nonlinearities

$$i\frac{\partial}{\partial t}u_j + \triangle u_j + \sum_{k=1}^m a_{jk}|u_k|^p|u_j|^{p-2}u_j = 0, \tag{1.2}$$

with $x \in \mathbb{R}^N$, and $j = 1, \ldots, m$, where u_j are complex-valued functions of $(x, t) \in \mathbb{R}^N \times \mathbb{R}$ and $a_{jk} = a_{kj}$ are positive real numbers, arises under conditions similar to those described by Eq. (1.1). CNLS models physical systems in which the field has more than one component. For example, in optical fibers and waveguides, the propagating electric field has two components that are transverse to the direction of propagation. The CNLS system also arises in the Hartree-Fock theory for a two-component Bose-Einstein condensate, i.e., a binary mixture of Bose-Einstein condensates in two distinct hyperfine states. In almost all of these applications, $N = 1$ and $p = 2$. Readers are referred to various other works[5,12,13,28,29] for the derivation and applications of this system.

The energy E and the component mass Q for the system (1.2) are defined, respectively, as

$$E(u_1, \ldots, u_m) = \frac{1}{2}\sum_{j=1}^m \int_{\mathbb{R}^N} |\nabla u_j(x,t)|^2\, \mathrm{d}x - \frac{1}{2p}\sum_{j,k=1}^m a_{jk}\int_{\mathbb{R}^N} |u_j(x,t)|^p|u_k(x,t)|^p\, \mathrm{d}x, \tag{1.3}$$

$$Q(u_j) = \int_{\mathbb{R}^N} |u_j(x,t)|^2\, \mathrm{d}x, \tag{1.4}$$

for $j = 1, 2, \ldots, m$. Their conservation is an important ingredient for our proof of the global existence of solutions.

Notation: For $1 \leq p \leq \infty$, we denote by $L^p = L^p(\mathbb{R}^N)$ the space of all measurable functions f on $\mathbb{R}^N$ for which the norm $\|f\|_p = \left(\int_{\mathbb{R}^N} |f|^p \, \mathrm{d}x\right)^{\frac{1}{p}}$ is finite for $1 \leq p < \infty$ and $\|f\|_\infty$ is the essential supremum of $|f|$ on $\mathbb{R}^N$. Whether we intend the functions in L^p to be real-valued or complex-valued will be clear from context. $H^1(\mathbb{R}^N)$ is the usual Sobolev space consisting of measurable functions such that both f and its first spatial derivative are in L^2. We define the space

$$\mathcal{H}^{(m)} = \underbrace{H^1(\mathbb{R}^N) \times \ldots \times H^1(\mathbb{R}^N)}_{m}.$$

If $T > 0$ and Y is a Banach space, we denote by $C([0, T], Y)$ the Banach space of continuous maps f: $[0, T] \to Y$, with norms $\|f\|_{C([0,T],Y)} = \sup_{t\in[0,T]} \|f(t)\|_Y$.

We study the well-posedness of the Cauchy problem for system (1.2) in the space $\mathcal{H}^{(m)}$. In other words, for a certain range of p, we examine the existence and uniqueness of solutions to

$$\begin{cases} i\dfrac{\partial}{\partial t}u_j + \triangle u_j + \displaystyle\sum_{k=1}^{m} a_{jk}|u_k|^p|u_j|^{p-2}u_j = 0, \\ u_j(x, 0) = \psi_{j0}(x), \end{cases} \tag{1.5}$$

where $u_j : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$, $\psi_{j0} : \mathbb{R}^N \to \mathbb{C}$ for $j = 1, 2, \ldots, m$ and $a_{jk} = a_{kj}$ are positive real numbers. This is analogous to the case of the single (focusing) NLS equation

$$\begin{cases} i\dfrac{\partial}{\partial t}v + \triangle v + |v|^{2\alpha}v = 0, \\ v(x, 0) = \psi(x), \end{cases} \tag{1.6}$$

for $v : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$ and $\psi : \mathbb{R}^N \to \mathbb{C}$. It is well known that the local critical exponent for the H^1 local well-posedness of Eq. (1.6) is $\alpha = 2/(N - 2)$.[6,7] Indeed, one can use a contraction mapping technique based on Strichartz estimates[6,7] or Kato's fixed-point method[14] to prove that Eq. (1.6) is locally well posed in $H^1(\mathbb{R}^N)$ for $0 \leq \alpha < 2/(N - 2)$, ($0 \leq \alpha < \infty$ if $N = 1, 2$). To establish a global result, the conservation of

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v(x, t)|^2 \, \mathrm{d}x - \frac{1}{2\alpha + 2}\int_{\mathbb{R}^N} |v(x, t)|^{2\alpha+2} \, \mathrm{d}x$$

is used to obtain an *a priori* estimate for extending the unique solution by a continuation argument in the case $0 \leq \alpha < 2/N$. In the critical power case $\alpha = 2/N$, Weinstein[27] showed that the local well-posedness result can be extended to a global one, provided that the L^2-norms of the initial data are small enough. More precisely, it was shown that the minimal constant $C = C_{\alpha,N}$ for the interpolation estimate due to Gagliardo and Nirenberg

$$\|f\|_{2\alpha+2}^{2\alpha+2} \leq C_{\alpha,N}^{2\alpha+2} \|\nabla f\|_{L^2}^{\alpha N} \|f\|_{L^2}^{2+\alpha(2-N)}, \tag{1.7}$$

for $0 \leq \alpha < 2/(N - 2)$ if $N \geq 3$, and $0 \leq \alpha < \infty$ if $N = 1, 2$ is

$$C_{\alpha,N} = \left(\frac{\alpha + 1}{\|\phi\|_{L^2}^{2\alpha}}\right)^{\frac{1}{2\alpha+2}}, \tag{1.8}$$

where ϕ is the ground state (positive solution of minimal L^2-norm[27]) solution of

$$\frac{\alpha N}{2}\triangle\phi - \left(1 + \frac{\alpha}{2}(2 - N)\right)\phi + \phi^{2\alpha+1} = 0.$$

Using this minimal constant, Weinstein showed that for $\alpha = 2/N$, a sufficient condition for the global existence of solutions to Eq. (1.6) is

$$\|\psi\|_{L^2} < \|\Phi\|_{L^2},$$

where Φ is the unique, positive, radial solution of

$$\triangle u - u + u^{\frac{4}{N}+1} = 0.$$

It is useful to point out that the constant $C_{\alpha,N}$ is related to the L^2-norm of any ground-state solution of Eq. (1.1), and that it depends only on the dimension N and power α. Thus, it is easily estimated numerically.

Another question relevant for nonlinear systems such as system (1.2) is the existence and stability of *nontrivial solutions* $(u_1, \ldots, u_m)$, that is, solutions with m nonzero components. Such solutions are referred to as co-existing or vector solutions. For system (1.2), there are many semitrivial (or collapsing) solutions, which are vector solutions with at least one, but not all, components identically zero. In these cases, the system collapses into a system with fewer components. For example, Nguyen and Wang[19] show that for a 2-component coupled system ($m = p = 2$ and $N = 1$), there are obstructions to the existence and stability of nontrivial solutions with all components positive (a solution of (1.2) is called positive if each component is of the form $e^{i\alpha t}R(x)$ with $R(x)$ a real-valued positive function). Roughly speaking, in order to have positive nontrivial solutions, the nonlinear couplings have to be either small or large. In this situation, multiple solutions of system (1.2) exist and it is nontrivial to classify and distinguish solutions. Many works[1–4,9,15–18,21,22,25,26] concern 2-component systems or systems with small couplings. Despite the partial progress made so far, many questions remain and little is known for m-component systems with $m \geq 3$.

It is the aim of this paper to establish the well-posedness result as well as to investigate the analogous properties of solutions mentioned above for the m-component coupled system (1.2). The case $m = 2$ has been studied[1,2,10,11,23] and standard scaling arguments suggest the local critical exponent for the local well-posedness in $\mathcal{H}^{(m)}$ is $p = N/(N - 2)$. Indeed, it is well-known that for the equation

$$iu_t + \triangle u + |u|^\alpha u = 0$$

the critical nonlinearity is $\alpha = 4/(N - 2)$. (See, for example, Refs. 6 and 7) Due to the L^2, H^1-scalings of u_j, exact same calculation gives $p = N/(N - 2)$ as the critical exponent for our system. (See also Remark 1.1 below.) Thus, the methods mentioned above for establishing local existence for Eq. (1.6) in H^1, namely, the contraction mapping technique based on Strichartz estimates,[6,7] or Kato's fixed point method,[14] can be used to establish local well-posedness of system (1.2) in $\mathcal{H}^{(m)}$ for $p < N/(N - 2)$. The following bounds for the nonlinear terms are crucial. Let $\mathbf{u} = (u_1, u_2, \ldots, u_m)$. Equate

$$g_j(\mathbf{u}) = \sum_{k=1}^{m} a_{jk} |u_k|^p |u_j|^{p-2} u_j.$$

For every $K > 0$, there exists $L(K) < \infty$ such that for almost all $x \in \mathbb{R}^N$ and all $\mathbf{u}, \mathbf{v}$ such that $|u_j|, |v_j| \leq K$,

$$|g_j(\mathbf{u}) - g_j(\mathbf{v})| \leq c_1 L(K) \sum_{k=1}^{m} |u_k - v_k|, \tag{1.9}$$

where c_1 is a positive constant and

$$\begin{cases} L(K) \in C([0,\infty)), & \text{with } 2 \leq p < \infty \quad \text{if} \quad N = 1, 2, \\ L(K) \leq K^{2p-2}, & \text{with} \quad 2 \leq p < 3 \quad \text{if} \quad N = 3. \end{cases}$$

To derive Eq. (1.9), we have used

$$\left||u_j|^{2p-2}u_j - |v_j|^{2p-2}v_j\right| \leq \max\{|u_j|^{2p-2}, |v_j|^{2p-2}\}|u_j - v_j|, \tag{1.10a}$$

$$\begin{aligned} \left||u_k|^p|u_j|^{p-2}u_j - |v_k|^p|v_j|^{p-2}v_j\right| &\leq \max\{|u_k|^p|u_j|^{p-2}, |v_k|^p|v_j|^{p-2}\}\left|u_j - v_j\right| \\ &\leq K^{2p-2}|u_j - v_j|. \end{aligned} \tag{1.10b}$$

Remark 1.1: The use of (1.10b) *necessitates* $2 \leq p$ *in order to use the contraction mapping technique based on Strichartz estimates;*[6,7] *and when coupled with* $p < N/(N-2)$, *it is implied that* $2 \leq p < 3$ *when* $N = 3$. *This condition puts a restriction on the applicable range of p for dimension* $1 \leq N \leq 3$ *for the proof of local existence and for* $N = 1$ *for the proof of global existence.*

Remark 1.2: It is worth pointing out that there are cases when $1 \leq p$ *is allowed. For example, if* $u_j = A_j u$ *for some real constants* A_j, *then the system* (1.2) *is uncoupled and the result follows directly from Cazenave,*[6,7] *provided the initial data are related accordingly.*

One technical point deserves some comments here. It was claimed by Fanelli and Montefusco[11] and Song[23] that the local well-posedness result for $m = 2$ follows from the contraction mapping argument for $1 \leq p < N/(N-2)$ (the power has been re-scaled here for comparison). While it is true that there are instances when $1 \leq p$ is acceptable as mentioned above, it appears the range for p cannot be extended to include $p < 2$ in general due to Remark 1.2 and thus the claim is doubtful. It may be possible that other methods allow for the local well-posedness when $1 \leq p < N/(N-2)$ in which case results in this paper hold for all dimensions N.

Recall the following inequality due to Gagliardo and Nirenberg:[20]

$$\begin{aligned} P(\psi_1, \psi_2, \ldots, \psi_m) :&= \frac{1}{2p} \sum_{j,k=1}^{m} a_{jk} \int_{\mathbb{R}^N} |\psi_j|^p |\psi_k|^p \, \mathrm{d}x \\ &\leq c \left(\sum_{j=1}^{m} \|\nabla \psi_j\|_2^2 \right)^{\frac{(p-1)N}{2}} \left(\sum_{j=1}^{m} \lambda_j \|\psi_j\|_2^2 \right)^{\frac{N-p(N-2)}{2}}. \end{aligned} \tag{1.11}$$

This allows the use of the conserved quantity $E(u_1, u_2, \ldots, u_m)$ associated with the system (1.2) to obtain an $\mathcal{H}^{(m)}$-bound on the solutions in order to obtain global existence in the case $p < 1 + 2/N$. (See, for example, Theorem 6.1.1 in Cazenave.[7])

Theorem 1.1: *Let* $p < 1 + 2/N$. *For any* $(\psi_{10}, \ldots, \psi_{m0}) \in \mathcal{H}^{(m)}$, *there exists a unique solution* $(u_1, \ldots, u_m) \in C([0, \infty); \mathcal{H}^{(m)})$ *and the Cauchy problem* (1.5) *is globally well posed in* $\mathcal{H}^{(m)}$.

Remark 1.3: To extend the local existence result to a global one, we require $p < 1 + 2/N$ *as the nonlinear terms can then be controlled by the* H^1 *–norm of the solution, a sufficient condition for a continuation argument (see, for example, Theorem 6.1.1 of Ref. 7). Recall that for the local existence, the contraction method we used above requires that* $2 \leq p$ *to have Lipschitz continuity. The condition* $p < 1 + 2/N$ *when coupled with* $2 \leq p < N/(N-2)$ *for local existence implies that* $N = 1$. *Thus, we refrain from putting the restriction* $2 \leq p$ *in the above Theorem since if other methods allow for* $1 \leq p$ *in local existence (which we know is true for at least one case as mentioned in the Remark 1.2) then global existence follows, so long as* $p < 1 + 2/N$.

The manuscript is organized as follows. Section II summarizes progress that has been made for system (1.2) as well as its associated elliptic system (2.1) and gives a statement of our contribution. A sharp form of the vector-valued Gagliardo-Nirenberg inequality is established in Sec. III. This yields an *a priori* estimate needed for global existence of solutions in the case $p < 1 + 2/N$, along with the minimal embedding constant for the Gagliardo-Nirenberg inequality. Using this minimal embedding constant, global existence for small initial data is shown for the case $p = 1 + 2/N$ in Sec. IV. Finite-time blow-up and stability of solutions in the global critical power case ($p = 1 + 2/N$) are discussed in Sec. V.

II. STATEMENT OF RESULTS

Consider standing wave solutions of the form $u_j(t,x) = e^{i\lambda_j t}\psi_j(x)$, with $\lambda_j > 0$. Substituting this into the first equation of (1.5), one obtains the associated elliptic system

$$-\Delta\psi_j + \lambda_j\psi_j = \sum_{k=1}^{m} a_{jk}|\psi_k|^p|\psi_j|^{p-2}\psi_j. \tag{2.1}$$

When $p = 2$ or $m = 2$, much is known about this system.[1,3,4,9,15–18,21,22,25,26] In these references, various methods were employed to construct solutions for various parameter regimes. In particular, Wei and Yao[24] studied the associated elliptic system when $p = m = 2$, $a_{11} = \mu_1$, $a_{12} = a_{21} = \beta$, $a_{22} = \mu_2$, $\lambda_1 = \lambda_2 = \lambda$:

$$\begin{cases} \triangle u - \lambda u + \mu_1 u^3 + \beta u v^2 = 0, \\ \triangle v - \lambda v + \mu_2 v^3 + \beta u^2 v = 0 \end{cases} \tag{2.2}$$

with $N \le 3$. Complete classifications in the case of $N = 1$ and partial answers in the case $N = 2, 3$ were given in by Wei and Yao.[24] When $p = m = 2$ and $N \le 3$, there exists $\Lambda > 0$, depending on λ_j, $a_{jk} > 0$, such that system (2.1) has a vector ground-state solution provided $a_{12} > \Lambda$.[2] The case $p = 2$, $\lambda_j = \lambda$, $N \le 3$ and general m were studied by Bartsch and Wang.[3]

Since we will discuss the elliptic system (2.1) for which the restriction $2 \le p$ is not required, and since, as pointed out in the Introduction, there are cases when $1 \le p$ allows for local existence, we will assume henceforth that $1 \le p$ unless otherwise stated.

First, we show that the minimal constant C_{N,p,a_{jk},λ_j} in the vector-valued Gagliardo-Nirenberg inequality(1.11) is achieved through a minimization problem and

$$C_{N,p,a_{jk},\lambda_j} = P(\psi_1^*, \psi_2^*, \ldots, \psi_m^*)$$

with $(\psi_1^*, \psi_2^*, \ldots, \psi_m^*)$ being minimizer (see Lemma 3.2). Using this minimal embedding constant, we deduce that the Cauchy problem (1.5) is also well posed in the critical power case $p = 1 + 2/N$ provided the initial data are sufficiently small. More precisely,

Theorem 2.1: *Let* $(\psi_{10}, \ldots, \psi_{m0}) \in \mathcal{H}^{(m)}$ *and* $C = C_{N,p,a_{jk},\lambda_j}$. *If* $p = 1 + 2/N$, *then there exists a unique solution* $(u_1, \ldots, u_m) \in C([0,\infty); \mathcal{H}^{(m)})$ *of the Cauchy problem* (1.5) *so long as*

$$\sum_{j=1}^{m} \lambda_j \|\psi_{j0}\|_2^2 < \left(\frac{1}{2C}\right)^{\frac{N}{2}}. \tag{2.3}$$

Theorem 2.2 gives sufficient conditions for blow-up of solutions of system (1.5).

Theorem 2.2: *When* $1 + 2/N \le p < N/(N - 2)$, *there exists* $0 < T < \infty$ such that

$$\lim_{t\to T} \sum_{j=1}^{m} \|\nabla u_j\|_2 = +\infty,$$

provided any of the following three holds:

(i) $E < 0$;

(ii) $E = 0$, *and* $\operatorname{Im} \int_{\mathbb{R}^N} \sum_{j=1}^{m} (x\cdot\nabla\psi_{j0})\overline{\psi}_{j0}\,\mathrm{d}x < 0$;

(iii) $E > 0$, *and* $\operatorname{Im} \int_{\mathbb{R}^N} \sum_{j=1}^{m} (x\cdot\nabla\psi_{j0})\overline{\psi}_{j0}\,\mathrm{d}x < -\sqrt{2E\cdot V(0)}$.

Even though the minimal constant C_{N,p,a_{jk},λ_j} can be expressed in terms of $P(\psi_1^*, \psi_2^*, \ldots, \psi_m^*)$, the description of minimizers in general is difficult. This is due to the fact that not much is known about the associated elliptic system (2.1) when $m \ge 3$ (even for the case $p = 2$).

We make the following two assumptions,

(P1): (*symmetry condition*) $\lambda_j \equiv \lambda$, $a_{jj} = \mu_j$, $a_{jk} \equiv \beta$ $(j \neq k)$ for $j, k = 1, 2, \ldots m$;
(P2): (*existence condition*) the coupling coefficient $\beta > 0$ is either

(i) sufficiently small, or
(ii) $\beta > \max\{\mu_1, \mu_2\}$, if $m = 2$, $p = 2$.

Notice that the assumptions (P1)-(P2) lead to a focusing model. Without loss of generality, assume that $\lambda = 1$. The system takes the form

$$-\Delta\psi_j + \psi_j = \mu_j|\psi_j|^{2p-2}\psi_j + \beta\sum_{k\neq j}|\psi_k|^p|\psi_j|^{p-2}\psi_j \tag{2.4}$$

with $m \geq 2$ and $N \leq 3$. Let ω be the unique positive radial solution of

$$-\Delta\omega + \omega = \omega^{2p-1}, \tag{2.5}$$

where $x \in \mathbb{R}^N$ and $\omega(|x|) \to 0$ as $|x| \to \infty$.

Theorem 2.3: *For $\beta > 0$ small enough, there exists a unique positive vector solution to system* (2.4).

Theorem 2.4: *Let ω be defined as above. Then system* (2.4) *has a positive vector solution Ψ^* that can be written in terms of ω, provided any of the following two hold:*

(i) $p = 2$ *and* $0 < \beta < \min_{1\leq j\leq m}\{\mu_j\}$ *or* $\beta > \max_{1\leq j\leq m}\{\mu_j\}$;
(ii) $1 < p \neq 2$, *for any* β.

Remark 2.1: The case $m = p = 2$ has been proved many times before.[2,11,24] *Theorems 2.3 and 2.4, proved in Sec. III guarantee existence of a positive vector solution whose components are constant multiples of ω. Moreover, when $\beta > 0$ is small enough, the positive vector solution is unique. Thus, any ground-state solution has identical components equal to the scalar solution.*

It follows immediately from Theorems 2.2, 2.3, and 2.4 in the critical power case $p = 1 + 2/N$ that the solutions for the m-system (1.2) are unstable. Therefore, the condition in Theorem 2.1 is sharp.

Theorem 2.5: *Assume (P1) holds and $\lambda = 1$, then the minimal constant in the vector Gagliardo-Nirenberg inequality* (1.11) *is*

1. *for $\beta > 0$ sufficiently small*

$$C_{N,p} = \min\{\mu_1, \ldots, \mu_m\}\frac{p(N - p(N-2))^{\frac{N(p-1)-2}{2}}}{2(N(p-1))^{\frac{N(p-1)}{2}}\|\omega\|_2^{2p-2}}$$

2. *when $m = p = 2$, $\beta > \max\{\mu_1, \mu_2\}$,*

$$C_N = \frac{\mu_1 + \mu_2 - 2\beta}{\mu_1\mu_2 - \beta^2}\frac{(4-N)^{\frac{N-2}{2}}}{N^{\frac{N}{2}}\|\omega\|_2^2}.$$

Remark 2.2: The above Theorem generalizes the results obtained by Fanelli and Montefusco[11] *in a couple of aspects. They only discussed the case $m = 2$ (the 2-system); moreover, the discussion was restricted to the super-symmetric situation where $\mu_1 = \mu_2 = 1$. Here, we study the general m –system where the μ_j are not necessarily the same. In particular, our constants $C_{N,p}$ generalize those obtained in Theorem 2 by Fanelli and Montefusco*[11] *for the super-symmetric 2-system.*

Theorem 2.6: *Let $p = 1 + 2/N$, and suppose that (P1) and (P2) hold. Then the $\mathcal{H}^{(m)}$-solution of* (2.4) *is unstable for the m-system* (1.2) *in the following sense. Let $\overline{\psi} = (\psi_1, \ldots, \psi_m) \in \mathcal{H}^{(m)}$, $(\overline{\psi} \neq 0)$ solve system* (2.4)*. Then, for any $\delta > 0$, there is an m-vector function $\overline{\xi}$ with $\|\overline{\xi} - \overline{\psi}\|_2 < \delta$,* *such that for* $(u_1(t,x),\ldots,u_m(t,x))$ *the solution of system* (1.5) *with* $u_j(x,0)=\xi_j$ *attains*

$$\lim_{t\to T^-}\sum_{j=1}^m\|\nabla u_j(t)\|_2=\infty,$$

for some $0<T<\infty$.

Note that this theorem follows immediately from Theorem 2.2.

III. MINIMAL CONSTANT FOR THE VECTOR-VALUED GAGLIARDO-NIRENBERG INEQUALITY

First, we present the proof of Theorem 2.3, which guarantees the uniqueness of a positive vector solution to system (2.4) when $\beta>0$ is sufficiently small.

Proof: Define the following functional $\Phi_\beta:\mathcal{H}^{(m)}\to\mathbb{R}$,

$$\Phi_\beta(\psi_1,\ldots,\psi_m)=\frac{1}{2}\sum_{j=1}^m(\|\nabla\psi_j\|_2^2+\|\psi_j\|_2^2)-\frac{1}{2p}\left(\sum_{j=1}^m\mu_j\|\psi_j\|_{2p}^{2p}+\beta\int_{\mathbb{R}^N}\sum_{j,k=1,j\neq k}|\psi_j\psi_k|^p\,\mathrm{d}x\right). \tag{3.1}$$

It is well known that Φ_0 has a unique positive and radial solution[6,7]

$$(\psi_1^*(x),\ldots,\psi_m^*(x))=\left(\left(\frac{1}{\mu_1}\right)^{\frac{1}{2p-2}}\omega(x),\ldots,\left(\frac{1}{\mu_m}\right)^{\frac{1}{2p-2}}\omega(x)\right),$$

where ω is the unique positive radial solution to equation (2.5). Moreover, the Hessian $\Phi_0''(\psi_1^*,\ldots,\psi_1^*)$ is invertible. By the implicit function theorem, there exist $\beta_0>0$, $r_0>0$ and a map $\phi:(-\beta_0,\beta_0)\to B_{r_0}(\psi_1^*,\ldots,\psi_m^*)\subset\mathcal{H}^{(m)}$ such that for any $\beta\in(-\beta_0,\beta_0)$, $\Phi_\beta(\psi_1,\ldots,\psi_m)=0$ has a unique solution $(\psi_1,\ldots,\psi_m)=\phi(\beta)$ in $B_{r_0}(\psi_1^*,\ldots,\psi_m^*)$.

To complete the argument, one needs to show that the set of positive radial solutions to system (2.4) is compact for bounded β. The method used here is borrowed from Dancer and Wei.[8] By standard regularity theory, $\psi_j^*\in C^2(\mathbb{R})$ for each $j=1,\ldots,m$. Here, we consider the ψ_j^*'s as functions of one variable. Thus, on any finite interval the solution sequence $\{(\psi_1^l,\ldots,\psi_m^l)\}_{l=1}^\infty$ has a convergent subsequence. In order to show compactness, it suffices to show the sequence of solutions uniformly vanishes as $r\to\infty$.

Claim: For any $\epsilon>0$, $1\le j\le m$, there exists $r_\epsilon>0$, such that

$$\sum_{j=1}^m\psi_j^l(r_\epsilon)<\epsilon,\qquad\forall l\in\mathbb{N}. \tag{3.2}$$

Note that ψ_j^l is positive and decreasing in r so (3.2) indicates the uniform decay of the sequence in L^∞. Suppose that the claim is false, then there exists a subsequence (still denoted by $\{(\psi_1^l,\ldots,\psi_m^l)\}$) and $\alpha>0$, such that

$$\alpha=\psi_1^l(r_j)+\psi_2^l(r_j)+\ldots+\psi_m^l(r_j). \tag{3.3}$$

By shifting the origin to r_j and letting $\epsilon\to0$, one obtains a nontrivial solution $(\psi_1,\psi_2,\ldots,\psi_m)$ on $\mathbb{R}$ of the system

$$-\psi_j''=-\psi_j+\mu_j\psi_j^{2p-1}+\sum_{k\neq j}^m\beta\psi_j^{p-1}\psi_k^p, \tag{3.4}$$

where $j, k = 1, 2, \ldots, m$, and $\sum_{j=1}^{m} \psi_j(0) = \alpha$. Note $\psi_j \geq 0$ are bounded and decreasing on $\mathbb{R}$. Denote

$$\psi_j^+ = \lim_{r\to\infty} \psi_j(r), \qquad \psi_j^- = \lim_{r\to-\infty} \psi_j(r).$$

Then $(\psi_1^+, \ldots, \psi_m^+)$ and $(\psi_1^-, \ldots, \psi_m^-)$ both solve the equations

$$\psi_j = \mu_j \psi_j^{2p-1} + \beta \sum_{k\neq j}^{m} \psi_j^{p-1}\psi_k^p, \quad j, k = 1, 2, \ldots, m.$$

Since $\sum_{j=1}^{m} \psi_j^+ \leq \alpha$, by choosing α arbitrarily small, we see that $\psi_j^+ = 0$. If ψ_1 does not vanish identically, then $\psi_1^- > 0$ and $1 = \mu_1(\psi_1^-)^{2p-2} + \beta \sum_{j=2}^{m} (\psi_j^-)^p$. Hence, $\psi_1(-1 + (\psi_1)^{2p-2} + \beta \sum_{j=2}^{m} (\psi_j)^p) < 0$ on $\mathbb{R}$ and therefore $\psi_1'' > 0$ on $\mathbb{R}$. Consequently, ψ_1 is strictly concave up and bounded on $\mathbb{R}$, which is impossible. Thus, the claim holds and Theorem 2.3 follows. □

Remark 3.1: Notice that Theorem 2.3 does not require $\mu_j \equiv \mu$.

Theorem 2.4 provides sufficient conditions for the existence of a positive vector solution to (2.4) whose components are constant multiples of ω, where ω is defined as in (2.5). We prove it now.

Proof: **Case (i)**. System (2.4) becomes

$$-\Delta\psi_j + \psi_j = \mu_j |\psi_j|^2 \psi_j + \beta\psi_j \sum_{k\neq j}^{m} |\psi_k|^2, \quad m \geq 2. \tag{3.5}$$

If $0 < \beta < \min\{\mu_j\}$ or $\beta > \max\{\mu_j\}$, a direct calculation shows the above system has solutions of the form

$$\Psi^* = (\psi_1, \psi_2, \ldots, \psi_m) = \left(\sqrt{(\Gamma^{-1}\Lambda)_1}w, \sqrt{(\Gamma^{-1}\Lambda)_2}w, \ldots, \sqrt{(\Gamma^{-1}\Lambda)_m}w\right), \tag{3.6}$$

where

$$\Gamma = \begin{pmatrix} \mu_1 & \beta & \ldots & \beta \\ \beta & \mu_2 & \ldots & \beta \\ \vdots & \vdots & \ddots & \vdots \\ \beta & \beta & \ldots & \mu_m \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \tag{3.7}$$

and $(\Gamma^{-1}\Lambda)_j$ denotes the jth component of the m-vector. Therefore, Lemma 2.4 is proved in this case.

Case (ii). We use mathematical induction. First, let us look at the case $m = 2$. Assume $\psi_1(x) = A\omega(x)$, $\psi_2(x) = B\omega(x)$ solve system (2.4) for some $A, B > 0$. Then, one has the following algebraic system:

$$\mu_1 A^{2p-2} + \beta A^{p-2} B^p = 1, \tag{3.8a}$$

$$\mu_2 B^{2p-2} + \beta A^p B^{p-2} = 1. \tag{3.8b}$$

Subtracting Eq. (3.8a) from Eq. (3.8b) and letting $t = B/A$, one arrives at the following continuous function:

$$g(t) := \mu_2 t^{2p-2} - \beta t^p + \beta t^{p-2} - \mu_1.$$

One can claim that for fixed $1 < p$ and $p \neq 2$, there exists $t_0 > 0$ such that $g(t_0) = 0$. Indeed:

(a) If $p > 2$, then $g(0) = -\mu_1 < 0$ and $g(1) = \mu_2 - \mu_1 > 0$, thus there exists $t_0 \in (0, 1)$ such that $g(t_0) = 0$.

(b) If $1 < p < 2$, then $g(1) = \mu_2 - \mu_1 > 0$ and $g(M) < 0$ for some $M > 0$, thus there exists $t_0 > 1$ such that $g(t_0) = 0$.

Substituting $B = t_0 A$ into Eq. (3.8)

$$A = \left(\frac{1}{\mu_1 + \beta t_0^p}\right)^{\frac{1}{2p-2}}, \text{ and}$$

$$B = t_0 \left(\frac{1}{\mu_1 + \beta t_0^p}\right)^{\frac{1}{2p-2}}.$$

Consequently, the solution of (2.4) can be represented as

$$\psi_1(x) = \frac{1}{(\mu_1 + \beta t_0^p)^{\frac{1}{2p-2}}}\omega(x),$$

$$\psi_2(x) = \frac{t_0}{(\mu_1 + \beta t_0^p)^{\frac{1}{2p-2}}}\omega(x).$$

Thus, case (ii) is proved for $m = 2$.

As it is shown above, the key step is finding solutions of (3.8). Notice that solving system (3.8) for A and B is the same as finding the positive zeroes of the auxiliary function g. This idea can be generalized to m equations.

In the case $m = 3$, one needs to solve

$$\begin{aligned} \mu_1 A^{2p-2} + \beta A^{p-2}(B^p + C^p) &= 1, \\ \mu_2 B^{2p-2} + \beta B^{p-2}(A^p + C^p) &= 1, \\ \mu_3 C^{2p-2} + \beta C^{p-2}(A^p + B^p) &= 1. \end{aligned} \tag{3.9}$$

Subtracting the first equation from the second and the third equation separately, and setting $t = B/A$, $s = C/A$,

$$\begin{aligned} \mu_2 t^{2p-2} + \beta t^{p-2}(1 + s^p) &= \mu_1 + \beta(t^p + s^p), \\ \mu_3 s^{2p-2} + \beta s^{p-2}(1 + t^p) &= \mu_1 + \beta(t^p + s^p). \end{aligned} \tag{3.10}$$

Denote

$$g_1(t, s) = \mu_2 t^{2p-2} + \beta t^{p-2}(1 + s^p) - \mu_1 - \beta(t^p + s^p),$$

$$g_2(t, s) = \mu_3 s^{2p-2} + \beta s^{p-2}(1 + t^p) - \mu_1 - \beta(t^p + s^p).$$

One needs to find $t_0, s_0 > 0$ such that $g_1(t_0, s_0) = 0$ and $g_2(t_0, s_0) = 0$.

Due to the symmetric structure of (3.9), without loss of generality, let us assume that:

(i) if $1 < p < 2$, then $\mu_1 > \max\{\mu_2, \mu_3\}$;
(ii) if $p > 2$, then $\mu_1 < \min\{\mu_2, \mu_3\}$.

In subcase (i), for each $s \in [0, 1]$,

$$g_1(1, s) = \mu_2 - \mu_1 < 0,$$

and $g(T_0, s) > 0$, where

$$T_0 = \left(\frac{\beta}{\mu_1 + \beta}\right)^{\frac{1}{2-p}}.$$

Thus, there exists $t_s \in (T_0, 1)$ such that $g_1(t_s, s) = 0$. Define

$$\mathcal{S} := \{(t_s, s) | s \in [T_0, 1]\},$$

then $\mathcal{S} \subset [T_0, 1] \times [T_0, 1]$ is a continuous curve, and

$$g_2(t_1, 1) = \mu_3 - \mu_1 < 0, \qquad g_2(t_{T_0}, T_0) > 0,$$

i.e., the function g_2 changes sign along $\mathcal{S}$. Since g_2 is also continuous, there exists a point on this line (t_{s_0}, s_0) such that $g_2(t_{s_0}, s_0) = 0$. Consequently, system (3.9) is solvable.

In subcase (ii), for each $s \in [0, 1]$, $g_1(0, s) = -\mu_1 - \beta s^p < 0$ and $g_1(1, s) = \mu_2 - \mu_1 \geq 0$. Thus, there exists $t_s \in (0, 1]$ such that $g_1(t_s, s) = 0$, and

$$\mathcal{S} := \{(t_s, s) | s \in [0, 1]\}$$

is a continuous curve. On the other hand, $g_2(t_0, 0) = -\mu_1 - \beta t_0^p < 0$ and $g_2(t_1, 1) = \mu_3 - \mu_1 > 0$. Hence, the function g_2 changes signs along $\mathcal{S}$. Since g_2 is also continuous, there exists a point on the line (t_{s_0}, s_0) such that $g_2(t_{s_0}, s_0) = 0$. Thus, system (3.9) is also solvable in this case.

Consider next the system of $(m + 1)$ equations,

$$\mu_j A_j^{2p-2} + \beta A_j^{p-2} \sum_{k \neq j} A_k^p = 1, \tag{3.11}$$

with $j = 1, 2, \ldots, m + 1$. The proofs for $1 < p < 2$ and $p > 2$ run parallel to the arguments shown above and thus, we omit the details for case $1 < p < 2$ and only discuss the case $p > 2$. Without loss of generality, assume that $\mu_1 < \min_{2 \leq j \leq m}\{\mu_j\}$. One now has at hand m continuous functions $g_1(t_1, \ldots, t_m), \ldots, g_m(t_1, \ldots, t_m)$. The zeros of these m functions will yield a solution of the system (3.11).

By induction, for each fixed $s \in [0, 1]$ and $1 \leq j \leq m - 1$, $g_j|_{t_m = s}$ has a root $((t_1, \ldots, t_{m-1})_s, s)$. Thus,

$$\mathcal{S} = \{((t_1, \ldots, t_{m-1})_s, s), s \in [0, 1]\}$$

is a continuous curve on which the first $(m - 1)$ functions are identically zero. On the other hand,

$$g_m((t_1, \ldots, t_{m-1})_0, 0) < 0,$$

and

$$g_m((t_1, \ldots, t_{m-1})_1, 1) \geq 0.$$

Continuity of g_m implies there exists $s_0 \in (0, 1]$ such that

$$g_m((t_1, \ldots, t_{m-1})_{s_0}, s_0) = 0.$$

Since $((t_1, \ldots, t_{m-1})_{s_0}, s_0) \in \mathcal{S}$,

$$g_j((t_1, \ldots, t_{m-1})_{s_0}, s_0) = 0 \quad \text{for } j = 1, \ldots, m-1.$$

Therefore, there exists a zero for the m functions, which in turn yields a solution to the system (3.11). Hence, case (ii) is proved. Thus, Theorem 2.4 is proved. □

Remark 3.2: As a special case of Theorem 2.4, when $\mu_j \equiv \mu$ for all j, the system takes the form

$$-\Delta \psi_j + \psi_j = \mu \sum_{j=1}^{m} |\psi_j|^{2p-2} \psi_j + \beta \psi_j \sum_{k \neq j}^{m} |\psi_k|^p |\psi_j|^{p-2} \psi_j$$

with $m \geq 2$, and straightforward calculations show that it has a positive vector solution

$$\left(\psi_1^*(x), \ldots, \psi_m^*(x)\right) = \left(\left(\frac{1}{\mu + (m-1)\beta} \right)^{\frac{1}{2p-2}} \omega(x), \ldots, \left(\frac{1}{\mu + (m-1)\beta} \right)^{\frac{1}{2p-2}} \omega(x) \right).$$

Next, we turn our attention to finding the minimal constant in the vector-valued Gagliardo-Nirenberg inequality that corresponds to the system (1.5), i.e., the constant C_{N,p,a_{jk},λ_j} in inequality (1.11). This is established using the framework outlined by Weinstein.[27]

Let $J : \mathcal{H}^{(m)} \to \mathbb{R}$ be defined as

$$J(\psi_1, \psi_2, \ldots, \psi_m) := \frac{\left(\sum_{j=1}^m \|\nabla \psi_j\|_2^2\right)^{(p-1)\frac{N}{2}} \left(\sum_{j=1}^m \lambda_j \|\psi_j\|_2^2\right)^{[N-p(N-2)]/2}}{P(\psi_1, \psi_2, \ldots, \psi_m)}. \tag{3.12}$$

A minimal constant is determined by the equation

$$\alpha := \frac{1}{C_{N,p,a_{jk},\lambda_j}} = \inf_{(\psi_1,\ldots,\psi_m)\in\mathcal{H}^{(m)}} J(\psi_1, \ldots, \psi_m). \tag{3.13}$$

Lemma 3.1: The minimum value for Eq. (3.13) is achieved and the minimizer $(\psi_1^*, \ldots, \psi_m^*)$ can be selected such that

$$\sum_{j=1}^m \|\nabla \psi_j^*\|_2^2 = 1 = \sum_{j=1}^m \lambda_j \|\psi_j^*\|_2^2. \tag{3.14}$$

Proof: For $\psi_j \in H^1$ and any $\nu, \mu > 0$ let $\psi_j^{\nu,\mu}(x) = \nu\psi_j(\mu x)$. Then,

$$\|\nabla \psi_j^{\nu,\mu}\|_2^2 = \nu^2 \mu^{2-N} \|\nabla \psi_j\|_2^2, \|\psi_j^{\nu,\mu}\|_2^2 = \nu^2 \mu^{-N} \|\psi_j\|_2^2,$$

$$\|\psi_j^{\nu,\mu}\|_{2p}^{2p} = \nu^{2p} \mu^{-N} \|\psi_j\|_{2p}^{2p}, \|\psi_j^{\nu,\mu} \psi_k^{\nu,\mu}\|_p^p = \nu^{2p} \mu^{-N} \|\psi_j \psi_k\|_p^p.$$

Therefore, $J(\psi_1^{\nu,\mu}, \ldots, \psi_m^{\nu,\mu}) = J(\psi_1, \psi_2, \ldots, \psi_m)$, for any $(\psi_1, \ldots, \psi_m) \in \mathcal{H}^{(m)}$.
Let $\{(\psi_1^s, \psi_2^s, \ldots, \psi_m^s)\}$ be a minimizing sequence for Eq. (3.13). Set

$$\nu_s = \frac{\left(\sum_{j=1}^m \lambda_j \|\psi_j^s\|_2^2\right)^{\frac{N-2}{4}}}{\left(\sum_{j=1}^m \|\nabla \psi_j\|_2^2\right)^{\frac{N}{4}}},$$

$$\mu_s = \left(\frac{\sum_{j=1}^m \lambda_j \|\psi_j^s\|_2^2}{\sum_{j=1}^m \|\nabla \psi_j\|_2^2}\right)^{\frac{1}{2}}.$$

By the above scaling invariance, $\{((\psi_1^s)^{\nu_s,\mu_s}, (\psi_2^s)^{\nu_s,\mu_s}, \ldots, (\psi_m^s)^{\nu_s,\mu_s})\}$ is also a minimizing sequence. Moreover,

$$\sum_{j=1}^m \|\nabla (\psi_j^s)^{\nu_s,\mu_s}\|_2^2 = 1,$$

$$\sum_{j=1}^m \lambda_j \|(\psi_j^s)^{\nu_s,\mu_s}\|_2^2 = 1$$

for each $s \in \mathbb{N}$.

By Schwarz symmetrization,[27] one can take $(\psi_j^s)^{\nu_s,\mu_s}(x) = (\psi_j^s)^{\nu_s,\mu_s}(|x|)$. According to the properties of symmetrization, the sequence of radial functions is also a minimizing sequence and is bounded in $\mathcal{H}^{(m)}$. Therefore, there exist $(\psi_1^*, \ldots, \psi_m^*) \in \mathcal{H}^{(m)}$ and a subsequence, denoted $\{((\psi_1^s)^{\nu_s,\mu_s}, \ldots, (\psi_m^s)^{\nu_s,\mu_s})\}$, such that

$$((\psi_1^s)^{\nu_s,\mu_s}, \ldots, (\psi_m^s)^{\nu_s,\mu_s}) \rightharpoonup (\psi_1^*, \ldots, \psi_m^*)$$

in $\mathcal{H}^{(m)}$. For $p < N/(N-2)$,

$$((\psi_1^s)^{\nu_s,\mu_s}, \ldots, (\psi_m^s)^{\nu_s,\mu_s}) \to (\psi_1^*, \ldots, \psi_m^*)$$

in $L^{2p} \times \ldots \times L^{2p}$.

Strong convergence is implied by Strauss's Compactness Lemma.[27] Since the L^2-norm is weakly lower semi-continuous,

$$\sum_{j=1}^{m} \|\nabla \psi_j^*\|_2^2 \le 1, \qquad \sum_{j=1}^{m} \lambda_j \|\psi_j^*\|_2^2 \le 1.$$

The strong convergence in L^{2p} implies

$$P((\psi_1^s)^{\nu_s,\mu_s}, (\psi_2^s)^{\nu_s,\mu_s}, \ldots, (\psi_m^s)^{\nu_s,\mu_s}) \to P(\psi_1^*, \ldots, \psi_m^*).$$

Hence,

$$\alpha \le J(\psi_1^*, \ldots, \psi_m^*) \le \frac{1}{P(\psi_1^*, \ldots, \psi_m^*)} = \lim_{s\to\infty} J((\psi_1^s)^{\nu_s,\mu_s}, (\psi_2^s)^{\nu_s,\mu_s}, \ldots, (\psi_m^s)^{\nu_s,\mu_s}) = \alpha.$$

Therefore,

$$\left(\sum_{j=1}^{m} \|\nabla \psi_j^*\|_2^2\right)^{(p-1)\frac{N}{2}} \left(\sum_{j=1}^{m} \lambda_j \|\psi_j^*\|_2^2\right)^{\frac{N-p(N-2)}{2}} = 1,$$

and consequently

$$\sum_{j=1}^{m} \|\nabla \psi_j^*\|_2^2 = 1, \qquad \sum_{j=1}^{m} \lambda_j \|\psi_j^*\|_2^2 = 1.$$

Combined with weak convergence, one concludes that

$$((\psi_1^s)^{\nu_s,\mu_s}, \ldots, (\psi_m^s)^{\nu_s,\mu_s}) \to (\psi_1^*, \ldots, \psi_m^*)$$

in $\mathcal{H}^{(m)}$. Thus, $\alpha = J(\psi_1^*, \ldots, \psi_m^*)$ and Eq. (3.13) holds. □

Remark 3.3: It follows from the above that the components of this minimizer are nonnegative. On the other hand, some, but not all, of its components may be identically zero. For instance, if $m = 2$, $\lambda_j = 1$, and $a_{12} = a_{21} = \beta \in (a_{11}, a_{22})$ then the Euler-Lagrange equations corresponding to Eq. (3.13) do not have a positive vector solution, i.e., a solution with two positive components.[3]

Indeed, the minimal embedding constant C_{N,p,a_{jk},λ_j} can be represented in terms of minimizer $(\psi_1^*, \ldots, \psi_m^*)$ as follows.

Lemma 3.2:

$$C_{N,p,a_{jk},\lambda_j} = P(\psi_1^*, \psi_2^*, \ldots, \psi_m^*).$$

Proof: Minimizer $(\psi_1^*, \ldots, \psi_m^*)$ satisfies the following Euler-Lagrange equations:

$$-N(p-1)\Delta\psi_j^* + (N - p(N-2))\lambda_j \psi_j^* = \alpha \sum_{k=1}^{m} a_{jk}|\psi_k^*|^p |\psi_j^*|^{p-2}\psi_j^*, \tag{3.15}$$

where $j = 1, \ldots, m$. Multiplying both sides of the jth equation by ψ_j^*, integrating over $\mathbb{R}^N$ and adding the resulting equations,

$$N(p-1)\sum_{j=1}^{m} \|\nabla \psi_j^*\|_2^2 + (N - p(N-2))\sum_{j=1}^{m} \lambda_j \|\psi_j^*\|_2^2 = \alpha \sum_{j,k=1}^{m} a_{jk}\|\psi_j^*\psi_k^*\|_p^p. \tag{3.16}$$

Similarly, multiplying both sides of the jth equation by $(x \cdot \nabla\psi_j^*)$, integrating over $\mathbb{R}^N$ and adding the resulting equations,

$$(2-N)(p-1)\sum_{j=1}^{m} \|\nabla \psi_j^*\|_2^2 - [N - p(N-2)]\sum_{j=1}^{m} \lambda_j \|\psi_j^*\|_2^2 = -\frac{\alpha}{p}\sum_{j,k=1}^{m} a_{jk}\|\psi_j^*\psi_k^*\|_p^p. \tag{3.17}$$

Combining (3.16) − (3.17), $1/\alpha = P(\psi_1^*, \psi_2^*, \ldots, \psi_m^*)$. Thus, the minimal embedding constant is

$$C_{N,p,a_{jk},\lambda_j} = P(\psi_1^*, \psi_2^*, \ldots, \psi_m^*),$$

and $(\psi_1^*, \psi_2^*, \ldots, \psi_m^*)$ is minimizer of Eq. (3.13). □

Even though the minimal constant C_{N,p,a_{jk},λ_j} can be expressed in terms of $P(\psi_1^*, \psi_2^*, \ldots, \psi_m^*)$, the description of minimizers in general proves to be a difficult task. This is due to the fact that not much is known about the associated elliptic system (2.1) when $m \geq 3$ (even for the case $p = 2$). The rest of this section is devoted to this.

The minimal constant of Gagliardo-Nirenberg inequality is explicitly calculated first under conditions (P1) and (P2). As the arguments being used are slightly different for the cases $p = 2$ and $p \neq 2$, the results are presented separately.

Lemma 3.3: Let ω be the unique positive radial solution of Eq. (2.5) *and $p = 2$.*

(a) *For any $m \geq 2$, if $\beta > 0$ is sufficiently small, the minimizer of Eq.* (3.13) *has exactly one nontrivial component and the minimal constant can be computed explicitly as*

$$C_N = \min(\mu_1, \mu_2, \cdots, \mu_m\} \frac{(4-N)^{\frac{N-2}{2}}}{N^{\frac{N}{2}} \|\omega\|_2^2}.$$

(b) *In the case $m = 2$ and $\beta > \max\{\mu_1, \mu_2\}$,*

$$C_N = \frac{\mu_1 + \mu_2 - 2\beta}{\mu_1\mu_2 - \beta^2} \frac{(4-N)^{\frac{N-2}{2}}}{N^{\frac{N}{2}} \|\omega\|_2^2}.$$

Proof: **Case (a)**. According to Theorem 2.3, when $\beta > 0$ is small enough Eq. (3.6) gives the unique positive solution of system (2.1). Thus, one needs only to compare the value of J at Ψ^* with its values at the semitrivial solutions of system (2.4).

Observe

$$(\Gamma^{-1}\Lambda)_j = |\Gamma|^{-1} \prod_{k\neq j}^{m} (\tilde{\mu}_k - \tilde{\beta}) = \frac{N\alpha}{|\Gamma|(4-N)} \prod_{k\neq j}^{m} (\mu_k - \beta).$$

Using the scaling invariance of J, one obtains

$$\begin{aligned} J(\psi_1, \ldots, \psi_m) &= \frac{\left(\sum_{j=1}^m \|\nabla\psi_j\|_2^2\right)^{\frac{N}{2}} \left(\sum_{j=1}^m \|\psi_j\|_2^2\right)^{\frac{4-N}{2}}}{P(\psi_1, \psi_2, \ldots, \psi_m)} \\ &= \frac{4\left(\sum_{j=1}^m \prod_{k\neq j}^m (\mu_k - \beta)\right)^2}{\sum_{j=1}^m \mu_j \prod_{k\neq j}^m (\mu_k - \beta)^2 + 2\beta \prod_{j=1}^m (\mu_j - \beta)} \cdot \frac{\|\nabla\omega\|_2^N \|\omega\|_2^{4-N}}{\|\omega\|_4^4}. \end{aligned}$$

Denote

$$f_m(\beta) = \frac{\left(\sum_{j=1}^m \prod_{k\neq j}^m (\mu_k - \beta)\right)^2}{\sum_{j=1}^m \mu_j \prod_{k\neq j}^m (\mu_k - \beta)^2 + 2\beta \prod_{j=1}^m (\mu_j - \beta)}.$$

One can rewrite $J(\Psi^*)$ as

$$J(\psi_1, \ldots, \psi_m) = f_m(\beta) \frac{4\|\nabla\omega\|_2^N \|\omega\|_2^{4-N}}{\|\omega\|_4^4}. \tag{3.18}$$

It is easy to see that

$$f_m(0) = \left.\frac{\left(\sum_{j=1}^m \prod_{k\neq j}^m (\mu_k - \beta)\right)^2}{\sum_{j=1}^m \mu_j \prod_{k\neq j}^m (\mu_k - \beta)^2 + 2\beta \prod_{j=1}^m (\mu_j - \beta)}\right|_{\beta=0} = \sum_{j=1}^m \frac{1}{\mu_j}.$$

On the other hand, let

$$\psi_j(x) = \sqrt{\frac{4-N}{\mu_j \alpha}}\omega\left(\sqrt{\frac{4-N}{N}}x\right),$$

where $j = 1, 2, \ldots, m$.

(Notice that as $2 = p < N/(N-2)$, one has $N < 4$ and the square roots are well defined.) Each $(0, \ldots, \psi_j, \ldots, 0)$ is a critical point of J, and

$$J(\psi_j, 0) = \frac{4\|\nabla\omega\|_2^N \|\omega\|_2^{4-N}}{\mu_j \|\omega\|_4^4}. \tag{3.19}$$

Comparing Eqs. (3.18) and (3.19),

$$J(\psi_1, \ldots, \psi_m) > \max\{J(\psi_1, 0, \ldots, 0), \cdots, J(0, \ldots, 0, \psi_m)\}.$$

By iterative arguments, one sees that

$$\tau_m > \max\{\tau_{m-1}, \tau_{m-2}, \ldots, \tau_2\} > \min\{\tau_{m-1}, \tau_{m-2}, \ldots, \tau_2\} > \tau_1,$$

where $\tau_j = \min\{J(\Psi^j)\}$ such that Ψ^j solves (3.5) with exactly j nontrivial components.

Therefore, the minimal constant is given by

$$C_N = \min\{\mu_1, \mu_2, \cdots, \mu_m\}\frac{(4-N)^{\frac{N-2}{2}}}{N^{\frac{N}{2}}\|\omega\|_2^2}.$$

Case (b). When $m = 2$, the unique positive solution of system (3.5) (Ref. 24) can be represented as

$$\Psi^*(x) = \left(\sqrt{\frac{\mu_2-\beta}{\mu_1\mu_2-\beta^2}}\omega\left(\sqrt{\frac{4-N}{N}}x\right), \sqrt{\frac{\mu_1-\beta}{\mu_1\mu_2-\beta^2}}\omega\left(\sqrt{\frac{4-N}{N}}x\right)\right). \tag{3.20}$$

Thus, one needs to compare the values of J at Ψ^* with the values at semitrivial critical points. It suffices to show that $f_2(\beta) < \min\{\mu_1^{-1}, \mu_2^{-1}\}$ when $\beta > \max\{\mu_1, \mu_2\}$. This point is clear since

$$f_2(\beta) - \frac{1}{\mu_1} = \frac{(\mu_1-\beta)^2}{\mu_1(\mu_1\mu_2-\beta^2)} < 0 \text{ and}$$

$$f_2(\beta) - \frac{1}{\mu_2} = \frac{(\mu_2-\beta)^2}{\mu_2(\mu_1\mu_2-\beta^2)} < 0.$$

□

Remark 3.4: The explicit representation of C_N when β is large and $m \geq 3$ remains an open question. Since results on the uniqueness of a positive solution are not known, the comparison above cannot be used.

Lemma 3.4: Let ω be the unique solution of (2.5), $p \neq 2$. Then provided $\beta > 0$ is sufficiently small, the minimizer of Eq. (3.13) must have exactly one nonzero component. Moreover, the minimal constant in this case is given by

$$C_{N,p} = \min\{\mu_1, \ldots, \mu_m\}\frac{p(N-p(N-2))^{\frac{N(p-1)-2}{2}}}{2(N(p-1))^{\frac{N(p-1)}{2}}\|\omega\|_2^{2p-2}}. \tag{3.21}$$

Proof: Notice that from Theorem 2.4, the Eq. (2.4) has a positive vector solution Ψ^* that can be written in terms of ω, provided that $\beta > 0$ is small enough.

First, consider the case $m = 2$. Since J is scaling invariant, we evaluate J at (ψ_1^*, ψ_2^*),

$$J(\psi_1^*, \psi_2^*) = 2p(A^2+B^2)^{p-1}\frac{\|\nabla\omega\|_2^{N(p-1)} \cdot \|\omega\|_2^{N-p(N-2)}}{\|\omega\|_{2p}^{2p}}.$$

where A, B are solutions of (3.8). Denote $f_{2,p}(\beta) = (A^2 + B^2)^{p-1}$. It is clear that

$$f_{2,p}(0) = \frac{(1+t_0^2)^{p-1}}{\mu_1 + \beta t_0^p}\bigg|_{\beta=0} = \frac{(1+t_0^2)^{p-1}}{\mu_1} > \frac{1}{\mu_1}.$$

Similarly, one has $f_{2,p}(0) > 1/\mu_2$, i.e., the unique positive solution has higher energy than the semitrivial solutions. Thus, the minimizer must have one trivial component. Straightforward calculation using the known semitrivial solution gives the advertised minimal constant.

Next, the general case $m \geq 3$ follows from induction. Consider the minimizer of J with m components $(\psi_1^*, \ldots, \psi_m^*)$.

$$J(\psi_1^*, \ldots, \psi_m^*) = 2pf_{m,p}(\beta)\frac{\|\nabla\omega\|_2^{N(p-1)} \cdot \|\omega\|_2^{N-p(N-2)}}{\|\omega\|_{2p}^{2p}},$$

where $\psi_j^*(x) = A_j\omega\left(\sqrt{\frac{N-p(N-2)}{N(p-1)}}x\right)$, with the A_j's as given in Theorem 2.4, and

$$f_{m,p}(\beta) = \frac{\left(\sum_{j=1} A_j^2\right)^p}{\sum_{j=1}\mu_j A_j^{2p} + \beta\sum_{j\neq k} A_j^p A_k^p}.$$

Denote $A_j/A_1 = t_j$ for $j = 2, \ldots, m$, then

$$f_{m,p}(\beta) = \frac{\left(1+\sum_{j=2} t_j^2\right)^p}{\mu_1\left(1+\sum_{j=2} t_j^2\right) + \beta F(t_2, \ldots, t_m)},$$

where F is polynomial in the t_j's. Therefore,

$$f_{m,p}(0) = \frac{\left(1+\sum_{j=2} t_j^2\right)^{p-1}}{\mu_1} > \frac{1}{\mu_1}.$$

Explicit calculation gives $f_{m,p}(0) > \max\{1/\mu_1, \ldots, 1/\mu_m\}$. For semitrivial solutions, similar inequalities can be established in a straightforward manner. Thus by the continuity of the $f_{m,p}$ for small β, the minimizer of J has only one nonzero component, and the minimal constant takes the form (3.21). □

IV. GLOBAL EXISTENCE IN THE GLOBAL CRITICAL POWER CASE

Theorem 2.1 assures that the Cauchy problem (1.5) is well posed in the critical power case $p = 1 + 2/N$ provided the initial data are sufficiently small. We present a proof of this now.

Proof: Denote the energy functional by

$$E(\psi_1, \ldots, \psi_m) = \frac{1}{2}\sum_{j=1}^{m}\|\nabla\psi_j(t)\|_2^2 - P(\psi_1(t), \ldots, \psi_m(t)). \tag{4.1}$$

Using the minimal constant $C = C_{N,p,a_{jk},\lambda_j}$ obtained above with $p = 1 + 2/N$,

$$\sum_{j=1}^{m} \|\nabla\psi_j(t)\|_2^2 = 2E + 2P(\psi_1(t), \ldots, \psi_m(t))$$

$$\leq 2E + 2C\left(\sum_{j=1}^{m}\|\nabla\psi_j\|_2^2\right)^{(p-1)\frac{N}{2}}\left(\sum_{j=1}^{m}\lambda_j\|\psi_j\|_2^2\right)^{\frac{N-p(N-2)}{2}}$$

$$= 2E + 2C\left(\sum_{j=1}^{m}\|\nabla\psi_j\|_2^2\right)\left(\sum_{j=1}^{m}\lambda_j\|\psi_j\|_2^2\right)^{\frac{2}{N}}.$$

This implies

$$\left(1 - 2C\left(\sum_{j=1}^{m}\lambda_j\|\psi_{j0}\|_2^2\right)^{\frac{2}{N}}\right)\sum_{j=1}^{m}\|\nabla\psi_j\|_2^2 \leq 2E. \tag{4.2}$$

Therefore, if the initial data $\sum_{j=1}^{m}\lambda_j\|\psi_{j0}\|_2^2$ are chosen small enough, namely

$$\sum_{j=1}^{m}\lambda_j\|\psi_{j0}\|_2^2 < \left(\frac{1}{2C}\right)^{\frac{N}{2}},$$

the $\mathcal{H}^1$-seminorm will be uniformly bounded. A standard continuation argument (see, for example, Theorem 6.1.1 in Cazenave[7]) establishes the global existence of solution for system (1.5). □

Remark 4.1: Under conditions (P1) and (P2), the bound on the initial data is the same as that given by Weinstein.[27]

Remark 4.2: The condition in Theorem 2.1 is sharp, that is, the bound given in Theorem 2.1 is the smallest possible. This can be seen by considering the case where the initial data are exactly a solitary wave. A direct calculation shows that $E(\psi_1, \ldots, \psi_m) = 0$.

V. FINITE TIME BLOW-UP OF SOLUTIONS AND INSTABILITY RESULT FOR $p = 1 + 2/N$

For $2/N \leq \alpha < 2/(N - 2)$, $N \geq 3$, it has been proved[27] that the L^2-norms of the gradient of solutions can blow up in finite time without restriction on the initial data. More precisely, we have:

Lemma 5.1: Let f be such that $|x|f$ *and* ∇f *belong to* $L^2(\mathbb{R}^N)$*. Then f is in* $L^2(\mathbb{R}^N)$ *and the following estimate holds:*

$$\|f\|_2^2 \leq \frac{2}{N}\|\nabla f\|_2\|xf\|_2.$$

To show the $\mathcal{H}^1$-semi-norm blow-up, it suffices to show that the functional variance

$$V(t) = \int_{\mathbb{R}^N}|x|^2\sum_{j=1}^{m}|\psi_j(t,x)|^2\,\mathrm{d}x \tag{5.1}$$

vanishes as $t \to t^*$ for some $t^* < \infty$.

Lemma 5.2: Let $(\psi_1, \ldots, \psi_m)$ *be a solution to the system (1.5) on an interval I. Then for each t* $\in$ *I, the variance satisfies the following identities:*

$$V'(t) = 4\,Im\int_{\mathbb{R}^N}\sum_{j=1}^{m}(x\cdot\nabla\psi_j)\overline{\psi}_j\,\mathrm{d}x, \tag{5.2}$$

$$V''(t) = 8\int_{\mathbb{R}^N}\sum_{j=1}^{m}|\nabla\psi_j|^2\,\mathrm{d}x - \frac{4N(p-1)}{p}\int_{\mathbb{R}^N}\sum_{j,k=1}^{m}a_{jk}|\psi_j|^p|\psi_k|^p\,\mathrm{d}x. \tag{5.3}$$

Proof: Multiplying the jth equation of (1.5) by $2\overline{\psi}_j$ and examining the imaginary parts,

$$\frac{\partial}{\partial t}|\psi_j|^2 = -2\mathrm{Im}(\overline{\psi}_j\Delta\psi_j) = -2\nabla\cdot(\mathrm{Im}\overline{\psi}_j\nabla\psi_j) \tag{5.4}$$

with $j = 1, \ldots, m$. Multiplying these m equations by $|x|^2$ and integrating by parts, one arrives at Eq. (5.2).

On comparing Eqs. (5.2) and (5.3), it is clear that one needs only prove

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{Im}\int_{\mathbb{R}^N}(x\cdot\nabla\psi_j)\overline{\psi}_j\,\mathrm{d}x\right) = 2\int_{\mathbb{R}^N}|\nabla\psi_j|^2\,\mathrm{d}x - \frac{N(p-1)}{p}\int_{\mathbb{R}^N}\sum_{k=1}^{m}a_{jk}|\psi_k|^p|\psi_j|^p, \tag{5.5}$$

for $1 \le j \le m$. Fixing j and integrating by parts,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{Im}\int_{\mathbb{R}^N}(x\cdot\nabla\psi_j)\overline{\psi}_j\,\mathrm{d}x\right) &= \mathrm{Im}\int_{\mathbb{R}^N}\left((x\cdot\nabla\psi_j\overline{\psi}_{jt} + (x\cdot\nabla\psi_{jt})\overline{\psi}_j\right)\,\mathrm{d}x\\ &= \mathrm{Re}\int_{\mathbb{R}^N}i(x\cdot\nabla\overline{\psi}_j)\psi_{jt} - N\mathrm{Im}\int_{\mathbb{R}^N}\overline{\psi}_j\psi_{jt}\,\mathrm{d}x - \mathrm{Im}\int_{\mathbb{R}^N}(x\cdot\nabla\overline{\psi}_j)\psi_{jt}\\ &= 2\mathrm{Re}\int_{\mathbb{R}^N}i(x\cdot\nabla\overline{\psi}_j)\psi_{jt} + N\mathrm{Re}\int_{\mathbb{R}^N}i\overline{\psi}_j\psi_{jt}.\end{aligned} \tag{5.6}$$

It is easy to see Eq. (5.3) holds by substituting the following two equalities into Eq. (5.6):

$$\begin{aligned}\mathrm{Re}\int_{\mathbb{R}^N}i(x\cdot\nabla\overline{\psi}_j)\psi_{jt} &= \mathrm{Re}\int_{\mathbb{R}^N}(x\cdot\nabla\overline{\psi}_j)\left(-\Delta\psi_j - \sum_{k=1}^{m}a_{jk}|\psi_k|^p|\psi_j|^{p-2}\psi_j\right)\\ &= -\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla\psi_j|^2 + \frac{N}{2p}\int_{\mathbb{R}^N}\sum_{k=1}^{m}a_{jk}|\psi_k|^p|\psi_j|^p.\end{aligned}$$

$$\begin{aligned}\mathrm{Re}\int_{\mathbb{R}^N}i\,\overline{\psi}_j\psi_{jt} &= \mathrm{Re}\int_{\mathbb{R}^N}\overline{\psi}_j\left(-\Delta\psi_j - \sum_{k=1}^{m}a_{jk}|\psi_k|^p|\psi_j|^{p-2}\psi_j\right)\\ &= \int_{\mathbb{R}^N}|\nabla\psi_j|^2 - \int_{\mathbb{R}^N}\sum_{k=1}^{m}a_{jk}|\psi_k|^p|\psi_j|^p.\end{aligned}$$

□

Remark 5.1: Equation (5.3) can be rewritten as

$$\begin{aligned}V''(t) &= 8\sum_{j=1}^{m}\|\nabla\psi_j\|_2^2 - 8N(p-1)P(\psi_1,\ldots,\psi_m)\\ &= 16E - 8(N(p-1)-2)P(\psi_1,\ldots,\psi_m).\end{aligned}$$

Moreover, in the global critical power case $p = 1 + 2/N$, one has $V''(t) = 16E$.

The blow-up result for solutions of system (1.5) is given in Theorem 2.2 and is now proved.

Proof: There exists $t^* < \infty$ such that $\lim_{t\to t^*}V(t) = 0$ provided ψ_j remains in $H^1(\mathbb{R}^N)$ and $V(t)$ is defined as in Eq. (5.1). In cases (i) and (ii), it is easy to see that the claim is true. In case (iii), Lemma 5.2 implies $V''(t) \le 16E$. Integrating twice,

$$V(t) \le 8Et^2 + V'(0)t + V(0). \tag{5.7}$$

That is, V is bounded above by a quadratic function in t. Under the assumptions

$$\mathrm{Im}\int_{\mathbb{R}^N}\sum_{j=1}^{m}(x\cdot\nabla\psi_{j0})\overline{\psi}_{j0}\,\mathrm{d}x < -\sqrt{2EV(0)},$$

one has $V'(0) \le -4\sqrt{2E\cdot V(0)}$. Then (5.7) has a nonnegative minimum value, which is attained at $\tilde{t} = -V'(0)/16E$. Therefore, there exists a $t^* \le \tilde{t}$ such that the claim holds. The conclusion follows from Lemma 5.1. □

It has been shown in Sec. III that under the assumption (P1) and $\lambda = 1$, the solutions of (2.4) must have exactly one nonzero component. Theorems 2.5 and 2.6 thus follow immediately.

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