Heat Conduction on the ring: Interface problems with periodic boundary conditions

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Abstract

The classical problem of heat conduction in one dimension on a composite ring is examined. The problem is formulated using the heat equation with periodic boundary conditions. We provide an explicit solution of this problem using the Method of Fokas. The location of the interfaces is known, but neither temperature nor heat flux are prescribed there. Instead, the physical assumption of continuity at the interface is imposed.

Keywords: Method of Fokas, Heat Equation, Periodic Boundary Conditions, Interface

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1. Introduction

Interface problems for partial differential equations (PDEs) are initial boundary value problems for which the solution of an equation in one domain prescribes boundary conditions for the equations in adjacent domains. In applications, precise interface conditions follow from conservations laws. Few interface problems allow for an explicit closed-form solution using classical solution methods. Using the Fokas method [1, 2] such solutions may be constructed.

In three recent papers [3, 4, 5] this was done for the classical problem of the heat equation. In [5] the main application considered is that of heat flow in composite walls or rods while in [3, 4] the heat equation is viewed as a simplified reaction-diffusion equation describing the spreading of tumors in the brain. Problems in both finite and infinite domains were investigated in [5] and the method was compared with classical solution approaches if such exist. Here we extend the method to consider the heat equation with an interface and periodic boundary conditions. The representation formulae for the solution can be evaluated numerically, hence the problem can be solved in practice using techniques presented in [6, 7].

Solutions of PDEs with periodic boundary conditions can be easily computed using Fourier series [8] and using the method of Fokas [9]. The addition of an interface makes finding an explicit solution to

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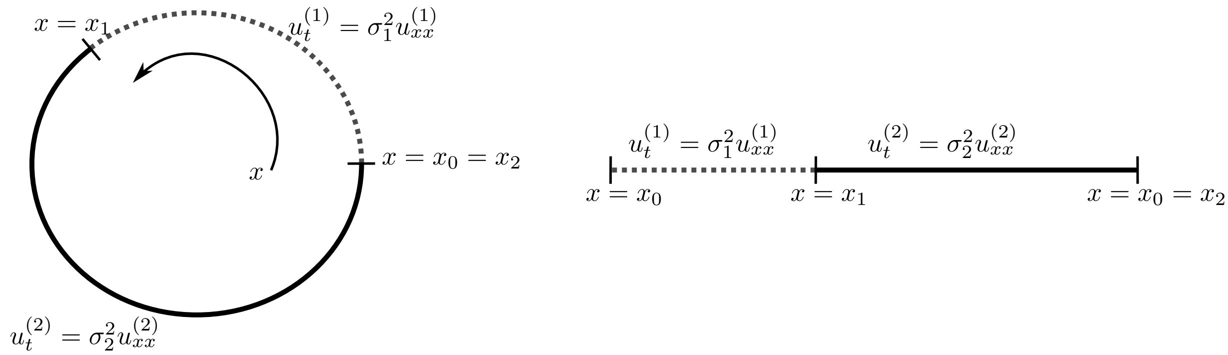

Figure 1: The heat equation with an interface posed on a ring.

this problem impossible using separation of variables. Rather, separation of variables results in a solution defined implicitly in terms of eigenvalues which satisfy a transcendental equation. In other words, although the solution is periodic, it is not useful to assume a solution of the form $\sum_{j=-\infty}^{\infty} a_j(t)e^{ijx}$ since the equation defined on the whole domain $u_t = \sigma^2(x)u_{xx}$ with $\sigma(x)$ piecewise constant is no longer diagonal in Fourier space since $\sigma(x)$ has an infinite-term Fourier series. However, the addition of an interface makes the problem only slightly more difficult when using the method of Fokas as presented in [5, 9]. Thus, the Fokas method provides an interesting alternative to this important classical problem.

2. Heat Conduction with Periodic Boundary Conditions

We consider the problem of heat conduction in a ring consisting of two different materials as in Figure 1. We seek two functions:

$$u^{(1)}(x,t), \quad x \in (x_0, x_1), \quad t \geq 0, \qquad\qquad u^{(2)}(x,t), \quad x \in (x_1, x_2), \quad t \geq 0,$$

satisfying the equations, initial, boundary, interface continuity conditions:

$$u_t^{(1)} = \sigma_1^2 u_{xx}^{(1)}, \qquad u^{(1)}(x,0) = u_0^{(1)}(x) \qquad x \in (x_0, x_1),\ t > 0, \tag{1a}$$

$$u_t^{(2)} = \sigma_2^2 u_{xx}^{(2)}, \qquad u^{(2)}(x,0) = u_0^{(2)}(x), \qquad x \in (x_1, x_2),\ t > 0, \tag{1b}$$

$$u^{(1)}(x_0,t) = u^{(2)}(x_2,t), \qquad u^{(1)}(x_1,t) = u^{(2)}(x_1,t), \qquad t > 0, \tag{1c}$$

$$\sigma_1^2 u_x^{(1)}(x_0,t) = \sigma_2^2 u_x^{(2)}(x_2,t), \qquad \sigma_1^2 u_x^{(1)}(x_1,t) = \sigma_2^2 u_x^{(2)}(x_1,t), \qquad t > 0. \tag{1d}$$

Following the Fokas method [1, 2, 5, 10] we have the local relations

$$(e^{-ikx+(\sigma_1 k)^2 t} u^{(1)})_t = (\sigma_1^2 e^{-ikx+(\sigma_1 k)^2 t}(u_x^{(1)} + iku^{(1)}))_x, \qquad x \in (x_0, x_1), \tag{2a}$$

$$(e^{-ikx+(\sigma_2 k)^2 t} u^{(2)})_t = (\sigma_2^2 e^{-ikx+(\sigma_2 k)^2 t}(u_x^{(2)} + iku^{(2)}))_x, \qquad x \in (x_1, x_2). \tag{2b}$$

Let $\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0\}$, $\mathbb{C}^- = \{z \in \mathbb{C} : \mathrm{Im}(z) \leq 0\}$. Define $D = \{k \in \mathbb{C} : \mathrm{Re}(k^2) < 0\} = D^+ \cup D^-$. We define the time transforms of the initial and boundary data and the spatial transforms of u for $k \in \mathbb{C}$ as follows:

$$
\begin{aligned}
\hat{u}_0^{(1)}(k) &= \int_{x_0}^{x_1} e^{-ikx} u_0^{(1)}(x)\,\mathrm{d}x, & \hat{u}^{(1)}(k,t) &= \int_{x_0}^{x_1} e^{-ikx} u^{(1)}(x,t)\,\mathrm{d}x,\\
\hat{u}_0^{(2)}(k) &= \int_{x_1}^{x_2} e^{-ikx} u_0^{(2)}(x)\,\mathrm{d}x, & \hat{u}^{(2)}(k,t) &= \int_{x_1}^{x_2} e^{-ikx} u^{(2)}(x,t)\,\mathrm{d}x,\\
g_0(\omega,t) &= \int_0^t e^{\omega s} u^{(1)}(x_1,s)\,\mathrm{d}s = \int_0^t e^{\omega s} u^{(2)}(x_1,s)\,\mathrm{d}s, & g_1(\omega,t) &= \int_0^t e^{\omega s} u_x^{(1)}(x_1,s)\,\mathrm{d}s = \frac{\sigma_2^2}{\sigma_1^2}\int_0^t e^{\omega s} u_x^{(2)}(x_1,s)\,\mathrm{d}s,\\
h_0(\omega,t) &= \int_0^t e^{\omega s} u^{(1)}(x_0,s)\,\mathrm{d}s = \int_0^t e^{\omega s} u^{(2)}(x_2,s)\,\mathrm{d}s, & h_1(\omega,t) &= \int_0^t e^{\omega s} u_x^{(1)}(x_0,s)\,\mathrm{d}s = \frac{\sigma_2^2}{\sigma_1^2}\int_0^t e^{\omega s} u_x^{(2)}(x_2,s)\,\mathrm{d}s.
\end{aligned}
$$

Using Green's Theorem on the domains $[x_0, x_1] \times [0,t]$, and $[x_1, x_2] \times [0,t]$ respectively, we have the global relations

$$
\begin{aligned}
e^{(\sigma_1 k)^2 t}\hat{u}^{(1)}(k,t) =&\sigma_1^2 e^{-ikx_1}\left(g_1((\sigma_1 k)^2,t) + ikg_0((\sigma_1 k)^2,t)\right)\\
&- \sigma_1^2 e^{-ikx_0}\left(h_1((\sigma_1 k)^2,t) + ikh_0((\sigma_1 k)^2,t)\right) + \hat{u}_0^{(1)}(k), \qquad (3a)\\
e^{(\sigma_2 k)^2 t}\hat{u}^{(2)}(k,t) =&e^{-ikx_2}\left(\sigma_1^2 h_1((\sigma_2 k)^2,t) + ik\sigma_2^2 h_0((\sigma_2 k)^2,t)\right)\\
&- e^{-ikx_1}\left(\sigma_1^2 g_1((\sigma_2 k)^2,t) + ik\sigma_2^2 g_0((\sigma_2 k)^2,t)\right) + \hat{u}_0^{(2)}(k). \qquad (3b)
\end{aligned}
$$

Both equations are valid for $k \in \mathbb{C}$ as is to be expected using the Fokas Method in bounded domains. Inverting the Fourier transforms in (3a)

$$
\begin{aligned}
u^{(1)}(x,t) =&\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-(\sigma_1 k)^2 t}\hat{u}_0^{(1)}(k)\,\mathrm{d}k\\
&+ \frac{\sigma_1^2}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x_1)-(\sigma_1 k)^2 t}(g_1((\sigma_1 k)^2,t) + ikg_0((\sigma_1 k)^2,t))\,\mathrm{d}k \qquad (4)\\
&- \frac{\sigma_1^2}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x_0)-(\sigma_1 k)^2 t}(h_1((\sigma_1 k)^2,t) + ikh_0((\sigma_1 k)^2,t))\,\mathrm{d}k.
\end{aligned}
$$

The integrand of the second integral is entire and decays as $k \to \infty$ for $k \in \mathbb{C}^- \setminus D^-$. The third integral has an integrand that is entire and decays as $k \to \infty$ for $k \in \mathbb{C}^+ \setminus D^+$. It is convenient to deform both contours away from $k = 0$ to avoid singularities in the integrands below. Initially, these singularities are removable, since the integrands are entire. Writing integrals of sums as sums of integrals, the singularities may, and do, cease to be removable. With the deformations away from $k = 0$, the apparent singularities are no cause for concern. In other words, we deform D^+ to D_0^+ and D^- to D_0^- as shown in Figure 2. Thus

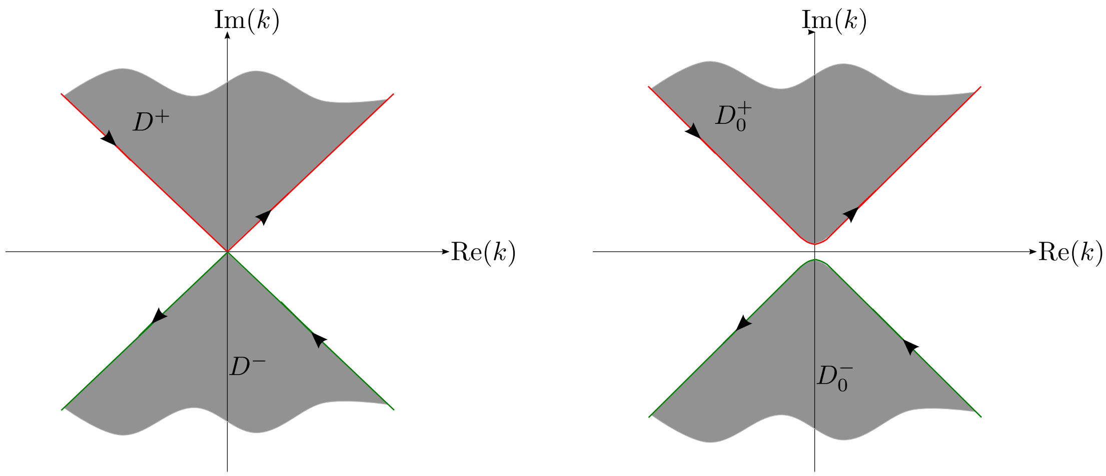

Figure 2: The domains D^+ and D^- as well as D_0^+ and D_0^- which are used for integration in (5) and (6).

$$\begin{aligned} u^{(1)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-(\sigma_1 k)^2 t}\hat{u}_0^{(1)}(k)\,\mathrm{d}k \\ &- \frac{\sigma_1^2}{2\pi}\int_{\partial D_0^-} e^{ik(x-x_1)-(\sigma_1 k)^2 t}(g_1((\sigma_1 k)^2,t) + ikg_0((\sigma_1 k)^2,t))\,\mathrm{d}k \\ &- \frac{\sigma_1^2}{2\pi}\int_{\partial D_0^+} e^{ik(x-x_0)-(\sigma_1 k)^2 t}(h_1((\sigma_1 k)^2,t) + ikh_0((\sigma_1 k)^2,t))\,\mathrm{d}k. \end{aligned} \tag{5}$$

To obtain the solution $u_2(x,t)$ for $x \in (x_1, x_2)$ we apply the inverse Fourier transform to (3b) and again deform where appropriate to find

$$\begin{aligned} u^{(2)}(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-(\sigma_2 k)^2 t}\hat{u}_0^{(2)}(k)\,\mathrm{d}k \\ &- \frac{1}{2\pi}\int_{\partial D_0^+} e^{ik(x-x_1)-(\sigma_2 k)^2 t}(\sigma_1^2 g_1((\sigma_2 k)^2,t) + ik\sigma_2^2 g_0((\sigma_2 k)^2,t))\,\mathrm{d}k \\ &- \frac{1}{2\pi}\int_{\partial D_0^-} e^{ik(x-x_2)-(\sigma_2 k)^2 t}(\sigma_1^2 h_1((\sigma_1 k)^2,t) + ik\sigma_2^2 h_0((\sigma_2 k)^2,t))\,\mathrm{d}k. \end{aligned} \tag{6}$$

The pair (3) and their evaluation at $-k$ (using the invariance of $\omega_1(k)$ and $\omega_2(k)$ under $k \to -k$) give four equations to solve for four unknowns, $g_0(\omega,t), g_1(\omega,t), h_0(\omega,t)$, and $h_1(\omega,t)$ where one must be careful to use all the symmetries of the set of dispersion relations, namely $k \to -k$, $k \to k\sigma_1/\sigma_2$ and $k \to k\sigma_2/\sigma_1$. Substituting these expressions into (5) we have equations for $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ which involve $\hat{u}^{(1)}(k,t)$ and $\hat{u}^{(2)}(k,t)$ evaluated at a variety of arguments but without the factor $e^{\omega t}$. Such integrands decay in the regions around whose boundaries they are integrated. Making extensive use of Jordan's Lemma and Cauchy's Theorem, these integrals are shown to vanish. Thus the final solution is given by

$$
\begin{aligned}
u^{(1)}(x,t)=&\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega_1 t}\hat{u}_0^{(1)}(k)\mathrm{d}k\\
&+\int_{\partial D_0^-}\frac{e^{ikx-\omega_1 t}}{2\Delta_1(k)}\left((\sigma_1+\sigma_2)^2 e^{ik(x_0-x_1)+\frac{ik\sigma_1}{\sigma_2}(x_1-x_2)}-4\sigma_1\sigma_2-(\sigma_1-\sigma_2)^2 e^{ik(x_0-x_1)+\frac{ik\sigma_1}{\sigma_2}(x_2-x_1)}\right)\hat{u}_0^{(1)}(k)\mathrm{d}k\\
&+\int_{\partial D_0^-}\frac{(\sigma_1^2-\sigma_2^2)e^{ik(x-x_0-x_1)-\omega_1 t}}{2\Delta_1(k)}\left(e^{\frac{ik\sigma_1}{\sigma_2}(x_1-x_2)}-e^{\frac{ik\sigma_1}{\sigma_2}(x_2-x_1)}\right)\hat{u}_0^{(1)}(-k)\mathrm{d}k\\
&+\int_{\partial D_0^-}\frac{\sigma_1(\sigma_1+\sigma_2)e^{ikx-\omega_1 t}}{\Delta_1(k)}\left(e^{-ikx_1+\frac{ik\sigma_1}{\sigma_2}x_1}-e^{-ikx_0+\frac{ik\sigma_1}{\sigma_2}x_2}\right)\hat{u}_0^{(2)}\left(\frac{k\sigma_1}{\sigma_2}\right)\mathrm{d}k\\
&+\int_{\partial D_0^-}\frac{\sigma_1(\sigma_1-\sigma_2)e^{ikx-\omega_1 t}}{\Delta_1(k)}\left(e^{-ikx_0-\frac{ik\sigma_1}{\sigma_2}x_2}-e^{-ikx_1-\frac{ik\sigma_1}{\sigma_2}x_1}\right)\hat{u}_0^{(2)}\left(-\frac{k\sigma_1}{\sigma_2}\right)\mathrm{d}k\\
&+\int_{\partial D_0^+}\frac{e^{ikx-\omega_1 t}}{2\Delta_1(k)}\left((\sigma_1-\sigma_2)^2 e^{ik(x_1-x_0)+\frac{ik\sigma_1}{\sigma_2}(x_2-x_1)}+4\sigma_1\sigma_2-(\sigma_1+\sigma_2)^2 e^{ik(x_0-x_1)+\frac{ik\sigma_1}{\sigma_2}(x_2-x_1)}\right)\hat{u}_0^{(1)}(k)\mathrm{d}k\\
&+\int_{\partial D_0^+}\frac{(\sigma_1^2-\sigma_2^2)e^{ik(x-x_0-x_1)-\omega_1 t}}{2\Delta_1(k)}\left(e^{\frac{ik\sigma_1}{\sigma_2}(x_1-x_2)}-e^{\frac{ik\sigma_1}{\sigma_2}(x_2-x_1)}\right)\hat{u}_0^{(1)}(-k)\mathrm{d}k\\
&+\int_{\partial D_0^+}\frac{\sigma_1(\sigma_1+\sigma_2)e^{ikx-\omega_1 t}}{\Delta_1(k)}\left(e^{-ikx_1+\frac{ik\sigma_1}{\sigma_2}x_1}-e^{-ikx_0+\frac{ik\sigma_1}{\sigma_2}x_2}\right)\hat{u}_0^{(2)}\left(\frac{k\sigma_1}{\sigma_2}\right)\mathrm{d}k\\
&+\int_{\partial D_0^+}\frac{\sigma_1(\sigma_2-\sigma_1)e^{ikx-\omega_1 t}}{\Delta_1(k)}\left(e^{-ikx_0-\frac{ik\sigma_1}{\sigma_2}x_2}-e^{-ikx_1-\frac{ik\sigma_1}{\sigma_2}x_1}\right)\hat{u}_0^{(2)}\left(-\frac{k\sigma_1}{\sigma_2}\right)\mathrm{d}k,
\end{aligned} \tag{7}
$$

for $x \in (x_0, x_1)$ where

$$
\begin{aligned}
\Delta_1(k) =&\pi\left(\sigma_1\left(e^{-ik\frac{\sigma_1}{\sigma_2}x_1}-e^{-ik\frac{\sigma_1}{\sigma_2}x_2}\right)\left(e^{-ikx_0}+e^{-ikx_1}\right)+\sigma_2\left(e^{-ik\frac{\sigma_1}{\sigma_2}x_1}+e^{-ik\frac{\sigma_1}{\sigma_2}x_2}\right)\left(e^{-ikx_0}-e^{-ikx_1}\right)\right)*\\
&\left(\sigma_1\left(e^{-ik\frac{\sigma_1}{\sigma_2}x_1}+e^{-ik\frac{\sigma_1}{\sigma_2}x_2}\right)\left(e^{-ikx_0}-e^{-ikx_1}\right)+\sigma_2\left(e^{-ik\frac{\sigma_1}{\sigma_2}x_1}-e^{-ik\frac{\sigma_1}{\sigma_2}x_2}\right)\left(e^{-ikx_0}+e^{-ikx_1}\right)\right).
\end{aligned}
$$

For $x \in (x_1, x_2)$, the solution $u^{(2)}(x,t)$ is found by switching (1) and (2) on σ_1, σ_2, $u^{(1)}(\cdot)$ and $u^{(2)}(\cdot)$, replacing $\Delta_1(k)$ with $\Delta_2(k) = -\Delta_1(k\sigma_2/\sigma_1)$, and interchanging the integration paths $\int_{D_0^+}$ and $-\int_{D_0^-}$.

Note that $\Delta_1(k) = 0$ whenever $k = 0$, $\cot\left(\frac{k(x_0-x_1)}{2}\right)\tan\left(\frac{k\sigma_1(x_1-x_2)}{2\sigma_2}\right) = -\frac{\sigma_2}{\sigma_1}$, or $\cot\left(\frac{k\sigma_1(x_1-x_2)}{2\sigma_2}\right)\tan\left(\frac{k(x_0-x_1)}{2}\right) = -\frac{\sigma_2}{\sigma_1}$ are satisfied. Observe that $\Delta_j(k) = 0$ only for real values of k for $j = 1, 2$. Thus, through the deformation to D_0^+ and D_0^- we have avoided any singularities and, on the contours, the quantities needed are evaluated without problem.

We have solved the classical problem of heat conduction on a ring with two domains using the Method of Fokas. It is straightforward to generalize this work to the problem of n domains with periodic boundary conditions by combining what was done in [5] and [3, 4] for the heat equation with multiple domains, with what we present here for periodic boundary conditions.

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