

Short-wave transverse instabilities of line solitons of the 2-D hyperbolic nonlinear Schrödinger equation

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Abstract

We prove that line solitons of the two-dimensional hyperbolic nonlinear Schrödinger equation are unstable with respect to transverse perturbations of arbitrarily small periods, *i.e.*, short waves. The analysis is based on the construction of Jost functions for the continuous spectrum of Schrödinger operators, the Sommerfeld radiation conditions, and the Lyapunov–Schmidt decomposition. Precise asymptotic expressions for the instability growth rate are derived in the limit of short periods.

1 Introduction

Transverse instabilities of line solitons have been studied in many nonlinear evolution equations (see the pioneering work [14] and the review article [10]). In particular, this problem has been studied in the context of the hyperbolic nonlinear Schrödinger (NLS) equation

$$i\psi_t + \psi_{xx} - \psi_{yy} + 2|\psi|^2\psi = 0, \quad (1)$$

which models oceanic wave packets in deep water. Solitary waves of the one-dimensional (y -independent) NLS equation exist in closed form. If all parameters of a solitary wave have been removed by using the translational and scaling invariance, we can consider the one-dimensional trivial-phase solitary wave in the simple form $\psi = \text{sech}(x)e^{it}$. Adding a small perturbation $e^{i\rho y + \lambda t + it}(U(x) + iV(x))$ to the one-dimensional solitary wave and linearizing the underlying equations, we obtain the coupled spectral stability problem

$$(L_+ - \rho^2)U = -\lambda V, \quad (L_- - \rho^2)V = \lambda U, \quad (2)$$

where λ is the spectral parameter, ρ is the transverse wave number of the small perturbation, and L_{\pm} are given by the Schrödinger operators

$$L_+ = -\partial_x^2 + 1 - 6\text{sech}^2(x), \quad L_- = -\partial_x^2 + 1 - 2\text{sech}^2(x).$$

Note that small ρ corresponds to long-wave perturbations in the transverse directions, while large ρ corresponds to short-wave transverse perturbations.

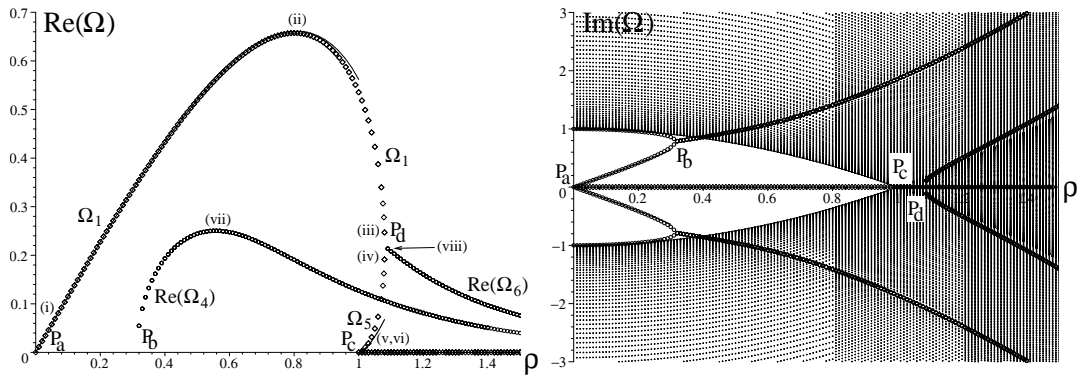


Figure 1: Numerical computations of the real (left panel) and imaginary (right panel) parts of the isolated eigenvalues and the continuous spectrum of the spectral stability problem (2) versus the transverse wave number ρ . Reprinted from [5].

Numerical approximations of unstable eigenvalues (positive real part) of the spectral stability problem (2) were computed in our previous work [5] and reproduced recently by independent numerical computations in [13, Fig. 5.27] and [3, Fig. 2]. Fig. 2 from [5] is reprinted here as Figure 1. The figure illustrates various bifurcations at P_a , P_b , P_c , and P_d , as well as the behavior of eigenvalues and the continuous spectrum in the spectral stability problem (2) as a function of the transverse wave number ρ .

An asymptotic argument for the presence of a real unstable eigenvalue bifurcating at P_a for small values of ρ was given in the pioneering paper [14]. The Hamiltonian Hopf bifurcation of a complex quartet at P_b for $\rho \approx 0.31$ was explained in [5] based on the negative index theory. That paper also proved the bifurcation of a new unstable real eigenvalue at P_c for $\rho > 1$, using Evans function methods. What is left in this puzzle is an argument for the existence of unstable eigenvalues for arbitrarily large values of ρ . This is the problem addressed in the present paper.

The motivation to develop a proof of the existence of unstable eigenvalues for large values of ρ originates from different physical experiments (both old and new). First, Ablowitz and Segur [1] predicted there are no instabilities in the limit of large ρ and referred to water wave experiments done in narrow wave tanks by J. Hammack at the University of Florida in 1979, which showed good agreement with the dynamics of the one-dimensional NLS equation. Observation of one-dimensional NLS solitons in this limit seems to exclude transverse instabilities of line solitons.

Second, experimental observations of transverse instabilities are quite robust in the context of nonlinear laser optics via a four-wave mixing interaction. Gorza *et al.* [6] observed the primary snake-type instability of line solitons at P_a for small values of ρ as well as the persistence of the instabilities for large values of ρ . Recently, Gorza *et al.* [7] demonstrated experimentally the presence of the secondary neck-type instability that bifurcates at P_b near $\rho \approx 0.31$.

In a different physical context of solitary waves in PT -symmetric waveguides, results on the transverse instability of line solitons were re-discovered by Alexeeva *et al.* [3]. (The authors of [3] did not notice that their mathematical problem is identical to the one for transverse instability of line solitons in the hyperbolic NLS equation.) Appendix B in [3] contains asymptotic results suggesting that if there are unstable eigenvalues of the spectral problem (2) in the limit of large ρ , the instability growth rate is exponentially small in terms of the large parameter ρ . No evidence

to the fact that these eigenvalues have *nonzero* instability growth rate was reported in [3].

Finally and even more recently, similar instabilities of line solitons in the hyperbolic NLS equation (1) were observed numerically in the context of the discrete nonlinear Schrödinger equation away from the anti-continuum limit [12].

The rest of this article is organized as follows. Section 2 presents our main results. Section 3 gives the analytical proof of the main theorem. Section 4 is devoted to computations of the precise asymptotic formula for the unstable eigenvalues of the spectral stability problem (2) in the limit of large values of ρ . Section 5 summarizes our findings and discusses further problems.

2 Main results

To study the transverse instability of line solitons in the limit of large ρ , we cast the spectral stability problem (2) in the semi-classical form by using the transformation

$$\rho^2 = 1 + \frac{1}{\epsilon^2}, \quad \lambda = \frac{i\omega}{\epsilon^2},$$

where ϵ is a small parameter. The spectral problem (2) is rewritten in the form

$$\begin{cases} (-\epsilon^2 \partial_x^2 - 1 - 6\epsilon^2 \operatorname{sech}^2(x)) U = -i\omega V, \\ (-\epsilon^2 \partial_x^2 - 1 - 2\epsilon^2 \operatorname{sech}^2(x)) V = i\omega U. \end{cases} \quad (3)$$

Note that we are especially interested in the spectrum of this problem for $\epsilon \rightarrow 0$, which corresponds to $\rho \rightarrow \infty$ in the original problem. Also, the real part of λ , which determines the instability growth rate for (2) corresponds, up to a factor of ϵ^2 , to the imaginary part of ω .

Next, we introduce new dependent variables which are more suitable for working with continuous spectrum for real values of ω :

$$\varphi := U + iV, \quad \psi := U - iV.$$

Note that φ and ψ are not generally complex conjugates of each other because U and V may be complex valued since the spectral problem (3) is not self-adjoint. The spectral problem (3) is rewritten in the form

$$\begin{cases} (-\epsilon^2 \partial_x^2 + \omega - 1 - 4\epsilon^2 \operatorname{sech}^2(x)) \varphi - 2\epsilon^2 \operatorname{sech}^2(x) \psi = 0, \\ (-\epsilon^2 \partial_x^2 - \omega - 1 - 4\epsilon^2 \operatorname{sech}^2(x)) \psi - 2\epsilon^2 \operatorname{sech}^2(x) \varphi = 0. \end{cases} \quad (4)$$

We note that the Schrödinger operator

$$L_0 = -\partial_x^2 - 4\operatorname{sech}^2(x) \quad (5)$$

admits exactly two eigenvalues of the discrete spectrum located at $-E_0$ and $-E_1$ [11], where

$$E_0 = \left(\frac{\sqrt{17}-1}{2} \right)^2, \quad E_1 = \left(\frac{\sqrt{17}-3}{2} \right)^2. \quad (6)$$

The associated eigenfunctions are

$$\varphi_0 = \operatorname{sech}^{\sqrt{E_0}}(x), \quad \varphi_1 = \tanh(x) \operatorname{sech}^{\sqrt{E_1}}(x). \quad (7)$$

In the neighborhood of each of these eigenvalues, one can construct a perturbation expansion for exponentially decaying eigenfunction pairs (φ, ψ) and a quartet of complex eigenvalues ω of the original spectral problem (4). This idea appears already in Appendix B of [3], where formal perturbation expansions are developed in powers of ϵ .

Note that the perturbation expansion for the spectral stability problem (4) is not a standard application of the Lyapunov–Schmidt reduction method [4] because the eigenvalues of the limiting problem given by the operator L_0 are embedded into a branch of the continuous spectrum. Therefore, to justify the perturbation expansions and to derive the main result, we need a perturbation theory that involves Fermi’s Golden Rule [9]. An alternative version of this perturbation theory can use the analytic continuation of the Evans function across the continuous spectrum, similar to the one in [5]. Additionally, one can think of semi-classical methods like WKB theory to be suitable for applications to this problem [2].

The main results of this paper are as follows. To formulate the statements, we are using the notation $|a| \lesssim \epsilon$ to indicate that for sufficiently small positive values of ϵ , there is an ϵ -independent positive constant C such that $|a| \leq C\epsilon$. Also, $H^2(\mathbb{R})$ denotes the standard Sobolov space of distributions whose derivatives up to order two are square integrable.

Theorem 1. *For sufficiently small $\epsilon > 0$, there exist two quartets of complex eigenvalues $\{\omega, \bar{\omega}, -\omega, -\bar{\omega}\}$ in the spectral problem (4) associated with the eigenvectors (φ, ψ) in $H^2(\mathbb{R})$.*

Let $(-E_0, \varphi_0)$ be one of the two eigenvalue–eigenvector pairs of the operator L_0 in (5). There exists an $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, the complex eigenvalue ω in the first quadrant and its associated eigenfunction satisfy

$$|\omega - 1 - \epsilon^2 E_0| \lesssim \epsilon^3, \quad \|\varphi - \varphi_0\|_{L^2} \lesssim \epsilon, \quad \|\psi\|_{L^\infty} \lesssim \epsilon, \quad (8)$$

while the positive value of $\text{Im}(\omega)$ is exponentially small in ϵ .

Proposition 1. *Besides the two quartets of complex eigenvalues in Theorem 1, no other eigenvalues of the spectral problem (4) exist for sufficiently small $\epsilon > 0$.*

Proposition 2. *The instability growth rates for the two complex quartets of eigenvalues in Theorem 1 are given explicitly as $\epsilon \rightarrow 0$ by*

$$\text{Re}(\lambda) = \frac{\text{Im}(\omega)}{\epsilon^2} \sim \frac{2^{p+\frac{3}{2}}\pi^2}{[\Gamma(p)]^2} \epsilon^{3-2p} e^{-\frac{\sqrt{2}\pi}{\epsilon}}, \quad \text{Re}(\lambda) = \frac{\text{Im}(\omega)}{\epsilon^2} \sim \frac{2^{p+\frac{5}{2}}\pi^2}{q^2[\Gamma(q)]^2} \epsilon^{1-2q} e^{-\frac{\sqrt{2}\pi}{\epsilon}}, \quad (9)$$

where $p = 2 + \sqrt{E_0}$ and $q = 2 + \sqrt{E_1}$.

Note that the result of Theorem 1 guarantees that the two quartets of complex eigenvalues that we can see on Figure 1 remain unstable for all large values of the transverse wave number ρ in the spectral stability problem (2).

3 Proof of Theorem 1

By the symmetry of the problem, we need to prove Theorem 1 only for one eigenvalue of each complex quartet, *e.g.*, for ω in the first quadrant of the complex plane. Let $\omega = 1 + \epsilon^2 E$ and rewrite the spectral problem (4) in the equivalent form

$$\begin{aligned} (-\partial_x^2 - 4\text{sech}^2(x))\varphi - 2\text{sech}^2(x)\psi &= -E\varphi, \\ -2\psi - \epsilon^2(\partial_x^2 + E + 4\text{sech}^2(x))\psi &= 2\epsilon^2\text{sech}^2(x)\varphi. \end{aligned} \quad (10)$$

At the leading order, the first equation of system (10) has exponentially decaying eigenfunctions (7) for $E = E_0$ and $E = E_1$ in (6). However, the second equation of system (10) does not admit exponentially decaying eigenfunctions for these values of E because the operator

$$L_\epsilon(E) := -2 - \epsilon^2 (\partial_x^2 + E + 4\text{sech}^2(x))$$

is not invertible for these values of E . The scattering problem for Jost functions associated with the continuous spectrum of the operator $L_\epsilon(E)$ admits solutions that behave at infinity as

$$\psi(x) \sim e^{ikx}, \quad \text{where} \quad k^2 = E + \frac{2}{\epsilon^2}.$$

If $\text{Im}(E) > 0$, then $\text{Re}(k)\text{Im}(k) > 0$. The Sommerfeld radiation conditions $\psi(x) \sim e^{\pm ikx}$ as $x \rightarrow \pm\infty$ correspond to solutions $\psi(x)$ that are exponentially decaying in x when k is extended from real positive values for $\text{Im}(E) = 0$ to complex values with $\text{Im}(k) > 0$ for $\text{Im}(E) > 0$. Thus we impose Sommerfeld boundary conditions for the component ψ satisfying the spectral problem (10):

$$\psi(x) \rightarrow a \begin{cases} e^{ikx}, & x \rightarrow \infty, \\ \sigma e^{-ikx}, & x \rightarrow -\infty, \end{cases} \quad k = \frac{1}{\epsilon} \sqrt{2 + \epsilon^2 E}, \quad (11)$$

where a is the radiation tail amplitude to be determined and $\sigma = \pm 1$ depends on whether ψ is even or odd in x . To compute a , we note the following elementary result.

Lemma 1. *Consider bounded (in $L^\infty(\mathbb{R})$) solutions $\psi(x)$ of the second-order differential equation*

$$\psi'' + k^2\psi = f, \quad (12)$$

where $k \in \mathbb{C}$ with $\text{Re}(k) > 0$ and $\text{Im}(k) \geq 0$, whereas $f \in L^1(\mathbb{R})$ is a given function, either even or odd. Then

$$\psi(x) = \frac{1}{2ik} \int_{-\infty}^x e^{ik(x-y)} f(y) dy + \frac{1}{2ik} \int_x^{+\infty} e^{-ik(x-y)} f(y) dy \quad (13)$$

is the unique solution of the differential equation (12) with the same parity as f that satisfies the Sommerfeld radiation conditions (11) with

$$a = \frac{1}{2ik} \int_{-\infty}^{+\infty} f(y) e^{-iky} dy. \quad (14)$$

Proof. Solving (12) using variation of parameters, we obtain

$$\psi(x) = e^{ikx} \left[u(0) + \frac{1}{2ik} \int_0^x f(y) e^{-iky} dy \right] + e^{-ikx} \left[v(0) - \frac{1}{2ik} \int_0^x f(y) e^{iky} dy \right],$$

where $u(0)$ and $v(0)$ are arbitrary constants. We fix these constants using the Sommerfeld radiation conditions (11), which yields

$$u(0) = \frac{1}{2ik} \int_{-\infty}^0 f(y) e^{-iky} dy, \quad v(0) = \frac{1}{2ik} \int_0^{+\infty} f(y) e^{iky} dy.$$

Using these expressions and the definition $a = \lim_{x \rightarrow \infty} \psi(x) e^{-ikx}$, we obtain (13) and (14). It is easily checked that ψ has the same parity as f . \square

To prove Theorem 1, we select one of the two eigenvalue–eigenvector pairs (E_0, φ_0) of the operator L_0 in (5) and proceed with the Lyapunov–Schmidt decomposition

$$E = E_0 + \mathcal{E}, \quad \varphi = \varphi_0 + \phi, \quad \phi \perp \varphi_0.$$

To simplify calculations, we assume that φ_0 is normalized to unity in the L^2 norm. The orthogonality condition $\phi \perp \varphi_0$ is used with respect to the inner product in $L^2(\mathbb{R})$ and $\phi \in L^2(\mathbb{R})$ is assumed in the decomposition.

The spectral problem (10) is rewritten in the form

$$\begin{aligned} (L_0 + E_0)\phi - 2\operatorname{sech}^2(x)\psi &= -\mathcal{E}(\varphi_0 + \phi), \\ L_\epsilon(E_0 + \mathcal{E})\psi &= 2\epsilon^2\operatorname{sech}^2(x)(\varphi_0 + \phi). \end{aligned} \quad (15)$$

Because $\phi \perp \varphi_0$, the correction term \mathcal{E} is uniquely determined by projecting the first equation of the system (15) onto φ_0 :

$$\mathcal{E} = 2 \int_{-\infty}^{\infty} \operatorname{sech}^2(x)\varphi_0(x)\psi(x)dx. \quad (16)$$

If $\psi \in L^\infty(\mathbb{R})$, then $|\mathcal{E}| = \mathcal{O}(\|\psi\|_{L^\infty})$. Let P be the orthogonal projection from $L^2(\mathbb{R})$ to the range of $(L_0 + E_0)$. Then, ϕ is uniquely determined from the linear inhomogeneous equation

$$P(L_0 + E_0 + \mathcal{E})P\phi = 2\operatorname{sech}^2(x)\psi - 2\varphi_0 \int_{-\infty}^{\infty} \operatorname{sech}^2(x)\varphi_0(x)\psi(x)dx, \quad (17)$$

where $P(L_0 + E_0)P$ is invertible with a bounded inverse and $\psi \in L^\infty(\mathbb{R})$ is assumed. On the other hand, $\psi \in L^\infty(\mathbb{R})$ is uniquely found using the linear inhomogeneous equation

$$\psi'' + k^2\psi = f, \quad \text{where } f = -2\operatorname{sech}^2(x)(\varphi_0 + \phi + 2\psi), \quad (18)$$

subject to the Sommerfeld radiation condition (11), where $\phi \in L^\infty(\mathbb{R})$ is assumed. Note that ψ is not real because of the Sommerfeld radiation condition (11) and depends on ϵ because of the ϵ -dependence of k in

$$k = \frac{1}{\epsilon} \sqrt{2 + \epsilon^2 E_0 + \epsilon^2 \mathcal{E}}. \quad (19)$$

We are now ready to prove Theorem 1.

Proof of Theorem 1. The function f on the right-hand-side of (18) is exponentially decaying as $|x| \rightarrow \infty$ if $\phi, \psi \in L^\infty(\mathbb{R})$. From the solution (13), we rewrite the equation into the integral form

$$\begin{aligned} \psi(x) &= \frac{i\epsilon}{\sqrt{2 + \epsilon^2 E_0 + \epsilon^2 \mathcal{E}}} \int_{-\infty}^x e^{ik(x-y)} \operatorname{sech}^2(y)(\varphi_0 + \phi + 2\psi)(y)dy \\ &\quad + \frac{i\epsilon}{\sqrt{2 + \epsilon^2 E_0 + \epsilon^2 \mathcal{E}}} \int_x^{+\infty} e^{-ik(x-y)} \operatorname{sech}^2(y)(\varphi_0 + \phi + 2\psi)(y)dy. \end{aligned} \quad (20)$$

The right-hand-side operator acting on $\psi \in L^\infty(\mathbb{R})$ is a contraction for small values of ϵ if $\phi \in L^\infty(\mathbb{R})$ and $\mathcal{E} \in \mathbb{C}$ are bounded as $\epsilon \rightarrow 0$, and for $\operatorname{Im}(\mathcal{E}) \geq 0$ (yielding $\operatorname{Im}(k) \geq 0$). By the Fixed Point Theorem [4], we have a unique solution $\psi \in L^\infty(\mathbb{R})$ of the integral equation (20) for small values of ϵ such that $\|\psi\|_{L^\infty} = \mathcal{O}(\epsilon)$ as $\epsilon \rightarrow 0$. This solution can be substituted into the inhomogeneous equation (17).

Since $|\mathcal{E}| = \mathcal{O}(\|\psi\|_{L^\infty}) = \mathcal{O}(\epsilon)$ as $\epsilon \rightarrow 0$ and the operator $P(L_0 + E_0)P$ is invertible with a bounded inverse, we apply the Implicit Function Theorem and obtain a unique solution $\phi \in H^2(\mathbb{R})$ of the inhomogeneous equation (17) for small values of ϵ such that $\|\phi\|_{H^2} = \mathcal{O}(\epsilon)$ as $\epsilon \rightarrow 0$. Note that by Sobolev embedding of $H^2(\mathbb{R})$ to $L^\infty(\mathbb{R})$, the earlier assumption $\phi \in L^\infty(\mathbb{R})$ for finding $\psi \in L^\infty(\mathbb{R})$ in (18) is consistent with the solution $\phi \in H^2(\mathbb{R})$.

This proves bounds (8). It remains to show that $\text{Im}(\mathcal{E}) > 0$ for small nonzero values of ϵ . If so, then the real eigenvalue $1 + \epsilon^2 E_0$ bifurcates to the first complex quadrant and yields the eigenvalue $\omega = 1 + \epsilon^2 E_0 + \epsilon^2 \mathcal{E}$ of the spectral problem (4) with $\text{Im}(\omega) > 0$. Persistence of such an isolated eigenvalue with respect to small values of ϵ follows from regular perturbation theory. Also, the eigenfunction ψ in (20) is exponentially decaying in x at infinity if $\text{Im}(\mathcal{E}) > 0$. As a result, the eigenvector (ϕ, ψ) is defined in $H^2(\mathbb{R})$ for small nonzero values of ϵ , although $\|\psi\|_{H^2}$ diverges as $\epsilon \rightarrow 0$.

To prove that $\text{Im}(\mathcal{E}) > 0$ for small but nonzero values of ϵ , we use (11) and (18), integrate by parts, and obtain the exact relation

$$\begin{aligned} 2 \int_{-\infty}^{\infty} \text{sech}^2(x)(\varphi_0 + \phi)\psi(x)dx &= \int_{-\infty}^{\infty} \bar{\psi}(x) (-\partial_x^2 - k^2 - 4\text{sech}^2(x)) \psi(x)dx \\ &= \left. (-\bar{\psi}\psi_x + \bar{\psi}_x\psi) \right|_{x \rightarrow -\infty}^{x \rightarrow +\infty} \\ &\quad + \int_{-\infty}^{\infty} \psi(x) (-\partial_x^2 - k^2 - 4\text{sech}^2(x)) \bar{\psi}(x)dx \\ &= 4ik|a(\epsilon)|^2 + 2 \int_{-\infty}^{\infty} \text{sech}^2(x)(\varphi_0 + \phi)\bar{\psi}(x)dx. \end{aligned}$$

By using bounds (8), definition (14), and projection (16), we obtain

$$\begin{aligned} \text{Im}(\mathcal{E}) &= 2\text{Im} \int_{-\infty}^{\infty} \text{sech}^2(x)\varphi_0(x)\psi(x)dx = 2k|a(\epsilon)|^2 (1 + \mathcal{O}(\epsilon)) \\ &= \frac{2}{k} \left| \int_{-\infty}^{+\infty} \text{sech}^2(x)\varphi_0(x)e^{-ikx} dx \right|^2 (1 + \mathcal{O}(\epsilon)), \end{aligned} \tag{21}$$

which is strictly positive. Note that this expression is referred to as Fermi's Golden Rule in quantum mechanics [9]. Since $k = \mathcal{O}(\epsilon^{-1})$ as $\epsilon \rightarrow 0$, the Fourier transform of $\text{sech}^2(x)\varphi_0(x)$ at this k is exponentially small in ϵ . Therefore, $\text{Im}(\omega) > 0$ is exponentially small in ϵ . The statement of the theorem is proved. \square

4 Proofs of Propositions 1 and 2

To prove Proposition 1, let us fix E_c to be ϵ -independent and different from E_0 and E_1 in (6). We write $E = E_c + \mathcal{E}$ for some small ϵ -dependent values of \mathcal{E} . The spectral problem (10) is rewritten as

$$\begin{aligned} (L_0 + E_c)\varphi - 2\text{sech}^2(x)\psi &= -\mathcal{E}\varphi, \\ L_\epsilon(E_c + \mathcal{E})\psi &= 2\epsilon^2\text{sech}^2(x)\varphi. \end{aligned} \tag{22}$$

Proof of Proposition 1. If E_c is real and negative, the system (22) has only oscillatory solutions, hence exponentially decaying eigenfunctions do not exist for values of E near E_c . Furthermore, note that the Schrödinger operator L_0 in (5) has no end-point resonances. Therefore no bifurcation of isolated eigenvalues may occur if $E_c = 0$. Thus, we consider positive values of E_c if E_c is real and values with $\text{Im}(E_c) > 0$ if E_c is complex.

By Lemma 1, we rewrite the second equation of the system (22) in the integral form

$$\begin{aligned} \psi(x) = & \frac{i\epsilon}{\sqrt{2 + \epsilon^2 E_c + \epsilon^2 \mathcal{E}}} \int_{-\infty}^x e^{ik(x-y)} \text{sech}^2(y) (\varphi + 2\psi)(y) dy \\ & + \frac{i\epsilon}{\sqrt{2 + \epsilon^2 E_c + \epsilon^2 \mathcal{E}}} \int_x^{+\infty} e^{-ik(x-y)} \text{sech}^2(y) (\varphi + 2\psi)(y) dy. \end{aligned} \quad (23)$$

Again, the right-hand-side operator on $\psi \in L^\infty(\mathbb{R})$ is a contraction for small values of ϵ if $\varphi \in L^\infty(\mathbb{R})$ and $\mathcal{E} \in \mathbb{C}$ are bounded as $\epsilon \rightarrow 0$, and for $\text{Im}(E_c + \mathcal{E}) \geq 0$ (yielding $\text{Im}(k) \geq 0$). By the Fixed Point Theorem, under these conditions we have a unique solution $\psi \in L^\infty(\mathbb{R})$ of the integral equation (23) for small values of ϵ such that $\|\psi\|_{L^\infty} = \mathcal{O}(\epsilon)$ as $\epsilon \rightarrow 0$. This solution can be substituted into the first equation of the system (22).

The operator $L_0 + E_c$ is invertible with a bounded inverse if E_c is complex or if E_c is real and positive but different from E_0 and E_1 . By the Implicit Function Theorem, we obtain a unique solution $\varphi = 0$ of this homogeneous equation for small values of ϵ and for any value of \mathcal{E} as long as \mathcal{E} is small as $\epsilon \rightarrow 0$ (since E_c is fixed independently of ϵ). Next, with $\varphi = 0$, the unique solution of the integral equation (23) is $\psi = 0$, hence $E = E_c + \mathcal{E}$ is not an eigenvalue of the spectral problem (10). \square

To prove Proposition 2, we compute $\text{Im}(\omega)$ in Theorem 1 explicitly in the asymptotic limit $\epsilon \rightarrow 0$. It follows from (19) and (21) that

$$\text{Im}(\omega) = \sqrt{2}\epsilon^3 \left| \int_{-\infty}^{+\infty} \text{sech}^2(x) \varphi_0(x) e^{-ikx} dx \right|^2 (1 + \mathcal{O}(\epsilon)),$$

where $k = \sqrt{2}\epsilon^{-1}(1 + \mathcal{O}(\epsilon^2))$.

Proof of Proposition 2. Let us consider the first eigenfunction φ_0 in (7) for the lowest eigenvalue in (6). Using integral **3.985** in [8], we obtain

$$I_0 = \int_{-\infty}^{+\infty} \text{sech}^2(x) \varphi_0(x) e^{-ikx} dx = 2 \int_0^\infty \text{sech}^p(x) \cos(kx) dx = \frac{2^{p-1}}{\Gamma(p)} \left| \Gamma\left(\frac{p+ik}{2}\right) \right|^2,$$

where $p = 2 + \sqrt{E_0} = (\sqrt{17} + 3)/2$. Since $k = \mathcal{O}(\epsilon^{-1})$ and $\epsilon \rightarrow 0$, we have use the asymptotic limit **8.328** in [8]:

$$\lim_{|y| \rightarrow \infty} |\Gamma(x + iy)| e^{\frac{\pi}{2}|y|} |y|^{\frac{1}{2}-x} = \sqrt{2\pi}, \quad (24)$$

from which we establish the asymptotic equivalence:

$$I_0 = \frac{2^{p-1}}{\Gamma(p)} \left| \Gamma\left(\frac{p+ik}{2}\right) \right|^2 \sim \frac{2\pi}{\Gamma(p)k^{1-p}} e^{-\frac{\pi}{2}k} \sim \frac{2^{\frac{p+1}{2}} \pi}{\Gamma(p)} \epsilon^{1-p} e^{-\frac{\pi}{\sqrt{2}\epsilon}}.$$

Therefore, the leading asymptotic order for $\text{Im}(\omega)$ is given by

$$\text{Im}(\omega) \sim \frac{2^{p+\frac{3}{2}}\pi^2}{[\Gamma(p)]^2} \epsilon^{5-2p} e^{-\frac{\sqrt{2}\pi}{\epsilon}}. \quad (25)$$

Next, let us consider the second eigenfunction φ_1 in (7) for the second eigenvalue in (6). Using integral **3.985** in [8] and integration by parts, we obtain

$$I_1 = \int_{-\infty}^{+\infty} \text{sech}^2(x)\varphi_1(x)e^{-ikx} dx = -\frac{2ik}{q} \int_0^{\infty} \text{sech}^q(x) \cos(kx) dx = -\frac{ik2^{q-1}}{q\Gamma(q)} \left| \Gamma\left(\frac{q+ik}{2}\right) \right|^2,$$

where $q = 2 + \sqrt{E_1} = (\sqrt{17} + 1)/2$. Using limit (24), we obtain

$$I_1 = -\frac{ik2^{q-1}}{q\Gamma(q)} \left| \Gamma\left(\frac{q+ik}{2}\right) \right|^2 \sim -\frac{2\pi ik}{q\Gamma(q)k^{1-q}} e^{-\frac{\pi}{2}k} \sim -\frac{i2^{\frac{p+2}{2}}\pi}{q\Gamma(q)} \epsilon^{-p} e^{-\frac{\pi}{\sqrt{2}\epsilon}}.$$

Therefore, the leading asymptotic order for $\text{Im}(\omega)$ is given by

$$\text{Im}(\omega) \sim \frac{2^{p+\frac{5}{2}}\pi^2}{q^2[\Gamma(q)]^2} \epsilon^{3-2q} e^{-\frac{\sqrt{2}\pi}{\epsilon}}. \quad (26)$$

In both cases (25) and (26), the expression for $\text{Im}(\omega)$ have the algebraically large prefactor in ϵ with the exponent $5 - 2p = 2 - \sqrt{17} < 0$ and $3 - 2q = 2 - \sqrt{17} < 0$. Nevertheless, $\text{Im}(\omega)$ is exponentially small as $\epsilon \rightarrow 0$. \square

5 Conclusion

We have proved that the spectral stability problem (2) has exactly two quartets of complex unstable eigenvalues in the asymptotic limit of large transverse wave numbers. We have obtained precise asymptotic expressions for the instability growth rate in the same limit.

It would be interesting to verify numerically the validity of our asymptotic results. The numerical approximation of eigenvalues in this asymptotic limit is a delicate problem of numerical analysis because of the high-frequency oscillations of the eigenfunctions for large values of λ , *i.e.*, small values of ϵ , as discussed in [5]. As we can see in Figure 1, the existing numerical results do not allow us to compare with the asymptotic results of our work. This numerical problem is left for further studies.

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