

# The Solution of Linear Constant-Coefficient Evolution PDEs With Periodic Boundary Conditions

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## Abstract:

We implement the new transform method for solving boundary value problems developed by Fokas for periodic boundary conditions. The approach presented here is not a replacement for classical methods nor is it necessarily an improvement. However, in addition to establishing that periodic problems can indeed be solve by the new transform method (which enhances further its scope and applicability), our implementation also has the advantage that it yields a new simpler approach to computing the limit from the periodic Cauchy problem to the Cauchy problem on the line.

## 1 Introduction

Recently, A. S. Fokas [3] has made great progress on the problem of solving boundary-value problems for nonlinear integrable equations. A surprising outcome of this work has been the discovery of new methods to explicitly solve boundary value problems for linear partial differential equations, specifically linear equations with constant coefficients [2, 3, 4, 5]. These new methods contain the classical methods as special cases, but they have also allowed for the solution of problems that cannot be solved using these classical methods. In this paper, we discuss the solution of initial-value problems for constant-coefficient linear partial differential equations with periodic boundary conditions. Such initial-value problems are easily solved using Fourier series. Nevertheless, we wish to investigate how the new methods can be applied to these problems. The first reason to investigate this is that the initial-value problem with periodic boundary conditions is not in actuality a boundary-value problem. Rather it is an initial-value problem posed on the circle. Thus it is not obvious that the methods of Fokas can be extended to this scenario. Establishing this extends the applicability of the method. The second reason is that in order to understand how the method might be applied to nonlinear problems with periodic boundary conditions, it is important to examine how it is applied to linear problems. We are not claiming that the new methods provide a more efficient solution for periodic problems. We shall see this is not the case. But we are demonstrating that the new methods incorporate yet another classical solution method, which is important from a unification point of view.

We begin by briefly discussing the boundary-value problem for equations on the finite interval using the new formalism. We use information from this to solve the periodic problem by eliminating the boundary conditions. At first we do this using two illustrative examples, one dissipative and one dispersive. We proceed to demonstrate the generality of the new approach, after which we use the new solution formalism to obtain the infinite-line solution formula (the Fourier transform) by considering increasingly larger periods.

## 2 The Cauchy Problem on the Finite Interval

We are considering the problem

$$\begin{aligned} q_t + w(-i\partial_x)q &= 0, & (x, t) \in [0, L] \times [0, T], \\ q(x, 0) &= q_0(x) \in C^\infty[0, L], \\ \partial_x^k q(0, t) &= g_k(t), & k \in \{j_1^0, \dots, j_N^0\} \subset \{0, \dots, n-1\}, \\ \partial_x^k q(L, t) &= h_k(t), & k \in \{j_1^L, \dots, j_{n-N}^L\} \subset \{0, \dots, n-1\}, \end{aligned} \quad (2.1)$$

where  $w(k)$  is a polynomial of degree  $n \geq 1$  with the condition that for  $k \in \mathbb{R}$ ,  $\text{Re}(w(k)) \geq 0$  for  $k$  large. This condition is necessary for well-posedness of the problem for general initial data. Define

$$N = \begin{cases} n/2, & \text{if } n \text{ is even,} \\ (n+1)/2, & \text{if } n \text{ is odd and } \text{Im}(\alpha_n) > 0, \\ (n-1)/2, & \text{if } n \text{ is odd and } \text{Im}(\alpha_n) < 0. \end{cases}$$

Then [8],

**Theorem 1.** Consider the boundary value problem (2.1). Assume that  $g_j(t), h_j(t) \in C^\infty[0, T]$  for all  $j$  and that they are compatible with the initial condition, that is,  $\partial_x^j q_0(0) = g_j(0)$  and  $\partial_x^j q_0(L) = h_j(0)$ . Then (2.1) has a unique solution  $q(x, t)$  such that  $t \rightarrow q(\cdot, t)$  is a  $C^\infty$  map from  $[0, T]$  into  $C^\infty[0, L]$  if  $N$  boundary conditions are specified at  $x = 0$  and  $n - N$  at  $x = L$ .

Thus (2.1) has a unique solution. The problem has an equivalent one-parameter divergence formulation:

$$\left( e^{-ikx+w(k)t} q \right)_t - \left[ e^{-ikx+w(k)t} \left( \sum_{j=0}^{n-1} c_j(k) \partial_x^j q \right) \right]_x = 0, \quad (x, t) \in [0, L] \times [0, T], \quad k \in \mathbb{C},$$

where the functions  $c_j(k)$  are defined by the operator identity

$$\sum_{j=0}^{n-1} c_j(k) \partial_x^j = i \frac{w(k) - w(l)}{k - l} \Big|_{l=-i\partial_x}. \quad (2.2)$$

The use of Green's Theorem in  $[0, L] \times [0, t]$  for  $0 < t \leq T$  is justified, resulting in

$$\hat{q}_0(k) - \tilde{g}(k, t) + e^{-ikL} \tilde{h}(k, t) = e^{w(k)t} \int_0^L e^{-ikx} q(x, t) dx = e^{w(k)t} \hat{q}(k, t), \quad k \in \mathbb{C}, \quad (2.3)$$

where

$$\tilde{h}(k, t) = \sum_{j=0}^{n-1} c_j(k) \int_0^t e^{w(k)s} h_j(s) ds, \quad \tilde{g}(k, t) = \sum_{j=0}^{n-1} c_j(k) \int_0^t e^{w(k)s} g_j(s) ds.$$

With  $t = T$ , we refer to (2.3) as the Global Relation. The inverse Fourier transform is used to invert this relationship, thus

$$\begin{aligned} q(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \hat{q}(k, t) dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-w(k)t} \hat{q}_0(k) dk + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-w(k)t} \tilde{g}(k, t) dk - \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik(L-x)-w(k)t} \tilde{h}(k, t) dk. \end{aligned}$$

Define  $D = \{k \in \mathbb{C} : \operatorname{Re}(w(k)) < 0\}$  and  $D^+ = \mathbb{C}^+ \cap D$ ,  $D^- = \mathbb{C}^- \cap D$ . It is shown in [3, 4] that these integrals can be deformed off the real line to the boundaries of  $D^+$  and  $D^-$ , resulting in

$$q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx - w(k)t} \hat{q}_0(k) dk + \frac{1}{2\pi} \int_{\partial D^+} e^{ikx - w(k)t} \tilde{g}(k, t) dk - \frac{1}{2\pi} \int_{\partial D^-} e^{-ik(L-x) - w(k)t} \tilde{h}(k, t) dk. \quad (2.4)$$

Heuristically, this can be explained by the fact we can deform the contours in the region where  $\operatorname{Re}(w(k)) > 0$  ( $\mathbb{C} \setminus D$ ) as we have strong exponential decay and  $\tilde{g}, \tilde{h}$  and  $\hat{q}_0$  are entire. Thus, we deform through this region until the boundary of  $D$  is reached.

**Theorem 2.** [8] Assume that  $q(x, t)$  is a sufficiently smooth solution of (2.1). Then  $q(x, t)$  is given by (2.4).

### Remarks

- Using Jordan's Lemma in  $D$  we can replace  $\tilde{g}(k, t)$  with  $\tilde{g}(k, T)$  to simplify the dependence on  $t$ .
- All the above can be done without the a priori assumption of the Fourier transform and its inverse by using a Riemann-Hilbert problem constructed from a Lax pair. See [3] for this approach.
- Were we looking to solve (2.1) we would use the invariance properties of  $w(k)$  in combination with (2.3) to eliminate the unknown boundary values. This is worked out in detail in [3]. Here we are interested in the periodic problem where these details are not needed.

## 3 The Periodic Problem

Consider the periodic problem

$$\begin{aligned} q_t + w(-i\partial_x)q &= 0 & (x, t) \in [0, L] \times [0, T], \\ q(x, 0) &= q_0(x), \\ q_0(x + L) &= q_0(x) \in C^\infty([0, L]). \end{aligned} \quad (3.1)$$

**Remark.** The method developed by Fokas applies to boundary value problems. In [2] the domain was the half-line and then extended to the a finite interval in [4]. The periodic problem is an initial value problem on  $S^1$  so it is not immediately clear that the method will extend.

The regularity of  $q_0(x)$  can be relaxed based on each problem. The assumptions imposed are for convenience. Analogous to Theorem 1, we have an existence theorem [7].

**Theorem 3.** The initial value problem (3.1) has a unique solution  $q(x, t)$  such that  $t \rightarrow q(\cdot, t)$  is a  $C^\infty$  map from  $[0, T]$  into  $C^\infty[0, L]$ .

Given  $q_0(x)$  there exists a solution  $q(x, t)$ . Furthermore there exist boundary values  $b_j(t)$  so that  $\partial_x^j q(0, t) = \partial_x^j q(L, t) = b_j(t)$ . Thus necessarily, we have  $\partial_x^j q(0, 0) = \partial_x^j q(L, 0) = b_j(0)$  and these boundary values are compatible in the sense of Theorem 1. Thus  $q(x, t)$  can be given by (2.4) with a subset of this set of boundary values. The interesting aspect to note is that  $\tilde{h} = \tilde{g}$  since the solution is periodic. The resulting global relation is easily solved for the unknowns:

$$\hat{q}_0(k) - \tilde{g}(k, T) + e^{-ikL} \tilde{h}(k, T) = e^{w(k)T} \hat{q}_T(k), \quad k \in \mathbb{C}. \quad (3.2)$$

Thus

$$\tilde{g}(k, T) = \tilde{h}(k, T) = \frac{1}{\Delta(k)} \left( \hat{q}_0(k) - e^{w(k)T} \hat{q}_T(k) \right), \quad (3.3)$$

where  $\Delta(k) = 1 - e^{-ikL}$ . We discuss a few examples to illustrate the successes and complications of this approach. First we solve the heat equation to show the success of this method on dissipative problems. Then we complicate matters using a dispersive problem, the Linear Schrödinger equation. Ideas from both these problems are used to solve any problem tractable with Fokas' method, posed on  $S^1$ .

### 3.1 The Heat Equation

For the heat equation  $q_t = q_{xx}$ , we have  $w(k) = k^2$ .

**Remark.** The solution using Fourier series is given by

$$q(x, t) = \sum_{n \in \mathbb{Z}} \hat{q}_n e^{i \frac{2\pi n}{L} x - \frac{4\pi^2 n^2}{L^2} t},$$

where

$$\hat{q}_n = \frac{1}{L} \int_0^L e^{-i \frac{2\pi n}{L} x} q_0(x) dx.$$

The integral representation we present below has the advantage over this representation of allowing one to use all the techniques of the asymptotic evaluation of integrals [1].

In this case the regions  $D^+$  and  $D^-$  are as in Figure 1(a). Notice that  $\Delta(0) = 0$  and  $k = 0$  is on the contour in Figure 1(a). From its definition,  $\tilde{g}$  is analytic in the finite complex plane and we can replace  $D$  with  $D_1$ , see Figure 1(b). The expression for the solution (2.4) becomes

$$\begin{aligned} q(x, t) = & \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx - w(k)t} \hat{q}_0(k) dk - \frac{1}{2\pi} \int_{\partial D_1^+} e^{ikx - w(k)t} \frac{\hat{q}_0(k)}{\Delta(k)} dk - \frac{1}{2\pi} \int_{\partial D_1^-} e^{-ik(L-x) - w(k)t} \frac{\hat{q}_0(k)}{\Delta(k)} dk \\ & + \frac{1}{2\pi} \int_{\partial D_1^+} e^{ikx + w(k)(T-t)} \frac{\hat{q}_T(k)}{\Delta(k)} dk + \frac{1}{2\pi} \int_{\partial D_1^-} e^{-ik(L-x) + w(k)(T-t)} \frac{\hat{q}_T(k)}{\Delta(k)} dk. \end{aligned} \quad (3.4)$$

In order to have an effective solution, we must eliminate the two terms involving the transform of the solution at a future time. Let us examine these integrands more closely. In the fourth integral, the integrand is

$$e^{ikx + w(k)(T-t)} \frac{\hat{q}_T(k)}{\Delta(k)}.$$

We wish to apply Jordan's Lemma. Since  $\text{Re}(w(k)) < 0$  in  $D_1$  we have that  $|e^{w(k)(T-t)}|$  is bounded on  $\partial D_1^+$ . The factor

$$\frac{\hat{q}_T(k)}{\Delta(k)} = \int_0^L \frac{e^{-ikx} q(x, T)}{1 - e^{-ikL}} dx,$$

is bounded on compact subsets of  $D_1^+$ , but if we examine the large  $k$  behavior for  $\text{Im}(k) > 0$  we see that,

$$\frac{e^{-ikx} q(x, T)}{1 - e^{-ikL}} = e^{ik(L-x)} \frac{q(x, T)}{e^{ikL} - 1} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Using integration by parts, we conclude that  $\hat{q}_T(k)/\Delta(k)$  decays uniformly in  $D_1^+$ . Applying Jordan's Lemma gives

$$\frac{1}{2\pi} \int_{\partial D_1^+} e^{ikx + w(k)(T-t)} \frac{\hat{q}_T(k)}{\Delta(k)} dk = 0.$$

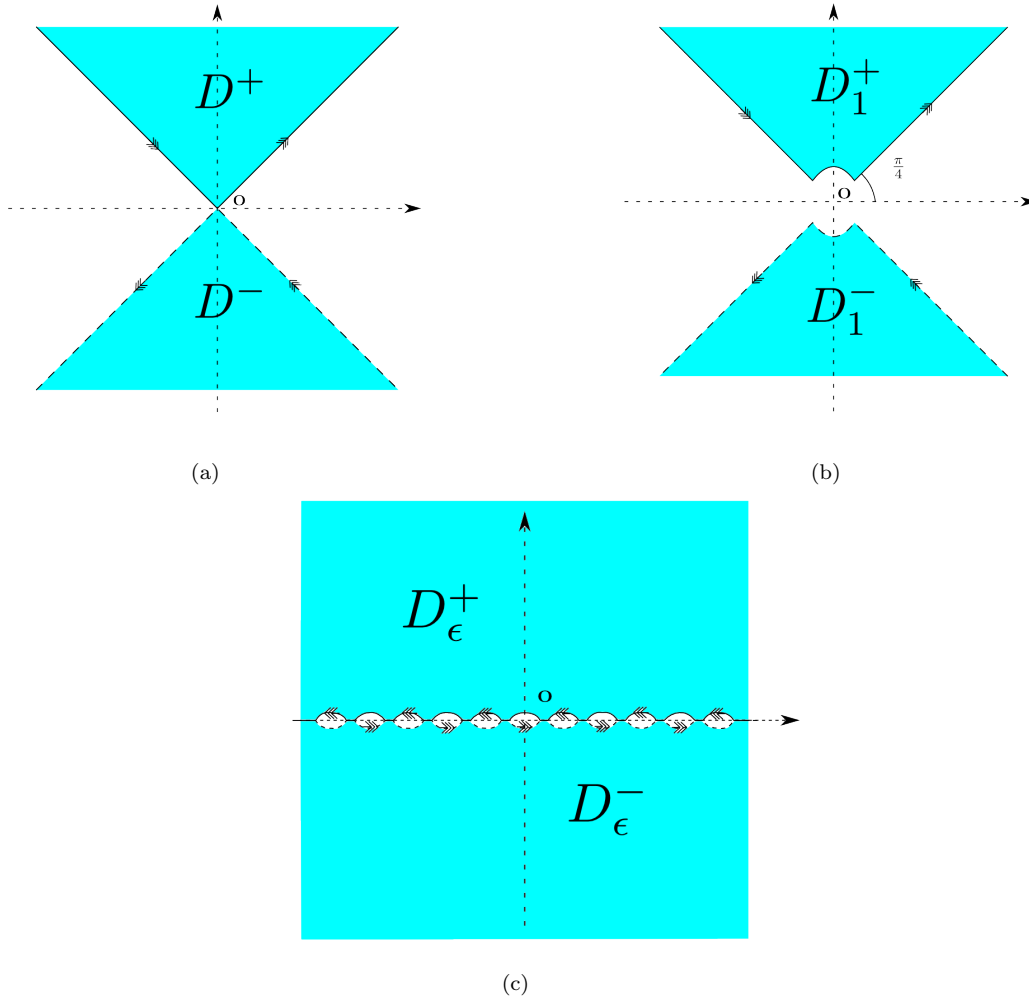


Figure 1. Deformations for the heat equation. (a) The regions  $D^+$  and  $D^-$  for heat equation. (b) Deformation of  $D^+$  and  $D^-$  to avoid the removable singularity at  $k = 0$ . (c) The regions  $D_\epsilon^+$  and  $D_\epsilon^-$  which both have oriented boundaries on the real line with deformations into  $\mathbb{C}^+$  or  $\mathbb{C}^-$  respectively, to avoid the zeros of  $\Delta(k)$ . Each of these deformations has an arc of radius  $\epsilon$ .

The final term in (3.4) is shown to be zero in a similar but even more straightforward way, as both  $1/\Delta(k)$  and  $\hat{q}_T(k)$  are bounded in  $D_1^-$ . Thus

$$q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-w(k)t} \hat{q}_0(k) dk - \frac{1}{2\pi} \int_{\partial D_1^+} e^{ikx-w(k)t} \frac{\hat{q}_0(k)}{\Delta(k)} dk - \frac{1}{2\pi} \int_{\partial D_1^-} e^{-ik(L-x)-w(k)t} \frac{\hat{q}_0(k)}{\Delta(k)} dk, \quad (3.5)$$

which gives an integral representation of the solution.

Using Jordan's Lemma in  $\mathbb{C}^+ \setminus D_1^+$  we deform the contour  $\partial D_1^+$  to  $\partial D_\epsilon^+$ , see Figure 1(c). This contour is deformed, using arcs of radius  $\epsilon$ , around the zeros of  $1/\Delta(k)$  which are at  $k = k_n = 2\pi n/L$  for  $n \in \mathbb{Z}$ . This is justified since the integrand  $e^{-w(k)t} \frac{\hat{q}_0(k)}{\Delta(k)}$  is exponentially decaying in this region. The same idea applies in  $\mathbb{C}^- \setminus D_1^-$  to deform  $\partial D_1^-$  to  $\partial D_\epsilon^-$ . In the limit as  $\epsilon \rightarrow 0$

$$\frac{1}{2\pi} \int_{\partial D_\epsilon^+} e^{ikx-w(k)t} \frac{\hat{q}_0(k)}{\Delta(k)} dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-w(k)t} \frac{\hat{q}_0(k)}{\Delta(k)} dk - \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{2\pi i}{2\pi} \text{Res} \left\{ e^{ikx-w(k)t} \frac{\hat{q}_0(k)}{\Delta(k)}, k_n \right\}.$$

Since  $1/\Delta(k)$  has only simple poles the residues are easy to compute:

$$\text{Res} \left\{ e^{ikx-w(k)t} \frac{\hat{q}_0(k)}{\Delta(k)}, k_n \right\} = e^{ik_n x - w(k_n)t} \frac{\hat{q}_0(k)}{\Delta'(k)} \Big|_{k=k_n} = \frac{1}{iL} e^{ik_n x - w(k_n)t} \hat{q}_0(k_n).$$

Similarly, on  $\partial D_\epsilon^-$ ,

$$\begin{aligned} \frac{1}{2\pi} \int_{\partial D_1^-} e^{-ik(L-x)-w(k)t} \frac{\hat{q}_0(k)}{\Delta(k)} dk &= \frac{1}{2\pi} \int_{-\infty}^{-\infty} e^{-ikL} e^{ikx-w(k)t} \frac{\hat{q}_0(k)}{\Delta(k)} dk \\ &- \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{2\pi i}{2\pi} \text{Res} \left\{ e^{-ik(L-x)-w(k)t} \frac{\hat{q}_0(k)}{\Delta(k)}, k_n \right\}. \end{aligned}$$

Since  $e^{ik_n L} = 1$ ,

$$\text{Res} \left\{ e^{-ik(L-x)-w(k)t} \frac{\hat{q}_0(k)}{\Delta(k)}, k_n \right\} = e^{-ik_n(L-x)-w(k_n)t} \frac{\hat{q}_0(k)}{\Delta'(k)} \Big|_{k=k_n} = \frac{1}{iL} e^{ik_n x - w(k_n)t} \hat{q}_0(k_n).$$

Summing the three components, we obtain

$$\begin{aligned} q(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-w(k)t} \hat{q}_0(k) dk - \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-w(k)t} \frac{\hat{q}_0(k)}{\Delta(k)} dk + \frac{1}{2L} \sum_{n=-\infty}^{\infty} e^{ik_n x - w(k_n)t} \hat{q}_0(k_n) \\ &+ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik(L-x)-w(k)t} \frac{\hat{q}_0(k)}{\Delta(k)} dk + \frac{1}{2L} \sum_{n=-\infty}^{\infty} e^{ik_n x - w(k_n)t} \hat{q}_0(k_n) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-w(k)t} q_0(k) dk - \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 - e^{-ikL}) e^{ikx-w(k)t} \frac{\hat{q}_0(k)}{\Delta(k)} dk + \frac{1}{L} \sum_{n=-\infty}^{\infty} e^{ik_n x - w(k_n)t} \hat{q}_0(k_n) \\ &= \sum_{n=-\infty}^{\infty} e^{ik_n x - w(k_n)t} \frac{\hat{q}_0(k_n)}{L}, \end{aligned}$$

which is the solution of (3.1) given in terms of Fourier series.

### 3.2 The Linear Schrödinger Equation

Consider the Linear Schrödinger equation without a potential

$$iq_t = -q_{xx},$$

so that,  $w(k) = ik^2$ . This example is significantly more complicated owing to the geometry of  $D$ , see Figure 2. In the case of the heat equation one zero of  $\Delta(k)$  is on the initial contour, whereas now an infinite number of zeros are on the contour. The same difficulty encountered in this case will arise when for every compact set  $K \subset \mathbb{R}$ ,  $D$  cannot be bounded away from  $\mathbb{R} \setminus K$ . In general, we abandon trying to get an integral representation and go directly to the Fourier series solution.

**Remark.** If we assume that  $q_0(x) \in C^\infty(\mathbb{R})$  and its support,  $\text{supp}(q_0(x)) \subset [0, L]$ , we can obtain an integral representation of the solution. Restricting to these initial conditions implies that  $\hat{q}_0(k)$  decays to all orders. This space is not convenient in applications, but does turn out be beneficial when investigating the large period limit, as we will see below.

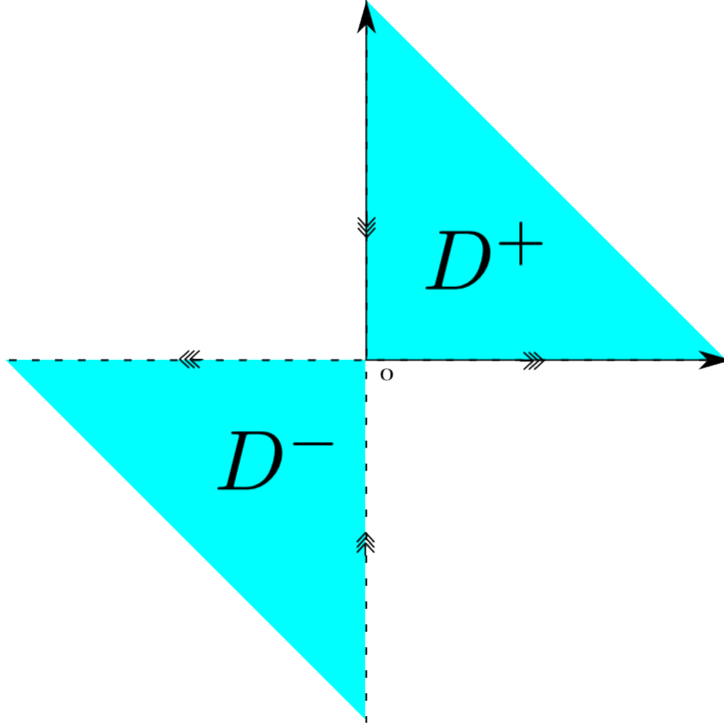


Figure 2. The regions  $D^+$  and  $D^-$  for the Linear Schrödinger equation.

We start with the expression

$$\begin{aligned}
 q(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-w(k)t} \hat{q}_0(k) dx - \frac{1}{2\pi} \int_{\partial D^+} e^{ikx-w(k)t} \frac{1}{\Delta(k)} \left( \hat{q}_0(k) - e^{w(k)T} \hat{q}_T(k) \right) dk \\
 &- \frac{1}{2\pi} \int_{\partial D^-} e^{-ik(L-x)-w(k)t} \frac{1}{\Delta(k)} \left( \hat{q}_0(k) - e^{w(k)T} \hat{q}_T(k) \right) dk,
 \end{aligned}$$

and examine terms individually. First consider the integral

$$\begin{aligned}
I_1 &= \frac{1}{2\pi} \int_{\partial D^+} e^{ikx-w(k)t} \frac{1}{\Delta(k)} \left( \hat{q}_0(k) - e^{w(k)T} \hat{q}_T(k) \right) dk \\
&= \frac{1}{2\pi} \int_{\partial D^+} e^{ikx-w(k)t} \frac{1}{\Delta(k)} \left( \hat{q}_0(k) - e^{w(k)T} \hat{q}_T(k) \right) dk \\
&\quad - \frac{i}{2} \sum'_{n \geq 0} \text{Res} \left\{ e^{ikx-w(k)t} \frac{1}{\Delta(k)} \left( \hat{q}_0(k) - e^{w(k)T} \hat{q}_T(k) \right), k = \frac{2\pi n}{L} \right\},
\end{aligned}$$

where  $\sum'$  denotes that the term with  $n = 0$  is halved. This may appear a trivial statement as the integrand is analytic, but it allows us to proceed as below.

$$\begin{aligned}
I_1 &= \frac{1}{2\pi} \int_{\partial D^+} \frac{e^{ikx-w(k)t}}{\Delta(k)} \hat{q}_0(k) dk - \frac{1}{2\pi} \int_{\partial D^+} \frac{e^{ikx+w(k)(T-t)}}{\Delta(k)} \hat{q}_T(k) dk \\
&\quad - \sum'_{n \geq 0} \frac{i}{2} \text{Res} \left\{ \frac{e^{ikx-w(k)t}}{\Delta(k)} \hat{q}_0(k), k = \frac{2\pi n}{L} \right\} + \sum'_{n \geq 0} \frac{i}{2} \text{Res} \left\{ \frac{e^{ikx+w(k)(T-t)}}{\Delta(k)} \hat{q}_T(k), k = \frac{2\pi n}{L} \right\},
\end{aligned}$$

since each one of the integrals converges and the sums converge absolutely. We collect the terms involving  $\hat{q}_T$  and claim

$$\begin{aligned}
\int_{C_\epsilon^+} \frac{e^{ikx+w(k)(T-t)}}{\Delta(k)} \hat{q}_T(k) dk &= \frac{1}{2\pi} \int_{\partial D^+} \frac{e^{ikx+w(k)(T-t)}}{\Delta(k)} \hat{q}_T(k) dk \\
&\quad - \sum'_{n \geq 1} \frac{i}{2} \text{Res} \left\{ \frac{e^{ikx+w(k)(T-t)}}{\Delta(k)} \hat{q}_T(k), k = \frac{2\pi n}{L} \right\},
\end{aligned}$$

where  $C_\epsilon^\pm$  are the contours shown in Figure 3.

Consider the factors in the integrand.  $\text{Re}(w(k)) < 0$  in  $D$  so  $e^{w(k)(T-t)}$  is bounded on  $C_\epsilon^\pm$ . Next, through a change of variables

$$\frac{\hat{q}_T(k)}{\Delta(k)} = \int_0^L e^{iks} \frac{q(L-s, T)}{e^{ikL} - 1} ds,$$

which is integrable, thus the integral on  $C_\epsilon^+$  converges. Deforming down to the real line, the claim follows.

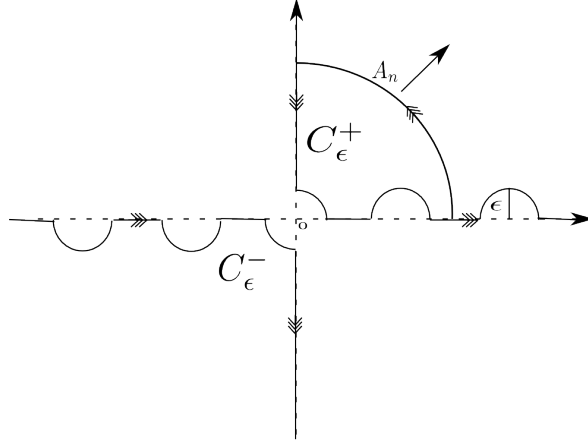


Figure 3.  $C_\epsilon^\pm$  for the Linear Schrödinger equation

Now close this contour with the arc  $A_n = \{k \in \mathbb{C} : k = \frac{2\pi n}{L} + \frac{\pi n}{L}, \arg k \in [0, \pi/2]\}$ , see Figure 3. Letting  $n \rightarrow \infty$  the integral on  $A_n$  vanishes by Jordan's Lemma, and thus the integral on  $C_\epsilon^+$  also vanishes. We



deform the integral involving  $\hat{q}_0$  to the real line,

$$\begin{aligned} I_1 &= \frac{1}{2\pi} \int_{\partial D^+} \frac{e^{ikx-w(k)t}}{\Delta(k)} \hat{q}_0(k) dk - \sum_{n \geq 0} \frac{i}{2} \operatorname{Res} \left\{ \frac{e^{ikx-w(k)t}}{\Delta(k)} \hat{q}_0(k), k = \frac{2\pi n}{L} \right\} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikx-w(k)t}}{\Delta(k)} \hat{q}_0(k) dk - \sum_{n=-\infty}^{\infty} \frac{i}{2} \operatorname{Res} \left\{ \frac{e^{ikx-w(k)t}}{\Delta(k)} \hat{q}_0(k), k = \frac{2\pi n}{L} \right\}, \end{aligned}$$

by using Jordan's Lemma in the second quadrant and collecting the appropriate residues. Repeating the same computation for the integral on  $\partial D^-$ , using  $C_\epsilon^-$  we obtain

$$\begin{aligned} &\frac{1}{2\pi} \int_{\partial D^-} \frac{e^{-ik(L-x)-w(k)t}}{\Delta(k)} (\hat{q}_0(k) - e^{w(k)T} \hat{q}_T(k)) dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ik(L-x)-w(k)t}}{\Delta(k)} dk - \sum_{n=-\infty}^{\infty} \frac{i}{2} \operatorname{Res} \left\{ \frac{e^{-ik(L-x)-w(k)t}}{\Delta(k)} \hat{q}_0(k), k = \frac{2\pi n}{L} \right\}. \end{aligned}$$

Summing all five components

$$\begin{aligned} q(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-w(k)t} \hat{q}_0(k) dk \\ &\quad - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikx-w(k)t}}{\Delta(k)} \hat{q}_0(k) dk - \sum_{n=-\infty}^{\infty} \frac{i}{2} \operatorname{Res} \left\{ \frac{e^{ikx-w(k)t}}{\Delta(k)} \hat{q}_0(k), k = \frac{2\pi n}{L} \right\} \\ &\quad - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ik(L-x)-w(k)t}}{\Delta(k)} dk + \sum_{n=-\infty}^{\infty} \frac{i}{2} \operatorname{Res} \left\{ \frac{e^{-ik(L-x)-w(k)t}}{\Delta(k)} \hat{q}_0(k), k = \frac{2\pi n}{L} \right\} \\ &= \sum_{n=-\infty}^{\infty} i \operatorname{Res} \left\{ \frac{e^{-ik(L-x)-w(k)t}}{\Delta(k)} \hat{q}_0(k), k = \frac{2\pi n}{L} \right\}, \end{aligned}$$

which is exactly the solution of (3.1) with  $w(k) = ik^2$  in terms of a Fourier series since the two sums of residues coincide.

### 3.3 Generalization

We show that these ideas apply to all PDEs of the type described above. Assume

$$w(k) = \sum_{j=1}^n \alpha_j k^j.$$

**1) Dissipative Problems:**  $\operatorname{Im} \left( \frac{\alpha_{n-1}}{\alpha_n} \right) \neq 0$  or  $\operatorname{Re}(\alpha_n) > 0$  and  $n$  even.

We use the fact that  $D$  can be replaced with its asymptotic form  $D_R$ , see [3]. Namely, if  $w(k) = \sum_{j=1}^n \alpha_j k^j$  then the asymptotic form  $D_R$  of  $D$  for large  $k$  is defined by

$$\operatorname{Re} \left( \alpha_n \left( k + \frac{\alpha_{n-1}}{n\alpha_n} \right)^n \right) = 0,$$

which is a collection of  $2n$  rays emanating from  $k = -\frac{\alpha_{n-1}}{n\alpha_n}$ . We impose conditions on  $w(k)$  so that  $\operatorname{Re}(w(k)) \geq 0$  for  $k \in \mathbb{R}$  as  $k \rightarrow \pm\infty$ . These combined conditions ensure the boundary of  $D$  does not approach the real line for large  $k$ . This implies that we have only a finite number of zeros of  $\Delta(k)$  in the closure of  $D_R$ . As in the case of the heat equation, we can deform to avoid all poles, obtaining an integral representation.

As an example consider Figure 4(a), which is deformed as in Figure 4(b), so that the contour is now bounded away from the real line. The dependence of the RHS on  $\hat{q}_T$  is removed by invoking Jordan's Lemma

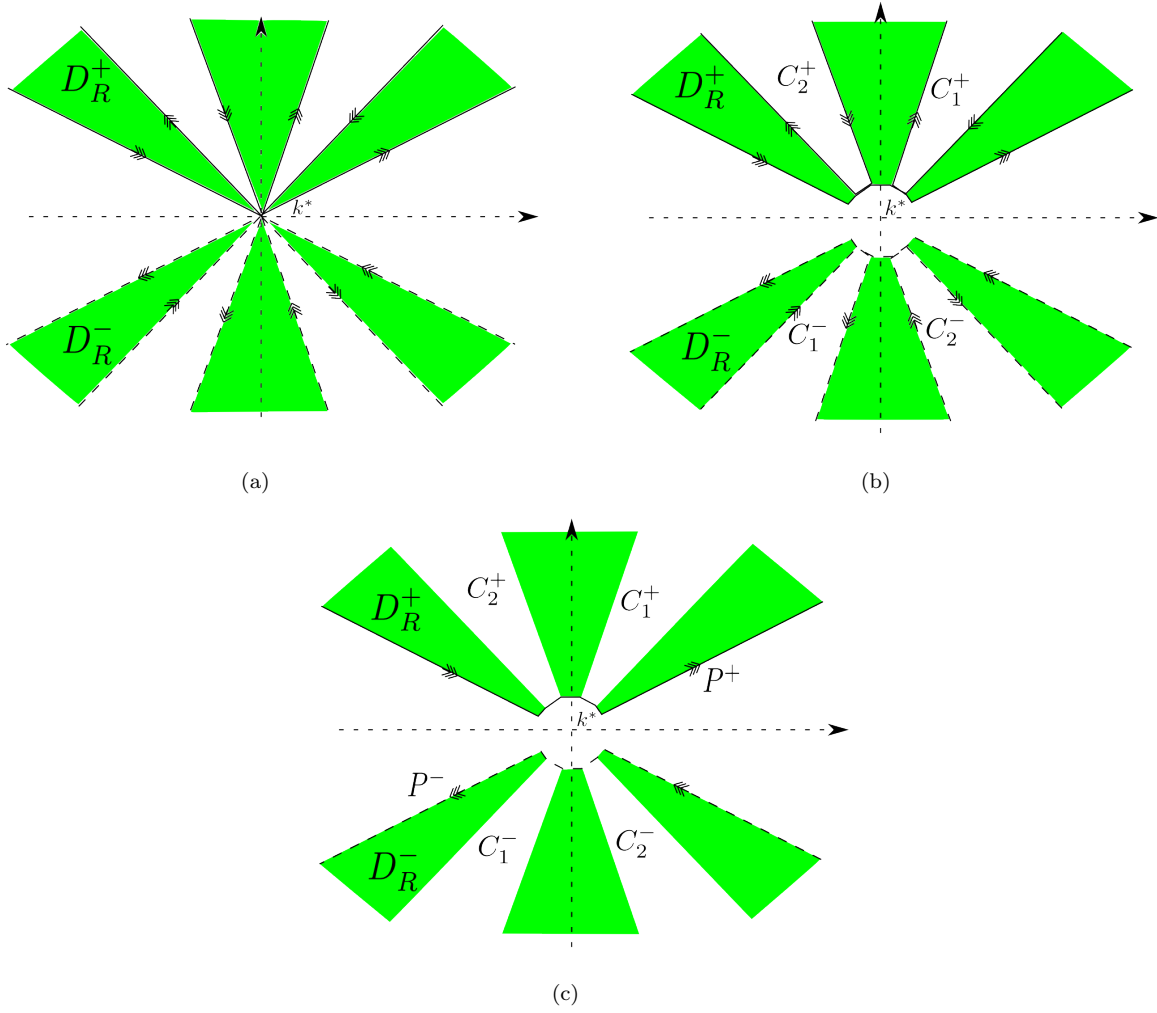


Figure 4. An example with its deformations. (a) The regions  $D_R^+$  and  $D_R^-$ . (b) Deformation of  $D_R^+$  and  $D_R^-$  to avoid poles and bound it away from the real line. (c) The final contours  $P^+$  and  $P^-$  for a dissipative problem.

in each component of  $D_R^c$ . The integrals around the boundaries of  $C_1^+$ ,  $C_2^+$ ,  $C_1^-$  and  $C_2^-$  involving  $\hat{q}_0(k)$  can be eliminated by using Jordan's Lemma in these regions. Thus the final solution is given in terms of the integrals along  $P^+$  and  $P^-$ , see Figure 4(c).

**2) Dispersive Problems:**  $\text{Im}\left(\frac{\alpha_{n-1}}{\alpha_n}\right) = 0$  and  $\text{Re}(\alpha_n) = 0$ , or  $\text{Im}\left(\frac{\alpha_{n-1}}{\alpha_n}\right) = 0$  and  $n$  odd.

This condition ensures that either  $\partial D_R^+ \cap \mathbb{R}$  or  $\partial D_R^- \cap \mathbb{R}$  are unbounded. Assume that  $\partial D_R^+ \cap \mathbb{R}$  is unbounded. If not, all deformations are justified using the argument in the previous section. In addition, assume that  $(-\infty, \infty) \subset \partial D_R^+$ , see Figure 5(a). We follow the same approach as for the Linear Schrödinger Equation. First, we replace all integrals with principal values and sums. Then, we deform the integrals involving  $\hat{q}_T(k)$  along with the sums to contours  $C_\epsilon^+$  with small arcs over each pole  $k_n$ . Closing contours inside  $D_R^+$  allows the use of Jordan's Lemma, so that

$$\int_{C_\epsilon^+} e^{ikx+w(k)(T-t)} \frac{\hat{q}_T(k)}{\Delta(k)} dk = 0.$$

This leaves only dependence on  $q_0(k)$ . At this point the problem is solved. We can simplify the solution representation by repeating the process above by using Jordan's Lemma in  $C_1^+$ ,  $C_2^+$ ,  $C_1^-$  and  $C_2^-$ .

Thus the final form of the solutions is, see Figure 5(b),

$$q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-w(k)t} \hat{q}_0(k) dk - \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-w(k)t} \frac{\hat{q}_0(k)}{\Delta(k)} dk + \sum_{n=-\infty}^{\infty} \frac{i}{2} \text{Res} \left\{ \frac{e^{ikx-w(k)t}}{\Delta(k)} \hat{q}_0(k), k = \frac{2\pi n}{L} \right\} + \frac{1}{2\pi} \int_{P^-} e^{-ik(L-x)-w(k)t} \frac{\hat{q}_0(k)}{\Delta(k)} dk.$$

The contour  $P^-$  in Figure 5(b) can be deformed to the real line in order to extract the Fourier series as in the case of the heat equation.

## 4 The Infinite-Line Limit

An integral representation is necessary to obtain the infinite-line case from the  $L \rightarrow \infty$  limit. This bypasses having to interpret a sum in the limit as a Riemann sum, as in the classical approach [6]. We shift the domain  $[0, L]$  to  $[-L/2, L/2]$ . Let  $\mathcal{S}(\mathbb{R})$  denote the Schwartz class on  $\mathbb{R}$ . For this section we assume  $q_0(x) \in \mathcal{S}(\mathbb{R})$ . Define the bump function  $b_L(x) \in \mathcal{S}(\mathbb{R})$ , with support  $\text{supp}(b_L(x))$ , with the following properties, valid for all  $x$ :

- $b_L(x) \leq 1$ ,
- $\lim_{L \rightarrow \infty} b_L(x) = 1$ ,
- $\text{supp}(b_L(x)) = [-L/2, L/2]$ ,
- $\partial_x^j b_L(\pm L/2) = 0$ ,  $j \in \mathbb{N}$ ,
- $\sup_{x \in \mathbb{R}} |\partial_x^j b_L(x)| \leq M_j$ ,  $j \in \mathbb{N}$ ,  $M_j \in \mathbb{R}$ .

**Remark.** The function  $e^{\frac{1}{x-L/2}} e^{\frac{-1}{x+L/2}} \chi_{[-L/2, L/2]}$  where  $\chi$  is the characteristic function is an example of such a function.

Fully stated, the problem is

$$\begin{aligned} q_t + w(-i\partial_x)q &= 0, & (x, t) \in [-L/2, L/2] \times [0, T], \\ q(x, 0) &= q_0(x)b_L(x), \\ q_0(x) &= q_0(x+L). \end{aligned}$$

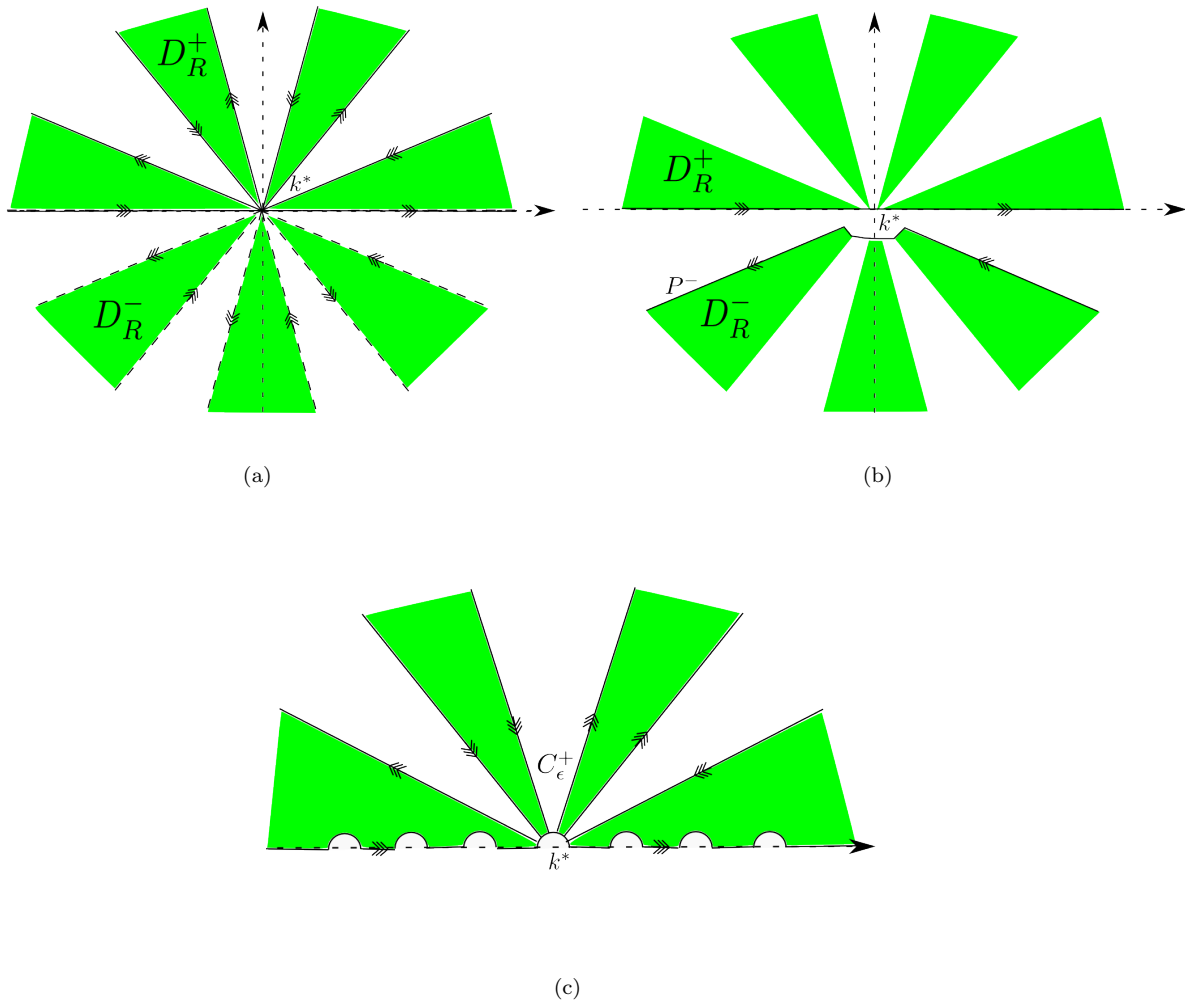


Figure 5. A dispersive example with its deformations. (a) The regions  $D_R^+$  and  $D_R^-$  for a typical dispersive problem. (b) Deformation of  $D_R^+$  and  $D_R^-$  to avoid poles on  $D_R^-$  but with a principal value integral still on some of  $D_R^+$ . (c) The contour  $C_\epsilon^+$  for a dispersive problem.

Denote the solution of this problem by  $q_L(x, t)$ , given by

$$\begin{aligned} q_L(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-w(k)t} \hat{q}_{0L}(k) dk - \frac{1}{2\pi} \int_{\partial D_R^+} e^{ik(L/2+x)-w(k)t} \tilde{g}_L(k) dk \\ &\quad - \frac{1}{2\pi} \int_{\partial D_R^-} e^{-ik(L/2-x)-w(k)t} \tilde{h}_L(k) dk. \end{aligned}$$

In this case

$$\tilde{g}_L(k) = \tilde{h}_L(k) = \frac{1}{\Delta_L(k)} \left( \hat{q}_0^L(k) - e^{w(k)T} \hat{q}_T^L(k) \right),$$

where

$$\Delta_L(k) = 2i \sin\left(\frac{kl}{2}\right), \quad \hat{q}_a^L(k) = \int_{-L/2}^{L/2} e^{-ikx} q(0, a) dx.$$

We require a few lemmas.

**Lemma 1.** Let  $S_j = \{s + is^{-j} : s \in [1, \infty)\}$ , then

$$\frac{1}{|\Delta_L(k)|} \sim \mathcal{O}(k^j), \quad k \in S_j, \quad |k| \rightarrow \infty.$$

**Proof.** With  $\Delta_L(k) = e^{ikL/2} - e^{-ikL/2}$ , we have

$$\begin{aligned} |\Delta_L(k)| &= |e^{iLs/2} e^{-Ls^{-j}/2} - e^{-iLs/2} e^{Ls^{-j}/2}| = |e^{-Ls^{-j}/2} - e^{-iLs} e^{Ls^{-j}/2}| \\ &\geq 1 - e^{-Ls^{-j}}, \end{aligned}$$

and  $\lim_{s \rightarrow \infty} \frac{s^{-j}}{1 - e^{-Ls^{-j}/2}} = \frac{2}{L}$ .

**Lemma 2.** Let  $f(x) \in \mathcal{S}(\mathbb{R})$  and let  $b_L(x)$  be the bump function as defined above. There exists a function  $g \in L^1(S_j)$ , independent of  $L$ , so that

$$\left| e^{ik(x+L/2)} \frac{\hat{f}_L(k)}{\Delta_L(x)} \right| \leq g, \quad \text{for all } k \in S_j,$$

where  $f_L(x) = f(x)b_L(x)$ .

**Proof.** Consider

$$\left| \frac{e^{ik(x+L/2)}}{e^{ikL/2} - e^{-ikL/2}} \right| \leq \frac{1}{|1 - e^{-Ls^{-j}/2}|} \leq \frac{2}{1 - e^{-s^{-j}/2}},$$

for  $L \geq 1$ . For the numerator, using integration by parts

$$|\hat{f}_L(k)| = \left| (ik)^{-m} \int_{-L/2}^{L/2} \sum_{j=0}^m c_j \partial_x^j f(x) \cdot \partial_x^{m-j} b_j(x) dx \right| \leq |k|^{-m} \int_{-\infty}^{\infty} \sum_{j=0}^m |c_j M_j| |\partial_x^j f(x)| dx$$

where  $c_j$  are real constants. Since we chose  $f$  to be Schwartz class,  $\int_{-\infty}^{\infty} \sum_{j=0}^m c_j M_j |\partial_x^j f(x)| dx = P_m < \infty$ . Let

$$g = \frac{P_m |k|^{-m}}{1 - e^{k^j/2}}, \quad \text{where } m > j + 1.$$

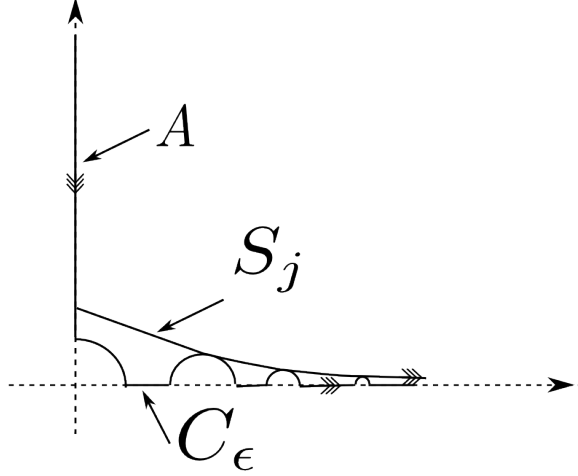


Figure 6. Deforming the principal value integral up into  $D_R^+$ .

By Lemma 1,  $g \in L^1(S_j)$ .

**Theorem 4.** Let  $A$  be a contour in the upper half plane that is bounded away from the real line or  $A = S_j$  for some  $j$ . Then

$$\lim_{L \rightarrow \infty} \int_A e^{ik(x+L/2)} \frac{\hat{q}_L(k)}{\Delta_L(k)} dk = 0.$$

**Proof.**

(a) The contour  $A$  is bounded away from the real line. So  $1/|e^{ikL} - 1| \leq B \in \mathbb{R}$  on  $A$  for  $L \geq 1$ . Next

$$\left| \frac{\hat{q}_L(k)}{\Delta_L(k)} \right| = \frac{1}{|e^{ikL} - 1|} \left| \int_{-L/2}^{L/2} e^{ik(L/2-x)} q_0(x) dx \right| \leq B \left| \int_{-L/2}^{L/2} e^{ik(L/2-x)} q_0(x) dx \right| \leq BP_m |k|^{-m},$$

using a simplified version of Lemma 2. Consider

$$\int_{-L/2}^{L/2} e^{ik(L/2-x)} q_0(x) dx.$$

We have  $e^{ik(L/2-x)} q_0(x) \leq q_0(x) \in L^1(\mathbb{R})$  and for fixed  $x$ ,  $e^{ik(L/2-x)} \rightarrow 0$  as  $L \rightarrow \infty$ . From dominated convergence,  $\lim_{L \rightarrow \infty} \int_{-L/2}^{L/2} e^{ik(L/2-x)} q_0(x) dx = 0$ . It follows that dominated convergence can be used again, and

$$\lim_{L \rightarrow \infty} \int_A e^{ik(x+L/2)} \frac{\hat{q}_L(k)}{\Delta_L(k)} dk = \int_A e^{ik(x+L/2)} \lim_{L \rightarrow \infty} \frac{\hat{q}_L(k)}{\Delta_L(k)} dk = 0.$$

(b) Assume  $A = S_j$  for some  $j$ . Lemma 2 gives a function  $g$  to dominate the first integral. Part (a) provides what we need to dominate the inner integral. So

$$\lim_{L \rightarrow \infty} \int_A e^{ik(x+L/2)} \frac{\hat{q}_L(k)}{\Delta_L(k)} dk = 0.$$

**Remark.** The same proof extends to integrals of the form  $\int_A e^{-ik(L/2-x)-w(k)t} \frac{\hat{q}_L(k)}{\Delta_L(k)} dk$  in the lower half plane.

Fix  $x$  and  $t$  and consider

$$I_L = \frac{1}{2\pi} \int_{\partial D_R^+} e^{ik(L/2+x)-w(k)t} \tilde{g}_L(k) dk.$$

We assume that  $[0, \infty) \subset \partial D_R^+$  but  $(-\infty, 0) \cap \partial D_R^+ = \emptyset$ . When this is not the case, the same calculations carry through for  $(-\infty, 0]$ . We follow the same process as above to remove dependence on  $\hat{q}_T(k)$ , leaving

$$I_L = \oint_{\partial D_R^+} e^{ik(L/2+x)-w(k)t} \frac{\hat{q}_0(k)}{\Delta_L(k)} dk - \sum_{n \geq 0} \frac{i}{2} \operatorname{Res} \left\{ \frac{e^{ikx-w(k)t}}{\Delta(k)} \hat{q}_0(k), k = \frac{2\pi n}{L} \right\}.$$

Let  $C_\epsilon$  be a contour with arcs over every zero of  $\Delta_L(k)$  such that each arc is tangent to  $S_j$ , see Figure 6. On  $C_\epsilon$ ,  $1/\Delta_L(k) \sim c(\epsilon)k^j$ , this is shown in the same way as Lemma 1. Using Lemma 2 we can deform this integral up to  $S_j$  using  $C_\epsilon$  as an intermediate step (Figure 6). We obtain

$$I_L = \int_A e^{ik(L/2+x)-w(k)t} \frac{\hat{q}_0(k)}{\Delta_L(k)} dk + \int_{S_j} e^{ik(L/2+x)-w(k)t} \frac{\hat{q}_0(k)}{\Delta_L(k)} dk,$$

where  $A$  is bounded away from the real line and  $j \geq n$  where  $n$  is the order of  $w(k)$ . This is done so that  $e^{-w(k)t}$  is bounded on  $S_j$ . From Theorem 4,  $\lim_{L \rightarrow \infty} I_L = 0$ . The same arguments apply to integrals on  $D_{\bar{R}}$ . Therefore

$$\begin{aligned} \lim_{L \rightarrow \infty} q_L(x, t) &= \frac{1}{2\pi} \lim_{L \rightarrow \infty} \int_{-\infty}^{\infty} e^{ikx-w(k)t} \hat{q}_{0_L}(k) dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-w(k)t} \lim_{n \rightarrow \infty} \hat{q}_{0_L}(k) dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-w(k)t} \hat{q}_0(k) dk, \end{aligned}$$

where the limit interchange was done in the same way as above. Thus the infinite-line limit directly gives rise to the Fourier transform solution.

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