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Real Lax spectrum implies spectral stability

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Abstract

We consider the dynamical stability of periodic solutions of integrable equations with 2×2 Lax pairs. We construct the eigenfunctions and hence the Floquet discriminant for such Lax pairs. The boundedness of the eigenfunctions determines the *Lax spectrum*. We use a connection between the Lax spectrum and the stability spectrum to show that the subset of the real line which gives rise to stable eigenvalues is contained in the Lax spectrum. This subset is the full spectrum for self-adjoint members of the AKNS hierarchy. For non-self-adjoint members of the AKNS hierarchy admitting a common reduction, the real line is always part of the Lax spectrum and maps to stable eigenvalues of the stability problem. We demonstrate our methods work for a variety of examples.

1 Introduction

A surprisingly large number equations of physical significance possess a Lax pair [3, 4]. An important feature of equations with a Lax pair is the Lax spectrum: the set of all Lax parameter values for which the solution of the Lax pair is bounded. For our purposes, this is important for determining the stability of solutions of a given integrable equation [7, 8, 9, 10, 12, 14, 15, 16, 19, 28, 31]. Until recently, the Lax spectrum has only been determined explicitly for decaying potentials on the whole line or for self-adjoint problems with a finite domain. For problems with periodic coefficients, the Floquet discriminant [2, 9, 18, 19, 23, 28] is a useful tool for numerically computing and giving a qualitative description of the Lax spectrum, but it is not used generally to get an explicit description of the Lax spectrum. A full description of the Lax spectrum can allow one to prove the stability of solutions to integrable equations with respect to certain classes of perturbations [16].

In this paper, we construct a function defined by an integral whose zero level set determines the Lax spectrum for 2×2 Lax pairs. We show that the subset of the real line which gives rise to stable eigenvalues in the stability problem is always contained in the Lax spectrum. Next, we show that the Lax spectrum of self-adjoint members of the AKNS hierarchy consists only of this subset of the real line. Conversely, for non-self-adjoint members of the AKNS hierarchy admitting a common reduction, we show that the real line is always part of the Lax spectrum and always maps to stable eigenvalues of the stability problem. Our construction determines all unbounded components of the Lax spectrum and of the stability spectrum. We finish by providing examples of equations that fit the framework presented as well as some examples that do not immediately fit, but for which similar conclusions may be drawn.

2 Setup

We consider an integrable evolution equation of the form

$$u_t = \mathcal{N}(u, u_x, \ldots, u_{Nx}), \tag{1}$$

where $\mathcal{N}$ is a nonlinear function of u and N of its spatial derivatives. The function u may be real or complex valued. Since (1) is integrable, it possesses a Lax pair (we use this as our definition of integrable). We focus on equations which possess a 2×2 Lax pair, *i.e.*, there exists a pair of two linear ordinary differential equations (ODEs),

$$\phi_x(x,t;\zeta) = \begin{pmatrix} \alpha(x,t;\zeta) & \beta(x,t;\zeta) \\ \gamma(x,t;\zeta) & -\alpha(x,t;\zeta) \end{pmatrix} \phi(x,t;\zeta) = X\phi, \tag{2a}$$

$$\phi_t(x,t;\zeta) = \begin{pmatrix} A(x,t;\zeta) & B(x,t;\zeta) \\ C(x,t;\zeta) & -A(x,t;\zeta) \end{pmatrix} \phi(x,t;\zeta) = T\phi, \tag{2b}$$

such that the compatibility of mixed derivatives $\partial_t \phi_x = \partial_x \phi_t$ holds if (1) holds. Here $\zeta \in \mathbb{C}$ is the *Lax parameter*, assumed to be independent of x and t. The compatibility of (2) defines the evolution equations [3]

$$\alpha_t - A_x = \gamma B - \beta C, \tag{3a}$$

$$\beta_t - B_x = 2\left(\beta A - \alpha B\right), \tag{3b}$$

$$\gamma_t - C_x = -2\left(\gamma A - \alpha C\right). \tag{3c}$$

Equation (1) is equivalent to (3) upon making specific choices for $\mathcal{P} = \{\alpha,\ \beta,\ \gamma,\ A,\ B,\ C\}$. We make the following assumptions on the elements of $\mathcal{P}$.

Assumption 1. *All elements of* $\mathcal{P}$ *are bounded for all* $x \in \mathbb{R}$.

Assumption 2. *The elements of* $\mathcal{P}$ *are autonomous in* t.

Assumptions 1 and 2 hold for many equations (see Section 6), including all frequently studied members of the AKNS hierarchy (Section 5).

We define the *Lax spectrum* σ_L as $\sigma_L = \{\zeta \in \mathbb{C} : \phi \text{ is bounded for } x \in \mathbb{R}, \text{ including as } x \to \pm\infty\}$. The Lax spectrum depends on the solution to (1), which determines the coefficients of X and T. Of course, σ_L depends on which norm of ϕ is used. In this paper, we are interested in the Lax spectrum when $|u|$ is periodic with period P. For studying the stability of u, one typically writes the equation in a frame in which u is stationary with respect to the temporal variable. In what follows, we assume that (1) has already been put in a frame such that the solutions of interest are independent of time. In the stationary frame, the conditions (3) become

$$A_x = \beta C - \gamma B, \tag{4a}$$

$$B_x = 2\left(\alpha B - \beta A\right), \tag{4b}$$

$$C_x = -2\left(\alpha C - \gamma A\right), \tag{4c}$$

by Assumption 2.

3 Motivation: the nonlinear Schrödinger equation

The nonlinear Schrödinger (NLS) equation is

$$i\Psi_t + \frac{1}{2}\Psi_{xx} - \kappa\Psi\left|\Psi\right|^2 = 0, \tag{5}$$

where $\Psi(x,t)$ is a complex-valued function and $\kappa = -1$ and $\kappa = 1$ correspond to the focusing and defocusing equations respectively. The Lax pair for the NLS equation [32] is given by (2) with

$$\alpha = -i\zeta, \qquad \beta = \Psi, \qquad \gamma = \kappa\Psi^*, \tag{6a}$$

$$A = -i\zeta^2 - i\kappa\left|\Psi\right|^2/2, \qquad B = \zeta\Psi + i\Psi_x/2, \qquad C = \zeta\kappa\Psi^* - i\kappa\Psi_x^*/2. \tag{6b}$$

Here and throughout Ψ^* is the complex conjugate of Ψ. Equating $\Psi(x,t) = e^{-i\omega t}\psi(x,t)$, where $\omega \in \mathbb{R}$ is constant, we obtain the NLS equation in a frame rotating with constant phase speed ω,

$$i\psi_t + \omega\psi + \frac{1}{2}\psi_{xx} - \kappa\psi\,|\psi|^2 = 0. \tag{7}$$

Equation (7) is obtained from the compatibility of the new t-equation,

$$\phi_t = \begin{pmatrix} -i\zeta^2 - i\kappa\,|\psi|^2/2 + i\omega/2 & \zeta\psi + i\psi_x/2 \\ \zeta\kappa\psi^* - i\kappa\psi_x^*/2 & i\zeta^2 + i\kappa\,|\psi|^2/2 - i\omega/2 \end{pmatrix}\phi = \left(T - \frac{\omega}{2}T_0\right)\phi, \tag{8}$$

where

$$T_0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \tag{9}$$

and the x equation, which is unchanged. The elliptic solutions of the NLS equation are solutions of (7) with periodic amplitude and $\psi_t = 0$.

The stability of the elliptic solutions of the defocusing and focusing NLS equation was studied in [8] and [16] respectively. To study the stability of the elliptic solutions, linearize (7) about a stationary solution $\tilde{\psi}(x)$ by letting $\psi(x,t) = \tilde{\psi}(x) + \epsilon u(x,t) + \mathcal{O}(\epsilon^2)$. This results in

$$U_t = \begin{pmatrix} u \\ \kappa u^* \end{pmatrix}_t = \begin{pmatrix} \frac{i}{2}\partial_x^2 - 2i\kappa|\tilde{\psi}|^2 + i\omega & -i\tilde{\psi}^2 \\ i\tilde{\psi}^{*2} & -\frac{i}{2}\partial_x^2 + 2i\kappa|\tilde{\psi}|^2 - i\omega \end{pmatrix}\begin{pmatrix} u \\ \kappa u^* \end{pmatrix} = \mathcal{L}_{\text{NLS}}U. \tag{10}$$

Since $\mathcal{L}_{\text{NLS}}$ does not depend explicitly on t, we may separate variables with

$$U(x,t) = \begin{pmatrix} u(x,t) \\ \kappa u^*(x,t) \end{pmatrix} = e^{\lambda t}\begin{pmatrix} v(x) \\ \kappa v^*(x) \end{pmatrix} = e^{\lambda t}V(x), \tag{11}$$

resulting in the spectral problem

$$\lambda V = \mathcal{L}_{\text{NLS}}V. \tag{12}$$

We define the *stability spectrum* $\sigma_{\mathcal{L}_{\text{NLS}}} = \{\lambda \in \mathbb{C} : V \text{ is bounded for } x \in \mathbb{R}\}$. Since the NLS equation is Hamiltonian [32], the stability spectrum has a quadrafold symmetry: if $\lambda \in \mathcal{L}_{\text{NLS}}$, then $-\lambda,\ \lambda^*,\ -\lambda^* \in \sigma_{\mathcal{L}_{\text{NLS}}}$. Therefore $\tilde{\psi}(x)$ is spectrally stable if $\sigma_{\mathcal{L}_{\text{NLS}}} \subset i\mathbb{R}$. For the NLS equation and other integrable equations, $\sigma_{\mathcal{L}_{\text{NLS}}}$ can be determined by using the *squared-eigenfunction connection.* The eigenfunctions of $\mathcal{L}_{\text{NLS}}$ are given by ([8], see [3] for the original ideas leading to this)

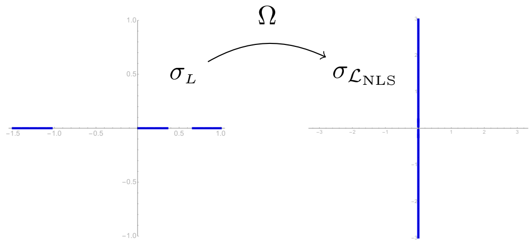
$$U(x,t) = \begin{pmatrix} \phi_1^2 \\ \phi_2^2 \end{pmatrix}, \tag{13}$$

where $\phi = (\phi_1,\ \phi_2)^{\intercal}$ is an eigenfunction of (8). Since (8) is homogeneous in t, it may be solved by separation of variables. With

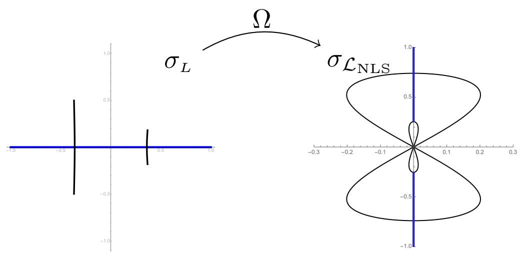
$$\phi(x,t) = e^{\Omega t}\varphi(x) \tag{14}$$

we obtain

$$\Omega\varphi = \left(T - \frac{\omega}{2}T_0\right)\varphi = T_2\varphi, \tag{15}$$



(a)



(b)

Figure 1: The real *vs.* imaginary part of the Lax and stability spectrum (left and right of each panel, respectively) for an elliptic solution of (a) the defocusing NLS equation and (b) the focusing NLS equation. The real component of σ_L and its image under Ω is colored blue. The rest of σ_L is in black. The Lax spectrum is computed analytically [8, 15] and the stability spectrum is the image under Ω.

and the eigenfunctions of $\mathcal{L}_{\mathrm{NLS}}$ are given by

$$U(x,t) = e^{\lambda t} V(x) = \begin{pmatrix} \phi_1^2 \\ \phi_2^2 \end{pmatrix} = e^{2\Omega t} \begin{pmatrix} \varphi_1^2(x) \\ \varphi_2^2(x) \end{pmatrix}. \tag{16}$$

The major insight from (16) is that $\lambda = 2\Omega(\zeta)$ when $\zeta \in \sigma_L$. Since $\lambda \in i\mathbb{R}$ corresponds to stable elements of the spectrum, we are interested in finding ζ such that $\Omega(\zeta) \in i\mathbb{R}$. Since Ω is defined by the 2×2 eigenvalue equation (15), $\Omega \in i\mathbb{R}$ if $\zeta \in \sigma_L$ and T_2 is skew adjoint. For both the focusing [16] and the defocusing cases [8], $\Omega(\sigma_L \cap \mathbb{R}) \subset i\mathbb{R}$, see Figure 2. The elliptic solutions of the defocusing NLS equation are stable since $\sigma_L = \{\zeta \in \mathbb{R} : \Omega(\zeta) \in i\mathbb{R}\}$ [7], see Figure 1a. The Lax spectrum for the elliptic solutions of the focusing NLS equation is more difficult to find since it is not just a subset of the real line (see Figure 1b and Appendix A). However, for the focusing NLS equation, the elliptic solutions are stable only with respect to a special class of perturbations which gives rise to only real ζ [16].

While investigating the stability for elliptic solutions of the focusing NLS equation in [16], we observed that the imaginary elements of the stability spectrum for other integrable equations also originate from the real part of the Lax spectrum. This includes the defocusing mKdV equation (see [14] and Section 6.1) and the sine-Gordon equation (see [12] and Section 6.2). This leads to the conjecture that solutions of integrable equations are stable if and only if $\zeta \in \mathbb{R}$. This conjecture is proven here for members of the AKNS hierarchy with $r = \pm q^*$ (see Corollary 9). In the rest of the paper, we generalize this idea for equations with more general Lax pairs (2). Although finding the Lax spectrum is an interesting and important problem in its own right, we are interested in finding the Lax spectrum to determine the stability of stationary solutions to integrable equations. Since the squared-eigenfunction connection gives

$$\lambda = 2\Omega(\zeta), \tag{17}$$

we are especially interested in finding σ_L when $\Omega(\zeta) \in i\mathbb{R}$, since integrable equations are Hamiltonian and the stability spectrum possesses a quadrafold symmetry. Thus the only option for stability is to have the spectrum confined to the imaginary axis.

4 Computing the Lax spectrum

With A, B, and C t-independent, (2b) may be solved by separation of variables. Equating

$$\phi(x,t) = e^{\Omega t} \varphi(x), \tag{18}$$

(2b) becomes a 2×2 eigenvalue equation for Ω:

$$\Omega\varphi = T\varphi. \tag{19}$$

Using the expression for T,

$$\Omega^2 = A^2 + BC. \tag{20}$$

Lemma 3. *Ω is independent of x and t.*

Proof. Independence of t is by Assumption 2. Multiplying (4b) by C and (4c) by B and adding the resulting equations yields

$$0 = \partial_x(BC) + 2A(\beta C - \gamma B). \tag{21}$$

Using (4a),

$$0 = \partial_x(BC + A^2) = \partial_x(\Omega^2), \tag{22}$$

so Ω^2 is independent of x. □

As a result of Lemma 3, $\Omega = \Omega(\zeta)$ is a function only of ζ and the solution parameters. The eigenfunctions of (2b) may be written as either

$$\phi(x,t) = e^{\Omega t} y_1(x) \begin{pmatrix} -B(x) \\ A(x) - \Omega \end{pmatrix}, \qquad \text{or} \qquad \phi(x,t) = e^{\Omega t} y_2(x) \begin{pmatrix} A(x) + \Omega \\ C(x) \end{pmatrix}. \tag{23}$$

Here the scalar functions $y_1(x)$ and $y_2(x)$ are determined by the requirement that $\phi(x,t)$ not only solves (2b), but also (2a), since (2a-b) have a common set of eigenfunctions. Substitution in (2a) gives

$$-By_1' - B_x y_1 = (-\alpha B + \beta(A - \Omega))y_1, \qquad (A-\Omega)y_1' + A_x y_1' = (-\gamma B - \alpha(A-\Omega))y_1, \tag{24a}$$

$$(A+\Omega)y_2' + A_x y_2 = (\alpha(A+\Omega) + \beta C)y_2, \qquad Cy_2' + C_x y_2 = (\gamma(A+\Omega) - \alpha C)y_2, \tag{24b}$$

so that different (but equivalent) representations for $y_1(x)$ and $y_2(x)$ are obtained from the first or second equation of (2a):

$$y_1 = \hat{y}_1 \exp\left(\int_x \frac{\alpha B - \beta(A-\Omega) - B_x}{B}\,\mathrm{d}x\right), \qquad y_1 = \tilde{y}_1 \exp\left(-\int_x \frac{\gamma B + \alpha(A-\Omega) + A_x}{A-\Omega}\,\mathrm{d}x\right), \tag{25a}$$

$$y_2 = \hat{y}_2 \exp\left(\int_x \frac{\alpha(A+\Omega) + \beta C - A_x}{A+\Omega}\,\mathrm{d}x\right), \qquad y_2 = \tilde{y}_2 \exp\left(\int_x \frac{\gamma(A+\Omega) - \alpha C - C_x}{C}\,\mathrm{d}x\right), \tag{25b}$$

where $\hat{y}_1,\ \tilde{y}_1,\ \hat{y}_2$, and $\tilde{y}_2$ are constants of integration. Since $A,\ B$, and C are bounded in x by Assumption 1, in order for $\zeta \in \sigma_L$, it must be that the exponential part of $\phi(x,t)$ is bounded in x. To bound the exponential growth, we consider the real part of the exponential. The integrands may be rewritten using (4):

$$\frac{\alpha B - \beta(A-\Omega) - B_x}{B} = -\frac{1}{2}\frac{B_x}{B} + \frac{\beta\Omega}{B}, \tag{26a}$$

$$\frac{\gamma B + \alpha(A-\Omega) + A_x}{A-\Omega} = \alpha + \frac{\beta C}{A-\Omega}, \tag{26b}$$

$$\frac{\alpha(A+\Omega) + \beta C - A_x}{A+\Omega} = \alpha + \frac{\gamma B}{A+\Omega}, \tag{26c}$$

$$\frac{\gamma(A+\Omega) - \alpha C - C_x}{C} = -\frac{1}{2}\frac{C_x}{C} + \frac{\gamma\Omega}{C}. \tag{26d}$$

In order for $\zeta \in \sigma_L$, each of the following should be bounded for $x \in \mathbb{R}$, including as $|x| \to \infty$,

$$I_1 = \operatorname{Re} \int \left(\alpha - \frac{\mathrm{d}}{\mathrm{d}x} \log(B) - \frac{\beta(A-\Omega)}{B} \right) \mathrm{d}x, \tag{27a}$$

$$I_2 = \operatorname{Re} \int \left(\alpha + \frac{\mathrm{d}}{\mathrm{d}x} \log(A-\Omega) + \frac{\gamma B}{A-\Omega} \right) \mathrm{d}x, \tag{27b}$$

$$I_3 = \operatorname{Re} \int \left(\alpha - \frac{\mathrm{d}}{\mathrm{d}x} \log(A-\Omega) + \frac{\beta C}{A+\Omega} \right) \mathrm{d}x, \tag{27c}$$

$$I_4 = \operatorname{Re} \int \left(-\alpha - \frac{\mathrm{d}}{\mathrm{d}x} \log(C) + \frac{\gamma(A+\Omega)}{C} \right) \mathrm{d}x, \tag{27d}$$

$$I_5 = \operatorname{Re} \int \left(-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x} \log(B) + \frac{\beta\Omega}{B} \right) \mathrm{d}x, \tag{27e}$$

$$I_6 = \operatorname{Re} \int \left(\alpha + \frac{\beta C}{A-\Omega} \right) \mathrm{d}x, \tag{27f}$$

$$I_7 = \operatorname{Re} \int \left(\alpha + \frac{\gamma B}{A+\Omega} \right) \mathrm{d}x, \tag{27g}$$

$$I_8 = \operatorname{Re} \int \left(-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x} \log(C) + \frac{\gamma\Omega}{C} \right) \mathrm{d}x. \tag{27h}$$

The boundedness of any of the expressions I_k (27) defines σ_L. For some problems, it is sufficient to analyze the boundedness integral condition without calculating the integral explicitly [7, 8, 14, 31]. For other problems, only part of the spectrum can be found without calculating the integral explicitly [12, 15, 16] (see Section 4.1). When the solutions of interest are given by elementary or elliptic functions, the integral may often be calculated explicitly [12, 15, 16] and used to determine σ_L [16].

4.1 The Lax spectrum for imaginary Ω

The results in this section hold only for those solutions and parameter values for which Assumptions 4 and 5 hold.

Assumption 4. *$\alpha \in i\mathbb{R}$.*

Assumption 5. *The quantities $\beta C/A$, and $\gamma B/A$ are periodic in x with the same period P as the solution.*

With Assumptions 4 and 5 the boundedness conditions for the integrals I_k (27) become

$$\operatorname{Re}\left\langle \frac{\beta(A-\Omega)}{B} \right\rangle = 0, \tag{28a}$$

$$\operatorname{Re}\left\langle \frac{\gamma B}{A-\Omega} \right\rangle = 0, \tag{28b}$$

$$\operatorname{Re}\left\langle \frac{\beta C}{A+\Omega} \right\rangle = 0, \tag{28c}$$

$$\operatorname{Re}\left\langle \frac{\gamma(A+\Omega)}{C} \right\rangle = 0, \tag{28d}$$

$$\operatorname{Re}\left\langle \frac{\beta\Omega}{B} \right\rangle = 0, \tag{28e}$$

$$\operatorname{Re}\left\langle \frac{\beta C}{A-\Omega} \right\rangle = 0, \tag{28f}$$

$$\operatorname{Re}\left\langle \frac{\gamma B}{A+\Omega} \right\rangle = 0, \tag{28g}$$

$$\operatorname{Re}\left\langle \frac{\gamma\Omega}{C} \right\rangle = 0, \tag{28h}$$

where $\langle \cdot \rangle = \frac{1}{P}\int_0^P \cdot \, \mathrm{d}x$ is the average over the period of the solution. Using (4) and (20), the conditions (28) are all equivalent. It follows immediately from (28h) that $\{\zeta \in \mathbb{C} : \Omega(\zeta) = 0\} \subset \sigma_L$.

We consider the special case of $\gamma = \kappa\beta^*$ where $\kappa = \pm 1$. We let

$$\beta(x;\zeta) = \eta(x;\zeta)e^{i\theta(x;\zeta)}, \qquad \gamma(x;\zeta) = \kappa\eta(x;\zeta)e^{-i\theta(x;\zeta)}, \tag{29}$$

where η and θ are real-valued functions with $\eta(x;\zeta) \geq 0$. Introducing the isospectral transformation

$$\Phi(x,t) = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} e^{-i\theta/2}\phi_1 \\ e^{i\theta/2}\phi_2 \end{pmatrix}, \tag{30}$$

the Lax pair becomes

$$\Phi_x = \begin{pmatrix} \hat{\alpha} & \hat{\beta} \\ \hat{\gamma} & -\hat{\alpha} \end{pmatrix}\Phi, \qquad \Phi_t = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & -\hat{A} \end{pmatrix}\Phi, \tag{31}$$

where

$$\hat{\alpha} = \alpha - i\theta_x/2, \qquad \hat{\beta} = \beta e^{-i\theta} = \eta, \qquad \hat{\gamma} = \gamma e^{i\theta} = \kappa\eta, \tag{32a}$$

$$\hat{A} = A, \qquad \hat{B} = e^{-i\theta}B, \qquad \hat{C} = e^{i\theta}C. \tag{32b}$$

Assumptions 1, 2, and 4 are satisfied in the hatted variables if they are satisfied in the original variables. Further, Ω is unchanged by this transformation, see (20).

Lemma 6. *If $\alpha \in i\mathbb{R}$, $A \in i\mathbb{R}$, and $\gamma = \kappa\beta^*$, where $\kappa = \pm 1$, then $C = \kappa B^*$.*

Proof. The compatibility conditions (4) in the hatted variables give

$$0 = \mathrm{Re}(\hat{A}_x) = \kappa\,\mathrm{Re}(\hat{C} - \kappa\hat{B}), \tag{33}$$

so that $\mathrm{Re}(\hat{C}) = \kappa\,\mathrm{Re}(\hat{B})$. Equations (4b) and (4c) give

$$\hat{C}_x - \kappa\hat{B}_x = -2\hat{\alpha}(\hat{C} + \kappa\hat{B}) + 4\kappa\eta\hat{A}. \tag{34}$$

Since $\alpha \in i\mathbb{R}$ implies $\hat{\alpha} \in i\mathbb{R}$, the only real term in the above equation is equated to zero:

$$\mathrm{Re}(-2\alpha(\hat{C} + \kappa\hat{B})) = 0. \tag{35}$$

As long as $\hat{\alpha} \neq 0$, $\mathrm{Im}(\hat{C}) = -\kappa\,\mathrm{Im}(\hat{B})$. If $\hat{\alpha} = 0$, then

$$\partial_x(\hat{C} + \kappa\hat{B}) = 0. \tag{36}$$

Since $\mathrm{Im}(\hat{C} + \kappa\hat{B}) = 0$ for $\hat{\alpha} \neq 0$, it must be zero for all x. Therefore $\hat{C} = \kappa\hat{B}^*$ and $C = \kappa B^*$.

□

Theorem 7. *Assume that A is periodic in x. Let $\mathcal{O} = \{\zeta \in \mathbb{C} : A \in i\mathbb{R} \text{ and } \beta^* = -\gamma\}$, then $\Omega(\mathcal{O}) \subset i\mathbb{R}$ and $\mathcal{O} \subset \sigma_L$.*

Proof. Let $\zeta \in \mathcal{O}$. By Lemma 6, $C^* = -B$ and

$$\Omega^2(\zeta) = A^2 - |B|^2 \leq 0, \tag{37}$$

thus $\Omega(\mathcal{O}) \subset i\mathbb{R}$. By (4a),

$$A_x = 2i\,\mathrm{Im}(\beta C), \tag{38}$$

so that

$$\operatorname{Re}\frac{\beta C}{A-\Omega} = -i\frac{\operatorname{Im}(\beta C)}{A-\Omega} = -\frac{A_x}{2(A+\Omega)} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\log(A+\Omega). \tag{39}$$

It follows from (28f) and the periodicity of A that $\mathcal{O} \subset \sigma_L$. □

Theorem 8. *Assume that A is periodic in x. Let $\mathcal{O} = \{\zeta \in \mathbb{C} : A \in i\mathbb{R}$ and $\beta = \gamma^*\}$, $\mathcal{S} = \{\zeta \in \mathbb{C} : \Omega(\zeta) \in i\mathbb{R}\}$, and $\mathcal{U} = \{\zeta \in \mathbb{C} : \Omega(\zeta) \in \mathbb{R} \setminus \{0\}\}$. Then $\mathcal{O} \subset \mathcal{S} \cup \mathcal{U}$ and $\mathcal{O} \cap \mathcal{S} \subset \sigma_L$. Additionally, if* $\operatorname{Arg}(\beta) = \operatorname{Arg}(B)$ *then $\mathcal{O} \cap \mathcal{U} \not\subset \sigma_L$.*

Proof. Let $\zeta \in \mathcal{O}$. By Lemma 6, $C^* = B$ and

$$\Omega^2(\zeta) = A^2 + |B|^2 \in \mathbb{R}, \tag{40}$$

so either $\zeta \in \mathcal{S}$ or $\zeta \in \mathcal{U}$. When $\zeta \in \mathcal{O} \cap \mathcal{S}$,

$$\begin{aligned}\operatorname{Re}\frac{\gamma B}{A-\Omega} &= \frac{1}{2}\left(\frac{\gamma B}{A-\Omega} + \frac{\gamma^* B^*}{A^*-\Omega^*}\right)\\ &= \frac{1}{2}\left(\frac{\gamma B - \beta C}{A-\Omega}\right) = -\frac{A_x}{2(A-\Omega)} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\log(A-\Omega).\end{aligned} \tag{41}$$

It follows from (28b) and the periodicity of A that $\mathcal{O} \cap \mathcal{S} \subset \sigma_L$.

Next, let $\zeta \in \mathcal{O} \cap \mathcal{U}$. The integral conditions induced by the Lax pair (32) are given by (28) in the hatted variables. Further, $C^* = B$ implies that $\hat{C}^* = \hat{B}^*$. Condition (28h) becomes

$$\operatorname{Re}\frac{\hat{\gamma}\Omega}{\hat{C}} = \frac{\kappa\eta\Omega}{|\hat{B}|^2}\operatorname{Re}\hat{B}. \tag{42}$$

In order for $\Omega \in \mathbb{R} \setminus \{0\}$,

$$0 \le A^2 < -|B|^2 = -|\hat{B}|^2. \tag{43}$$

If $\operatorname{Arg}(B) = \operatorname{Arg}(\beta)$, then $\hat{(B)} \in \mathbb{R}$. Since $|\hat{B}|$ is nonzero, $\operatorname{Re}(\hat{B})$ is nonzero. Equations (42) and (28h) imply that $\zeta \notin \sigma_L$, so that $\mathcal{O} \cap \mathcal{U} \not\subset \sigma_L$. □

5 The AKNS hierarchy

The AKNS hierarchy [3] contains a variety of physically important nonlinear evolution equations. For the Lax pair (2) which are members of the AKNS hierarchy

$$\alpha = -i\zeta, \qquad \beta = q(x,t), \qquad \gamma = r(x,t), \tag{44}$$

where $r(x,t)$ and $q(x,t)$ may be complex-valued functions. The compatibility conditions (3) become

$$A_x = qC - rB, \tag{45a}$$

$$q_t = B_x + 2i\zeta B + 2Aq, \tag{45b}$$

$$r_t = C_x - 2i\zeta C - 2Ar. \tag{45c}$$

Equation (1) is equivalent to (45b) and (45c) by relating q and r to u. In what follows we examine stationary solutions so that $q_t = r_t = 0$ and the compatibility conditions (45) are

$$A_x = qC - rB, \tag{46a}$$
$$B_x = -2i\zeta B - 2Aq, \tag{46b}$$
$$C_x = 2i\zeta C + 2Ar. \tag{46c}$$

The x-equation of the Lax pair (2a,44) may be written in terms of a problem with spectral parameter ζ,

$$\zeta\phi = \begin{pmatrix} i\partial_x & -iq \\ ir & -i\partial_x \end{pmatrix} \phi = L\phi. \tag{47}$$

The operator L is self adjoint if and only if $r = q^*$. When L is self adjoint, $\sigma_L \subset \mathbb{R}$. Determining σ_L when L is not self adjoint is significantly more complicated.

Assumptions 1 and 2 are satisfied by α, β and γ, by construction. Also, $\alpha \in i\mathbb{R}$ is constant with respect to x for $\zeta \in \mathbb{R}$, which satisfies Assumption 4 and the first part of Assumption 5. The t-equation for equations in the AKNS hierarchy may be found by assuming A, B and C have power series representations. A recursion operator is used to find A, B, and C [21, Chapter 2]. The recursion operator depends and acts only on homogeneous combinations of q and r. Therefore, it gives rise only to homogeneous combinations of q, r, and their derivatives, *i.e.*, terms of the form $q_{nx}^j q_{mx}^k$ where $n,\ m,\ j,\ k \in \mathbb{N}$ and $j+k$ is even. Thus Assumptions 1 and 2 are also satisfied for A, B and C. If q and r are related by

$$r = \kappa q^*, \tag{48}$$

where $\kappa = \pm 1$, then

$$q(x) = e^{i\theta(x)} Q(x), \quad \text{and} \quad r(x) = \kappa e^{-i\theta(x)} Q(x), \tag{49}$$

where Q and θ are real-valued functions and $|q| = Q$. Assuming $|q|$ is periodic with period P, it follows that A has period P by the above arguments. Using (46), Assumption 5 is satisfied. The reduction (48) is a common reduction for members of the AKNS hierarchy (see Sections 3 and 6.1). The condition (48) is the same as that studied in Section 4.1.

Theorem 8 applies to the entire Lax spectrum for problems with self-adjoint Lax pairs since $\sigma_L \subset \mathbb{R}$. In particular,

$$\sigma_L = \{\zeta \in \mathbb{R} : \Omega(\zeta) \in i\mathbb{R}\}. \tag{50}$$

Theorem 7 applies to non self-adjoint Lax pairs with $r = -q^*$ and for $\zeta \in \sigma_L \cap \mathbb{R}$ such that $\alpha(\zeta) \in i\mathbb{R}$. In particular, $\mathbb{R} \subset \sigma_L$ and $\Omega(\mathbb{R}) \subset i\mathbb{R}$.

It is clear that the real line is an important part of the Lax spectrum for members of the AKNS hierarchy. In fact, we have the following corollary.

Corollary 9. $\Omega(\sigma_L \cap \mathbb{R}) \subset i\mathbb{R}$ *for members of the AKNS hierarchy satisfying* (48).

In other words, **any eigenmodes of the stability problem of** (1) **that are parameterized by real** $\zeta \in \sigma_L$ **do not give rise to instabilities.**

6 Examples

Many equations fit the above description, *i.e.*, Theorems 7 and 8 can be applied directly. For other equations, Theorems 7 and 8 cannot be applied directly, but the spectrum has a similar structure and can be found in a similar way. We begin with two equations in the AKNS hierarchy for which the results in Section 5 apply directly. We provide one example which is not in the AKNS hierarchy but for which Theorems 7 and 8 apply directly. Finally, we provide three examples for which Theorems 7 and 8 do not apply directly but for which the spectrum is analyzed in a similar way.

6.1 The modified Korteweg-de Vries equation

The modified Korteweg-de Vries equation (mKdV) is given by

$$u_t - 6\kappa u^2 u_x + u_{xxx} = 0, \tag{51}$$

where u is real valued and $\kappa = -1$ and $\kappa = 1$ correspond to the focusing and defocusing cases respectively. Equation (51) is a member of the AKNS hierarchy (Section 5) with [3]

$$A = -4i\zeta^3 - 2i\zeta qr, \qquad B = 4\zeta^2 q + 2q^2 r + 2i\zeta q_x - q_{xx}, \qquad C = 4\zeta^2 r + 2qr^2 - 2i\zeta r_x - r_{xx}, \tag{52}$$

and $r = \kappa q = \kappa u$. Letting $(y, \tau) = (x - ct, t)$, where $c \in \mathbb{R}$ is constant, gives mKdV in the traveling frame,

$$u_\tau - cu_y - 6\kappa u^2 u_y + u_{yyy} = 0. \tag{53}$$

The Lax pair (53) given by

$$\phi_y = X\phi, \qquad \phi_\tau = (T + cX)\phi. \tag{54}$$

The elliptic solutions of mKdV are found upon equating $u_\tau = 0$ [14]. Theorem 7 applies when $\kappa = -1$ and Theorem 8 applies when $\kappa = 1$. The squared-eigenfunction connection for the mKdV equation is the same as for NLS (13). Since $\lambda = 2\Omega(\zeta)$ where $\zeta \in \sigma_L = \{\zeta \in \mathbb{R} : \Omega(\zeta) \in i\mathbb{R}\}$, the elliptic solutions for the defocusing mKdV equation are spectrally stable [14]. For the focusing mKdV equation, the elliptic solutions are stable with respect to perturbations which correspond to $\zeta \in \mathbb{R}$. To determine the full stability spectrum, one must compute σ_L off the real line. To do so, one may use the integral condition(27) or the Floquet discriminant (Section A). Plots of the Lax spectrum for the defocusing case can be found in [14, Figures 7 and 8] and for the focusing case in [14, Figures 9 and 13]. Plots of the stability spectrum for the defocusing case can be found in [14, Figures 3 and 5] and for the focusing case in [14, Figures 10 and 14].

6.2 The sine- and sinh-Gordon equations

The sine-Gordon (s-G) equation in light-cone coordinates is given by

$$u_{\xi\eta} = \sin u, \tag{55}$$

where $u(\xi, \eta)$ is real valued. Equation (55) is a member of the AKNS hierarchy (Section 5) with [3]

$$A = \frac{i}{4\zeta}\cos(u), \qquad B = \frac{i}{4\zeta}\sin(u), \qquad C = \frac{i}{4\zeta}\sin(u). \tag{56}$$

We write (ξ, η) instead of (x, t) to distinguish between the light-cone coordinates (ξ, η) and the space-time coordinates (x, t). Equation (55) is equivalent to the compatibility of mixed derivatives, $\partial_\eta v_\xi = \partial_\xi v_\eta$, by requiring that $r = -q = u_x/2$. Since $r = -q$, (55) is not self adjoint. A self-adjoint variant of s-G is the sinh-Gordon (sh-G) equation,

$$u_{\xi\eta} = \sinh u. \tag{57}$$

Equation (57) is a member of the AKNS hierarchy (Section 5) with [3]

$$A = \frac{i}{4\zeta}\cosh(u), \qquad B = -\frac{i}{4\zeta}\sinh(u), \qquad C = \frac{i}{4\zeta}\sinh(u), \tag{58}$$

and is equivalent to the compatibility of mixed derivatives under the reduction $r = q = u_x/2$.

Assumptions 1-4 hold for both the s-G equation and the sh-G equation if $\zeta \in \mathbb{R}$ and u is periodic. Since $r = q$, the sh-G equation is a self-adjoint member of the AKNS hierarchy, so $\sigma_L \subset \mathbb{R}$ and Theorem 8 applies. Theorem 7 applies to the s-G equation, *i.e.*, $\mathbb{R} \subset \sigma_L$. To transform (55) from light-cone to laboratory coordinates, we let $(x,t) = (\eta + \xi, \eta - \xi)$ to obtain

$$u_{tt} - u_{xx} + \sin(u) = 0. \tag{59}$$

The same coordinate transformation on (57) gives

$$u_{tt} - u_{xx} + \sinh(u) = 0. \tag{60}$$

The Lax pair for both systems is

$$w_x = \frac{1}{2}(T+X)w = \hat{X}w, \qquad w_t = \frac{1}{2}(T-X)w = \hat{T}w. \tag{61}$$

We move the s-G equation and the sh-G equation to a traveling frame by letting $(z,\tau) = (x - Vt, t)$ for constant $V \in \mathbb{R}$ and find

$$(V^2-1)u_{zz} - 2Vu_{z\tau} + u_{\tau\tau} + \sin(u) = 0, \tag{62}$$

and

$$(V^2-1)u_{zz} - 2Vu_{z\tau} + u_{\tau\tau} + \sinh(u) = 0, \tag{63}$$

respectively. The new Lax pair is given by

$$w_z = \hat{X}w, \qquad w_\tau = (\hat{T} + V\hat{X})w = \tilde{T}w. \tag{64}$$

Stationary solutions are found by letting $u_\tau = 0$. The s-G equation and the sh-G equation in lab coordinates do not fit into the AKNS hierarchy. However, when $\zeta \in \mathbb{R}$ and u is periodic, Assumptions 1-4 hold for both equations in lab coordinates. Theorems 7 and 8 apply to (59) and (60) respectively. Theorems 7 and 8 apply in both light-cone and lab coordinates since the transform to lab coordinates is isospectral: the eigenfunctions of (64) are bounded if and only if the eigenfunctions of the problem in light-cone coordinates are bounded. We continue to study the spectral problem for pedagogical purposes.

The Lax pair (61) defines a quadratic eigenvalue problem (QEP),

$$Q(\zeta)w = (M\zeta^2 + N\zeta + K)w = 0. \tag{65}$$

There are two choices for M, N, and K:

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad N_1 = \begin{pmatrix} -2i\partial_x & iq \\ -ir & 2i\partial_x \end{pmatrix}, \qquad K_1 = \begin{pmatrix} ia & ib \\ ic & -ia \end{pmatrix}, \tag{66a}$$

$$M_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad N_2 = \begin{pmatrix} -2i\partial_x & iq \\ ir & -2i\partial_x \end{pmatrix}, \qquad K_2 = \begin{pmatrix} ia & ib \\ -ic & ia \end{pmatrix}, \tag{66b}$$

where

$$a = \zeta A, \quad b = \zeta B, \quad c = \zeta C. \tag{67}$$

Then $\zeta \in \mathbb{C}$ is an eigenvalue of Q if $Q(\zeta)w = 0$ for all bounded w. A QEP is classified as self adjoint if M, N, and K are self adjoint [33]. The eigenvalues for self-adjoint QEPs are either real or come in complex-conjugate pairs. If M_1, N_1, and K_1 are chosen, then $Q(\lambda)$ is self adjoint if $r = q^*$, $a^* = -a$, and $c^* = b$, which is the case for the sh-G equation. If M_2, N_2, and K_2 are chosen, then $Q(\lambda)$ is self adjoint if $r = -q^*$, $a^* = -a$, and $c^* = -b$, which is the case for the s-G equation. It follows that for either equation, the Lax spectrum consists of real or complex-conjugate spectral elements. This confirms what we know from the isospectral transform to light-cone coordinates. The whole real line is part of the Lax spectrum for the s-G equation and the Lax spectrum for the sh-G equation is a subset of the real line. To determine the subset of σ_L off the real line, one may use the integral condition (27) or the Floquet discriminant (Section A) (see [12]). Plots of the Lax spectrum for the s-G equation can be found in [12, Figures 7 and 8]. Plots of the stability spectrum for the s-G equation can be found in [12, Figures 6 and 8].

6.3 Derivative nonlinear Schrödinger equation

The derivative NLS (dNLS) equation,

$$iq_t = -q_{xx} + i\kappa(|q|^2 q)_x, \qquad \kappa = \pm 1, \tag{68}$$

was first solved on the whole line using the Inverse Scattering Transform in [24]. The Lax pair for (68) is given by (2) with [24]

$$\alpha = -i\zeta^2, \qquad \beta = q\zeta, \qquad \gamma = r\zeta, \tag{69a}$$

$$A = -2i\zeta^4 - i\zeta^2 rq, \qquad B = 2\zeta^3 q + i\zeta q_x + \zeta r q^2, \qquad C = 2\zeta^3 r - i\zeta r_x + \zeta r^2 q. \tag{69b}$$

where $r = \kappa q^* \in \mathbb{C}$. Using $q(x,t) \mapsto e^{-i\omega t} q(x,t)$ where ω is a real constant, (68) becomes

$$iq_t = -q_{xx} + i\kappa(|q|^2 q)_x - \omega q, \tag{70}$$

and $A \mapsto A + i\omega/2$; otherwise (69) remains the same. Stationary solutions satisfy

$$-q_{xx} + i\kappa(|q|^2 q)_x - \omega q = 0. \tag{71}$$

Quasi-periodic elliptic solutions to the stationary problem were found in [22].

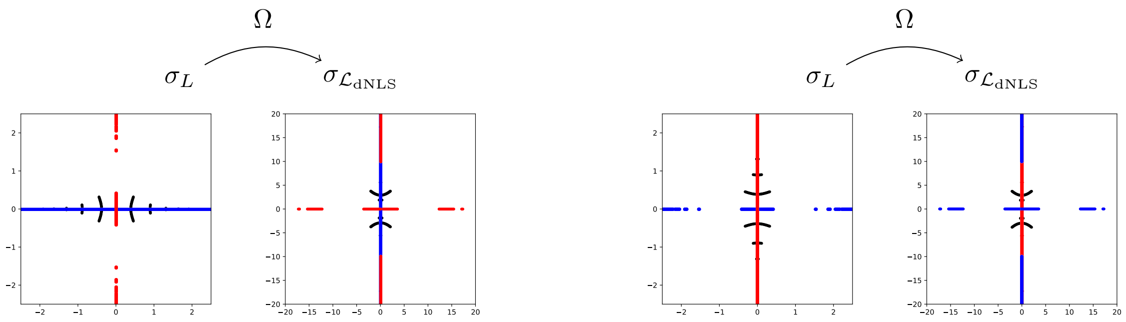
The Lax pair defines a QEP (65). There are two choices for M, N, and K,

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad N_1 = \begin{pmatrix} 0 & iq \\ -ir & 0 \end{pmatrix}, \qquad K_1 = \begin{pmatrix} -i\partial_x & 0 \\ 0 & i\partial_x \end{pmatrix}, \tag{72a}$$

$$M_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad N_2 = \begin{pmatrix} 0 & iq \\ ir & 0 \end{pmatrix}, \qquad K_2 = \begin{pmatrix} -i\partial_x & 0 \\ 0 & -i\partial_x \end{pmatrix}. \tag{72b}$$

If M_1, N_1, and K_1 are chosen, then $Q(\lambda)$ is self adjoint if $r = q^*$. If M_2, N_2, and K_2 are chosen, then $Q(\lambda)$ is self adjoint if $r = -q^*$. It follows that eigenvalues are real or come in complex-conjugate pairs for either choice of κ.

Since Assumptions 1-4 hold for $\zeta \in \mathbb{R} \cup i\mathbb{R}$ and $|q|$ periodic, Theorems 7 and 8 apply in some cases. Theorem 7 applies when $\beta^* = -\gamma$, *i.e.*, when $\beta^* = -\gamma$ if $\zeta \in \mathbb{R}$ and $\kappa = -1$ or $\zeta \in i\mathbb{R}$ and $\kappa = 1$. Theorem 8 applies when $\beta^* = \gamma$, *i.e.*, when $\beta^* = \gamma$ if $\zeta \in \mathbb{R}$ and $\kappa = 1$ or $\zeta \in i\mathbb{R}$ and $\kappa = -1$. Defining $\Omega_i = \{\zeta \in \mathbb{C} : \Omega(\zeta) \in i\mathbb{R}\}$, $\mathbb{R} \cup (\Omega_i \cap i\mathbb{R}) \subset \sigma_L$ and $\Omega(\mathbb{R} \cup (\Omega_i \cap i\mathbb{R})) \subset i\mathbb{R}$ for $\kappa = -1$ (see Figure 2a). If $\kappa = 1$, $i\mathbb{R} \cup (\mathbb{R} \cap \Omega_i) \subset \sigma_L$ and $\Omega(i\mathbb{R} \cup (\mathbb{R} \cap \Omega_i)) \subset i\mathbb{R}$ (see Figure 2b). To compute the spectrum off of the



(a) (b)

Figure 2: The real *vs.* imaginary part of the Lax and stability spectrum (left and right respectively) for a periodic solution of (a) the focusing dNLS equation and (b) the defocusing dNLS equation. The Lax spectrum is computed numerically using [11] and the stability spectrum is the image under Ω. The Lax spectrum on the real and imaginary axes are colored blue and red respectively. Their image under the map Ω is colored in the stability spectrum accordingly. The Lax Spectrum off the real and imaginary axes is black.

real or imaginary axes, one must examine the integral conditions (27) or construct the Floquet discriminant (Appendix A).

6.4 The Korteweg-de Vries equation

Perhaps the best-known integrable equation is the Korteweg-de Vries (KdV) equation [4],

$$u_t + 6uu_x + u_{xxx} = 0, \tag{73}$$

where u is a real-valued function of x and t. The KdV equation is a singular member of the AKNS hierarchy: it is obtained using the AKNS Lax pair (44) with $r = -1$ [3]. When r is constant, the Lax pair can be rewritten as an eigenvalue problem with spectral parameter ζ^2. With $r = -1$,

$$\zeta^2 v_2 = -(q + \partial_x^2)v_2 = L_{\text{KdV}}\, v. \tag{74}$$

The operator L_{KdV} is self adjoint, so the KdV equation is said to have a self-adjoint Lax pair even though the Lax operator L (47) is not. Since L_{KdV} is self adjoint, $\zeta^2 \in \mathbb{R}$.

Defining $(y, \tau) = (x - ct, t)$, we obtain the KdV equation in the traveling frame,

$$u_\tau - cu_y + 6uu_y + u_{yyy} = 0. \tag{75}$$

The periodic stationary solutions of (75) are the cnoidal waves which are defined in terms of Jacobi elliptic functions [1, 10]. In the traveling frame, the Lax pair is given by (2) with

$$\alpha = -i\zeta, \qquad \beta = u, \qquad \gamma = -1, \tag{76a}$$

$$A = -4i\zeta^3 + 2iu\zeta - u_x - ic\zeta, \qquad B = 4u\zeta^2 + 2i\zeta u_x - 2u^2 - u_{xx} + cu, \qquad C = -4\zeta^2 + 2u - c. \tag{76b}$$

Using (20) and (75) with $u_\tau = 0$,

$$\Omega^2 = 2a - (c + 4\zeta^2)(-k + c\zeta^2 + 4\zeta^4), \tag{77}$$

where

$$k = 3u^2 - cu + u_{xx} \quad \text{and} \quad 2a = u_x - 2ku + 2u^3 - cu^2 \tag{78}$$

are constants found from integrating (75) with $u_\tau = 0$. Assumptions 1-4 hold for $\zeta \in \mathbb{R}$ if u is periodic. Since γ and C are real valued for $\zeta \in \mathbb{R}$, $\zeta \in \mathbb{R} \cap \sigma_L$ if $\Omega(\zeta) \in i\mathbb{R}$ (28h). The analysis of the Lax spectrum for the cnodial waves of the KdV equation can be found in [10] including plots of the real Lax spectrum [10, Figure 4] and of the imaginary stability spectrum [10, Figure 1].

6.5 Vector and matrix nonlinear Schrödinger equations

The *Manakov system* or two-component Vector NLS (VNLS) equation is given by

$$\begin{aligned} i\frac{\partial q_1}{\partial t} + \frac{\partial^2 q_1}{\partial x^2} + 2(|q_1|^2 + |q_2|^2)q_1 = 0, \\ i\frac{\partial q_2}{\partial t} + \frac{\partial^2 q_2}{\partial x^2} + 2(|q_1|^2 + |q_2|^2)q_2 = 0, \end{aligned} \tag{79}$$

where q_1 and q_2 are complex-valued functions. The system (79) was shown to be integrable in [30]. Its finite-genus solutions (including its elliptic solutions) were explicitly constructed in [17]. Its Lax pair is

$$v_x = \begin{pmatrix} \alpha & \beta^\intercal \\ \gamma & \rho \end{pmatrix} v = Xv, \qquad v_t = \begin{pmatrix} A & B^\intercal \\ C & D \end{pmatrix} v = Tv, \tag{80}$$

with

$$\alpha = -i\zeta, \qquad \beta = q, \qquad \gamma = -q^*, \qquad \rho = i\zeta I_2, \tag{81a}$$

$$A = -2i\zeta^2 + iq^\intercal q^*, \qquad B = 2\zeta q + iq_x, \qquad C = -2\zeta q^* + iq_x^*, \qquad D = 2i\zeta^2 I_2 - iq^* q^\intercal, \tag{81b}$$

and $q = (q_1, q_2)^\intercal$. The compatibility conditions are

$$A_x = \beta^\intercal C - B^\intercal \gamma, \tag{82a}$$

$$B_x = 2\alpha B^\intercal - 2A\beta^\intercal, \tag{82b}$$

$$C_x = 2\rho C + 2\gamma A, \tag{82c}$$

$$D_x = \gamma B^\intercal - C\beta^\intercal. \tag{82d}$$

As before, Ω is found by separation of variables and satisfies

$$\begin{pmatrix} A - \Omega & B^\intercal \\ C & D - \Omega I_2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0, \tag{83}$$

for nontrivial eigenvectors $\phi = (\phi_1, \phi_2)^\intercal$. Note that Ω does not have the form (20). Instead, Ω satisfies

$$0 = \det \begin{pmatrix} A - \Omega & B^\intercal \\ C & D - \Omega I_2 \end{pmatrix} = \begin{cases} (A-\Omega)\det\left((D - \Omega I_2) - CB^\intercal/(A-\Omega)\right), & \Omega \notin \sigma(A), \\ \det(D - \Omega I_2)\left((A - \Omega) - B^\intercal (D - \Omega I_2)^{-1} C\right), & \Omega \notin \sigma(D), \end{cases} \tag{84}$$

where $\sigma(L)$ represents the spectrum of L. As usual, $\Omega(\zeta)$ defines a Riemann surface. In the genus-one case [17], it is represented by

$$
\begin{aligned}
f(\zeta,\Omega) &= (\Omega+2i\zeta^2)(\Omega-2i\zeta^2)^2+(2\lambda_2\zeta+\lambda_3)(\Omega-2i\zeta^2)+\mu_0=0, && \text{(85a)}\\
\lambda_2 &= -i(q^\intercal \bar{q}_x - q_x^\intercal \bar{q}), && \text{(85b)}\\
\lambda_3 &= q_x^\intercal \bar{q}_x + (q^\intercal \bar{q})^2, && \text{(85c)}\\
\mu_0 &= i\left|q_{1,x}q_2 - q_{2,x}q_1\right|^2. && \text{(85d)}
\end{aligned}
$$

Since

$$
\det(D-\Omega I_2) = (\Omega-2i\zeta^2)(A+\Omega), \tag{86}
$$

we use the second expression in (84) only when $\Omega = 2i\zeta^2$ or $A+\Omega=0$. But $\Omega=2i\zeta^2$ satisfies (85a) only if q_2 and q_1 are proportional, and $\Omega = A$ satisfies (85a) only if $|q_1|^2+|q_2|^2$ is constant. In the first case, (79) reduces to two uncoupled NLS equations, for which the spectrum is known. In the second case (79) reduces to two uncoupled linear NLS equations, for which the spectrum is known. Therefore, we assume that $D-\Omega I_2$ is invertible.

The eigenfunctions of (80) are

$$
v(x,t) = e^{\Omega t} y_1(x) \begin{pmatrix} a \\ -(D-\Omega I_2)^{-1}Ca \end{pmatrix}, \tag{87}
$$

where $a \in \mathbb{C}$ is an arbitrary scalar. The scalar function $y_1(x)$ is determined by substitution in the x equation (80):

$$
y_1' = \left(\alpha - \beta^\intercal (D-\Omega I_2)^{-1}C\right) y_1, \tag{88}
$$

so that

$$
y_1 = \hat{y}_1 \exp\left(\int_x \left(\alpha - \beta^\intercal (D-\Omega I_2)^{-1}C\right)\,\mathrm{d}x\right), \tag{89}
$$

where $\hat{y}_1$ is a constant. Thus, $\zeta \in \sigma_L$ provided that

$$
\left|\mathrm{Re}\int_x \left(\alpha - \beta^\intercal (D-\Omega I_2)^{-1}C\right)\,\mathrm{d}x\right| < \infty \tag{90}
$$

is bounded for all $x \in \mathbb{R}$. For periodic potentials and $\zeta \in \mathbb{R}$, this becomes

$$
\mathrm{Re}\left\langle \beta^\intercal (D-\Omega I_2)^{-1}C\right\rangle = 0. \tag{91}
$$

For $\zeta \in \mathbb{R}$,

$$
A^\dagger = -A, \qquad C^\dagger = -B^\intercal, \qquad D^\dagger = -D, \qquad (\beta^\intercal)^\dagger = -\gamma, \tag{92}
$$

where $F^\dagger = (F^*)^\intercal$ is the conjugate transpose of F. It follows that T defined by (80) is skew-adjoint and $\Omega \in i\mathbb{R}$. Further,

$$
(D-\Omega I_2)^\dagger = -(D-\Omega I_2). \tag{93}
$$

It follows that

$$\begin{aligned}
\operatorname{Re}\beta^{\intercal}(D-\Omega I_2)^{-1}C &= \frac{1}{2}\left[\beta^{\intercal}(D-\Omega I_2)^{-1}C + C^{\dagger}((D-\Omega I_2)^{-1})^{\dagger}(\beta^{\intercal})^{\dagger}\right] \\
&= \frac{1}{2}\left[\operatorname{Tr}\left(\beta^{\intercal}(D-\Omega I_2)^{-1}C - B^{\intercal}(D-\Omega I_2)^{-1}\gamma\right)\right] \\
&= \frac{1}{2}\left[\operatorname{Tr}\left((D-\Omega I_2)^{-1}C\beta^{\intercal} - (D-\Omega I_2)^{-1}\gamma B^{\intercal}\right)\right] \\
&= \frac{1}{2}\left[\operatorname{Tr}\left((D-\Omega I_2)^{-1}\left(C\beta^{\intercal} - \gamma B^{\intercal}\right)\right)\right] \\
&= \frac{1}{2}\left[\operatorname{Tr}\left((D-\Omega I_2)^{-1}(-D_x)\right)\right] \\
&= -\frac{1}{2}\frac{\partial_x \det(D-\Omega I_2)}{\det(D-\Omega I_2)} \\
&= -\frac{1}{2}\partial_x \log\det(D-\Omega I_2),
\end{aligned} \tag{94}$$

so that

$$\operatorname{Re}\left\langle \beta^{\intercal}(D-\Omega I_2)^{-1}C\right\rangle = 0, \tag{95}$$

and $\Omega(\zeta) \in i\mathbb{R}$ for $\zeta \in \mathbb{R}$, as illustrated in Figure 3.

The work above can be generalized to the Matrix NLS (MNLS) equation,

$$iU_t + U_{xx} - 2\kappa U U^* U = 0, \tag{96}$$

where U is an $\ell_1 \times \ell_2$ matrix, and $\kappa = -1$ and $\kappa = 1$ correspond to the focusing and defocusing cases respectively. The Lax pair for the MNLS equation is given by

$$\Psi_x = \begin{pmatrix} -i\zeta I_{\ell_1} & Q \\ R & i\zeta I_{\ell_2} \end{pmatrix} \Psi = X\Psi, \tag{97a}$$

$$\Psi_t = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Phi = T\Psi, \tag{97b}$$

$$A = -2i\zeta^2 I_{\ell_1} - iQR, \qquad B = 2\zeta Q + iQ_x, \qquad C = 2\zeta R - iR_x, \qquad D = 2i\zeta^2 I_{\ell_2} + iRQ. \tag{97c}$$

Here, Q and R are $\ell_1 \times \ell_2$ and $\ell_2 \times \ell_1$ complex-valued matrices respectively and I_n is the $n \times n$ identity matrix. The x equation may be written as a spectral problem,

$$\zeta\Psi = \begin{pmatrix} iI_{\ell_1}\partial_x & -iQ \\ iR & -iI_{\ell_2}\partial_x \end{pmatrix} \Psi = L\Psi. \tag{98}$$

L is self adjoint if $R^* = Q$, hence $\sigma_L \subset \mathbb{R}$ if $R^* = Q$. The compatibility conditions are the same as (82d).

6.6 PT-symmetric reverse space nonlocal NLS equation

The PT-symmetric reverse space nonlocal NLS equation is given by [5]

$$i\Psi_t(x,t) + \frac{1}{2}\Psi_{xx}(x,t) - \kappa\Psi(x,t)^2\Psi^*(-x,t) = 0, \tag{99}$$

where $\kappa = \pm 1$. This equation is a member of the AKNS hierarchy (Section 5) with [5]

$$A(x,t) = -i\zeta^2 - iqr/2, \qquad B = \zeta q + iq_x/2, \qquad C = \zeta r - ir_x/2, \tag{100}$$

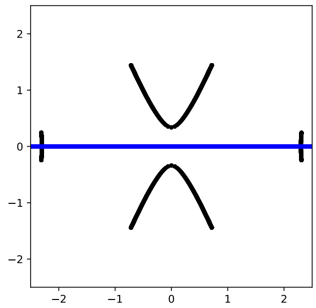


Figure 3: The Lax spectrum (Re(ζ) *vs.* Im(ζ)) for the Manakov system (80) with initial condition $q_1(x,0) = 4\cos(x),\ q_2(x,0) = 10\cos(x)$. The real line is colored blue. The Lax spectrum is computed numerically using the method described in [11].

and $r(x,t) = \kappa q^*(-x,t) = \kappa \Psi^*(-x,t) \in \mathbb{C}$. Solutions of (5) with

$$\Psi(x,t) = \Psi(-x,t), \tag{101}$$

are also solutions of (99). Thus all even solutions of 5 examined in [8] for $\kappa = 1$ and in [16] for $\kappa = -1$ are solutions to (99). These solutions, and other periodic and quasi-periodic solutions of (99), were first reported in [26]. Almost every solution found in [26] is even in x, except for one which is odd. For the even solutions, the Lax spectrum remains unchanged since $\Psi(x,t) = \Psi(-x,t)$. For $\kappa = 1$, the Lax pair is not self adjoint, and σ_L is not necessarily a subset of $\mathbb{R}$. With the equation written in a uniformly rotating frame so that all components are time independent, we see that for $\zeta \in \mathbb{R}$, Assumptions 1-4 hold,

$$A^*(-x) = -A(x), \qquad \text{and} \qquad C^*(-x) = \kappa B(x). \tag{102}$$

It follows that $\zeta \in \sigma_L$ if any of the conditions (28) hold. With $\zeta \in \mathbb{R}$,

$$\begin{aligned} \Omega^2(\zeta) &= A(x)^2 + B(x)C(x) = A(-x)^2 + B(-x)C(-x) \\ &= (A^2 + BC)^* = \left(\Omega^2(\zeta)\right)^*, \end{aligned} \tag{103}$$

so $\Omega(\mathbb{R}) \subset \mathbb{R} \cup i\mathbb{R}$.

If $\Omega(\zeta) \in i\mathbb{R}$,

$$\begin{aligned} \operatorname{Re} \int_0^P \left(\frac{q(x)C(x)}{A(x) - \Omega} \right) \mathrm{d}x &= \frac{1}{2} \left(\int_0^P \frac{q(x)C(x)}{A(x) - \Omega} \,\mathrm{d}x + \int_0^P \frac{q^*(x)C^*(x)}{A^*(x) - \Omega^*} \,\mathrm{d}x \right) \\ &= \frac{1}{2} \left(\int_0^P \frac{q(x)C(x)}{A(x) - \Omega} \,\mathrm{d}x - \int_0^{-P} \frac{q^*(-x)C^*(-x)}{A^*(-x) - \Omega^*} \,\mathrm{d}x \right) \\ &= \frac{1}{2} \left(\int_0^P \frac{q(x)C(x)}{A(x) - \Omega} \,\mathrm{d}x + \int_{-P}^0 \frac{q^*(-x)C^*(-x)}{-A(x) + \Omega} \,\mathrm{d}x \right) \\ &= \frac{1}{2} \left(\int_0^P \frac{q(x)C(x)}{A(x) - \Omega} \,\mathrm{d}x + \int_0^P \frac{q^*(-x)C^*(-x)}{-A(x) + \Omega} \,\mathrm{d}x \right) \\ &= \frac{1}{2} \left(\int_0^P \frac{q(x)C(x) - q^*(-x)C^*(-x)}{A(x) - \Omega} \,\mathrm{d}x \right) \\ &= \frac{1}{2} \int_0^P \frac{A_x}{A(x) - \Omega} \,\mathrm{d}x = 0, \end{aligned} \tag{104}$$

since A is periodic when q and r are. It follows that $\{\zeta \in \mathbb{R} : \Omega(\zeta) \in i\mathbb{R}\} \subset \sigma_L$ for $\kappa = \pm 1$.

7 Conclusion

The stability spectrum and the Lax spectrum for solutions of many integrable equations on the whole line have been characterized for some time. The same level of understanding for the periodic problem does not exist. One reason the whole line problem is more straightforward to study is the ability to do spatial asymptotics to find the essential spectrum which contains the unbounded components of the spectrum. In this work, **we have given a complete characterization of all unbounded components of the Lax spectrum for a number of integrable equations.**

We have shown that for a number of members of the AKNS hierarchy, real Lax spectra corresponds to stable modes of the linearization. We have also shown how this might occur for non-AKNS equations. We have given two theorems with easily verifiable assumptions (7 and 8) for determining this. The methods described in this paper can be applied to other equations not mentioned in Section 6. Some examples include: Hirota's equation (or the mixed generalized NLS-generalized mKdV equation) [20, 27], the Modified Vector dNLS equation [13], the Massive Thirring Model [25], the O_4 nonlinear σ-model [29], the complex reverse space-time nonlocal mKdV equation [6], and the reverse space-time nonlocal generalized sine-Gordon equations [6].

Appendix A The Floquet discriminant

A common tool for characterizing the Lax spectrum for periodic potentials is the Floquet discriminant [2, 9, 18, 28]. The Floquet discriminant is typically approximated numerically since the eigenfunctions of the x equation are unknown for generic potentials. In our setting, we have explicit expressions for the eigenfunctions (23). Since $\Omega(\zeta)$ is defined by its square, (20) defines two different values of Ω for every value of ζ for which $\Omega(\zeta) \neq 0$. Hence (23) defines the two linearly independent solutions of (2) except for when $\Omega(\zeta) = 0$. When $\Omega(\zeta) = 0$, only one solution is generated by (23) and a second solution is found using the method of reduction of order. The solution found by reduction of order is algebraically unbounded so it is not an eigenfunction. For $\Omega(\zeta) \neq 0$, the two eigenfunctions of (2) are

$$\phi_\pm(x,t) = e^{\pm\Omega t} y_\pm(x) \begin{pmatrix} -B(x) \\ A(x) - \Omega_\pm \end{pmatrix}. \tag{105}$$

We use one choice of (23) and one choice of y_1. The following computations proceed similarly for the other choices. A fundamental matrix solution (FMS) of (2a) is given by

$$M(x) = \begin{pmatrix} -B(x)y_+(x) & -B(x)y_-(x) \\ (A(x) - \Omega_+)y_+(x) & (A(x) - \Omega_-)y_-(x) \end{pmatrix}, \tag{106}$$

where dependence on t has been omitted. The FMS normalized to the identity is given by

$$\tilde{M}(x; x_0) = M^{-1}(x_0)M(x). \tag{107}$$

To simplify notation, we define

$$I_\pm(x;\zeta) = -\int \left(\alpha + \frac{\beta C}{A - \Omega_\pm}\right) \, \mathrm{d}x. \tag{108}$$

In this section we use Assumption 10 instead of Assumption 4.

Assumption 10. *α is periodic in x with the same period P as the solution.*

Under Assumptions 5 and 10, each of the integrands in (27) is P-periodic, and we may use any of the representations for I. It follows that

$$I_{\pm}(x+P;\zeta) = I_{\pm}(x;\zeta) + I_{\pm}(P;\zeta). \tag{109}$$

Then

$$y_{\pm}(x+P) = y_{\pm}(x)e^{I_{\pm}(x;\zeta)}e^{I_{\pm}(P;\zeta)} = y_{\pm}(x)\Gamma_{\pm}(P), \tag{110}$$

and

$$\tilde{M}(x+P;x_0) = M^{-1}(x_0)M(x+P) = M^{-1}(x_0)M(x)\Gamma(P) = \tilde{M}(x;x_0)\Gamma(P), \tag{111}$$

where

$$\Gamma(P) = \begin{pmatrix} \Gamma_+(P) & 0 \\ 0 & \Gamma_-(P) \end{pmatrix} \tag{112}$$

is the transfer matrix. In order for solutions to be bounded in space, it must be that the eigenvalues of the transfer matrix have unit modulus. Thus,

$$\operatorname{Re}\left(I_{\pm}(P;\zeta)\right) = 0. \tag{113}$$

If Assumption 10 holds, this is equivalent to (27). The Floquet discriminant is defined by

$$\Delta(\zeta) = \operatorname{tr}(\Gamma(P)) = \Gamma_+(P) + \Gamma_-(P), \tag{114}$$

and

$$\sigma_L = \left\{\zeta \in \mathbb{C} : \operatorname{Im}(\Delta(\zeta)) = 0 \quad \text{and} \quad |\Delta(\zeta)| \leq 2\right\}. \tag{115}$$

Both definition (115) and (27) require numerical computation or the use of special functions. We prefer working with (27) directly, but we present the Floquet discriminant because of its popularity.

References

[1] *NIST Digital Library of Mathematical Functions.* http://dlmf.nist.gov/, Release 1.0.14 of 2016-12-21. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B. V. Saunders, eds.

[2] M. Ablowitz, B. Herbst, and C. Schober, *Computational chaos in the nonlinear Schrödinger equation without homoclinic crossings*, Physica A: Statistical Mechanics and its Applications, 228 (1996), pp. 212–235.

[3] M. Ablowitz, D. Kaup, A. Newell, and H. Segur, *The inverse scattering transform-Fourier analysis for nonlinear problems*, Studies in Appl. Math., 53 (1974), pp. 249–315.

[4] M. Ablowitz and H. Segur, *Solitons and the inverse scattering transform*, vol. 4 of SIAM Studies in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa., 1981.

[5] M. J. Ablowitz and Z. H. Musslimani, *Inverse scattering transform for the integrable nonlocal nonlinear Schrödinger equation*, Nonlinearity, 29 (2016), pp. 915–946.

[6] ———, *Integrable nonlocal nonlinear equations*, Stud. Appl. Math., 139 (2017), pp. 7–59.

[7] N. Bottman and B. Deconinck, *KdV cnoidal waves are spectrally stable*, Discrete and Continuous Dynamical Systems-Series A (DCDS-A), 25 (2009), pp. 1163–1180.

[8] N. Bottman, B. Deconinck, and M. Nivala, *Elliptic solutions of the defocusing NLS equation are stable*, Journal of Physics A: Mathematical and Theoretical, 44 (2011), p. 285201.

[9] A. Calini, S. F. Keith, and S. Lafortune, *Squared eigenfunctions and linear stability properties of closed vortex filaments*, Nonlinearity, 24 (2011), pp. 3555–3583.

[10] B. Deconinck and T. Kapitula, *The orbital stability of the cnoidal waves of the Korteweg–de Vries equation*, Physics Letters A, 374 (2010), pp. 4018–4022.

[11] B. Deconinck and J. Kutz, *Computing spectra of linear operators using the Floquet-Fourier-Hill method*, J. Comput. Phys., 219 (2006), pp. 296–321.

[12] B. Deconinck, P. McGill, and B. L. Segal, *The stability spectrum for elliptic solutions to the sine-Gordon equation*, Phys. D, 360 (2017), pp. 17–35.

[13] B. Deconinck, P. Meuris, and F. Verheest, *Oblique nonlinear Alfvén waves in strongly magnetized beam plasmas. part 1. nonlinear vector evolution equation*, Journal of plasma physics, 50 (1993), pp. 445–455.

[14] B. Deconinck and M. Nivala, *The stability analysis of the periodic traveling wave solutions of the mKdV equation*, Stud. Appl. Math., 126 (2011), pp. 17–48.

[15] B. Deconinck and B. L. Segal, *The stability spectrum for elliptic solutions to the focusing NLS equation*, Phys. D, 346 (2017), pp. 1–19.

[16] B. Deconinck and J. Upsal, *The orbital stability of elliptic solutions of the Focusing Nonlinear Schrödinger Equation*, arXiv preprint arXiv:1901.08702, (2019).

[17] J. Elgin, V. Enolski, and A. Its, *Effective integration of the nonlinear vector Schrödinger equation*, Physica D: Nonlinear Phenomena, 225 (2007), pp. 127–152.

[18] N. M. Ercolani and D. W. McLaughlin, *Toward a topological classification of integrable PDEs*, in The geometry of Hamiltonian systems (Berkeley, CA, 1989), vol. 22 of Math. Sci. Res. Inst. Publ., Springer, New York, 1991, pp. 111–129.

[19] M. G. Forest and D. W. McLaughlin, *Spectral theory for the periodic sine-Gordon equation: a concrete viewpoint*, J. Math. Phys., 23 (1982), pp. 1248–1277.

[20] J. Frenkel and T. Kontorova, *On the theory of plastic deformation and twinning*, Acad. Sci. U.S.S.R. J. Phys., 1 (1939), pp. 137–149.

[21] V. Gerdjikov, G. Vilasi, and A. B. Yanovski, *Integrable hamiltonian hierarchies: Spectral and geometric methods*, vol. 748, Springer Science & Business Media, 2008.

[22] T. Hada, C. Kennel, and B. Buti, *Stationary nonlinear Alfvén waves and solitons*, Journal of Geophysical Research: Space Physics, 94 (1989), pp. 65–77.

[23] T. Ivey and S. Lafortune, *Spectral stability analysis for periodic traveling wave solutions of NLS and CGL perturbations*, Physica D: Nonlinear Phenomena, 237 (2008), pp. 1750–1772.

[24] D. Kaup and A. Newell, *An exact solution for a derivative nonlinear Schrödinger equation*, Journal of Mathematical Physics, 19 (1978), pp. 798–801.

[25] D. J. Kaup and T. I. Lakoba, *The squared eigenfunctions of the massive Thirring model in laboratory coordinates*, J. Math. Phys., 37 (1996), pp. 308–323.

[26] A. Khare and A. Saxena, *Periodic and hyperbolic soliton solutions of a number of nonlocal nonlinear equations*, J. Math. Phys., 56 (2015), pp. 032104, 27.

[27] Y. Kodama and A. Hasegawa, *Nonlinear pulse propagation in a monomode dielectric guide*, IEEE Journal of Quantum Electronics, 23 (1987), pp. 510–524.

[28] J. J. Lee, *The inverse spectral solution, modulation theory and linearized stability analysis of n-phase, quasi-periodic solutions of the nonlinear Schrodinger equation*, PhD thesis, The Ohio State University, 1986.

[29] F. Lund, *Example of a relativistic, completely integrable, hamiltonian system*, Physical Review Letters, 38 (1977), p. 1175.

[30] S. V. Manakov, *On the theory of two-dimensional stationary self-focusing of electromagnetic waves*, Soviet Physics-JETP, 38 (1974), pp. 248–253.

[31] M. Nivala and B. Deconinck, *Periodic finite-genus solutions of the KdV equation are orbitally stable*, Phys. D, 239 (2010), pp. 1147–1158.

[32] A. Shabat and V. Zakharov, *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media*, Soviet physics JETP, 34 (1972), p. 62.

[33] F. Tisseur and K. Meerbergen, *The quadratic eigenvalue problem*, SIAM review, 43 (2001), pp. 235–286.