Well-posedness of boundary-value problems for the linear Benjamin-Bona-Mahony equation

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Abstract

A new method due to Fokas for explicitly solving boundary-value problems for linear partial differential equations is extended to equations with mixed partial derivatives. The Benjamin-Bona-Mahony equation is used as an example: we consider the Robin problem for this equation posed both on the half line and on the finite interval. For specific cases of the Robin boundary conditions the boundary-value problem is found to be ill posed.

1 Introduction

Recently, the work of Fokas and others [6] has demonstrated that some ideas arising in the analysis of nonlinear so-called integrable partial differential equations (PDEs) may be applied to solve boundary-value problems (BVPs) for linear PDEs. The method, referred herein as the method of Fokas (MoF), has been applied with great success to solve problems previously beyond the reach of conventional analytic techniques [6, 3]. The MoF provides explicit solutions to boundary-value problems for linear evolution equations with variable coefficients [5, 13], systems of linear constant coefficient evolution equations [12] and moving-boundary-value problems [8]. The MoF has been extended to incorporate two-dimensional problems in non-separable domains [10] as well as elliptic PDEs [11].

In this paper, we apply the MoF to the linear Benjamin-Bona-Mahony (BBM) equation [1]:

$$u_t - u_{xxt} + u_x = 0. \tag{1}$$

The BBM equation is a well-known model for long waves in shallow water. Due to the presence of the mixed derivative, the linear BBM equation has a rational dispersion relation which tends to zero for large wave numbers. This non-polynomial nature of the dispersion relation means that a modification of the MoF is required in order to use the method. Although we do not consider examples other than the linear BBM equation, the procedure presented applies to problems of the form

$$M(-i\partial_x)u_t + L(-i\partial_x)u = 0, \tag{2}$$

where $M(k)$ and $L(k)$ are polynomials. Using the MoF, it is possible to find explicit solutions for various boundary-value problems associated with such PDEs. In fact, such equations may be analyzed with the same efficiency as those without mixed derivatives. Although, mixed derivative PDEs do not often appear in the classroom, they are important in many applications [4]. Indeed, inspired by problems in water-waves, Fokas and Pelloni discuss boundary-value problems for Boussinesq type systems [7]. However, those mixed-partial derivative equations do not demonstrate the behavior we report. The only investigation of boundary-value problems for linear BBM we are aware of, considers the Dirichlet problem on the half-line [9]. Our work includes explicit solutions for Dirichlet, Neumann and Robin boundary conditions for the half-line and for the finite interval.

As already stated, the non-polynomial nature of the dispersion relation for the linear BBM equation has a significant impact on the application of the MoF. It is the cause of the essential singularities that arise in the analysis which obstruct the deformation of contour integrals. Further, the so-called global relation (see below) ceases to be valid at certain points in the finite complex plane. This results in the PDE being ill-posed for specific cases of the Robin boundary condition. Specifically, for these cases of the Robin boundary condition, initial and boundary conditions may not be arbitrarily specified. Instead, there exists a discontinuous relation between the initial and boundary conditions. As a final remark, we point out that the results presented extend to the forced equation

$$u_t - u_{xxt} + u_x = F(x,t).$$

For small time, we may regard the nonlinear term in the full BBM equation as a forcing. Since BBM is a semi-linear equation, we conjecture that the ill-posedness results apply to the nonlinear problem as well.

2 The local relation and Lax pairs

In this section, an algorithmic procedure for deriving one-parameter divergence forms associated with linear PDEs is presented. Such a divergence form, referred to as the *local relation* is the starting point for the MoF. Further, this section highlights the connection of the method with techniques for nonlinear integrable PDEs through the existence of Lax pairs for linear PDEs.

Consider the linear constant-coefficient differential equation

$$u_t + \omega(-i\partial_x)u = 0. \tag{3}$$

This PDE has a Lax pair of the form

$$\begin{aligned} \mu_x - ik\mu &= u, \\ \mu_t + \omega(k)\mu &= Xu, \end{aligned}$$

where X is a differential operator acting on $u(x,t)$ with coefficients depending on the spectral parameter k. Imposing the compatibility of these two equations and assuming that $u(x,t)$ solves (3) one obtains

$$X = i\,\frac{\omega(l)-\omega(k)}{l-k}\bigg|_{l=-i\partial_x}.$$

As an example, for the heat equation with "dispersion relation"[1] $\omega(k) = k^2$, we obtain the Lax pair

$$\mu_x - ik\mu = u,$$
$$\mu_t + k^2\mu = u_x + iku.$$

A one-parameter family of equations in divergence form, referred to as the *local relation* follows immediately. Indeed the above equations may be written as

$$(e^{-ikx+k^2t}\mu)_x = e^{-ikx+k^2t}u,$$
$$(e^{-ikx+k^2t}\mu)_t = e^{-ikx+k^2t}(u_x + iku),$$

which implies

$$\left[e^{-ikx+k^2t}u\right]_t - \left[e^{-ikx+k^2t}(u_x + iku)\right]_x = 0.$$

Equation (3) represents the evolution of $u(x,t)$ with dispersion relation $\omega(k)$. Similarly, one may consider (2) to represent the evolution of a quantity $u(x,t)$ with dispersion relation $L(k)/M(k)$. Consequently, we seek a Lax pair of the form

$$\mu_x - ik\mu = M(-i\partial_x)u,$$
$$\mu_t + \frac{L(k)}{M(k)}\mu = Xu.$$

Further, it is assumed that $M(k) > 0$ for $k \in \mathbb{R}$ and $L(k), M(k)$ do not share any roots. As before, imposing compatibility of the two ordinary differential equations for μ and imposing that $u(x,t)$ solves (2), we obtain that the appropriate differential operator is given by

$$X = i\left.\frac{L(l)M(k) - L(k)M(l)}{M(k)(l-k)}\right|_{l=-i\partial_x}.$$

For the specific case of the linear BBM equation with $L(k) = ik$ and $M(k) = 1 + k^2$, we obtain

$$X = \frac{-1 - ik\partial_x}{1 + k^2},$$

resulting in the local relation

$$\left[e^{-ikx+\frac{ik}{1+k^2}t}(u - u_{xx})\right]_t - \left[e^{-ikx+\frac{ik}{1+k^2}t}\left(\frac{-u - iku_x}{1+k^2}\right)\right]_x = 0. \tag{4}$$

3 Solutions of the linear BBM equation on the half-line

The BBM equation models long waves in shallow water. It is natural to consider the boundary-value problem for the equation posed on the semi-infinite line:

$$u_t - u_{xxt} + u_x = 0, \qquad x \geq 0,\ t \in (0,T], \tag{5a}$$
$$u(x,0) = u_0(x), \qquad x \geq 0, \tag{5b}$$
$$\alpha u(0,t) + \beta u_x(0,t) = g(t), \qquad t \in (0,T),\ \alpha, \beta \in \mathbb{R}. \tag{5c}$$

[1] We follow the notation for the dispersion relation that is typically used in the literature where the MoF is used. This differs by a factor of i from what is used in the literature on dispersive wave equations.

In what follows, we start with formal calculations assuming a smooth solution exists which has sufficiently rapid decay at infinity. These calculations lead us to a solution expression $u(x,t)$ for most values of α and β. Assuming sufficient regularity of the initial and boundary condition functions, we claim the solution expression is a classical solution to the above boundary-value problem. For the particular case when $\alpha = \beta$, the arguments below lead us to conclude the problem is ill posed in the sense that the initial and boundary condition functions may not be chosen arbitrarily. Instead there exists a discontinuous relation between them in order for the problem to be solvable.

Integrating the local relation (4) over the region $\{(x,s) : x \geq 0, s \in (0,t)\}$, applying Green's Theorem and integrating by parts, we obtain the global relation

$$U_0(k) + \frac{\tilde{g}(k,t)}{1+k^2} = e^{\omega t} U(k,t), \quad \operatorname{Im}(k) \leq 0, k \neq -i, \tag{6}$$

with

$$\begin{aligned} U_0(k) &= (1+k^2)\hat{u}_0(k) + u_x(0,0) + iku(0,0), \\ U(k,t) &= (1+k^2)\hat{u}(k,t) + u_x(0,t) + iku(0,t), \\ \tilde{g}(k,t) &= \tilde{g}_0(\omega,t) + ik\tilde{g}_1(\omega,t), \\ \omega &= \frac{ik}{1+k^2}, \end{aligned}$$

where $\hat{u}_0(k)$ and $\hat{u}(k,t)$ represent the Fourier transform of the initial condition and of the solution at time t respectively. They are defined by

$$\hat{u}_0(k) = \int_0^\infty e^{-ikx} u_0(x)dx, \quad \hat{u}(k,t) = \int_0^\infty e^{-ikx} u(x,t)dx,$$

with inverses

$$u_0(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{u}_0(k)dk, \quad u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{u}(k,t)dk,$$

where $u_0(x)$ and $u(x,t)$ are defined to be zero for $x < 0$. Similarly, $\tilde{g}_0(\omega,t)$ and $\tilde{g}_1(\omega,t)$ are the finite-time transforms of the boundary data given by

$$\tilde{g}_0(\omega,t) = \int_0^t e^{\omega s} u(0,s)ds, \quad \tilde{g}_1(\omega,t) = \int_0^t e^{\omega s} u_x(0,s)ds.$$

Solving the global relation (6) for $\hat{u}(k,t)$ and applying the inverse Fourier transform, we obtain the following integral expression for the solution

$$\begin{aligned} u(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega t}\frac{U_0(k)}{1+k^2}dk + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega t}\frac{\tilde{g}(k,t)}{(1+k^2)^2}dk \\ &- \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\frac{u_x(0,t)+iku(0,t)}{1+k^2}dk, \\ \Rightarrow u(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega t}\hat{u}_0(k)dk + \frac{1}{2\pi}\int_{\mathcal{C}} e^{ikx-\omega t}\frac{\tilde{g}(k,t)}{(1+k^2)^2}dk \\ &+ \frac{1}{2\pi}\int_{\mathcal{C}} e^{ikx-\omega t}\frac{u_x(0,0)+iku(0,0)}{1+k^2}dk - \frac{1}{2\pi}\int_{\mathcal{C}} e^{ikx}\frac{u_x(0,t)+iku(0,t)}{1+k^2}dk, \end{aligned} \tag{7}$$

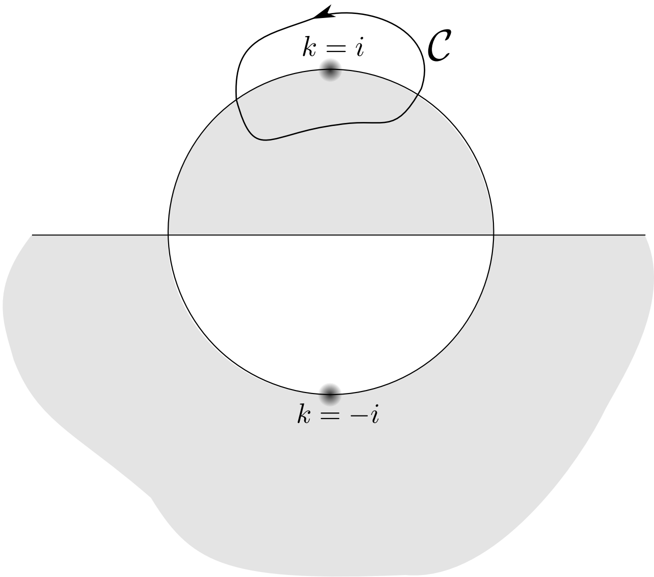

Figure 1: The closed contour $\mathcal{C}$ and the region $D = \{k : \text{Re}(\omega) < 0\}$, indicated in grey.

where the integral on the real line has been deformed to a closed contour $\mathcal{C}$ around $k = i$ as shown in Figure 1. This is possible since the respective integrands are analytic functions which decay in the upper-half plane for large k.

The integral expression (7) depends on unprescribed boundary data. As is usual in the MoF, this is resolved by using the invariances of $\omega(k)$. Note that $\omega(k)$ is invariant under the transform $k \to 1/k$. Using this transformation, the global relation becomes

$$U_0(1/k) + \frac{\tilde{g}(1/k,t)}{1+1/k^2} = e^{\omega t}U(1/k,t), \quad \text{Im}(k) \geq 0,\ k \neq 0, i,$$
$$\tilde{g}(1/k,t) = \tilde{g}_0(\omega,t) + \frac{i}{k}\tilde{g}_1(\omega,t).$$

The time transform of the boundary condition (5c) is given by

$$\alpha\tilde{g}_0(\omega,t) + \beta\tilde{g}_1(\omega,t) = G(\omega,t) = \int_0^t e^{\omega s}g(s)ds.$$

Solving the above equations for $\tilde{g}_0(\omega,t)$ and $\tilde{g}_1(\omega,t)$, and substituting the result in (7) we obtain

$$\begin{aligned} u(x,t) =& S(x,t) + \frac{1}{2\pi}\int_{\mathcal{C}} \frac{e^{ikx}}{k^3}\left(\frac{\alpha ik-\beta}{\alpha i - k\beta}\right)\hat{u}(1/k,t)dk \\ &+ \frac{1}{2\pi}\int_{\mathcal{C}} \frac{e^{ikx}}{1+k^2}\left[(u_x(0,t) + \frac{i}{k}u(0,t))\left(\frac{\alpha ik-\beta}{\alpha ik - k^2\beta}\right) - (u_x(0,t) + iku(0,t))\right]dk \end{aligned} \tag{8}$$

where

$$
\begin{aligned}
S(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega t}\hat{u}_0(k)dk + \frac{1}{2\pi}\int_{\mathcal{C}} \frac{ie^{ikx-\omega t}}{(1+k^2)^2}\left(\frac{1-k^2}{\alpha i - k\beta}\right) G(\omega,t)dk \\
&- \frac{1}{2\pi}\int_{\mathcal{C}} \frac{e^{ikx-\omega t}}{k^3}\left(\frac{\alpha ik-\beta}{\alpha i - k\beta}\right)\hat{u}_0(1/k)dk \\
&+ \frac{1}{2\pi}\int_{\mathcal{C}} \frac{e^{ikx-\omega t}}{1+k^2}\left[(u_x(0,0)+iku(0,0)) - \left(\frac{\alpha ik-\beta}{\alpha ik - k^2\beta}\right)(u_x(0,0)+\frac{i}{k}u(0,0))\right]dk.
\end{aligned}
$$

The right-hand side of (8) depends on $u(x,t)$ itself, through the presence of $\hat{u}(1/k,t)$ in the second term. As such, (8) does not represent an explicit solution formula, and more work is required. We consider two possible cases.

- **Case 1.** $\boldsymbol{\alpha \neq \beta}$**.** By deforming the contour $\mathcal{C}$ as indicated in Figure 1 *before* substitution of the transforms of the unknown boundary terms, we assure that the only singularity of the integrals in (8) enclosed by $\mathcal{C}$ is at $k=i$. Thus

$$
\int_{\mathcal{C}} \frac{e^{ikx}}{k^3}\left(\frac{\alpha ik-\beta}{\alpha i - k\beta}\right)\hat{u}(1/k,t)dk = 0,
$$

since the integrand is analytic in a neighborhood of $k=i$. Further, an application of the Residue Theorem for the third term in (8) leads to

$$
\frac{1}{2\pi}\int_{\mathcal{C}} \frac{e^{ikx}}{1+k^2}\left[(u_x(0,t)+\frac{i}{k}u(0,t))\left(\frac{\alpha ik-\beta}{\alpha ik-k^2\beta}\right) - (u_x(0,t)+iku(0,t))\right]dk = \frac{e^{-x}g(t)}{\alpha-\beta}.
$$

We conclude that the solution $u(x,t)$ is given by

$$
u(x,t) = S(x,t) + \frac{e^{-x}g(t)}{\alpha-\beta}. \tag{9}
$$

In particular, the Dirichlet ($\beta=0$) and Neumann ($\alpha=0$) problems for this PDE have solutions given by the above expression. Since the term proportional to the exponential satisfies the boundary condition, it is necessary that

$$
\alpha S(0,t) + \beta S_x(0,t) = 0,
$$

which is assured by taking $g(0) = u_0(0) = 0$. Thus, the existence of a classical solution requires the compatibility of the boundary and initial conditions at the corner point $(x,t)=(0,0)$.

- **Case 2.** $\boldsymbol{\alpha = \beta = 1}$**.** The solution to the boundary-value problem given above is not valid when $\alpha=\beta$. Without loss of generality, we take $\alpha=\beta=1$. In this case, both integrals possess singularities at $k=i$. The residues are calculated to obtain

$$
u(x,t) = S(x,t) + e^{-x}g(t)\left(x+\frac{1}{2}\right) + 2e^{-x}u(0,t) - 2e^{-x}\hat{u}(-i,t). \tag{10}
$$

This may be interpreted as a linear integral equation for the solution $u(x,t)$ that depends on known initial-boundary data as well as on the unknowns $u(0,t)$ and

$\hat{u}(-i,t)$. If it is possible to express these unknowns in terms of the known initial- and boundary-condition functions then the above expression represents the solution to the problem. However, since by assumption $u(x,t)$ solves the PDE, we may substitute (10) into (5a) to obtain

$$g'(t) + \hat{u}(-i,t) = u(0,t). \tag{11}$$

If this relation holds for all time, we obtain the following solution to the problem on the half-line

$$u(x,t) = S(x,t) + e^{-x} g(t)\left(x + \frac{1}{2}\right) + 2e^{-x} g'(t). \tag{12}$$

Further, since the terms proportional to e^{-x} satisfy the boundary condition, the expression $S(x,t)$ has the same interpretation as in the case $\alpha \neq \beta$. Consquently we require $g(0) = u_0(0) = 0$. However, the initial condition is not satisfied unless

$$g'(0) + \hat{u}_0(-i) = 0, \tag{13}$$

as is readily seen by letting $t = 0$ in (12). It is noted that this expression may be obtained from (11) also.

Given a continuously differentiable function $g(t)$, the constraint (13) requires the initial condition $u_0(x)$ to be of the form

$$u_0(x) = -4xe^{-x} g'(0) + w(x), \tag{14}$$

where $w(x)$ is a function orthogonal to e^{-x} using the standard inner product on $[0,\infty)$, and $w(0) = 0$ with sufficient smoothness and decay for large x to justify the contour deformations in the previous section. Alternatively, given an initial condition $u_0(x)$ with $u_0(0) = 0$ and sufficient decay and smoothness, the above is a constraint on permissible boundary conditions $g(t)$.

The relation between initial and boundary conditions evidently restricts our freedom to arbitrarily choose initial and boundary functions for the case $\alpha = \beta$. Interestingly, if $g(t)$ is identically zero, then $u_0(x)$ must be identically zero too, as we show below. Indeed, if $g(t) \equiv 0$, the solution expression is given by

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\omega t}\hat{u}_0(k)dk - \int_{\mathcal{C}} \frac{e^{ikx-\omega t}}{k^3}\left(\frac{ik-1}{i-k}\right)\hat{u}_0(1/k)dk. \tag{15}$$

Substituting this expression into (11), we find

$$\int_0^\infty e^{-x} u(x,t)dx = u(0,t) \;\Rightarrow$$

$$\int_{-\infty}^{\infty} \frac{e^{-\omega t}\hat{u}_0(k)}{1-ik}dk + \int_{\mathcal{C}} \frac{e^{-\omega t}\hat{u}_0(1/k)}{i-k}dk = \int_{-\infty}^{\infty} e^{-\omega t}\hat{u}_0(k)dk - \int_{\mathcal{C}} \frac{e^{-\omega t}(ik-1)\hat{u}_0(1/k)}{k^3(i-k)}dk. \tag{16}$$

Using the transformation $k \to 1/k$ and replacing all contours $\mathcal{C}$ to contours $\tilde{\mathcal{C}}$ around $k = -i$, we obtain

$$\int_{-\infty}^{\infty} \frac{e^{-\omega t}\hat{u}_0(k)}{1-ik}dk + \int_{\tilde{\mathcal{C}}} \frac{e^{-\omega t}\hat{u}_0(k)}{k(1-ik)}dk = \int_{-\infty}^{\infty} e^{-\omega t}\hat{u}_0(k)dk + \int_{\tilde{\mathcal{C}}} \frac{e^{-\omega t}k(i-k)\hat{u}_0(k)}{(ik-1)}dk$$
$$\Rightarrow \quad -\int_{\tilde{\mathcal{C}}} \frac{e^{-\omega t}\hat{u}_0(k)}{1-ik}dk + \int_{\tilde{\mathcal{C}}} \frac{e^{-\omega t}\hat{u}_0(k)}{k(1-ik)}dk = -\int_{\tilde{\mathcal{C}}} e^{-\omega t}\hat{u}_0(k)dk + \int_{\tilde{\mathcal{C}}} \frac{e^{-\omega t}k(i-k)\hat{u}_0(k)}{(ik-1)}dk$$
$$\Rightarrow \quad \int_{\tilde{\mathcal{C}}} \frac{e^{-\omega t}\hat{u}_0(k)(1-k)(k^2+k+1)}{k(1-ik)}dk = 0, \tag{17}$$

Since this statement is valid for all t we may expand the exponential in a Taylor series in t, interchange order of summation and integration and equate the coefficient of every power of t to zero. This imposes the following set of conditions on the Fourier transform of the initial condition:

$$\text{Res}\left[\left(\frac{ik}{1+k^2}\right)^n \frac{(1-k)(k^2+k+1)}{(1-ik)k}\hat{u}_0(k)\right]_{k=-i} = 0, \quad n = 0, 1, 2\ldots,$$

hence

$$\left.\frac{d^n}{dk^n}\hat{u}_0(k)\right|_{k=-i} = 0, \quad n = 0, 1, 2\ldots.$$

However, $\hat{u}_0(k)$ is an analytic function of k in the lower-half plane and is continuous up to the real line. Consequently, $\hat{u}_0(k) \equiv 0$ for k real and the initial condition $u_0(x)$ is identically zero. It follows that the initial condition is not continuous as a function of the boundary data $g(t)$, at $g(t) \equiv 0$, since (14) shows that for $g(t) \not\equiv 0$, $u_0(x)$ can be arbitrarily large by choosing $w(x)$ large.

Finally we show that (11) holds for any classical solution to the PDE. Assuming $u(x,t)$ is a solution to the problem (5a-5c), we have

$$u_t - u_{xxt} + u_x = 0.$$

Multiplying this equation by e^{-x} and integrating over the domain $[0, \infty)$ yields after a few integrations by part

$$\frac{d}{dt}\left(u_x(0,t) + u(0,t)\right) + \int_0^{\infty} e^{-y}u(y,t)dy = u(0,t).$$

Noting that the first term on the left-hand side is the time derivative of the boundary condition, we obtain (13).

We conclude with the remark that the identity (13), valid for any solution to the differential equation (11) became a constraint on the initial and boundary conditions. Indeed it may be verified that also when $\alpha \neq \beta$, (11) is readily satisfied by the solution (9) to the problem.

4 Uniqueness of solutions to linear BBM on the half-line

In the previous section we constructed solutions to the linear BBM equation. The procedure was algorithmic. Depending on the values of the coefficients α and β, we obtained a solution expression (case 1) or we concluded additional constraints were required on the initial and boundary condition functions (case 2). In either case, it is necessary to investigate the uniqueness of the solution of the equation. If we can prove uniqueness, then in case 1, we conclude that (9) is the unique solution to the given boundary-value problem. In particular, it follows that the Dirichlet and Neumann problems have unique solutions. Similarly, in case 2, where the initial and boundary conditions are required to satisfy the constraint (13), we conclude that the problem is ill-posed in the sense that we cannot choose initial and boundary conditions arbitrarily.

It is typical to address uniqueness of solutions to evolution PDEs using energy integral arguments. The energy integral for the linear BBM equation is found as shown below.

$$
\begin{aligned}
& & u_t - u_{xxt} + u_x &= 0 \\
\Rightarrow & & uu_t - uu_{xxt} + uu_x &= 0 \\
\Rightarrow & & \int_0^\infty (uu_t - uu_{xxt} + uu_x)dx &= 0 \\
\Rightarrow & & \frac{d}{dt}\int_0^\infty \frac{u^2}{2}dx - [uu_{xxt}]_0^\infty + \int_0^\infty u_x u_{xt} dx + \left[\frac{u^2}{2}\right]_0^\infty &= 0 \\
\Rightarrow & & \frac{d}{dt}\int_0^\infty \frac{1}{2}\left(u^2 + u_x^2\right)dx &= \frac{u^2(0,t)}{2} - u(0,t)u_{xt}(0,t).
\end{aligned}
$$

In the above expression, we think of $u(x,t)$ as the difference of two distinct solutions to the BVP. Since the given initial and boundary conditions are linear, uniqueness follows if the right-hand side of the above expression is non-positive for homogeneous boundary conditions. Clearly this is the case for the Dirichlet problem. However, no direct conclusion is obtained for the Neumann or the Robin problem. We may consider the right-hand side of the last line above as a map from known boundary conditions to unknown values at the boundary. For the Neumann case, if the Neumann-to-Dirichlet map exists and maps zero to zero, we may yet conclude uniqueness. Similarly, for the Robin boundary condition. In what follows we construct explicit Robin-to-Dirichlet and Neumann-to-Dirichlet maps which do indeed map homogeneous data to zero. Consequently, we claim there is a unique solution to the BVP (5a-5c).

4.1 Robin-Dirichlet boundary-value map

The assumption of a classical solution to the BVP (5a-5c) leads to the global relation (6). In this section we show how the global relation may be used to derive a map from known initial-boundary data to unknown boundary data. Let us consider the Robin condition

$$\alpha u(0,t) + \beta u_x(0,t) = g(t),$$

which upon taking the time transform becomes

$$\alpha\tilde{g}_0(\omega,t)+\beta\tilde{g}_1(\omega,t)=G(\omega,t).$$

Solving for $u_x(0,t)$ and $\tilde{g}_1(\omega,t)$ (assuming $\beta\neq 0$) in the above equations and substituting in the global relation (6) we obtain

$$\begin{aligned}\frac{1}{(1+k^2)^2}&\left[\left(\frac{\beta-\alpha ik}{\beta}\right)\tilde{g}_0(\omega,t)+\frac{ik}{\beta}G(\omega,t)\right]+\hat{u}_0+\frac{1}{(1+k^2)^2}\left[\frac{g(0)-\alpha u(0,0)}{\beta}+iku(0,0)\right]\\&=\frac{e^{\omega t}}{1+k^2}\left[\frac{g(t)-\alpha u(0,t)}{\beta}+iku(0,t)+(1+k^2)\hat{u}(k,t)\right]\\&=e^{\omega t}\left[\hat{u}(k,t)+\frac{g(t)}{\beta(1+k^2)}+\frac{ik-\alpha/\beta}{1+k^2}u(0,t)\right].\end{aligned}\tag{18}$$

Multiplying this expression by $e^{-\omega t}\dfrac{1-k^2}{\beta-\alpha ik}$ and integrating over a sufficiently small contour C containing $k=-i$ we obtain the required map. The contribution from $\tilde{g}_0(\omega,t)$ vanishes since

$$\begin{aligned}\oint_C e^{-\omega t}\frac{1-k^2}{(1+k^2)^2}\tilde{g}_0(\omega,t)dk&=\oint_{\tilde{C}}e^{-ilt}\tilde{g}_0(il,t)dl,\\&=\oint_{\tilde{C}}e^{-ilt}\int_0^t e^{ils}u(0,s)dsdl,\\&=0.\end{aligned}$$

The first equality is obtained using the change of coordinates $l=k/(1+k^2)$, where $\tilde{C}$ is the image of C under this map. The second equality uses the definition of the time transform. Evidently the integrand is an entire function of l, from which the conclusion follows. If $\alpha\neq\beta$ the contribution from $\hat{u}(k,t)$ also vanishes since

$$\oint_C\frac{1-k^2}{\beta-\alpha ik}\hat{u}(k,t)dk=0,$$

due to analyticity of the integrand. An application of the Residue Theorem leads to the following expression for the Dirichlet data

$$\begin{aligned}\frac{-2\pi}{\beta}u(0,t)=\frac{2\pi}{\beta(\beta-\alpha)}g(t)&+\oint_C\frac{ike^{-\omega t}(1-k^2)}{(1+k^2)^2(\beta-\alpha ik)}G(\omega,t)dk+\oint_C\frac{e^{-\omega t}(1-k^2)}{\beta-\alpha ik}\hat{u}_0(k)dk\\&+\oint_C\frac{e^{-\omega t}(1-k^2)}{(1+k^2)^2(\beta-\alpha ik)}\left[\frac{g(0)}{\beta}+\left(ik-\frac{\alpha}{\beta}\right)u_0(0)\right]dk.\end{aligned}$$

This expression is also valid for the Neumann problem ($\beta=1$, $\alpha=0$). The expression above simplifies if we impose the corner condition $u_0(0)=g(0)=0$, which was required for existence of smooth solutions in the previous sections. Since the Robin-to-Dirichlet map maps the homogeneous problem to zero Dirichlet data, we obtain uniqueness for the case $\alpha\neq\beta$.

The argument above applies with little modification for the case $\alpha = \beta = 1$. We now obtain a contribution from $\hat{u}(k,t)$ given by

$$\begin{aligned}\oint_C \frac{1-k^2}{1-ik}\hat{u}(k,t)dk &= -4\pi\hat{u}(-i,t)\\ &= -4\pi u(0,t) + 4\pi g'(t),\end{aligned}$$

where we have used the integral relation (11). We obtain the following expression for the Robin-to-Dirichlet map

$$\begin{aligned}-2\pi u(0,t) = -4\pi g'(t) - \pi g(t) + \oint_C \frac{ike^{-\omega t}(1-k^2)}{(1+k^2)^2(1-ik)}G(\omega,t)dk + \oint_C \frac{e^{-\omega t}(1-k^2)}{1-ik}\hat{u}_0(k)dk\\ + \oint_C \frac{e^{-\omega t}(1-k^2)}{(1+k^2)^2(1-ik)}\left[g(0) + (ik-1)\,u_0(0)\right]dk,\end{aligned}$$

from which the same conclusion is obtained.

5 Linear BBM on the finite interval

Next, we consider the finite-interval BVP. The MoF is even more advantageous in this setting. Classical methods would lead one to consider separation of variables, *i.e.*, we start with $u(x,t) = f(x)s(t)$. Substituting into the differential equation, we obtain

$$f(x)s'(t) - f''(x)s'(t) + f'(x)s(t) = 0.$$

Dividing by $f'(x)s'(t)$ results in a generalized eigenvalue problem in the spatial variable requiring significant analysis in order to establish whether or not the eigenfunctions obtained form a complete set. The MoF not only provides a solution without this additional effort, but it also indicates which boundary-value problems may be ill posed in the same sense we used for the BVP on the half-line. Since the procedure is somewhat similar to what was done for the BVP on the half line, we present less detail than before.

By integrating the local relation over the region $\{(x,s) : 0 \le x \le L,\ 0 < s < t\}$ and applying Green's Theorem, we obtain the global relation for the finite interval:

$$U_0(k) + \frac{\tilde{g}(k,t)}{1+k^2} - e^{-ikL}\frac{\tilde{h}(k,t)}{1+k^2} = e^{\omega t}U(k,t),$$

where

$$\begin{aligned}U_0(k) &= (1+k^2)\hat{u}_0(k) + u_x(0,0) + iku(0,0) - e^{-ikL}u_x(L,0) - ike^{-ikL}u(L,0),\\ U(k,t) &= (1+k^2)\hat{u}(k,t) + u_x(0,t) + iku(0,t) - e^{-ikL}u_x(L,t) - ike^{-ikL}u(L,t),\\ \tilde{g}(k,t) &= \tilde{g}_0(\omega,t) + ik\tilde{g}_1(\omega,t),\\ \tilde{h}(k,t) &= \tilde{h}_0(\omega,t) + ik\tilde{h}_1(\omega,t),\\ \omega &= \frac{ik}{1+k^2},\end{aligned}$$

and $\tilde{h}(k,t)$ is defined similarly to $\tilde{g}(k,t)$ but with $u(x,t)$ evaluated at $x=L$. The Fourier transform is given by

$$\hat{u}_0(k)=\int_0^L e^{-ikx}u_0(x)dx, \quad \hat{u}(k,t)=\int_0^L e^{-ikx}u(x,t)dx.$$

Applying the inverse Fourier transform, we obtain the integral expression

$$\begin{aligned}u(x,t)=&\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx-\omega t}\frac{U_0(k)}{1+k^2}dk+\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx-\omega t}\frac{\tilde{g}(k,t)}{(1+k^2)^2}dk\\&-e^{-ikL}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx-\omega t}\frac{\tilde{h}(k,t)}{(1+k^2)^2}dk-\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx}\frac{u_x(0,t)+iku(0,t)}{1+k^2}dk\\&+\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx-ikL}\frac{u_x(L,t)+iku(L,t)}{1+k^2}dk.\end{aligned}$$

The integral of the boundary terms may be deformed off the real line to appropriate contours $\mathcal{C}_1$ and $\mathcal{C}_2$ which are closed curves around $k=i$ and $k=-i$ respectively.

$$\begin{aligned}u(x,t)=&\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx-\omega t}\hat{u}_0(k)dk+\frac{1}{2\pi}\int_{\mathcal{C}_1}e^{ikx-\omega t}\frac{\tilde{g}(k,t)}{(1+k^2)^2}dk-\frac{1}{2\pi}\int_{\mathcal{C}_2}e^{-ik(L-x)-\omega t}\frac{\tilde{h}(k,t)}{(1+k^2)^2}dk\\&+\frac{1}{2\pi}\int_{\mathcal{C}_1}e^{ikx-\omega t}\frac{u_x(0,0)+iku(0,0)}{1+k^2}dk-\frac{1}{2\pi}\int_{\mathcal{C}_1}e^{ikx}\frac{u_x(0,t)+iku(0,t)}{1+k^2}dk\\&-\frac{1}{2\pi}\int_{\mathcal{C}_2}e^{ikx-ikL-\omega t}\frac{u_x(L,0)+iku(L,0)}{1+k^2}dk+\frac{1}{2\pi}\int_{\mathcal{C}_2}e^{ikx-ikL}\frac{u_x(L,t)+iku(L,t)}{1+k^2}dk.\end{aligned} \tag{19}$$

Assume we are given Robin boundary conditions at both $x=0$ and $x=L$.

$$\begin{aligned}&\alpha u(0,t)+\beta u_x(0,t)=g(t), \;\; \gamma u(L,t)+\delta u_x(L,t)=h(t)\\ \Rightarrow\quad &\alpha\tilde{g}_0(\omega,t)+\beta\tilde{g}_1(\omega,t)=G(\omega,t), \;\; \gamma\tilde{h}_0(\omega,t)+\delta\tilde{h}_1(\omega,t)=H(\omega,t),\end{aligned} \tag{20}$$

where α, β, γ, and δ are real. Combining these last equations with the transformed version of the global relation

$$U_0(1/k)+\frac{\tilde{g}(1/k,t)}{1+1/k^2}-e^{-iL/k}\frac{\tilde{h}(1/k,t)}{1+1/k^2}=e^{\omega t}U(1/k,t),$$

as well as with the original global relation, we obtain a system of equations for the unknown boundary terms. For instance, solving (20) for $\tilde{g}_0$ and $\tilde{h}_0$, we obtain a system of equations for the two remaining boundary terms. To solve this system of equations we are required to invert the matrix

$$\begin{pmatrix} ik-\beta/\alpha & -e^{-ikL}\left(ik-\delta/\gamma\right)\\ i/k-\beta/\alpha & -e^{-iL/k}\left(i/k-\delta/\gamma\right)\end{pmatrix}.$$

For a given set $\{\alpha,\beta,\gamma,\delta\}$, the zeros in the complex k plane of the determinant of this matrix are the singularities which appear in the final expression for the solution. Since the entries of the matrix are analytic functions of k in $\mathbb{C}\backslash\{0\}$, the zeros of the determinant are isolated. Hence, by deforming the contours $\mathcal{C}_1, \mathcal{C}_2$ suitably *before* substitution of the expressions for $\tilde{g}_i, \tilde{h}_i,\ i = 0, 1$, it is possible to ensure the integrals contain at most one singularity, namely at $k = i$ or at $k = -i$. Consequently, if there exists a set $\{\alpha, \beta, \gamma, \delta\} \in \mathbb{R}^4$ such that for $k = \pm i$ the determinant is zero, then the problem is presumed to be ill posed in a sense similar to that observed in the half-line case. Indeed, setting $k = \pm i$ the determinant has a zero only if $(\alpha, \gamma) = \pm(\beta, \delta)$. Hence either one of the following boundary conditions leads to an ill-posed problem:

$$u(0,t) + u_x(0,t) = g(t), \tag{21a}$$
$$u(L,t) + u_x(L,t) = h(t), \tag{21b}$$

or

$$u(0,t) - u_x(0,t) = g(t), \tag{22a}$$
$$u(L,t) - u_x(L,t) = h(t). \tag{22b}$$

In all other cases, the solution proceeds in a manner similar to Case 1 ($\alpha \neq \beta$) of the half-line problem. In the following, we present details of the boundary-value problem with boundary conditions (21a-b). In this case, the global relation is

$$(1+k^2)U_0(k) + ikG - ike^{-ikL}H + (1-ik)(\tilde{g}_0 - e^{-ikL}\tilde{h}_0) = (1+k^2)e^{\omega t}U(k,t), \tag{23}$$

where $G, H, \tilde{g}_0, \tilde{h}_0$ are the time transforms of the Robin and Dirichlet data at the left and right boundaries defined analogously as to the half-line case. Looking ahead, we suppress the dependence of these terms on ω and t since the time transforms are invariant under the symmetries of the dispersion relation unlike the functions U and U_0, which are given by

$$\begin{aligned} U_0(k) &= (1+k^2)\hat{u}_0(k) + g(0) - h(0)e^{-ikL} + (ik-1)(u_0(0) - e^{-ikL}u_0(L)), \\ U(k,t) &= (1+k^2)\hat{u}(k,t) + g(t) - h(t)e^{-ikL} + (ik-1)(u(0,t) - e^{-ikL}u(L,t)). \end{aligned}$$

The global relation (23) is valid for all $k \in \mathbb{C}\backslash\{-i, i\}$. Using the symmetry $k \to 1/k$ we obtain another version of the global relation valid for $k \in \mathbb{C}\backslash\{-i, 0, i\}$,

$$\frac{1+k^2}{k^2}U_0\left(\frac{1}{k}\right) + \frac{i}{k}G - \frac{i}{k}e^{-iL/k}H + \frac{k-i}{k}(\tilde{g}_0 - e^{-iL/k}\tilde{h}_0) = \frac{1+k^2}{k^2}e^{\omega t}U\left(\frac{1}{k}, t\right). \tag{24}$$

Hence we obtain a system of equations for the unknown Dirichlet data which may be solved to obtain

$$\begin{pmatrix} \tilde{g}_0 \\ \tilde{h}_0 \end{pmatrix} = \Delta^{-1}(P + R), \tag{25}$$

where

$$\Delta^{-1} = \frac{1}{i(1+k^2)\delta(k)} \begin{pmatrix} -e^{-iL/k}(k-i) & e^{-iLk}k(1-ik) \\ -(k-i) & k(1-ik) \end{pmatrix}, \quad P = \frac{e^{\omega t}(1+k^2)}{k^2} \begin{pmatrix} k^2 U(k,t) \\ U(1/k,t) \end{pmatrix},$$

$$R = \begin{pmatrix} -(1+k^2)U_0(k) - ikG + ike^{-ikL}H \\ \\ -\frac{1+k^2}{k^2}U_0(\frac{1}{k}) - \frac{i}{k}G + \frac{i}{k}e^{-ikL}H \end{pmatrix}$$

and

$$\delta(k) = e^{-iL/k} - e^{-ikL}.$$

The global relation (23) is solved for $\hat{u}(k,t)$. Taking the inverse Fourier transform of the resulting expression leads to

$$\begin{aligned} u(x,t) = &- \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ikx}}{1+k^2}(g(t) - h(t)e^{-ikL})dk - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{ie^{ikx}}{k-i}(u(0,t) - u(L,t)e^{-ikL})dk \\ &+ \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ikx-\omega t}}{1+k^2}U_0(k)dk + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{ike^{ikx-\omega t}}{(1+k^2)^2}Gdk - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{ike^{ikx-ikL-\omega t}}{(1+k^2)^2}Hdk \\ &- \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i(k+i)e^{ikx-\omega t}}{(1+k^2)^2}\tilde{g}_0 dk + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i(k+i)e^{ikx-ikL-\omega t}}{(1+k^2)^2}\tilde{h}_0 dk. \end{aligned}$$

The integrals in the above expression involving boundary data may be deformed off the real axis to circular contours around $k = \pm i$ denoted by $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively, leading to

$$\begin{aligned} u(x,t) = &- \frac{e^{-x}}{2}g(t) + \frac{e^{x-L}}{2}h(t) + e^{-x}u(0,t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ikx-\omega t}}{1+k^2}U_0(k)dk \\ &+ \frac{1}{2\pi}\int_{\mathcal{C}_1} \frac{ike^{ikx-\omega t}}{(1+k^2)^2}Gdk + \frac{1}{2\pi}\int_{\mathcal{C}_2} \frac{ike^{ikx-ikL-\omega t}}{(1+k^2)^2}Hdk \\ &- \frac{1}{2\pi}\int_{\mathcal{C}_1} \frac{i(k+i)e^{ikx-\omega t}}{(1+k^2)^2}\tilde{g}_0 dk - \frac{1}{2\pi}\int_{\mathcal{C}_2} \frac{i(k+i)e^{ikx-ikL-\omega t}}{(1+k^2)^2}\tilde{h}_0 dk. \end{aligned}$$

Substituting for $\tilde{g}_0$ and $\tilde{h}_0$ from (25) and taking appropriate residues we obtain

$$\begin{aligned} u(x,t) = &- \frac{e^{-x}}{2}g(t) + \frac{e^{x-L}}{2}h(t) + e^{-x}u(0,t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ikx-\omega t}}{1+k^2}U_0(k)dk \\ &+ \frac{1}{2\pi}\int_{\mathcal{C}_1} \frac{ike^{ikx-\omega t}}{(1+k^2)^2}Gdk + \frac{1}{2\pi}\int_{\mathcal{C}_2} \frac{ike^{ikx-ikL-\omega t}}{(1+k^2)^2}Hdk \\ &- \frac{1}{2\pi}\int_{\mathcal{C}_1} \frac{ie^{ikx-\omega t}(k+i)}{(1+k^2)^2}(\Delta^{-1}R)_1 dk - \frac{1}{2\pi}\int_{\mathcal{C}_2} \frac{ie^{ikx-ikL-\omega t}(k+i)}{(1+k^2)^2}(\Delta^{-1}R)_2 dk \\ &- e^{x-L}\frac{g(t) - e^{-L}h(t)}{2(e^L - e^{-L})} + \frac{e^{-x}}{e^L - e^{-L}}(e^L(x+1)g(t) - (x+1-L)h(t)) \\ &- \frac{e^{-x}}{2(e^L - e^{-L})}(e^{-L}g(t) - h(t)) - 2\frac{e^{-x+L}}{e^L - e^{-L}}\hat{u}(-i,t) \\ &+ e^{-x+L}\frac{u(0,t) - u(L,t)e^{-L}}{e^L - e^{-L}} + e^{-x-L}\frac{u(0,t) - u(L,t)e^{L}}{e^L - e^{-L}}, \end{aligned} \tag{26}$$

where $(\Delta^{-1}R)_1$ and $(\Delta^{-1}R)_2$ refer to the first and second components of the vector $\Delta^{-1}R$. As in the half-line case, we notice that the above expression depends on the unknown Dirichlet data and the function $\hat{u}(-i,t)$. Since by assumption $u(x,t)$ is a solution to the

PDE, we may substitute the right-hand side in the BBM equation, which results in the following necessary condition relating the unknown functions $u(0,t), u(L,t)$ and $\hat{u}(-i,t)$

$$\hat{u}(-i,t) + g'(t) - e^{-L}h'(t) = u(0,t) - e^{-L}u(L,t). \tag{27}$$

It may be shown that this constraint is satisfied by any solution to the PDE. Indeed it is the finite interval analogue of (11). We may obtain two additional equations between the three unknowns from (25) by multiplying by $e^{-\omega t}(1-k^2)/(1+k^2)^2$ and integrating along a small contour $\mathcal{C}$ around $k = i$. This leads to

$$\int_{\mathcal{C}} e^{-\omega t} \frac{1-k^2}{(1+k^2)^2}\, \Delta^{-1}\,(P+R)\, dk = 0. \tag{28}$$

Evidently, equations (28) and (27) are three equations for three unknowns. This system of equations may be solved to obtain the Robin-to-Dirichlet map as well as a map from known initial and boundary data to the function $\hat{u}(-i,t)$. Using these definitions for the unknown functions, (26) represents the solution to the boundary-value problem on the finite interval. Indeed it is the unique solution since the Robin-to-Dirichlet map for the finite interval maps the homogeneous problem to trivial Dirichlet data, and an energy argument can be used to imply uniqueness. Note that (25), which was obtained from the global relation (23), and (27) are obtained from the assumption of existence of a smooth solution.

In order to satisfy the initial-boundary conditions we require additional compatibility conditions, namely $u_0(0) = u_0(L) = g(0) = h(0) = 0$ and that the relation (27) is satisfied at $t = 0$. We obtain the following constraint on the allowable initial and boundary data

$$\hat{u}_0(-i) + g'(0) - e^{-L}h'(0) = 0.$$

A calculation similar to that for the half-line case indicates that the only solution with this particular boundary condition and $g(t) = h(t) = 0$ is the trivial one.

Conclusions

We have demonstrated the use of the MoF to PDEs with mixed partial derivatives in two independent variables by dicussing BVPs for the linear BBM equation in detail. The application of these ideas to other mixed-derivative equations is straightforward. For the linear BBM equation, we have obtained two theorems.

Theorem 1. The BVP

$$u_t - u_{xxt} + u_x = 0, \qquad x \geq 0,\ t \in (0,T], \tag{29a}$$

$$u(x,0) = u_0(x), \qquad x \geq 0, \tag{29b}$$

$$\alpha u(0,t) + \beta u_x(0,t) = g(t), \qquad t \in (0,T),\ \alpha, \beta \in \mathbb{R}. \tag{29c}$$

has a unique solution when $\alpha \neq \beta$. Further, the existence of a smooth solution requires $u_0(0) = g(0) = 0$. If $\alpha = \beta$ the problem is ill-posed in the sense that the initial and boundary conditions may not be prescribed arbitrarily. If homogeneous boundary conditions are imposed, the initial condition must be identically zero for a smooth solution to exist.

By uniqueness, this implies the only solution with homogeneous boundary conditions is the trivial one. If $\alpha = \beta$ and non-homogeneous boundary conditions are prescribed, then the constraint between initial and boundary conditions

$$\hat{u}_0(-i) + g'(0) = 0,$$

must hold for a solution to exist.

Theorem 2. The BVP

$$\begin{aligned} u_t - u_{xxt} + u_x &= 0, && x \in [0, L]\,,\ t \in (0, T], && \text{(30a)}\\ u(x,0) &= u_0(x), && x \in [0, L]\,, && \text{(30b)}\\ \alpha u(0,t) + \beta u_x(0,t) &= g(t), && t \in (0,T),\ \alpha, \beta \in \mathbb{R}, && \text{(30c)}\\ \gamma u(L,t) + \delta u_x(0,t) &= h(t), && t \in (0,T),\ \gamma, \delta \in \mathbb{R}. && \text{(30d)} \end{aligned}$$

has a unique solution when $(\alpha, \gamma) \neq \pm(\beta, \delta)$. The existence of a smooth solution requires $u_0(0) = g(0) = 0$ and $u_0(L) = h(0) = 0$. If $(\alpha, \gamma) = \pm(\beta, \delta)$ the problem is ill-posed in the sense that the initial and boundary conditions may not be prescribed arbitrarily. If homogeneous boundary conditions are imposed, the initial condition must be identically zero for a smooth solution to exist. By uniqueness, this implies the only solution with homogeneous boundary conditions is the trivial one. If $(\alpha, \gamma) = \pm(\beta, \delta)$ and non-homogeneous boundary conditions are prescribed, then the constraint between initial and boundary conditions

$$\hat{u}_0(-i) + g'(0) - e^{-L} h'(0) = 0,$$

must hold for a solution to exist.

It is possible to anticipate the non-trivial behavior mentioned in these theorems. For linear BBM, the differential operator acting on the time derivative has the symbol $M(k) = 1 + k^2$, with a null-space spanned by $\{e^x, e^{-x}\}$. In order to invert this operator, it is necessary for the null-space of the operator to be empty. This is achieved by selecting the boundary condition. To see this consider linear BBM on the finite interval as a second-order forced ordinary differential equation in u_t (see [2]). For all boundary conditions except the cases $(\alpha, \gamma) = \pm(\beta, \delta)$, it is possible to construct a Green's function uniquely. Further, the functions e^x and e^{-x} are in the null-space of the exceptional boundary conditions. This is the cause of the ill-posedness. This form of ill-posedness should be a generic feature of PDEs with mixed partial derivatives.

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