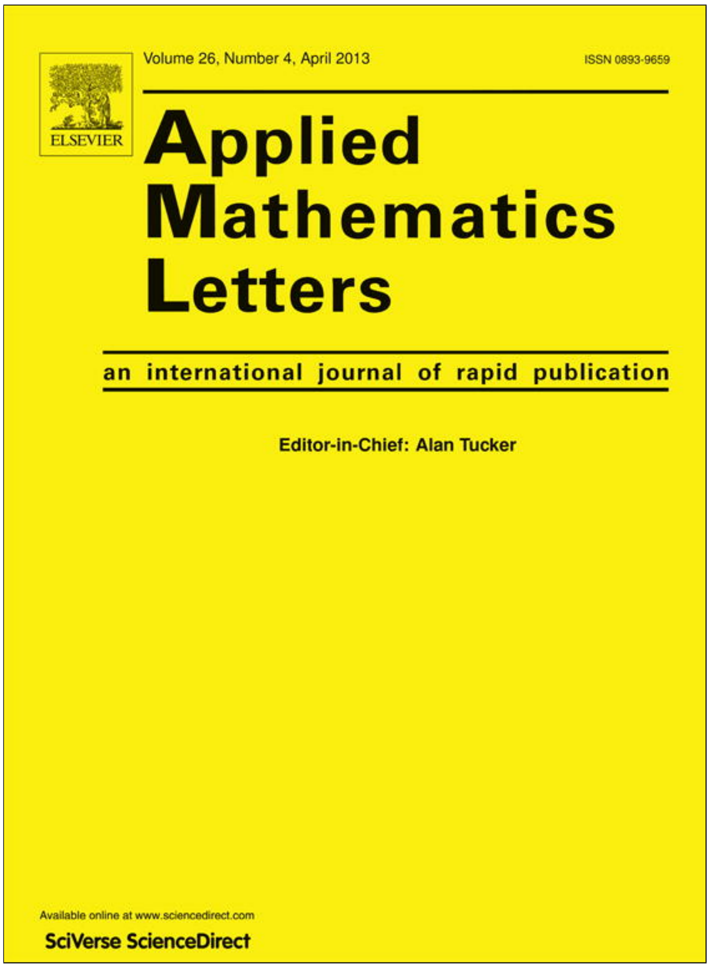
Volume 26, Number 4, April 2013
ISSN 0893-9659
ELSEVIER
Applied Mathematics Letters
an international journal of rapid publication
Editor-in-Chief: Alan Tucker
Available online at www.sciencedirect.com
SciVerse ScienceDirect

Applied Mathematics Letters 26 (2013) 515–519

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

The Bernoulli boundary condition for traveling water waves

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ARTICLE INFO

Article history:
Received 10 December 2012
Accepted 10 December 2012

Keywords:
Water waves

ABSTRACT

The Bernoulli boundary condition for traveling water waves is obtained from Euler's equation for inviscid flow by employing two key reductions: (i) the traveling wave assumption, (ii) the introduction of a velocity potential. Depending on the order of these reductions, the Bernoulli boundary condition may or may not contain an arbitrary constant. This note shows the equivalence of the two formulations. Further, we arrive at a physical interpretation for the Bernoulli constant, namely, that it is associated with an average current. Last, we show that the Bernoulli constant and the average current cannot simultaneously be zero for non-trivial traveling waves.

1. The Bernoulli boundary condition

The purpose of this note is to discuss some differences in the literature regarding the Bernoulli boundary condition for two-dimensional irrotational inviscid flow. In particular we present a physical interpretation for the Bernoulli constant for traveling water waves: it is related to an average horizontal fluid velocity *i.e.*, a current. Indeed, we show that the Bernoulli constant and the current cannot both be zero for non-trivial periodic traveling waves.

There are, effectively, two schools of thought concerning the Bernoulli condition typified by Constantin and Strauss [1] and Deconinck and Oliveras [2]. The difference lies in whether or not one includes an arbitrary constant in the formulation of periodic traveling waves (referred to a Bernoulli constant here). This choice leads to two different formulations of the traveling wave problem employing either one of the Bernoulli conditions. In this note, we demonstrate the equivalence of the two formulations. Further we show that the choice made in [2] leads to a flow with a current.

Below we present a derivation of the Bernoulli condition for inviscid irrotational fluid flow. The assumption of irrotationality is equivalent to assuming the existence of a velocity potential ϕ such that $\nabla\phi$ represents the fluid velocity vector. As shown below, the order of the reductions: (i) introduction of ϕ, and (ii) traveling waves, can lead to different Bernoulli conditions. In particular, the Bernoulli constant may be eliminated from the problem. In summation, we connect the traveling wave assumption at the level of the velocity potential with that made at the level of the velocities.

The Bernoulli condition is obtained as an integral of the equations of motion of the fluid domain and typically involves an arbitrary function of time. The Euler equations for such a fluid flow are given as (see Fig. 1)

$$\begin{aligned} &u_x + v_z = 0, \\ &u_t + uu_x + vu_z = -P_x, \\ &v_t + uv_x + vv_z = -P_z - g, \end{aligned}$$

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doi:10.1016/j.aml.2012.12.008

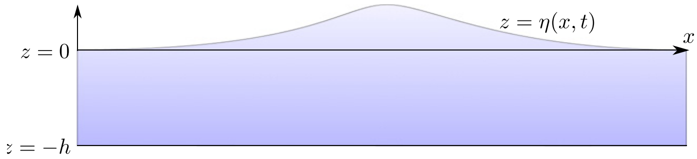

Fig. 1. Fluid domain.

where u and v are the horizontal and vertical components of the fluid velocity, $P(x, z, t)$ is the fluid pressure and g is the acceleration due to gravity. The boundary conditions for the fluid are

$$v = 0, \quad \text{at } z = -h,$$
$$P = 0, \quad \text{at } z = \eta(x, t),$$
$$\eta_t = v - u\eta_x \quad \text{at } z = \eta(x, t),$$

where $\eta(x, t)$ is the free surface elevation. We do not specify horizontal boundary conditions yet. As the flow is assumed to be irrotational ($u_z = v_x$) the equations for the time evolution of the fluid velocities can be written as

$$u_t + \frac{1}{2}\partial_x(u^2 + v^2) = -P_x, \tag{1}$$

$$v_t + \frac{1}{2}\partial_z(u^2 + v^2) = -P_z - g. \tag{2}$$

Introducing the velocity potential $[u, v]^T = \nabla\phi$ we obtain the following condition upon integrating along an arbitrary path in the fluid domain:

$$\phi_t + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_z^2 + gz = -P + c(t). \tag{3}$$

Here $c(t)$ is an arbitrary function of time. The above condition, when taken together with the dynamic boundary condition at the free surface $P = 0$ at $z = \eta(x, t)$, leads to the Bernoulli boundary condition

$$\phi_t + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_z^2 + g\eta = c(t), \quad \text{at } z = \eta(x, t). \tag{4}$$

For fluid flow in unbounded domains, the no-flow boundary condition at infinity implies the function $c(t)$ must be identically zero. In the case of periodic motion, boundary conditions alone do not eliminate this arbitrariness. In this case one can eliminate the arbitrariness by suitably redefining the velocity potential. The fluid velocities are defined as the spatial derivative of the potential ϕ. Thus ϕ is uniquely defined up to a function of time. Following the standard procedure [3,4] we define a new potential $\phi_c = \phi - \int_0^t c(s)ds$, thereby absorbing the function $c(t)$.

If, in addition, one considers either steady flow or traveling wave solutions, we obtain two different Bernoulli boundary conditions depending on when the traveling wave assumption is made. Introducing the velocity potential to the time-dependent problem (1)–(2), as above, eliminating $c(t)$ and then considering traveling wave solutions: $\phi(x, z, t) = \phi(\xi, z)$, $\eta(x, t) = \eta(\xi)$ where $\xi = x - ct$ one obtains the following Bernoulli condition

$$-c\phi_\xi + \frac{1}{2}(\phi_\xi^2 + \phi_z^2) + g\eta = 0, \quad \text{at } z = \eta(\xi). \tag{5}$$

However, if one introduces the traveling wave assumption in Eqs. (1)–(2), *i.e.* $u(x, z, t) = u(\xi, z)$, $v(x, z, t) = v(\xi, z)$ and then defines the potential, the Bernoulli condition is given as

$$-c\phi_\xi + \frac{1}{2}(\phi_\xi^2 + \phi_z^2) + g\eta = E, \quad \text{at } z = \eta(\xi), \tag{6}$$

where E is an arbitrary constant. We refer to E as the Bernoulli constant. Irrespective of the order of the reductions, the remaining equations for the velocity potential are

$$\phi_{\xi\xi} + \phi_{zz} = 0, \qquad D = \{(\xi, z) : 0 < \xi < L, -h < z < \eta(\xi)\}, \tag{7}$$

$$\phi_z = 0, \quad \text{at } z = -h, \tag{8}$$

$$-c\eta_\xi = \phi_z - \phi_\xi\eta_\xi, \quad \text{at } z = \eta(\xi). \tag{9}$$

Here we consider ϕ_ξ, ϕ_z and η to be periodic in ξ with period L. Evidently, depending on the order of the reductions made, we obtain two different boundary conditions: (5) or (6). Both of the conditions are used in the literature and there is some confusion regarding which is appropriate. We address this issue by showing they are equivalent. In addition we give a physical interpretation of the constant E.

A partial resolution is stated in Landau–Lifschitz [5, p. 17, Chapter 1, Section 19]: "The function $c(t)\cdots$ can be put to zero without loss of generality, because the potential is not uniquely defined: since the velocity is the space derivative of ϕ, we can add to ϕ any function of time. For steady flow we have (taking the potential ϕ to be independent of time) $\phi_t = 0$, $c(t) =$ Constant and Bernoulli's equation becomes:

$$\frac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta = \text{Constant.''}.$$

This passage hints at the root of the problem: in passing to a traveling frame, we have made an additional assumption on the function $c(t)$.

The two Bernoulli conditions (5) and (6) can be shown to be equivalent up to a redefinition of the velocity field. Indeed, defining

$$\tilde{\phi}_\xi = \phi_\xi - c \mp \sqrt{c^2 + 2E}, \qquad \tilde{\phi}_z = \phi_z.$$

Eq. (6) becomes

$$(\pm\sqrt{c^2 + 2E})\tilde{\phi}_\xi + \frac{1}{2}(\tilde{\phi}_\xi^2 + \tilde{\phi}_z^2) + g\eta = 0,$$

which is condition (5) for a different wave speed. Thus the two boundary conditions are equivalent up to the addition of a uniform horizontal velocity. In other words, eliminating the Bernoulli constant corresponds to introducing a current and changing the speed of the traveling wave. Note that the remaining equations of motion remain unchanged except for the appropriate change in the speed of the wave. The change of variables employed above indicates that, when using condition (5), the current may be non-zero. In the next section we establish this must be the case, *i.e.*, introducing the additional velocity $c \pm \sqrt{c^2 + 2E}$ (or equivalently considering condition 5) cannot lead to a flow with zero current.

2. The Bernoulli constant and currents

As the velocity potential is harmonic in the bulk of the fluid, it is possible to relate the norm of the velocities at the surface $z = \eta(\xi)$ and at the bottom boundary $z = -h$. Indeed, since $\phi_{\xi\xi} + \phi_{zz} = 0$ the vector

$$\left[2(\phi_\xi - c)\phi_z, \phi_z^2 - (\phi_\xi - c)^2\right]^T$$

is divergence free. Thus, applying Green's Theorem over the fluid domain D, and noting that the velocities are periodic in x we obtain (using that $dz = \eta_x dx$ along the free surface)

$$\oint_{\partial D} \left((\phi_\xi - c)^2 - \phi_z^2\right) dx + 2(\phi_x - c)\phi_z dz = 0$$
$$\Rightarrow \int_0^L \left((\phi_\xi - c)^2 - \phi_z^2 + 2(\phi_x - c)\phi_z\eta_x\right)_{z=\eta} dx + \int_L^0 \left((\phi_\xi - c)^2 - \phi_z^2 + 2(\phi_x - c)\phi_z\right)_{z=-h} dx = 0,$$
$$\Rightarrow \frac{1}{L}\int_0^L \left((\phi_\xi - c)^2 + \phi_z^2\right)_{z=\eta} dx = \frac{1}{L}\int_0^L (\phi_\xi - c)^2_{z=-h} dx. \tag{10}$$

In obtaining the result above we have used the boundary conditions

$$\phi_z(\xi, -h) = 0, \qquad \phi_z(\xi, \eta) = (\phi_\xi(\xi, \eta) - c)\eta_\xi.$$

The additional factor $1/L$ is added so that we may interpret the above as an equality of averages. Consider now the Bernoulli condition (6) at the surface $z = \eta(\xi)$

$$-c\phi_\xi + \frac{1}{2}\phi_\xi^2 + \frac{1}{2}\phi_z^2 + g\eta = E$$
$$\Rightarrow \frac{1}{2}(\phi_\xi - c)^2 + \frac{1}{2}\phi_z^2 + g\eta = E + \frac{1}{2}c^2. \tag{11}$$

Taking an average of (11) over one period, we obtain

$$\frac{1}{2L}\int_0^L \left((\phi_\xi - c)^2 + \phi_z^2\right)_{z=\eta} d\xi = E + \frac{1}{2}c^2, \tag{12}$$

where we assume η has zero average, without loss of generality. The corresponding Bernoulli condition at the bottom boundary $z = -h$ takes the form

$$-c\phi_\xi + \frac{1}{2}\phi_\xi^2 - gh = -P_b(\xi) + E$$
$$\Rightarrow \frac{1}{2}(\phi_\xi - c)^2 - gh = -P_b(\xi) + E + \frac{1}{2}c^2, \tag{13}$$

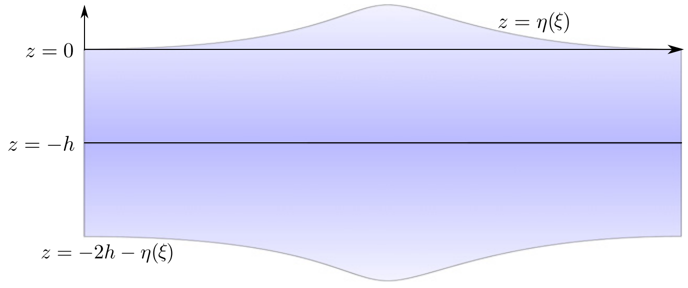

Fig. 2. The extended fluid domain.

since the vertical velocity $\phi_z = 0$ at $z = -h$. Here $P_b(\xi)$ is the pressure at the bottom boundary. Once again taking averages, we obtain

$$\frac{1}{2L}\int_0^L (\phi_\xi - c)^2_{z=-h} d\xi - gh = -\frac{1}{L}\int_0^L P_b(\xi) d\xi + E + \frac{1}{2}c^2.$$

The above relation with Eqs. (10) and (12) implies

$$\frac{1}{L}\int_0^L P_b(\xi) d\xi = gh. \tag{14}$$

Last, we show that if the average current and Bernoulli constant E are simultaneously zero then the free surface η is identically zero. The average current κ is given by

$$\kappa = \frac{1}{L}\int_0^L \phi_\xi(\xi, -h) d\xi = \frac{1}{L}\int_0^L (\phi_\xi + \eta_\xi \phi_z)_{z=\eta} d\xi.$$

The first equality follows the standard definition [6] and the second equality is a consequence of Stokes' Theorem for the vector $\left[\phi_\xi, \phi_z\right]^T$. The integrand on the right-hand side of the second equality can be interpreted as the tangential velocity at the surface $z = \eta(\xi)$.

Assume $E = 0$ in (13). Averaging the resulting expression and taking into account Eq. (14),

$$\frac{1}{L}\int_0^L (\phi_\xi - c)^2_{z=-h} d\xi = \frac{c^2}{2}.$$

If the current $\kappa = 0$, we obtain

$$\frac{1}{L}\int_0^L \phi_\xi^2(\xi, -h) d\xi = 0,$$

so that $\phi_\xi(\xi, -h) = 0$.

The velocity potential ϕ is harmonic in the region $-h < z < \eta(\xi)$ and $\phi_z(\xi, -h) = 0$. Thus ϕ has a harmonic extension to $z < -h$ through a reflection and ϕ is harmonic in the larger extended domain $-2h - \eta(\xi) < z < \eta(\xi)$ (see Fig. 2). Additionally, ϕ has a harmonic conjugate ψ which is harmonic in the same domain and

$$\phi_\xi = \psi_z, \qquad \phi_z = -\psi_\xi,$$

throughout this domain. As shown above, when $E = \kappa = 0$

$$\phi_\xi(\xi, -h) = \phi_z(\xi, -h) = 0.$$

Thus $\psi_\xi(\xi, -h) = \psi_z(\xi, -h) = 0$. Hence the complex analytic function

$$f(\xi + iz) = \phi(\xi, z) + i\psi(\xi, z),$$

is constant along $z = -h$. But this implies $f(\xi + iz)$ is a constant. Hence $\phi_\xi = \phi_z = 0$ for all x, z. By the Bernoulli condition (11), for $E = 0$, we have that the free surface $\eta \equiv 0$. Thus the Bernoulli constant and the average current cannot simultaneously be zero for non-trivial traveling waves. However, it is always possible to eliminate either the current or the Bernoulli constant.

Acknowledgments

We wish to acknowledge comments made by an anonymous referee for The Journal of Fluid Mechanics. It was in correspondence for a different manuscript with this referee that the central issue discussed here, namely the use in the literature of two seemingly distinct Bernoulli boundary conditions, was highlighted. Further, we are grateful for support from the National Science Foundation under grant NSF-DMS-1008001. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the funding sources.

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