Canonical Variables for multiphase solutions of the KP equation

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Submitted for publication

October 27, 1998

Abstract

The KP equation has a large family of quasiperiodic multiphase solutions. These solutions can be expressed in terms of Riemann-theta functions. In this paper, a finite-dimensional canonical Hamiltonian system depending on a finite number of parameters is given for the description of each such solution. The Hamiltonian systems are completely integrable in the sense of Liouville. In effect, this provides a solution of the initial-value problem for the theta-function solutions. Some consequences of this approach are discussed.

1 Introduction

In 1970, Kadomtsev and Petviashvili [1] derived two equations as generalizations of the Korteweg-de Vries (KdV) equation to two spatial dimensions:

$$(-4u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0, \qquad \text{(KP)}$$

where $\sigma^2 = \pm 1$ and the subscripts denote differentiation.

Depending on the physical situation, one derives the equation either with $\sigma^2 = -1$ or $\sigma^2 = +1$. The resulting partial differential equations are referred to as (KP1) and (KP2) respectively. For real-valued solutions, the two equations have different physical meaning and different properties [1]. Nevertheless, for some purposes the sign of σ^2 is irrelevant and we equate $\sigma \equiv 1$. In this case, we simply refer to "the KP equation" or to "(KP)". This amounts to working with (KP2). If necessary, (KP1) is obtained through a complex scaling of the y-variable.

The KP equation can be written as the compatibility condition of two linear equations for an auxiliary wave function Ψ [2, 3]:

$$\sigma \Psi_y = \Psi_{xx} + u\Psi, \qquad \text{(1.1 a)}$$

$$\Psi_t = \Psi_{xxx} + \frac{3}{2}u\Psi_x + \frac{3}{4}\left(u_x + w\right)\Psi. \qquad \text{(1.1 b)}$$

Expressing the compatibility of (1.1 a) and (1.1 b), $\Psi_{yt} \equiv \Psi_{ty}$, and eliminating w results in (KP), if we assume that a complete basis for the wave function Ψ exists. Note that if the KP

1

solution is independent of y, the above Lax pair (1.1 a, 1.1 b) reduces to the Lax pair for (KdV) [4] by simple separation of variables of the wave function Ψ.

The two equations (1.1 a), (1.1 b) are only two equations of an infinite hierarchy of linear evolution equations for the wave function Ψ with respect to *higher-order time variables* t_n [5]. We refer to this collection of linear flows as the *linear KP hierachy*:

$$\Psi_{t_k} = A_k \Psi, \quad \text{for } k = 1, 2, 3, \ldots \tag{1.2}$$

with A_k a linear differential operator in x of order k with (usually) nonconstant coefficients. We refer to the t_k, $k = 1, 2, 3, 4, \ldots$ as higher-order time variables. A consequence of our definition of the KP hierarchy given in Section 3 is that t_1 can be identified with x. Furthermore, y and t are related to t_2 and t_3 respectively.

By expressing the compatibility of these different linear flows, $\Psi_{t_{k_1} t_{k_2}} = \Psi_{t_{k_2} t_{k_1}}$, and assuming the existence of a complete basis for Ψ, we obtain an infinite number of nonlinear partial differential equations for the evolution of u, w (and other functions referred to as potentials) with respect to the t_k [5], called the *KP hierarchy*:

$$\frac{\partial A_{k_1}}{\partial t_{k_2}} - \frac{\partial A_{k_2}}{\partial t_{k_1}} = [A_{k_2}, A_{k_1}], \tag{1.3}$$

where $[A, B] \equiv AB - BA$. The linear KP hierarchy (1.2) and the KP hierarchy (1.3) are the fundamental ingredients for the methods presented in this paper.

A large family of quasiperiodic solutions of the KP equation was found by Krichever [6, 7]. Each of these solutions has a finite number of phases. They are given by

$$u = \tilde{u} + 2\partial_x^2 \ln \Theta_g(\phi_1, \phi_2, \ldots, \phi_g | \boldsymbol{B}), \tag{1.4 a}$$

$$\phi_j = k_j x + l_j y + \omega_j t + \phi_{0j}, \quad \text{for } j = 1, 2, \ldots, g, \tag{1.4 b}$$

for some constants $\tilde{u}, k_j, l_j, \omega_j$ and ϕ_{0j}, $j = 1, 2, \ldots, g$. Θ_g is a *Riemann theta function* of *genus* g, parametrized by a $g \times g$ *Riemann matrix B*:

$$\Theta_g(\boldsymbol{\phi} | \boldsymbol{B}) \equiv \sum_{\boldsymbol{m} \in \boldsymbol{Z}^g} \exp\left(\frac{1}{2}\boldsymbol{m} \cdot \boldsymbol{B} \cdot \boldsymbol{m} + i\boldsymbol{m} \cdot \boldsymbol{\phi}\right). \tag{1.5}$$

Here $\boldsymbol{\phi} \equiv (\phi_1, \ldots, \phi_g)$. The vector $\boldsymbol{m}$ runs over all g-dimensional vectors with integer components. The Riemann matrix $\boldsymbol{B}$ is a symmetric $g \times g$ matrix with negative definite real part. Whenever the matrix $\boldsymbol{B}$ is obtained from a compact, connected Riemann surface in a standard way [8], (1.4 a) defines a solution of the KP equation [6, 7]. In what follows, the dependence of the theta function Θ_g on the Riemann matrix $\boldsymbol{B}$ and the index g denoting the number of phases will be surpressed for notational simplicity. By construction, $\Theta(\boldsymbol{\phi})$ is periodic in each component of $\boldsymbol{\phi}$. Hence the restriction to the linear winding (1.4 b) makes (1.4 a) by definition a quasiperiodic function in x or y or t. A solution of the form (1.4 a) is said to have genus g. In the terminology of Krichever and Novikov [9, 10], all solutions of the form (1.4 a) have *rank 1*.

The problem addressed in this paper is to find $u(x, y, t)$ such that:

$$\begin{cases} (-4u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0 \\ u(x, y, 0) = \text{rank 1, finite-genus solution of KP,} \\ \qquad \text{evaluated at } t = 0. \end{cases} \tag{1.6}$$

The initial data are purposely not written in the form (1.4 a). The problem is not only to determine the frequencies ω_j, for $j = 1, 2, \ldots, g$, but also to fix the other parameters $g, \boldsymbol{B}, \tilde{u}, k_j, l_j, \phi_{0j}$, for $j = 1, 2, \ldots, g$: the acceptable class of initial data consists of all finite-genus solutions of the KP equations evaluated at a fixed time, with an unspecified but finite number of phases.

A solution to this problem was offered in [11], where a seven-step algorithm was presented. This algorithm was a mixture of new ideas and known work by Krichever [12, 6, 7] and Previato [13]. The main idea of the algorithm is to provide the ingredients required for Krichever's inverse procedure for the reconstruction of a finite-genus solution of the KP equation [6, 7], *i.e.* a compact connected Riemann surface and a divisor on this surface.

In the present paper, a different algorithm to solve the problem (1.6) is presented. This algorithm shares some steps with that in [11]. However, in contrast to the first algorithm, the second algorithm does not work towards Krichever's inverse procedure [6, 7]. The main idea here is to examine the structure of a set of ordinary differential equations obtained in step 5 of [11]. In this paper, we show the following:

- The rank 1, finite-genus solutions of the KP equation are governed by a system of ordinary differential equations. This system is constructed explicitly.

- This system of ordinary differential equations is Lagrangian.

- With some care, the Lagrangian equations are written as a Hamiltonian system of ordinary differential equations in x.

- This Hamiltonian system of ordinary differential equations is completely integrable in the sense of Liouville [14]. A complete set of conserved quantities in involution under the Poisson bracket is explicitly constructed.

- From these conserved quantities, one easily constructs completely integrable Hamiltonian systems of ordinary differential equations describing the evolution of the given initial condition of (1.6) under any of the higher-order time variables t_k, including $k = 1, 2, 3$. This provides a solution of (1.6).

In the next section, it is shown how the information listed above is used in an algorithm to solve problem (1.6). As with the algorithm in [11], most of the steps of the algorithm in this paper are due to others. The work of Bogoyavlenskii and Novikov [15] provided the main inspiration: the algorithm presented here is a generalization to the KP equation of their approach to solve problem (1.6) for the KdV equation. The work by Gel'fand and Dikii [16], Veselov [17], Adler [18] and Dikii [19] was used to execute some of the steps of the algorithm. Although all of the above authors have considered parts of the problem considered here, to the best of our knowledge a complete solution of problem (1.6) using ordinary differential equations to solve the posed initial-value problem was never given.

The algorithm presented here offers an alternative approach to that in [11]. There are however some natural consequences of the new approach that do not appear from the approach in [11]. These include

- **Canonical variables** for the rank 1, finite-genus solutions of the KP equation. Any rank 1, finite-genus solution of the KP equation satisfies a Hamiltonian system of ordinary differential equations. The Poisson bracket on the phase space of this Hamiltonian system is the canonical Poisson bracket, resulting in a description of the rank 1, finite-genus solutions of the KP equation in terms of canonical (Darboux) variables.

- **Conserved Quantities** for rank 1, finite-genus solutions (1.4 a) of the KP equation. The Hamiltonian system of ordinary differential equations is completely integrable in the sense of Liouville. A sufficient number of conserved quantities is constructed explicitly. These conserved quantities are mutually in involution under the canonical Poisson bracket.

- **Parameter count** of the theta-function solutions (1.4 a) of the KP equation. It is known [8] that a generic[1] solution of genus g of the KP equation is characterized by $4g + 1$ independent parameters. We reconfirm this result, and extend it, by providing a parameter count for nongeneric solutions of the KP equation as well. Furthermore, the parameters naturally divide in two classes: parameters with "dynamical significance" and other parameters. The dynamically significant parameters are the initial conditions of the Hamiltonian system of ordinary differential equations describing the solution. The other parameters foliate the phase space of the Hamiltonian system.

- **Minimal characterization of the initial data**. The approach presented here demonstrates that a finite-genus solution $u(x, y, t)$ of the KP equation is completely determined by a one-dimensional slice of the initial condition $u(x, y = 0, t = 0)$. In other words, it suffices to specify the initial data of (1.6) at a single y-value, say at $y = 0$.

Krichever [20] proposed another method to solve an initial-value problem for the KP2 equation with initial data that are spatially periodic (in both x and y). Krichever's method is not restricted to initial data of finite genus, hence it is in that sense more general than the algorithm presented here. On the other hand, the methods of this paper require no restriction to periodic initial data.

2 Overview of the algorithm

A solution for problem (1.6) is obtained using a seven-step algorithm. In this section, an overview of this algorithm is given, along with references to earlier work.

1. **Determine the genus of the initial data** [11] Let us rewrite (1.4 b) in the form

$$\phi_j = \boldsymbol{\kappa_j} \cdot \boldsymbol{x} + \omega_j t + \phi_{0j}, \quad j = 1, 2, \ldots, g, \tag{2.1}$$

with $\boldsymbol{\kappa_j} = (k_j, l_j)$ and $\boldsymbol{x} = (x, y)$. If all wave vectors $\boldsymbol{\kappa_j}$ are incommensurable, *i.e.*, if there is no relationship

$$\sum_{i=1}^{g} n_i \boldsymbol{\kappa_i} = 0 \tag{2.2}$$

for integers n_i not all zero, then a two-dimensional Fourier transform of the initial data resolves the vectors $\boldsymbol{\kappa_j}$. Because the initial data contain only a finite number of phases, the Fourier transform is necessarily discrete; *i.e.*, it consists of isolated spikes. Since the condition (2.2) almost never holds, we can almost always find the genus of the initial condition by counting the number of spikes in the Fourier transform, modulo harmonics.

If condition (2.2) holds, then the prescribed method finds only a lower bound on the genus of the initial data. The method fails especially dramatically in one important special case: if the

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[1] for a precise definition, see section 2

initial data are spatially periodic, then (2.2) holds automatically for any two wave vectors $\boldsymbol{\kappa_i}$ and $\boldsymbol{\kappa_j}$. The lower bound obtained in this case for the number of phases is 1. This problem was already pointed out in [11]. If the initial data are spatially periodic, it is most convenient to impose that the genus of the initial data also be given, as part of the initial data.

The method of Fourier transform to determine the genus of the initial condition has been used in [21, 22].

2. **Determine two stationary flows of the KP hierarchy: find (r, n)**

Mulase [23] and later Shiota [24] showed that a rank 1, finite-genus solution of the KP equation (1.4 a) is a simultaneous solution to all flows of the KP hierarchy by using (1.4 a) with

$$\phi_j = \sum_{i=1}^{\infty} k_{j,i} t_i, \tag{2.3}$$

for $j = 1, 2, \ldots, g$, instead of (1.4 b). Mulase and Shiota demonstrated that the corresponding rank 1, finite-genus solutions are stationary with respect to all but a finite number of the higher-order times in the KP hierarchy. A rank 1, finite-genus solution of the KP equation is said to be stationary with respect to t_k if

$$\sum_{i=1}^{k} d_i \frac{\partial u}{\partial t_i} = 0, \tag{2.4}$$

with all the d_i constant and $d_k = 1$.

The algorithm presented here requires the knowledge of two independent higher-order times of the KP hierarchy t_r and t_n, such that u is stationary with respect to both t_r and t_n. First, r is the minimal k for which (2.4) holds for $k = r$. For this r, n corresponds to the lowest-order higher-order time t_n, such that the t_n-flow is independent of the t_r-flow and (2.4) holds for $k = n$.

In [11], a recipe was presented to find (r, n), given the genus g of the initial data. Actually, a finite number of pairs (r, n) is determined for any given g. As we will see in step 4, each one of the resulting pairs (r, n) gives rise to a set of ordinary differential equations, one of which the initial condition necessarily satisfies. The pairs (r, n) for which the initial condition does not satisfy the differential equations need to be rejected. Hence, only at step 4 do we nail down a pair of stationary flows of the KP hierarchy for the given initial data. Here, at step 2, the numbers of pairs (r, n) is reduced to a finite number.

For initial data with g phases, the following constraints on (r, n) are known [11]:

- All values of r with $2 \leq r \leq g + 1$ are allowed.
- For each r, let $n_j(r)$ be the j-th integer greater than r that is coprime with r. The lowest $(g - r + 2)$ of these integers are possible values of n.
- Exclude from the list of pairs (r, n) obtained above the values of n for which $(r - 1)(n - 1)/2 < g$.
- The remaining pairs (r, n) are all possible for genus g.

5

Remarks

(a) When $r = 2$, the only possibility for n is $n = 2g + 1$. This is the case corresponding to one-dimensional solutions.

(b) A solution of the KP equation of genus g is called *generic* if the first g vectors $\boldsymbol{k_i} = (k_{1,i}, k_{2,i}, \ldots, k_{g,i})$ are linearly independent. For a generic solution of genus g of the KP equation, $(r, n) = (g + 1, g + 2)$.

(c) We have the following confusing situation: to find a generic genus g KP solution, we need $(r, n) = (g + 1, g + 2)$. This choice leads to a Hamiltonian system of ordinary differential equations (in step 5). However, a generic solution of genus g is not a typical solution of this Hamiltonian system (*i.e.*, it does not depend on a maximal number of parameters for this system). Typical solutions of the Hamiltonian system are nongeneric solutions of higher genus. Since these higher-genus solutions depend on more parameters, one must search carefully to find the generic genus g solutions among them. This is further discussed in Sections 11 and 12.

3. **Impose the r-reduction**

For a given value of r, obtained from step 2, we impose on the KP hierarchy (1.3) the reduction that the KP solution is independent of t_r. Hence, the coefficients of A_k, for all k are independent of t_r. Following Gel'fand and Dikii [16], Adler [18] and Strampp and Oevel [25], this allows us to rewrite the *r-reduced KP hierarchy* as an infinite ($r \times r$ matrix) hierarchy of partial differential equations, each with one space dimension, and all of them mutually commuting. In other words, (1.3) is replaced by a hierarchy of the form

$$\frac{\partial B}{\partial t_k} = [B_k, B], \quad r \text{ does not divide } k. \tag{2.5}$$

The matrices B and B_k contain $r - 1$ unknown functions. The higher-order time variables of the hierarchy in (2.5) are inherited from the KP hierarchy (1.3). Only t_r and the higher-order times of the form $t_{(ir)}$ for integer i do not appear any more. In particular $t_1 \equiv x$. Each equation of the hierarchy (2.5) is Hamiltonian, as is shown in Section 4, where the details of the r-reduction are given.

4. **Impose the n-reduction**

After imposing stationarity of the KP solution with respect to t_r, we now impose stationarity of the KP solution with respect to t_n as well. Imposing the n-reduction in addition to the r-reduction leads to the (r, n)-*reduced KP equation*. The (r, n)-reduced KP equation is a system of $(r - 1)$ ordinary differential equations in x for $(r - 1)$ unknown functions $\boldsymbol{u}$.

Again, following Gel'fand and Dikii [16], Adler [18] and Strampp and Oevel [25], we write the (r, n)-reduced KP equation in Lagrangian form:

$$\frac{\delta \mathcal{L}}{\delta \boldsymbol{u}} = \boldsymbol{0}, \tag{2.6}$$

where $\delta \mathcal{L}/\delta \boldsymbol{u}$ denotes the variational derivative of $\mathcal{L}$ with respect to a certain vector function $\boldsymbol{u} = (f_1, f_2, \ldots, f_{r-1})$, which is explicitly determined in terms of the solution of (KP):

6

$$\frac{\delta \mathcal{L}}{\delta \boldsymbol{u}} \equiv \left(\frac{\delta \mathcal{L}}{\delta f_1}, \frac{\delta \mathcal{L}}{\delta f_2}, \ldots, \frac{\delta \mathcal{L}}{\delta f_{r-1}} \right)^T \tag{2.7}$$

and for any function f, the *variational derivative* of $\mathcal{L}$ with respect to f is defined as

$$\frac{\delta}{\delta f} \mathcal{L}(u, u_x, u_{xx}, \ldots) \equiv \sum_{k \geq 0} (-1)^k \frac{\partial^k}{\partial x^k} \frac{\partial \mathcal{L}}{\partial f^{(k)}}. \tag{2.8}$$

Here, $f^{(k)}$ denotes the k-th derivative of f with respect to x.

Equations (2.6) are a set of ordinary differential equations that the initial condition needs to satisfy. This constitutes a test on the validity of the pair (r, n), chosen after step 2.

The details of imposing the n-reduction in addition to the r-reduction are found in Section 5

5. **The Ostrogradskii transformation, canonical variables and the Hamiltonian system**

In Section 7, the Lagrangian system of ordinary differential equations in x is transformed to a Hamiltonian system of ordinary differential equations in x with canonical variables. Since the Lagrangian $\mathcal{L}$ depends on more than the first derivatives of $\boldsymbol{u}$, an extention of the Legendre transformation is needed. This is the *Ostrogradskii transformation* [26, 27], defined in Section 7. It defines *canonical variables* $\boldsymbol{q}$ and $\boldsymbol{p}$ in terms of the Lagrangian variables $\boldsymbol{u}$:

$$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{u}, \boldsymbol{u}_x, \boldsymbol{u}_{xx}, \ldots), \quad \boldsymbol{p} = \boldsymbol{p}(\boldsymbol{u}, \boldsymbol{u}_x, \boldsymbol{u}_{xx}, \ldots). \tag{2.9}$$

The Lagrangian $\mathcal{L}$ is called *nonsingular* [28, 29] if the Ostrogradskii transformation is invertible, *i.e.*, if the transformation (2.9) can be solved for the Lagrangian variables $\boldsymbol{u}$ and their derivatives in terms of the canonical variables $\boldsymbol{q}$ and $\boldsymbol{p}$. If the Lagrangian is nonsingular, the Euler-Lagrange equations corresponding to the Lagrangian $\mathcal{L}$ are equivalent to the Hamiltonian system

$$\frac{\partial \boldsymbol{q}}{\partial x} = \frac{\partial H}{\partial \boldsymbol{p}}, \quad \frac{\partial \boldsymbol{p}}{\partial x} = -\frac{\partial H}{\partial \boldsymbol{q}} \tag{2.10}$$

where the Hamiltonian H is determined explicitly in terms of the Lagrangian.

If both r and n are odd, the Lagrangian $\mathcal{L}$ is shown to be singular in Section 10. Nevertheless, the dynamics in terms of the Lagrangian variables is still well-posed, as shown by Veselov [17]. In Section 10, the singular Lagrangians are further investigated. We indicate how one might be able to avoid dealing with singular Lagrangians: a simple invertible transformation on the Lagrangian variables should be able to transform the singular Lagrangian into a nonsingular one. Otherwise, one can always resort to the more general methods of Krupkova [28] or to the theory of constraint Hamiltonian systems [30].

6. **Complete integrability of the Hamiltonian system**

The Hamiltonian system (2.10) is shown to be *completely integrable in the sense of Liouville* in Section 8. If the dimension of the vectors $\boldsymbol{q}$ and $\boldsymbol{p}$ is N, the Hamiltonian system is $2N$-dimensional. A set of N functionally independent conserved quantities T_k is constructed.

Generalizing the work of Bogoyavlenskii and Novikov [15], these conserved quantities are shown to be mutually *in involution*, *i.e.*,

$$\{T_k, T_l\} \equiv 0, \tag{2.11}$$

where $\{f, g\}$ denotes the *Poisson bracket* of the functions f and g:

$$\{f, g\} \equiv \frac{\partial f}{\partial \boldsymbol{q}} \frac{\partial g}{\partial \boldsymbol{p}} - \frac{\partial f}{\partial \boldsymbol{p}} \frac{\partial g}{\partial \boldsymbol{q}}. \tag{2.12}$$

A consequence of proving the involutivity of the conserved quantities T_k is that $T_k = -H_k$, where H_k is the Hamiltonian describing the evolution of the canonical variables along the higher-order time variable t_k:

$$\frac{\partial \boldsymbol{q}}{\partial t_k} = \frac{\partial H_k}{\partial \boldsymbol{p}}, \quad \frac{\partial \boldsymbol{p}}{\partial t_k} = -\frac{\partial H_k}{\partial \boldsymbol{q}}. \tag{2.13}$$

The canonical variables are related to the dependent variable u of the KP equation. Hence, we have constructed a set of ordinary differential Hamiltonian systems, each one of which describes the evolution of a rank 1, finite-genus solution of the KP equation according to a different higher-order time variable. Since all these Hamiltonian systems are $2N$-dimensional and share a common set of N functionally independent conserved quantities T_k, mutually in involution, they are all completely integrable in the sense of Liouville.

7. **Solve the Hamiltonian system; reconstruct the solution of the KP equation**

The final step of the algorithm is to integrate explicitly the Hamiltonian systems obtained in the previous step. From Liouville's theorem [14] it is known that the Hamiltonian equations of motion can be solved in terms of quadratures.

This last step is not executed in this paper. For the KdV equation it can be found in [19]. Some partial results for the KP equation are also discussed there.

3   The KP hierarchy

In this section, the KP hierarchy is redefined, using the terminology of Gel'fand and Dikii [16], Adler [18] and others. More specifically, the notation of Strampp and Oevel [25] is used.

Consider the pseudo-differential operator

$$L = \partial + u_2 \partial^{-1} + u_3 \partial^{-2} + u_4 \partial^{-3} + \ldots = \sum_{j=-\infty}^{1} u_{1-j} \partial^j, \tag{3.1}$$

with $u_0 \equiv 1$, $u_1 \equiv 0$. We have used the notation $\partial = \partial_x$. The coefficients u_j can be functions of x. The u_j are referred to as *potentials*. This term is also used for any other set of functions, related to the u_j by an invertible transformation.

Remark In order to compare with the results in [11], we need $u_1 \neq 0$, but constant, extending the definition of the pseudo-differential operator L. Although this changes some of the formulas in this section, the added results for the KP equation are minor. In [19] and [31], it is shown that this

amounts to assigning a fixed value to the constant $\tilde{u}$ in (1.4 a). In the remainder of the paper, we assume $u_1 \equiv 0$, unless stated otherwise.

The action of the operator ∂^j is defined by the generalized Leibniz rule:

$$\partial^j f = \sum_{i=0}^{\infty} \binom{j}{i} f^{(i)} \partial^{j-i}, \tag{3.2}$$

where f is a function, $f^{(i)}$ is its i-th derivative with respect to x, and the binomial coefficients are defined as

$$\binom{j}{i} = \frac{j(j-1)\cdots(j-i+1)}{i!}, \quad \text{for } i > 0, \quad \text{and} \quad \binom{j}{0} = 1. \tag{3.3}$$

Note that this definition makes sense for negative integers j. For non-negative integers j, (3.2) is a finite sum. Otherwise, (3.2) results in an infinite series.

Next, consider positive integer powers of the pseudo-differential operator L:

$$
\begin{aligned}
L^r &= \left(\partial + u_2 \partial^{-1} + u_3 \partial^{-2} + u_4 \partial^{-3} + \ldots\right)^r \\
&= \sum_{j=-\infty}^{r} \alpha_j(r) \partial^j = \sum_{j=-\infty}^{r} \partial^j \beta_j(r).
\end{aligned} \tag{3.4}
$$

The last two equalities define the functions $\alpha_j(r)$ and $\beta_j(r)$, for $j \leq r$, $r > 0$. These are in general functions of x. One has

$$\alpha_1(1) = 1, \alpha_0(1) = 0, \alpha_j(1) = u_{j+3}, \quad \text{for } j = -1, -2, -3, \ldots, \tag{3.5}$$

and

$$\alpha_r(r) = 1, \quad \beta_r(r) = 1, \quad \text{and} \quad \alpha_{r-1}(r) = 0, \quad \beta_{r-1}(r) = 0. \tag{3.6}$$

Clearly the functions $\alpha_j(r)$ and $\beta_j(r)$ are related. Using (3.2), we get from (3.4) that

$$\alpha_j(r) = \sum_{k=0}^{r-j} \binom{j+k}{k} \beta_{j+k}^{(k)}(r). \tag{3.7}$$

This triangular system can be solved to obtain the functions $\beta_j(r)$ in terms of the functions $\alpha_j(r)$, if so desired. Note in particular that $\alpha_{-1}(r) = \beta_{-1}(r)$, since the binomial coefficient (3.3) vanishes for positive j less than i.

The functions $\alpha_j(r)$ can be determined explicitly in terms of the potentials $(u_2, u_3, \ldots)$. A convenient way to do this is to use a recursion relationship obtained from $L^r = LL^{r-1}$:

$$\alpha_j(r) = \alpha_{j-1}(r-1) + \frac{\partial}{\partial x}\alpha_j(r-1) + u_{r-j} + \sum_{k=j-r+3}^{-1} u_{1-k} \sum_{m=j}^{k+r-3} \binom{k}{m-j} \alpha_{m-k}^{(m-j)}(r-1). \tag{3.8}$$

It is possible to obtain an explicit formula which expresses $\alpha_j(r)$ in terms of only the potentials $(u_2, u_3 \ldots)$ [31], but such a formula is not as practical as the recursion relationship (3.8). However, the following result will be used. It extends a result of Date *et al* [32]:

9

$$\alpha_j(r) = \sum_{k=1}^{r-j-1} \binom{r}{k} \frac{\partial^{k-1}}{\partial x^{k-1}} u_{r-j-k+1} + \hat{\alpha}_j(r), \tag{3.9}$$

and $\hat{\alpha}_j(r)$ is a differential polynomial in $(u_2, u_3, \ldots, u_{r-j-2})$ containing only nonlinear terms. This follows easily from (3.8).

The differential part (including the purely multiplicative term) of the operator L^r is denoted by L^r_+:

$$L^r_+ = \sum_{j=0}^{r} \alpha_j(r) \partial^j = \sum_{j=0}^{r} \partial^j \beta_j(r). \tag{3.10}$$

Observe from (3.7) that the purely differential part of L^r is independent of the representation (3.4) used for L^r. This is also true for

$$L^r_- = L^r - L^r_+ = \sum_{j=-\infty}^{-1} \alpha_j(r) \partial^j = \sum_{j=-\infty}^{-1} \partial^j \beta_j(r). \tag{3.11}$$

Having introduced the above notation, the KP hierarchy is expressed quite easily. Consider the linear evolution equation for the wave function Ψ

$$\frac{\partial \Psi}{\partial t_r} = L^r_+ \Psi, \quad \text{for } r = 1, 2, \ldots. \tag{3.12}$$

This is the *linear KP hierarchy* (*cf.* (1.2)). For $r = 1$, this equation given $\Psi_{t_1} = \Psi_x$, hence the identification $t_1 \equiv x$. Assuming completeness of states, the *KP hierarchy* is obtained from the compatibility of the equations in (3.12) (*cf.* (1.3)):

$$\frac{\partial^2 \Psi}{\partial t_{r_1} \partial t_{r_2}} = \frac{\partial^2 \Psi}{\partial t_{r_2} \partial t_{r_1}} \Rightarrow \frac{\partial L^{r_1}_+}{\partial t_{r_2}} - \frac{\partial L^{r_2}_+}{\partial t_{r_1}} = \left[L^{r_2}_+, L^{r_1}_+ \right]. \tag{3.13}$$

These equations determine how the potentials depend on the higher-order time variables t_r so that the equations (3.12) are compatible. Again assuming completeness of states, equations (3.13) can also be obtained from the compatibility of the following sequence of Lax-like equations

$$\frac{\partial L}{\partial t_r} = [L^r_+, L] = [L, L^r_-], \quad r \geq 1. \tag{3.14}$$

The last equality is a consequence of $L^r = L^r_+ + L^r_-$.

Introducing the KP hierarchy as in (3.14) is equivalent to the approach used in [11]. Below, we use that (3.12) is essentially equivalent to (3.14). Our approach consists mainly of rewriting (3.14) and its reductions.

Each time we increase r by one, another potential appears in $L^r_+ = L^r_+(u_2, u_3, \ldots, u_r)$. Furthermore, u_r appears only in $\alpha_0(r)$: $\alpha_0(r) = r u_r + \tilde{\alpha}_0(r; u_2, u_3, \ldots, u_{r-1})$, as is seen from (3.8). As a consequence, there is a one-to-one correspondence between the potentials $u, w_1, w_2, \ldots$ appearing in the KP hierarchy as it is defined in [11] and the set of potentials $u_2, u_3, u_4, \ldots$ appearing in (3.1).

As an example, consider equations (1.1 a, 1.1 b). These are contained in the formulation of the KP hierarchy given here: writing out (3.12) for $r = 2$ and $r = 3$ and equating coefficients with (1.1 a) and (1.1 b) respectively gives $u_2 = u/2$ and $u_3 = w/4 - u_x/4$.

The explicit form of the Lax equations (3.14) is needed later on. We have [25]

$$\frac{\partial u_i}{\partial t_r} = \sum_{j=1}^{i} M_{i,j}\beta_{-j}(r), \quad i = 0, 1, 2, \ldots, \tag{3.15}$$

where the differential operator $M_{i,j}$ is given by

$$M_{i,j} = \sum_{k=0}^{i-j} \left(\begin{pmatrix} 1-j-k \\ i-j-k \end{pmatrix} u_k \partial^{i-j-k} - \begin{pmatrix} -j \\ i-j-k \end{pmatrix} \partial^{i-j-k} u_k \right). \tag{3.16}$$

Here and in what follows, the contribution of a sum is assumed to be zero if its upper limit is less than its lower limit, as happens in (3.16) when $i = 0$. Note that this immediately gives $\partial u_0 / \partial t_r = 0$ and $\partial u_1 / \partial t_r = 0$, for all r, as expected. Furthermore, $M_{i,i} = 0, M_{i,i-1} = \partial$. The differential equations (3.15) determine a first-order system for the t_r evolution of the infinite-dimensional vector of potentials $(u_2, u_3, u_4, \ldots)$.

4 Impose the r-reduction

Next we obtain a closed first-order system of partial differential equations for finitely many of the potentials, by imposing an *r-reduction*. This is the first reduction step in our scheme towards our goal of finding a set of ordinary differential equations describing the rank 1, finite-genus solutions of (KP).

The r-reduction of the operator L is obtained by imposing that the r-th power of L is purely differential:

$$L^r = L^r_+ \quad \text{or} \quad L^r_- = 0 \;\Rightarrow\; \beta_k(r) \equiv 0 \text{ for } k < 0 \;\Rightarrow\; \alpha_k(r) \equiv 0 \text{ for } k < 0. \tag{4.1}$$

Notice that the r-reduction implies immediately that all potentials are independent of t_r, from (3.15) or (3.14). The r-reduction determines the potentials $u_{r+1}, u_{r+2}, u_{r+3}, \ldots$ as differential polynomials of the potentials $u_2, u_3, \ldots, u_r$. This is a consequence of the triangular structure of the system relating the potentials $(u_2, u_3, u_4, \ldots)$ to the potentials $(\alpha_{r-2}(r), \alpha_{r-3}(r), \ldots)$.

Remark If we impose an r-reduction, for some positive integer number r, then we have automatically achieved an rk reduction, for any positive integer k. If L^r is purely differential, then so is $L^{rk} = (L^r)^k$.

Under r-reduction, the infinite system of evolution equations in (3.15) reduces to a finite number of equations for the independent potentials $(u_2, u_3, \ldots, u_r)$. We write this finite system in Hamiltonian form. First, we write the system in matrix-operator form. The matrices involved are now finite-dimensional, as there are only $r - 1$ independent potentials.

For a given n, define the $(r-1)$-dimensional vectors

$$\boldsymbol{U}(r) = \begin{pmatrix} u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_r \end{pmatrix}, \quad \boldsymbol{\beta}(r,n) = \begin{pmatrix} \beta_{-1}(n) \\ \beta_{-2}(n) \\ \beta_{-3}(n) \\ \vdots \\ \beta_{-r+1}(n) \end{pmatrix}, \tag{4.2}$$

and the operator-valued matrix

$$
\boldsymbol{M}(r) = \begin{pmatrix}
M_{2,1} & 0 & 0 & \cdots & 0 \\
M_{3,1} & M_{3,2} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
M_{r-1,1} & M_{r-1,2} & M_{r-1,3} & \cdots & 0 \\
M_{r,1} & M_{r,2} & M_{r,3} & \cdots & M_{r,r-1}
\end{pmatrix},
\tag{4.3}
$$

with the operators $M_{i,j}$ defined by (3.16). Then under r-reduction (3.15) can be written as

$$
\frac{\partial \boldsymbol{U}(r)}{\partial t_n} = \boldsymbol{M}(r)\boldsymbol{\beta}(r,n).
\tag{4.4}
$$

In order to write the system of equations (4.4) in Hamiltonian form, we first introduce new coordinates on the phase space of the system. Define

$$
\boldsymbol{\alpha}(r) = \begin{pmatrix}
\alpha_0(r) \\
\alpha_1(r) \\
\vdots \\
\alpha_{r-2}(r)
\end{pmatrix}.
\tag{4.5}
$$

The Jacobian matrix of the transformation from the coordinates $\boldsymbol{U}(r) \to \boldsymbol{\alpha}(r)$ is the Fréchet derivative of the transformation (which depends also on the spatial derivatives of the original coordinates, see for instance (3.8)). The Jacobian for such a transformation is an operator-valued matrix, whose action on an arbitrary vector of functions $\boldsymbol{v} = (v_2, v_3, \ldots, v_n)^T$ is given by

$$
\begin{aligned}
\boldsymbol{D}(r)\boldsymbol{v} &= \frac{\partial \boldsymbol{\alpha}(r)}{\partial \boldsymbol{U}(r)}\boldsymbol{v} \\
&= \frac{\partial}{\partial \epsilon} \begin{pmatrix}
\alpha_0(r)(u_2 + \epsilon v_2, u_3 + \epsilon v_3, \ldots, u_r + \epsilon v_r) \\
\alpha_1(r)(u_2 + \epsilon v_2, u_3 + \epsilon v_3, \ldots, u_r + \epsilon v_r) \\
\vdots \\
\alpha_{r-2}(r)(u_2 + \epsilon v_2, u_3 + \epsilon v_3, \ldots, u_r + \epsilon v_r)
\end{pmatrix}\Bigg|_{\epsilon=0}.
\end{aligned}
\tag{4.6}
$$

This Jacobian matrix is upper-triangular. This is a direct consequence of the triangular structure of the system relating the potentials u_j to the potentials $\alpha_j(r)$, $j = 2, 3, \ldots, r$.

We rewrite the equations (4.4) in terms of the coordinates $\boldsymbol{\alpha}(r)$:

$$
\frac{\partial \boldsymbol{\alpha}(r)}{\partial t_n} = \frac{\partial \boldsymbol{\alpha}(r)}{\partial \boldsymbol{U}(r)}\frac{\partial \boldsymbol{U}(r)}{\partial t_n} = \boldsymbol{D}(r)\boldsymbol{M}(r)\boldsymbol{\beta}(r,n) = \boldsymbol{J}(r)\boldsymbol{\beta}(r,n),
\tag{4.7}
$$

where we introduce the operator-valued matrix $\boldsymbol{J}(r) = \boldsymbol{D}(r)\boldsymbol{M}(r)$. Note that $\boldsymbol{J}(r)$ is always upper-triangular. This follows from the upper-triangular structure of $\boldsymbol{D}(r)$ and of the lower-triangular structure of $\boldsymbol{M}(r)$. Next we rewrite the vector $\boldsymbol{\beta}(r,n)$. We use an identity from the calculus of exterior derivatives for pseudo-differential operators [33]:

$$
d\beta_{-1}(r+n) = \frac{r+n}{r} \sum_{j=-1-n}^{r-2} \beta_{-1-j}(n) d\alpha_j(r),
\tag{4.8}
$$

which gives

$$
\beta_{-j}(n) = \frac{r}{r+n} \frac{\delta \beta_{-1}(r+n)}{\delta \alpha_{j-1}(r)},
\tag{4.9}
$$

where $\delta/\delta\alpha_{j-1}(r)$ is the *variational derivative* with respect to $\alpha_{j-1}(r)$. Hence

$$\boldsymbol{\beta}(r,n) = \frac{r}{r+n}\frac{\delta}{\delta\boldsymbol{\alpha}(r)}\beta_{-1}(r+n). \tag{4.10}$$

Equations (4.7) become

$$\frac{\partial\boldsymbol{\alpha}(r)}{\partial t_n} = \frac{r}{r+n}\boldsymbol{J}(r)\frac{\delta}{\delta\boldsymbol{\alpha}(r)}\beta_{-1}(r+n). \tag{4.11}$$

This set of equations is Hamiltonian [25], with Hamiltonian

$$H(r,n) = \frac{r}{r+n}\beta_{-1}(r+n) = \frac{r}{r+n}\alpha_{-1}(r+n). \tag{4.12}$$

(We have used the observation that $\alpha_{-1}(r) = \beta_{-1}(r)$.) It suffices to prove that the operator $J(r)$ is Hamiltonian [34], *i.e.*, that the operator $\boldsymbol{J}(r)$ defines a Poisson bracket. This Poisson bracket is given by [25]

$$\{S,T\} = \left(\frac{\delta S}{\delta\boldsymbol{\alpha}(r)}\right)^T \boldsymbol{J}(r)\left(\frac{\delta T}{\delta\boldsymbol{\alpha}(r)}\right). \tag{4.13}$$

Denote by $\mathcal{H}$ the quotient space of all smooth functionals of the potentials in $\boldsymbol{\alpha}(r)$, modulo total derivatives with respect to x. For (4.13) to define a Poisson bracket on $\mathcal{H}\times\mathcal{H}$, we need three properties:

1. **bilinearity:** This is obvious.

2. **skew-symmetry:** This is less obvious. Notice that the functions appearing in the bracket (4.13) appear only through their variational derivatives. Hence, these functions are only defined up to total derivatives with respect to x, *i.e.*, they are elements of $\mathcal{H}$. The skew-symmetry of the Poisson bracket (4.13) operating on $\mathcal{H}\times\mathcal{H}$ is then easily obtained by integration by parts.

3. **Jacobi identity:** This is also not obvious. The proof can be found in [25]. There it is shown that the above bracket is essentially the bracket Adler defines in [18]. The proof of the Jacobi identity then reduces to Adler's proof.

The Hamiltonian system of PDE's (4.11) describes a whole hierarchy of Hamiltonian PDEs for a fixed r. All the members of the hierarchy have the same Poisson structure with different Hamiltonians: for each n coprime with r, a different system of Hamiltonian partial differential equations is obtained, describing the t_n-evolution of the potentials. Note that the first member of every hierarchy is trivial. From the first flow of (3.14), we get

$$\frac{\partial L}{\partial t_1} = [L_+, L] = [\partial, L] = \partial L - L\partial = \frac{\partial L}{\partial x} + L\partial - L\partial = \frac{\partial L}{\partial x}. \tag{4.14}$$

which is the same for all r-reductions. Hence, the first member of every hierarchy is $\partial\boldsymbol{\alpha}(r)/\partial t_1 = \partial\boldsymbol{\alpha}(r)/\partial x$.

For example, choosing $r = 2$ results in the KdV hierarchy with Poisson operator $J(2) = 2\partial$. Choosing $r = 3$ gives the Boussinesq hierarchy [35, 36] with Poisson operator

$$J(3) = \left(\begin{array}{cc} 0 & 3\partial \\ 3\partial & 0 \end{array}\right). \tag{4.15}$$

Some remarks are in order about the Hamiltonian PDE's

$$\frac{\partial \boldsymbol{\alpha}(r)}{\partial t_n} = \boldsymbol{J}(r)\frac{\delta H(r,n)}{\delta \boldsymbol{\alpha}(r)}. \tag{4.16}$$

Remarks

(a) Note that in contrast with the Poisson brackets in [37, 35, 36], the bracket (4.13) is local, *i.e.*, it does not involve integrations. This is an immediate consequence of working in the quotient space $\mathcal{H}$. The Poisson bracket (4.13) for $r = 2$ and $r = 3$ are the integrands of the brackets introduced in [37] and [35, 36] respectively.

(b) Bogoyavlenskii and Novikov [15] considered only the Korteweg-deVries equation. As a consequence, none of the algebraic machinery of this and the previous section was required for their approach. Their starting point was the KdV equivalent of (4.16). This same starting point is used if one considers any other integrable partial differential equation which has only one spatial dimension, such as the nonlinear Schrödinger equation, the modified Korteweg-deVries equation, etc. By starting with a one-dimensional partial differential equation, the first step (imposing the r-reduction) is skipped. It is for this step that the algebraic language of the previous two sections is required.

(c) An infinite number of conserved quantities exists for each member of the hierarchy given by (4.16). This is a necessary condition for the integrability of these partial differential equations. Adler [18] showed that the different members of the r-reduced hierarchy define mutually commuting flows. The infinite set of Hamiltonians $\{H(r,n) : n \geq 1\}$ is a set of conserved densities for every member of the hierarchy. That different members of the hierarchy (4.16) commute is expressed as the commutation of their respective Hamiltonians under the Poisson bracket (4.13):

$$\{H(r,k_1), H(r,k_2)\} = 0, \tag{4.17}$$

for a fixed r and all k_1, k_2.

Denote the solution operator of the n-th member of the hierarchy (4.16) by $\boldsymbol{K}_n(t_n)$. In other words, given initial conditions $\boldsymbol{\alpha}(r)(x, t_n = 0)$, the solution $\boldsymbol{\alpha}(r)(x, t_n)$ for any t_n is written as

$$\boldsymbol{\alpha}(r)(x, t_n) = \boldsymbol{K}_n(t_n)\boldsymbol{\alpha}(r)(x, 0). \tag{4.18}$$

Adler's statement [18] that different flows in the hierarchy (4.16) commute is then expressed as

$$\boldsymbol{K}_n(t_n)\boldsymbol{K}_m(t_m) = \boldsymbol{K}_m(t_m)\boldsymbol{K}_n(t_n). \tag{4.19}$$

(d) The Hamiltonian operator $\boldsymbol{J}(r)$ is usually degenerate, *i.e.*, its kernel is not empty. Adler [18] showed that the variational derivatives of the elements of the set $\{\alpha_{-1}(r+n) : -r+1 \leq n \leq -1\}$ are all annihilated by $\boldsymbol{J}(r)$. In other words, these elements are Casimir functionals for the flows generated by the r-reduction. It is easy to see from the triangular form of $\boldsymbol{J}(r)$ that the dimension of its kernel is exactly $r-1$. This implies that the set of Casimir functionals found by Adler forms a complete basis for the kernel of $\boldsymbol{J}(r)$ (see also [17]).

5 Impose the n-reduction

Next, we consider stationary solutions of the system (4.16), for the n value determined in step 2. Hence, from (2.4),

$$\sum_{k=1}^{n} d_k \boldsymbol{J}(r) \frac{\delta H(r,k)}{\delta \boldsymbol{\alpha}(r)} = 0 \;\Rightarrow\; \boldsymbol{J}(r) \frac{\delta}{\delta \boldsymbol{\alpha}(r)} \sum_{k=1}^{n} d_k H(r,k) = 0, \tag{5.1}$$

with $d_n = 1$. Furthermore, without loss of generality, d_{n-1} can be equated to 0, as was shown in [38, 11].

The following theorem was first proved for the KdV equation by Lax [39] and Novikov [40].

Theorem 5.1 *The set of stationary solutions with respect to t_n is invariant under the action of any of the other higher-order flows.*

Proof Consider the hierarchy of mutually commuting Hamiltonian systems

$$\frac{\partial \boldsymbol{\alpha}(r)}{\partial \tilde{t}_n} = \boldsymbol{J}(r) \frac{\delta}{\delta \boldsymbol{\alpha}(r)} \left(\sum_{k=1}^{n} d_k H(r,k) \right). \tag{5.2}$$

Denote the solution operator of the n-th member of this hierarchy as $\tilde{\boldsymbol{K}}_n(\tilde{t}_n)$. Clearly these solution operators commute with the solution operators of the higher-order flows, $\boldsymbol{K}_m(t_m)$:

$$\tilde{\boldsymbol{K}}_n(\tilde{t}_n) \boldsymbol{K}_m(t_m) = \boldsymbol{K}_m(t_m) \tilde{\boldsymbol{K}}_n(\tilde{t}_n). \tag{5.3}$$

A stationary solution with respect to t_n is a fixed point of the operator $\tilde{\boldsymbol{K}}_n(\tilde{t}_n)$: $\tilde{\boldsymbol{K}}_n(\tilde{t}_n)\boldsymbol{\alpha}(r)(x) = \boldsymbol{\alpha}(r)(x)$. Hence

$$\boldsymbol{K}_m(t_m)\tilde{\boldsymbol{K}}_n(\tilde{t}_n)\boldsymbol{\alpha}(r)(x) = \boldsymbol{K}_m(t_m)\boldsymbol{\alpha}(r)(x) \tag{5.4}$$

$$\Rightarrow\; \tilde{\boldsymbol{K}}_n(\tilde{t}_n)\boldsymbol{K}_m(t_m)\boldsymbol{\alpha}(r)(x) = \boldsymbol{K}_m(t_m)\boldsymbol{\alpha}(r)(x), \tag{5.5}$$

since the two operators commute. Hence $\boldsymbol{K}_m(t_m)\boldsymbol{\alpha}(r)(x)$ is a fixed point of $\tilde{\boldsymbol{K}}_n(\tilde{t}_n)$ and hence a stationary solution with respect to t_n. ∎

Let us examine the structure of the equations (5.1), determining the stationary solutions with respect to t_n. From (5.1), $\frac{\delta}{\delta \boldsymbol{\alpha}(r)} \sum_{k=1}^{n} d_k H(r,k)$ is in the kernel of the Poisson operator $\boldsymbol{J}(r)$. Hence it is a linear combination of the Casimir functionals:

$$\frac{\delta}{\delta \boldsymbol{\alpha}(r)} \sum_{k=1}^{n} d_k H(r,k) + \sum_{k=1}^{r-1} h_k \frac{r}{k} \frac{\delta \alpha_{-1}(k)}{\delta \boldsymbol{\alpha}(r)} = 0; \tag{5.6}$$

the coefficient of the Casimir functional $\alpha_{-1}(r)$ has been written as $h_k r/k$ for convenience. Equation (5.6) is a system of Euler-Lagrange equations, with Lagrangian depending on the $r-1$ potentials $\boldsymbol{\alpha}(r)$ and their derivatives:

$$\begin{aligned}
\mathcal{L}(r,n) &=& H(r,n) + \sum_{k=1}^{n-2} d_k H(r,k) + \sum_{k=1}^{r-1} h_k \frac{r}{k} \alpha_{-1}(k) \\
&=& H(r,n) + \sum_{k=1}^{n-2} d_k H(r,k) + \sum_{k=1}^{r-1} h_k H(r,k-r),
\end{aligned} \tag{5.7}$$

since $d_{n-1} \equiv 0$. The last term in this equation is a slight abuse of notation. It is to be interpreted using (4.12).

The set of $r - 1$ Euler-Lagrange equations (5.6)

$$\frac{\delta \mathcal{L}(r, n)}{\delta \boldsymbol{\alpha}(r)} = 0 \tag{5.8}$$

will be referred to as the (r, n)-th (stationary) KP equation. This system of Euler-Lagrange equations is extremely important: it is a finite-dimensional system of *ordinary differential equations* describing how solutions of the (r, n)-th KP equation depend on x. At this point, the study of rank 1, finite-genus solutions of (KP) is immensely simplified: it is reduced to the study of a finite-dimensional set of ordinary differential equations (5.8) that are derived from one scalar quantity, the Lagrangian (5.7). The remainder of this paper examines the special structure of the set of equations (5.8).

Remarks

(a) As pointed out before, the first flow of the KP hierarchy defines t_1 to be x. This first flow imposes no constraints on the x-dependence of the potentials. After one imposes the r- and n-reductions this x-dependence is determined by the Lagrangian system (5.8).

(b) The Euler-Lagrange equations are a *minimal* set of differential equations the potentials in $\boldsymbol{\alpha}(r)$ have to satisfy to make the t_r-flow and the t_n-flow stationary. In step 5 of [11], a system of differential equations was proposed which the initial conditions of the KP equation needs to satisfy. Because of the way the r- and n-reductions were performed, it was not clear in [11] that these differential equations are in fact *ordinary* differential equations. Furthermore, the differential equations obtained in [11] were not necessarily functionally independent. The equations (5.8) are functionally independent, as was shown by Veselov [17].

(c) Since the order of imposing the r-reduction and the n-reduction can be reversed (in [11] they were executed simultaneously), the remark on page 11 can be repeated with n instead of r: if we impose an n-reduction, we automatically achieve an nk reduction for any positive integer k. Imposing an n-reduction implies that L^n is purely differential. In that case $L^{nk} = (L^n)^k$ is also purely differential.

(d) At this point, *Krichever's criterion [41]* surfaces[2]: for a finite-genus solution of rank 1 of the KP equation, r and n need to be coprime. Non-coprime r and n result in higher-rank solutions. We show next that to determine a rank 1, finite-genus solution completely, non-coprime r and n are not allowed.

Imposing the r- and n-reductions amounts (up to including lower order flows) to imposing that both L^r and L^n are purely differential operators. If r and n are not coprime, let $r = k\hat{r}$ and $n = k\hat{n}$, for integers $k, \hat{r}$ and $\hat{n}$. So, r and n have the common factor k. If $L^k = L^k_+$ (i.e., L^k is purely differential) then the solution is stationary with respect to t_k. Since $L^r = (L^k)^{\hat{r}}$ and $L^n = (L^k)^{\hat{n}}$ are purely differential, the solution is trivially stationary with respect to t_r and t_n. Thus, imposing stationarity with respect to t_k implies stationarity with respect to t_r and t_n. Imposing stationarity with respect to only one higher order flow t_k however, does not provide enough information for the determination of the solution using our methods. Therefore, r and n are required to be coprime.

---

[2]There is some discussion in the literature about the necessity of this criterion. See for instance [42]

6   The explicit dependence of the Lagrangian on the potentials and their derivatives

We want to examine the explicit functional dependence of the Lagrangian $\mathcal{L}(r, n)$ on the potentials $\boldsymbol{\alpha}(r) = (\alpha_0(r), \alpha_1(r), \ldots, \alpha_{r-2}(r))^T$ and their derivatives. We are especially interested in the order of the highest derivative of $\alpha_i(r)$, for $i = 0, 1, \ldots, r - 2$. This information is necessary in order to carry out the generalization of the Legendre transformation in Section 7.

Definition: The weight of an operator f(x), W[f(x)], is defined to be an integer-valued functional with the following properties:

1. $W[fg] = W[f] + W[g]$, and $W[f^N] = NW[f]$ for integer N.

2. If $W[f] = W[g]$ then $W[f \pm g] = W[f] = W[g]$.

3. $W[L] = 1$.

The usefulness of this definition is connected with the scaling symmetry of the KP equation. This symmetry is shared by the whole KP hierarchy through its definition using the operator L. Introducing the weight function turns the algebra of operators used here into a so-called graded algebra [19]. Essentially, the weight function introduced here is identical with the 'rank' introduced by Miura, Gardner and Kruskal in [43]. We use the name 'weight' because 'rank' has a very different meaning in KP theory [44, 41].

Using the defining properties of the weight function, we calculate the weight of some quantities we have used:

Examples

- Since $W[L] = 1$, any term in L has weight 1. In particular $W[\partial] = 1$.

- Hence $W[u_k \partial^{-k+1}] = 1 \Rightarrow W[u_k] + W[\partial^{-k+1}] = 1 \Rightarrow$
  $W[u_k] + (-k + 1)W[\partial] = 1 \Rightarrow W[u_k] = k$.

- $W[L^r] = r$, hence $W[\alpha_k(r)\partial^k] = r \Rightarrow W[\alpha_k(r)] + W[\partial^k] = r \Rightarrow W[\alpha_k(r)] = r - k$.
  Analogously $W[\beta_k(r)] = r - k$.

- $W[\partial/\partial t_r] = W[L^r] = r$.

- $W[H(r, n)] = r + n + 1$, from (4.12). Then also, $W[\mathcal{L}(r, n)] = r + n + 1$.

- $W[d_k] = n - k$ and $W[h_k] = r + n - k$.

We now use the weight function to calculate how the Lagrangian depends on the $r - 1$ potentials in $\alpha(r)$ and their derivatives. For $j = 2, 3, \ldots, r$, let us denote by N_j the highest order of differentiation of the potential $\alpha_{r-j}(r)$ in the Lagrangian (5.7). We say a term in the Lagrangian is of degree M if it contains M factors (counting multiplicities) that are linear in one of the potentials or in one of their derivatives. The Lagrangian has linear terms (*i.e.*, terms of degree one), quadratic terms (*i.e.*, terms of degree two), cubic terms (*i.e.*, terms of degree three), terms of degree four

and so on. Clearly the linear terms can not depend on the derivatives of the potentials: such a term would be a total derivative and would be disregarded.

All terms in the Lagrangian have the same weight, as a consequence of the scaling invariance of the KP hierarchy. In order to find the highest derivative of a potential, we need only consider the quadratic terms of the Lagrangians. All higher degree terms have contributions from more than one other potential. The nontrivial potential with the lowest weight is $\alpha_{r-2}(r)$: $W[\alpha_{r-2}(r)] = 2$. Every time a potential appears in a term of the Lagrangian, the number of x-derivatives of the other potentials in that term decreases by at least 2. Since the linear terms cannot contain derivatives, it follows that the highest derivatives of the potentials are found in the quadratic terms of the Lagrangian.

Similarly, every time one of the coefficients h_k, for $k = 1, 2, \ldots, r-1$ or d_j for $j = 1, 2, \ldots, n-2$ appears in a term of the Lagrangian, the number of x-derivatives in that term has to decrease by the weight of the coefficient h_k or d_j involved. It follows that the highest derivatives of the potentials are found in the quadratic terms of the Lagrangian, not containing any of these coefficients, *i.e.*, in the quadratic terms of $H(r, n)$.

We use these observations to find how the Lagrangian depends on the highest derivatives of the potentials, *i.e.*, to find N_j, for $j = 2, 3, \ldots, r$.

Theorem 6.1 *The order of differentiation in the Lagrangian $\mathcal{L}(r, n)$ of $\alpha_{r-2}(r)$ is $[(r + n - 3)/2]$, i.e., $N_2 = [(r + n - 3)/2]$. The order of differentiation in the Lagrangian $\mathcal{L}(r, n)$ of any other potential $\alpha_{r-i}(r)$ is $[(r + n - 2i + 2)/2]$, i.e., $N_i = [(r + n - 2i + 2)/2]$ for $i = 3, 4, \ldots, r$. The square brackets denote the integer part of the expression inside.*

Proof

The proof is a tedious check on all the possibilities of combinations of derivatives of the potentials appearing in the Lagrangian $\mathcal{L}(r, n)$. We start with a few specific cases before examining the general case.

Dependence of $\mathcal{L}(r, n)$ on $\alpha_{r-2}(r)$

We consider terms of the form $\alpha_{r-2}^{(k_1)}(r)\alpha_{r-j}^{(k_2)}(r)$. We want to find the maximal allowable value for k_1, *i.e.*, for the highest derivative of $\alpha_{r-2}(r)$ appearing in the Lagrangian. We have

$$W[\alpha_{r-2}^{(k_1)}(r)\alpha_{r-j}^{(k_2)}(r)] = k_1 + k_2 + 2 + j = r + n + 1 = W[\mathcal{L}(r, n)],$$

hence

$$k_1 + k_2 = r + n - 1 - j.$$

Only values of k_1, k_2 with $|k_1 - k_2| \leq 1$ need to be considered in this case. Other values are reduced to these cases using integration by parts. If $j = 2$, then necessarily $k_1 = k_2$. Otherwise the term we are considering is a total derivative, equivalent to $([\alpha_{r-2}^{(k_2)}(r)]^2/2)'$. In this case we find

$$k_1 = \frac{r + n - 3}{2}.$$

This value is not necessarily an integer. If it is, when $r + n - 3$ is even, it would be the maximum value for k_1. Otherwise, this term does not appear in $\mathcal{L}(r, n)$, so we consider next $j = 3$. If $k_1 = k_2 + 1$, we find $k_1 = (r + n - 5)/2$. If this is an integer, than so is $(r + n - 3)/2$, hence this does

not raise the order of differentiation with which $\alpha_{r-2}(r)$ appears. On the other hand, if $k_1 = k_2$, we find

$$k_1 = \frac{r+n-4}{2}.$$

Either $(r+n-3)/2$ or $(r+n-4)/2$ is guaranteed to be an integer. This integer is the maximal order of differentiation with which $\alpha_{r-2}(r)$ appears in the Lagrangian: hence

$$N_2 = \left[\frac{r+n-3}{2}\right], \tag{6.1}$$

where the square brackets denote the integer part of the expression inside. If $r+n-3$ is even, this results in the first value we obtained for k_1. If $r+n-3$ is odd, we find the second expression.

Dependence of $\mathcal{L}(r,n)$ on $\alpha_{r-3}(r)$

Next consider terms of the form $\alpha_{r-3}^{(k_1)}(r)\alpha_{r-j}^{(k_2)}(r)$. We want to find the maximal allowable value for k_1, i.e., N_3. We have

$$W[\alpha_{r-3}^{(k_1)}(r)\alpha_{r-j}^{(k_2)}(r)] = k_1 + k_2 + 3 + j = r + n + 1 = W[\mathcal{L}(r,n)],$$

or

$$k_1 + k_2 = r + n - 2 - j.$$

If $j = 2$, then for the case $k_1 = k_2$, we find

$$k_1 = \frac{r+n-4}{2}.$$

In the other case, $k_2 = k_1 + 1$ (we can always write the Lagrangian such that the potentials with the lowest weight have the higher order of differentiation), we obtain

$$k_1 = \frac{r+n-5}{2}.$$

In this case, $k_2 = (r+n-3)/2$, which corresponds to N_2 (if $r+n-3$ is even), found in the previous section.

The analysis for $j > 2$ does not increase the possible values of k_1. Either $(r+n-4)$ or $(r+n-5)$ is guaranteed to be even, so

$$N_3 = \left[\frac{r+n-4}{2}\right]. \tag{6.2}$$

Dependence of $\mathcal{L}(r,n)$ on $\alpha_{r-4}(r)$

Consider terms of the form $\alpha_{r-4}^{(k_1)}(r)\alpha_{r-j}^{(k_2)}(r)$. We want to find the maximal allowable value for k_1, i.e., N_4. We have

$$W[\alpha_{r-4}^{(k_1)}(r)\alpha_{r-j}^{(k_2)}(r)] = k_1 + k_2 + 4 + j = r + n + 1 = W[\mathcal{L}(r,n)],$$

or

$$k_1 + k_2 = r + n - 3 - j.$$

Consider the case when $j = 2$. If $k_1 = k_2$, then $k_1 = (r + n - 5)/2$ and $k_2 = (r + n - 5)/2 = (r + n - 3)/2 - 1$. If $r + n - 5$ is an integer, then so is $r + n - 3$. In this case we can use integration by parts to decrease k_1 by 1 and increase k_2 by 1. Therefore this possibility is not allowed. If $k_2 = k_1 + 1$, then we obtain

$$k_1 = \frac{r + n - 6}{2} \quad \text{and} \quad k_2 = \frac{r + n - 4}{2}.$$

This possibility is allowed. Also, if we let $k_2 = k_1 + 1$, we find

$$k_1 = \frac{r + n - 7}{2} \quad \text{and} \quad k_2 = \frac{r + n - 3}{2}.$$

This possibility is also allowed. Examining the other possibilities for j, k_2 does not improve the allowed values for k_1, therefore

$$N_4 = \left[\frac{r + n - 6}{2} \right]. \tag{6.3}$$

Dependence of $\mathcal{L}(r, n)$ on $\alpha_{r-i}(r)$, $3 \leq i \leq r$

We now state the general case. Using arguments as above (we want potentials with lower weight to have a higher order of differentiation in each term of the Lagrangian), we have

$$W[\alpha_{r-i}^{(k_1)}(r)\alpha_{r-j}^{(k_2)}(r)] = k_1 + k_2 + i + j = r + n + 1 = W[\mathcal{L}(r, n)],$$

or

$$k_1 + k_2 = r + n + 1 - i - j.$$

Consider the case when $j = 2$. Then if $k_2 = k_1 + m$,

$$k_1 = \frac{r + n - 1 - i - m}{2} \quad \text{and} \quad k_2 = \frac{r + n - 1 - i + m}{2}.$$

Using the above argument, we obtain an allowed possibility if either $k_2 = (n + r - 3)/2$ or $k_2 = (n + r - 4)/2$. This gives two possible values for m:

$$m = i - 2 \quad \text{or} \quad m = i - 3.$$

These respectively give

$$k_1 = \frac{r + n - 2i + 1}{2} \quad \text{and} \quad k_2 = \frac{r + n - 2i + 2}{2}.$$

The other possibilities for j give no additional information, hence

$$N_i = \left[\frac{n + r - 2i + 2}{2} \right]. \tag{6.4}$$

This formula is valid for all i, except $i = 2$, the first value. ∎
Table 6.1 gives an overview of the possibilities.

Remark In [11], it was argued that a generic rank 1, genus g solution of the KP equation corresponds to a solution of the (r, n)-th KP equation with $r = g + 1$ and $n = g + 2$. The dependence of the Lagrangian $\mathcal{L}(r, n) = \mathcal{L}(g + 1, g + 2)$ on the potentials for this case is found from Table 6.1 by using these values for r and n. It follows that in the generic case, the potential $\alpha_{r-j}(r)$ appears with $g - j + 2$ derivatives, for $j = 2, 3, \ldots, g + 1$.

Table 6.1: **The order of differentiation with which the potentials appear in the Lagrangian**

i	N_i
2	$\left[\frac{r+n-3}{2}\right]$
3	$\left[\frac{r+n-4}{2}\right]$
$\vdots$	$\vdots$
j	$\left[\frac{r+n-2j+2}{2}\right]$
$\vdots$	$\vdots$
r	$\left[\frac{n-r+2}{2}\right]$

7 The Ostrogradskii transformation, canonical variables and the Hamiltonian system

If we have a Lagrangian system, where the Lagrangian only depends on the potentials and their first derivatives, then under certain conditions we can use the Legendre transformation [14] to write the Lagrangian system in first-order form as a Hamiltonian system with canonical variables $(\boldsymbol{q}, \boldsymbol{p})$. Here the variables $\boldsymbol{q}$ are the potentials appearing in the Lagrangian and all of their derivatives, except the highest derivative. Their *conjugate variables* $\boldsymbol{p}$ are defined as the partial derivatives of the Lagrangian with respect to the $\boldsymbol{q}$ variables [14].

The Lagrangian system (5.8) we constructed from the KP hierarchy, assuming two of its flows are stationary, depends on more than just the first derivatives of the potentials $\boldsymbol{\alpha}(r) = (\alpha_0(r), \alpha_1(r), \ldots, \alpha_{r-2}(r))^T$. The Legendre transformation is generalized to write the Lagrangian system (5.8) in Hamiltonian form. This is achieved by Ostrogradskii's theorem, given later in this section. Consider the Ostrogradskii transformation (see [29, 27] for a simpler version)

$$q_{ij} = \frac{\partial^{j-1}}{\partial x^{j-1}}\alpha_{r-i-1}(r), \quad p_{ij} = \frac{\delta\mathcal{L}(r,n)}{\delta\alpha_{r-i-1}^{(j)}(r)} \tag{7.1}$$

for $i = 1, 2, \ldots, r-1$ and $j = 1, 2, \ldots, N_{i+1}$.

Note that when all the $N_j = 1$, for $j = 2, 3, \ldots, r$ (*i.e.*, when the Lagrangian only depends on first derivatives), then the Ostrogradskii transformation (7.1) reduces to the Legendre transformation.

Using the definition of the variational derivative, we establish the recurrence relations

$$p_{ij} = \frac{\partial\mathcal{L}(r,n)}{\partial\alpha_{r-i-1}^{(j)}(r)} - \frac{\partial p_{i(j+1)}}{\partial x}, \quad \text{for} \quad j = 1, 2, \ldots, N_{i+1}. \tag{7.2}$$

Here we have defined $p_{i(N_{i+1}+1)}$ to be zero. These recurrence relations will be useful later on. Furthermore, from the definition of the Ostrogradskii transformation (7.1), we obtain

$$W[q_{ij}] = i + j \quad \text{and} \quad W[p_{ij}] = r + n - (i + j). \tag{7.3}$$

Weight relationships such as these provide a quick check for the validity of large expressions, such as the ones encountered below. Many typographical errors are easily avoided by checking that only terms with the same weight are added.

21

The Lagrangian needs to fulfill a nonsingularity requirement for the Ostrogradskii transformation to be invertible:

Definition [29]: The Lagrangian $\mathcal{L}(r,n)$ is **(strongly) nonsingular** if the Ostrogradskii transformation (7.1) can be solved in the form

$$\alpha^{(j)}_{r-i}(r) = \alpha^{(j)}_{r-i}(r)(\boldsymbol{q}, \boldsymbol{p}), \quad \text{for } i = 2, 3, \ldots, r \ \text{ and } j = 1, 2, \ldots, 2N_i - 1.$$

Here $\boldsymbol{q}$ and $\boldsymbol{p}$ denote the vector of all variables q_{ij} and p_{ij} respectively. In other words, the Ostrogradskii transformation (7.1) is invertible if and only if the Lagrangian is nonsingular. Then the Euler-Lagrange equation (5.8) can be written in first-order form using the variables $\boldsymbol{q}$ and $\boldsymbol{p}$.

Define the vector $\boldsymbol{X} = (\alpha^{(N_2)}_{r-2}(r), \alpha^{(N_3)}_{r-3}(r), \ldots, \alpha^{(N_r)}_0(r))^T$. This is the vector containing the highest derivatives of the potentials. We already know that the highest derivatives of the potentials are found in the quadratic terms of the Lagrangian. The Lagrangian is conveniently written in the following form:

$$\mathcal{L}(r,n) = \frac{1}{2}\boldsymbol{X}^T \boldsymbol{G}(r,n)\boldsymbol{X} + \boldsymbol{A}^T(r,n)\boldsymbol{X} + \tilde{\mathcal{L}}(r,n), \tag{7.4}$$

with $\boldsymbol{G}(r,n)$, $\boldsymbol{A}(r,n)$, $\tilde{\mathcal{L}}(r,n)$ independent of $\boldsymbol{X}$. $\boldsymbol{G}(r,n)$ is a constant symmetric $(r-1) \times (r-1)$ matrix. $\boldsymbol{A}(r,n)$ is an $(r-1)$-dimensional vector. In the classical case of the Legendre transformation $\boldsymbol{G}(r,n)$ can be regarded as either a metric tensor or as the inverse of a mass tensor [14].

The following theorem generalizes a well-known result for the Legendre transformation.

Theorem 7.1 *The Lagrangian $\mathcal{L}(r,n)$ is nonsingular if and only if the matrix $\boldsymbol{G}(r,n)$ in (7.4) is nonsingular.*

Proof The proof is an extension of the proof in the case when the Lagrangian depends on only one potential [29]. We demonstrate that under the assumption that $\boldsymbol{G}(r,n)$ is nonsingular, the Ostrogradskii transformation (7.1) is invertible. Then by definition the Lagrangian is nonsingular.

First note that the variables $\boldsymbol{q}$ are expressed in terms of the potential and its derivatives, by their definition (7.1). Furthermore, the Lagrangian $\mathcal{L}(r,n)$ is a function of only $\boldsymbol{q}$ and $\boldsymbol{X}$: it follows from the Ostrogradksii transformation (7.1) that all derivatives of the potentials in the Lagrangian are $\boldsymbol{q}$-variables, except the highest derivative of the potentials. These are the components of $\boldsymbol{X}$.

We now construct the vector

$$
\begin{aligned}
\boldsymbol{P} &= \begin{pmatrix} p_{1N_2}, & p_{2N_3}, & \cdots, & p_{(r-1)N_r} \end{pmatrix}^T \\
&= \left(\frac{\delta\mathcal{L}(r,n)}{\delta\alpha^{(N_2)}_{r-2}(r)}, \frac{\delta\mathcal{L}(r,n)}{\delta\alpha^{(N_3)}_{r-3}(r)}, \cdots, \frac{\delta\mathcal{L}(r,n)}{\delta\alpha^{(N_r)}_0(r)} \right)^T \\
&= \frac{\delta\mathcal{L}(r,n)}{\delta\boldsymbol{X}}.
\end{aligned}
$$

Since $\boldsymbol{X}$ contains the highest derivatives of the potentials, by definition of the variational derivative we get

$$
\begin{pmatrix} p_{1N_2} \\ p_{2N_3} \\ \vdots \\ p_{(r-1)N_r} \end{pmatrix} = \frac{\partial\mathcal{L}(r,n)}{\partial\boldsymbol{X}} = \boldsymbol{G}(r,n)\boldsymbol{X} + \boldsymbol{A}(r,n). \tag{7.5}
$$

22

Now we solve (7.5) for $\boldsymbol{X}$, since by assumption $\boldsymbol{\mathcal{G}}(r,n)$ is nonsingular. We denote $\boldsymbol{\mathcal{G}}^{-1}(r,n) = \boldsymbol{\mathcal{M}}(r,n)$. Since $\boldsymbol{\mathcal{G}}(r,n)$ is symmetric, so is $\boldsymbol{\mathcal{M}}(r,n)$.

$$
\begin{aligned}
\boldsymbol{X} &= \begin{pmatrix} \alpha^{(N_2)}_{r-2}(r) \\ \alpha^{(N_3)}_{r-3}(r) \\ \vdots \\ \alpha^{(N_r)}_{0}(r) \end{pmatrix} = \boldsymbol{\mathcal{M}}(r,n) \begin{pmatrix} p_{1N_2} \\ p_{2N_3} \\ \vdots \\ p_{(r-1)N_r} \end{pmatrix} - \boldsymbol{\mathcal{M}}(r,n)\boldsymbol{\mathcal{A}}(r,n) \\
&= \boldsymbol{\mathcal{M}}(r,n)\left(\boldsymbol{P} - \boldsymbol{\mathcal{A}}(r,n)\right).
\end{aligned}
\tag{7.6}
$$

We have expressed $\boldsymbol{X}$ in terms of the coordinates $\boldsymbol{q}$ and $\boldsymbol{p}$. We want to do the same with its derivatives. Now consider the following set of Ostrogradskii equations, using the recurrence relations (7.2)

$$
\begin{aligned}
\begin{pmatrix} p_{1(N_2-1)} \\ p_{2(N_3-1)} \\ \vdots \\ p_{(r-1)(N_r-1)} \end{pmatrix} &= \begin{pmatrix} \frac{\partial \mathcal{L}(r,n)}{\partial \alpha^{(N_2-1)}_{r-2}(r)} - \frac{\partial p_{1N_2}}{\partial x} \\ \frac{\partial \mathcal{L}(r,n)}{\partial \alpha^{(N_3-1)}_{r-3}(r)} - \frac{\partial p_{2N_3}}{\partial x} \\ \vdots \\ \frac{\partial \mathcal{L}(r,n)}{\partial \alpha^{(N_r-1)}_{0}(r)} - \frac{\partial p_{(r-1)N_r}}{\partial x} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial \mathcal{L}(r,n)}{\partial \alpha^{(N_2-1)}_{r-2}(r)} \\ \frac{\partial \mathcal{L}(r,n)}{\partial \alpha^{(N_3-1)}_{r-3}(r)} \\ \vdots \\ \frac{\partial \mathcal{L}(r,n)}{\partial \alpha^{(N_r-1)}_{0}(r)} \end{pmatrix} - \frac{\partial}{\partial x} \begin{pmatrix} p_{1N_2} \\ p_{2N_3} \\ \vdots \\ p_{(r-1)N_r} \end{pmatrix}.
\end{aligned}
$$

Note that the first term depends only on $\boldsymbol{q}$ and $\boldsymbol{X}$. The last term is the derivative of $\boldsymbol{\mathcal{G}}(r,n)\boldsymbol{X} + \boldsymbol{\mathcal{A}}(r,n)$. Since $\boldsymbol{\mathcal{A}}(r,n)$ depends only on $\boldsymbol{q}$, the derivative of $\boldsymbol{\mathcal{A}}(r,n)$ depends only on $\boldsymbol{q}$ and on $\boldsymbol{X}$. Since $\boldsymbol{\mathcal{G}}(r,n)$ is constant, we can solve this relationship for $\boldsymbol{X}'$:

$$
\boldsymbol{X}' = \boldsymbol{\mathcal{M}}(r,n) \begin{pmatrix} \frac{\partial \mathcal{L}(r,n)}{\partial \alpha^{(N_2-1)}_{r-2}(r)} \\ \frac{\partial \mathcal{L}(r,n)}{\partial \alpha^{(N_3-1)}_{r-3}(r)} \\ \vdots \\ \frac{\partial \mathcal{L}(r,n)}{\partial \alpha^{(N_r-1)}_{0}(r)} \end{pmatrix} - \boldsymbol{\mathcal{M}}(r,n) \begin{pmatrix} p_{1(N_2-1)} \\ p_{2(N_3-1)} \\ \vdots \\ p_{(r-1)(N_r-1)} \end{pmatrix} - \boldsymbol{\mathcal{M}}(r,n)\frac{\partial \boldsymbol{\mathcal{A}}(r,n)}{\partial x},
$$

and we have expressed $\boldsymbol{X}'$ in terms of $\boldsymbol{q}, \boldsymbol{p}$ and $\boldsymbol{X}$. Since in the previous steps, $\boldsymbol{X}$ was expressed in terms of $\boldsymbol{q}$ and $\boldsymbol{p}$, this proves that $\boldsymbol{X}'$ is expressible in terms of $\boldsymbol{q}$ and $\boldsymbol{p}$. Continued use of the recursion relation (7.2) allows us to do the same with higher derivatives of $\boldsymbol{X}$, which are needed as the Euler-Lagrange equations (5.8) depend on twice as many derivatives of the potentials as the Lagrangian. This proves that the Ostrogradskii transformation (7.1) can be inverted. Hence the Lagrangian is nonsingular if $\boldsymbol{\mathcal{G}}(r,n)$ is nonsingular.

The converse statement is clearly also true: if the matrix $\boldsymbol{\mathcal{G}}(r,n)$ is singular, then the Ostrogradksii transformation is not invertible (step 1 in the proof fails, since (7.5) cannot be solved for $\boldsymbol{X}$). Hence the Lagrangian is singular if the matrix $\boldsymbol{\mathcal{G}}(r,n)$ is singular. ∎

In sharp contrast to the Korteweg-deVries hierarchy [44], the KP hierarchy contains both singular and nonsingular Lagrangians. This is an indication that the set of potentials $\boldsymbol{\alpha}(r) =$

$(\alpha_0(r), \alpha_1(r), \ldots, \alpha_{r-2}(r))^T$ is not a good set of variables to describe the dynamics of the system. These points are further explained in Section 10.

Denote by

$$N = \sum_{i=2}^{r} N_i = N_2 + N_3 + \ldots + N_r. \tag{7.7}$$

We have the following theorem:

Theorem 7.2 (Ostrogradskii [29, 26]) *If the Lagrangian $\mathcal{L}(r,n)$ is nonsingular, then the first-order system obtained by rewriting the Euler-Lagrange equations in terms of the new variables $\boldsymbol{q}$ and $\boldsymbol{p}$ is Hamiltonian with Hamiltonian*

$$H(\boldsymbol{q}, \boldsymbol{p}) = \sum_{i=1}^{r-1} \sum_{j=1}^{N_{i+1}} p_{ij} \alpha_{r-i-1}^{(j)}(r) - \mathcal{L}(r,n). \tag{7.8}$$

Here the inverse Ostrogradskii transformation is to be used in order to express the Hamiltonian in terms of $\boldsymbol{q}$ and $\boldsymbol{p}$ only. The Euler-Lagrange equations are equivalent to the N-dimensional Hamiltonian system

$$\frac{\partial q_{ij}}{\partial x} = \frac{\partial H}{\partial p_{ij}}, \quad \frac{\partial p_{ij}}{\partial x} = -\frac{\partial H}{\partial q_{ij}}, \tag{7.9}$$

for $i = 1, 2, \ldots, r-1$ and $j = 1, 2, \ldots, N_{i+1}$.

Proof The proof is identical to the proof in [29], except that more than one potential appears in the Lagrangian. This slight modification does not change the proof in any fundamental way.

In other words, the variables q_{ij} and p_{ij} are canonically conjugate variables under the classical symplectic structure where the Poisson bracket is given by

$$\{f, g\} = \sum_{i=1}^{r-1} \sum_{j=1}^{N_{i+1}} \left(\frac{\partial f}{\partial q_{ij}} \frac{\partial g}{\partial p_{ij}} - \frac{\partial f}{\partial p_{ij}} \frac{\partial g}{\partial q_{ij}} \right) = (\boldsymbol{\nabla} f)^T J (\boldsymbol{\nabla} g), \tag{7.10}$$

and

$$\boldsymbol{J} = \begin{pmatrix} \boldsymbol{0}_N & \boldsymbol{I}_N \\ -\boldsymbol{I}_N & \boldsymbol{0}_N \end{pmatrix},$$

where $\boldsymbol{0}_N$ is the $N \times N$-null matrix, $\boldsymbol{I}_N$ is the $N \times N$-identity matrix, and

$$\boldsymbol{\nabla} f = \left(\frac{\partial f}{\partial q_{11}}, \ldots, \frac{\partial f}{\partial q_{1N_2}}, \frac{\partial f}{\partial q_{21}}, \ldots, \frac{\partial f}{\partial q_{2N_3}}, \ldots, \frac{\partial f}{\partial q_{(r-1)N_r}}, \right.$$
$$\left. \frac{\partial f}{\partial p_{11}}, \ldots, \frac{\partial f}{\partial p_{1N_2}}, \frac{\partial f}{\partial p_{21}}, \ldots, \frac{\partial f}{\partial p_{2N_3}}, \ldots, \frac{\partial f}{\partial p_{(r-1)N_r}} \right)^T \tag{7.11}$$

is a $2N$-dimensional vector.

Theorem 7.3 *The Hamiltonian in (7.8) can be rewritten in the form*

$$H(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{2} \left(\boldsymbol{P} - \boldsymbol{\mathcal{A}}(r,n)(\boldsymbol{q}) \right)^T \boldsymbol{\mathcal{M}}(r,n) \left(\boldsymbol{P} - \boldsymbol{\mathcal{A}}(r,n)(\boldsymbol{q}) \right) +$$
$$\sum_{i=1}^{r-1} \sum_{j=1}^{N_{i+1}-1} p_{ij} q_{i(j+1)} - \tilde{\mathcal{L}}(r,n)(\boldsymbol{q}), \tag{7.12}$$

with $\boldsymbol{P} = (p_{1N_2}, p_{2N_3}, \ldots, p_{(r-1)N_r})^T$.

Proof (using (7.1), (7.4) and (7.6))

$$
\begin{aligned}
H(\boldsymbol{q},\boldsymbol{p}) \;=\;& \sum_{i=1}^{r-1}\sum_{j=1}^{N_{i+1}} p_{ij}\alpha^{(j)}_{r-i-1}(r) - \mathcal{L}(r,n) \\[4pt]
=\;& \sum_{i=1}^{r-1}\sum_{j=1}^{N_{i+1}-1} p_{ij}\alpha^{(j)}_{r-i-1}(r) + \sum_{i=1}^{r-1} p_{iN_{i+1}}\alpha^{(N_{i+1})}_{r-i-1}(r) - \frac{1}{2}\boldsymbol{X}^T\boldsymbol{\mathcal{G}}(r,n)\boldsymbol{X} - \boldsymbol{\mathcal{A}}^T(r,n)(\boldsymbol{q})\boldsymbol{X} - \\[2pt]
& \tilde{\mathcal{L}}(r,n)(\boldsymbol{q}) \\[4pt]
=\;& \sum_{i=1}^{r-1}\sum_{j=1}^{N_{i+1}-1} p_{ij}q_{i(j+1)} + \boldsymbol{P}^T\boldsymbol{X} - \frac{1}{2}\boldsymbol{X}^T\boldsymbol{\mathcal{G}}(r,n)\boldsymbol{X} - \boldsymbol{\mathcal{A}}^T(r,n)(\boldsymbol{q})\boldsymbol{X} - \tilde{\mathcal{L}}(r,n)(\boldsymbol{q}) \\[4pt]
=\;& \sum_{i=1}^{r-1}\sum_{j=1}^{N_{i+1}-1} p_{ij}q_{i(j+1)} + \boldsymbol{P}^T\boldsymbol{\mathcal{M}}(r,n)\left(\boldsymbol{P}-\boldsymbol{\mathcal{A}}(r,n)(\boldsymbol{q})\right) - \\[2pt]
& \frac{1}{2}\left(\boldsymbol{P}-\boldsymbol{\mathcal{A}}(r,n)(\boldsymbol{q})\right)^T\boldsymbol{\mathcal{M}}^T(r,n)\boldsymbol{\mathcal{G}}(r,n)\boldsymbol{\mathcal{M}}(r,n)\left(\boldsymbol{P}-\boldsymbol{\mathcal{A}}(r,n)(\boldsymbol{q})\right) - \\[2pt]
& \boldsymbol{\mathcal{A}}^T(r,n)(\boldsymbol{q})\boldsymbol{\mathcal{M}}(r,n)\left(\boldsymbol{P}-\boldsymbol{\mathcal{A}}(r,n)(\boldsymbol{q})\right) - \tilde{\mathcal{L}}(r,n)(\boldsymbol{q}) \\[4pt]
=\;& \sum_{i=1}^{r-1}\sum_{j=1}^{N_{i+1}-1} p_{ij}q_{i(j+1)} + \frac{1}{2}\left(\boldsymbol{P}-\boldsymbol{\mathcal{A}}(r,n)(\boldsymbol{q})\right)^T\boldsymbol{\mathcal{M}}(r,n)\left(\boldsymbol{P}-\boldsymbol{\mathcal{A}}(r,n)(\boldsymbol{q})\right) - \tilde{\mathcal{L}}(r,n)(\boldsymbol{q}) \\[4pt]
=\;& \frac{1}{2}\left(\boldsymbol{P}-\boldsymbol{\mathcal{A}}(r,n)(\boldsymbol{q})\right)^T\boldsymbol{\mathcal{M}}(r,n)\left(\boldsymbol{P}-\boldsymbol{\mathcal{A}}(r,n)(\boldsymbol{q})\right) + \sum_{i=1}^{r-1}\sum_{j=1}^{N_{i+1}-1} p_{ij}q_{i(j+1)} - \tilde{\mathcal{L}}(r,n)(\boldsymbol{q}).
\end{aligned}
$$

∎

Note that if the middle term were missing from (7.12), the Hamiltonian would be of the form: Kinetic Energy plus Potential Energy. Such Hamiltonians are called natural [14]. (Note that $\tilde{\mathcal{L}}(r,n)$ plays the role of minus the potential energy by its definition (7.4).) However, the term natural is conventionally only used if the mass tensor $\boldsymbol{\mathcal{M}}(r,n)$ is positive definite. This is not the case here, as the examples will illustrate.

The next theorem is well known [29] for a Lagrangian depending on one potential.

Theorem 7.4

$$
\frac{\partial H}{\partial x} = -\boldsymbol{\alpha}'^T(r)\frac{\delta\mathcal{L}(r,n)}{\delta\boldsymbol{\alpha}(r)}. \tag{7.13}
$$

Proof The proof is by direct calculation.

$$
\begin{aligned}
\frac{\partial H}{\partial x} \;=\;& \frac{\partial}{\partial x}\left(\sum_{i=1}^{r-1}\sum_{j=1}^{N_{i+1}} p_{ij}\alpha^{(j)}_{r-i-1}(r) - \mathcal{L}(r,n)\right) \\[4pt]
=\;& \sum_{i=1}^{r-1}\sum_{j=1}^{N_{i+1}}\left[\frac{\partial p_{ij}}{\partial x}\alpha^{(j)}_{r-i-1}(r) + p_{ij}\frac{\partial\alpha^{(j)}_{r-i-1}(r)}{\partial x}\right] - \frac{\partial\mathcal{L}(r,n)}{\partial x} \\[4pt]
=\;& \sum_{i=1}^{r-1}\frac{\partial p_{i1}}{\partial x}\frac{\partial\alpha_{r-i-1}(r)}{\partial x} + \sum_{i=1}^{r-1}\sum_{j=2}^{N_{i+1}}\frac{\partial p_{ij}}{\partial x}\alpha^{(j)}_{r-i-1}(r) +
\end{aligned}
$$

$$\sum_{i=1}^{r-1}\sum_{j=1}^{N_{i+1}} p_{ij}\alpha_{r-i-1}^{(j+1)}(r) - \sum_{i=1}^{r-1}\sum_{j=0}^{N_{i+1}} \frac{\partial\mathcal{L}(r,n)}{\partial\alpha_{r-i-1}^{(j)}(r)}\alpha_{r-i-1}^{(j+1)}(r)$$

$$= \sum_{i=1}^{r-1}\frac{\partial p_{i1}}{\partial x}\frac{\partial\alpha_{r-i-1}(r)}{\partial x} + \sum_{i=1}^{r-1}\sum_{j=2}^{N_{i+1}}\frac{\partial p_{ij}}{\partial x}\alpha_{r-i-1}^{(j)}(r) + \sum_{i=1}^{r-1}\sum_{j=1}^{N_{i+1}} p_{ij}\alpha_{r-i-1}^{(j+1)}(r) -$$

$$\sum_{i=1}^{r-1}\sum_{j=1}^{N_{i+1}}\frac{\partial\mathcal{L}(r,n)}{\partial\alpha_{r-i-1}^{(j)}(r)}\alpha_{r-i-1}^{(j+1)}(r) - \sum_{i=1}^{r-1}\frac{\partial\mathcal{L}(r,n)}{\partial\alpha_{r-i-1}(r)}\frac{\partial\alpha_{r-i-1}(r)}{\partial x}$$

$$\underset{\text{using (7.2)}}{=} \sum_{i=1}^{r-1}\left[\frac{\partial p_{i1}}{\partial x} - \frac{\partial\mathcal{L}(r,n)}{\partial\alpha_{r-i-1}(r)}\right]\frac{\partial\alpha_{r-i-1}(r)}{\partial x} + \sum_{i=1}^{r-1}\sum_{j=2}^{N_{i+1}}\left[\frac{\partial\mathcal{L}(r,n)}{\partial\alpha_{r-i-1}^{(j-1)}(r)} - p_{i(j-1)}\right]\alpha_{r-i-1}^{(j)}(r) +$$

$$\sum_{i=1}^{r-1}\sum_{j=1}^{N_{i+1}} p_{ij}\alpha_{r-i-1}^{(j+1)}(r) - \sum_{i=1}^{r-1}\sum_{j=1}^{N_{i+1}}\frac{\partial\mathcal{L}(r,n)}{\partial\alpha_{r-i-1}^{(j)}(r)}\alpha_{r-i-1}^{(j+1)}(r)$$

$$\underset{\text{using (7.1)}}{=} -\sum_{i=1}^{r-1}\left[\frac{\partial\mathcal{L}(r,n)}{\partial\alpha_{r-i-1}(r)} - \frac{\partial}{\partial x}\frac{\delta\mathcal{L}(r,n)}{\delta\alpha_{r-i-1}'(r)}\right]\frac{\partial\alpha_{r-i-1}(r)}{\partial x} + \sum_{i=1}^{r-1}\sum_{j=1}^{N_{i+1}-1}\frac{\partial\mathcal{L}(r,n)}{\partial\alpha_{r-i-1}^{(j)}(r)}\alpha_{r-i-1}^{(j+1)}(r) -$$

$$\sum_{i=1}^{r-1}\sum_{j=1}^{N_{i+1}-1} p_{ij}\alpha_{r-i-1}^{(j+1)}(r) + \sum_{i=1}^{r-1}\sum_{j=1}^{N_{i+1}-1} p_{ij}\alpha_{r-i-1}^{(j+1)}(r) + \sum_{i=1}^{r-1} p_{iN_{i+1}}\alpha_{r-i-1}^{(N_{i+1}+1)}(r) -$$

$$\sum_{i=1}^{r-1}\sum_{j=1}^{N_{i+1}-1}\frac{\partial\mathcal{L}(r,n)}{\partial\alpha_{r-i-1}^{(j)}(r)}\alpha_{r-i-1}^{(j+1)}(r) - \sum_{i=1}^{r-1}\frac{\partial\mathcal{L}(r,n)}{\partial\alpha_{r-i-1}^{(N_{i+1})}(r)}\alpha_{r-i-1}^{(N_{i+1}+1)}(r)$$

$$= -\sum_{i=1}^{r-1}\frac{\delta\mathcal{L}(r,n)}{\delta\alpha_{r-i-1}(r)}\frac{\partial\alpha_{r-i-1}(r)}{\partial x} + \sum_{i=1}^{r-1}\left[p_{iN_{i+1}} - \frac{\partial\mathcal{L}(r,n)}{\partial\alpha_{r-i-1}^{(N_{i+1})}(r)}\right]\alpha_{r-i-1}^{(N_{i+1}+1)}(r)$$

$$= -\boldsymbol{\alpha}'^{T}(r)\frac{\delta\mathcal{L}(r,n)}{\delta\boldsymbol{\alpha}(r)}.$$

∎

The simplest consequence of Theorem 7.4 is that the Hamiltonian is conserved along trajectories of the system (*i.e.*, where the Euler-Lagrange equations (5.8) are satisfied). But this result is a direct consequence of Ostrogradskii's theorem: since the resulting canonical Hamiltonian system is autonomous, the Hamiltonian is conserved. However, Theorem 7.4 will be useful when we construct more conserved quantities in the next section. It shows how the Hamiltonian fits in with the other conserved quantities.

8   Complete integrability of the Hamiltonian system

Denote
$$S(r,n) = \{H(r,k)\,|\,k=1,2,\ldots,\text{with } k \text{ not an integer multiple of } r \text{ or } n\}. \tag{8.1}$$

We know that in the quotient space where the Poisson bracket (2.12) is defined, using (4.17) we have
$$\{H(r,k_1),H(r,k_2)\} = \left(\frac{\delta H(r,k_1)}{\delta\boldsymbol{\alpha}(r)}\right)^{T}\boldsymbol{J}(r)\left(\frac{\delta H(r,k_2)}{\delta\boldsymbol{\alpha}(r)}\right) = 0. \tag{8.2}$$

In particular, in the quotient space (*i.e.*, up to total derivatives)
$$\{\mathcal{L}(r,n),H(r,k)\} = 0, \tag{8.3}$$

for $H(r,k) \in S(r,n)$. In other words, the Poisson bracket of these two quantities is a total derivative,

$$\{\mathcal{L}(r,n), H(r,k)\} = \frac{\partial T_k}{\partial x}. \tag{8.4}$$

Along the trajectories of the Euler-Lagrange equations, $i.e.$, along the trajectories of the Hamiltonian system (7.9), the quantities T_k are conserved. A list of conserved quantities for the system (7.9) is hence obtained from

$$T_k(\boldsymbol{q}, \boldsymbol{p}) = \int \left(\frac{\delta \mathcal{L}(r,n)}{\delta \boldsymbol{\alpha}(r)}\right)^T \boldsymbol{J}(r) \left(\frac{\delta H(r,k)}{\delta \boldsymbol{\alpha}(r)}\right) dx, \tag{8.5}$$

where k is not an integer multiple of r or n. The inverse Ostrogradskii transformation has to be used to express the right-hand side of this equation in terms of $\boldsymbol{q}$ and $\boldsymbol{p}$. Note that (8.5) is analogous to the expression for the conserved quantities in [15].

From the previous section, we know the Hamiltonian (7.8) is a conserved quantity for the system (7.9). The theorem below shows that up to a sign, the Hamiltonian is the first member of the newly constructed list of conserved quantities, (8.5).

Theorem 8.1

$$H(\boldsymbol{q}, \boldsymbol{p}) = -T_1(\boldsymbol{q}, \boldsymbol{p}) \tag{8.6}$$

Proof From Theorem 7.4,

$$\frac{\partial H}{\partial x} = -\boldsymbol{\alpha}'^T(r) \frac{\delta \mathcal{L}(r,n)}{\delta \boldsymbol{\alpha}(r)} = -\left(\frac{\delta \mathcal{L}(r,n)}{\delta \boldsymbol{\alpha}(r)}\right)^T \frac{\partial \boldsymbol{\alpha}(r)}{\partial x}.$$

But the first flow of each hierarchy is the x-flow, therefore

$$\begin{aligned}
\frac{\partial H}{\partial x} &= -\left(\frac{\delta \mathcal{L}(r,n)}{\delta \boldsymbol{\alpha}(r)}\right)^T \boldsymbol{J}(r) \frac{\partial H(r,1)}{\partial \boldsymbol{\alpha}(r)} \\
&= -\frac{\partial T_1}{\partial x},
\end{aligned}$$

from which we obtain the theorem. ∎

The N-dimensional Hamiltonian system (7.9) and a list of conserved quantities (8.5) for it have been constructed. Complete integrability in the sense of Liouville [14] can be concluded under the following conditions:

- In the phase space spanned by the independent coordinates $\boldsymbol{q}$ and $\boldsymbol{p}$ there are N nontrivial functionally independent conserved quantities.

- These N conserved quantities are mutually in involution with respect to the Poisson bracket (7.10), $i.e.$, their mutual Poisson brackets vanish.

The list of conserved quantities (8.5) for k not an integer multiple of r or n is infinite. It is clearly impossible for all of these conserved quantities to be functionally independent: a dynamical system in a $2N$-dimensional phase space has at most $2N$ independent conserved quantities. In particular, a $2N$-dimensional autonomous Hamiltonian system has at most N independent integrals of the motion that are mutually in involution. Below it is shown that all the conserved quantities $T_k(\boldsymbol{q}, \boldsymbol{p})$

27

are mutually in involution; therefore at most N of them are functionally independent. We wait until Theorem 8.3 to show that exactly N different $T_k(\boldsymbol{q}, \boldsymbol{p})$ are nontrivial and functionally independent.

In the rest of this section, it is proved that all the $T_k(\boldsymbol{q}, \boldsymbol{p})$ are in involution. This is not done by direct inspection of the mutual Poisson brackets. Instead, we follow the approach of Bogoyavlenskii and Novikov [15]: we know from Adler [18] that all flows of the hierarchy (4.16) commute. If we denote by X_{t_k} the vector field that evolves the phase space variables in the direction t_k, then the fact that all the flows in (4.16) commute can be equivalently stated as the mutual commutation of the vector fields X_{t_k}. Consider the Hamiltonian system with canonical variables $(\boldsymbol{q}, \boldsymbol{p})$, Poisson bracket (7.10) and Hamiltonian $H_k(\boldsymbol{q}, \boldsymbol{p}) = -T_k(\boldsymbol{q}, \boldsymbol{p})$. We show below that the vector field of this system, X_{H_k}, is the restriction of X_{t_k} to the phase space $(\boldsymbol{q}, \boldsymbol{p})$ of the finite-dimensional Hamiltonian system. So, the different t_k-flows commute, even when they are restricted to the phase space consisting of the $(\boldsymbol{q}, \boldsymbol{p})$ variables. In particular the t_k- and the t_1-flow (*i.e.*, the x-flow) commute. Hence we have a family of mutually commuting Hamiltonian systems, all with the same Poisson bracket (7.10). In [14], it is proved that then the family of Hamiltonians have mutually vanishing Poisson brackets. As a consequence, the $T_k(\boldsymbol{q}, \boldsymbol{p})$ are mutually in involution and the system (7.9) is completely integrable in the sense of Liouville, which is what we set out to prove.

We remark however, that this way of proving complete integrability also provides us with N-dimensional Hamiltonian systems for the evolution of the phase space variables, not only in x, but in all 'time' variables t_k. These different Hamiltonian systems have a common list of conserved quantities in involution. Hence each is completely integrable in the sense of Liouville. This will be spelt out in more detail once we finish proving that all the $T_k(\boldsymbol{q}, \boldsymbol{p})$ are in involution.

The following lemma is used in the proof of the Bogoyavlenskii-Novikov theorem.

Lemma 8.1

$$\frac{\partial^2 H}{\partial p_{ij} \partial q_{i(j+1)}} \quad = \quad 1, \qquad \text{for } j \leq N_{i+1} - 1 \tag{8.7}$$

$$\frac{\partial^2 H}{\partial p_{ij} \partial q_{is}} \quad = \quad 0, \qquad \text{for } s \neq j+1, j \leq N_{i+1} - 1 \tag{8.8}$$

$$\frac{\partial^2 H}{\partial p_{i_1 j_1} \partial q_{i_2 j_2}} \quad = \quad 0, \qquad \text{for } i_1 \neq i_2 \tag{8.9}$$

Proof We use the form (7.12) of the Hamiltonian. We get

$$\frac{\partial H}{\partial p_{ij}} = q_{i(j+1)}, \qquad \text{for } j \leq N_{i+1} - 1,$$

from which (8.7) and (8.8) easily follow. Also, (8.9) follows from this if $j_1 \leq N_{i_1+1} - 1$ and $j_2 \leq N_{i_2+1} - 1$. For other values of j_1, j_2, (8.9) follows immediately from (7.12). ∎

The following theorem generalizes the fundamental idea of Bogoyavlenskii and Novikov [15]:

Theorem 8.2 (Bogoyavlenskii-Novikov) *On the solution space of the (r, n)-th stationary KP equation, the action of the k-th time variable t_k can be written as an N-dimensional Hamiltonian system with Hamiltonian $H_k = -T_k$ and the same canonical variables as determined by Ostrogradksii's theorem for the (r, n)-th stationary KP equation.*

Proof The proof consists of four steps:

1. Prove that for $i = 1, 2, \ldots, r - 1$

$$\frac{\partial q_{i1}}{\partial t_k} = \frac{\partial H_k}{\partial p_{i1}}. \tag{8.10}$$

2. This is an induction step. Assuming the validity of step 1, prove that for $i = 1, 2, \ldots, r - 1$ and $j = 1, 2, \ldots, N_{i+1} - 1$

$$\frac{\partial q_{i(j+1)}}{\partial t_k} = \frac{\partial H_k}{\partial p_{i(j+1)}}. \tag{8.11}$$

3. Prove that for $i = 1, 2, \ldots, r - 1$

$$\frac{\partial p_{iN_{i+1}}}{\partial t_k} = -\frac{\partial H_k}{\partial q_{iN_{i+1}}}. \tag{8.12}$$

4. The last step is a 'backwards' induction step: assuming step 3 is valid, show that for $i = 1, 2, \ldots, r - 1$ and $j = N_{i+1} + 1, \ldots, 2, 1$

$$\frac{\partial p_{i(j-1)}}{\partial t_k} = -\frac{\partial H_k}{\partial q_{i(j-1)}}. \tag{8.13}$$

Proof of step 1:

During the course of this step, the index i can attain any value from $1, 2, \ldots, r - 1$.

Using the definition of the variational derivative,

$$
\begin{aligned}
\frac{\delta \mathcal{L}(r, n)}{\delta \alpha_{r-i-1}(r)} &= \frac{\partial \mathcal{L}(r, n)}{\partial \alpha_{r-i-1}(r)} - \frac{\partial}{\partial x} \frac{\delta \mathcal{L}(r, n)}{\delta \alpha'_{r-i-1}(r)} \\
&= \frac{\partial \mathcal{L}(r, n)}{\partial \alpha_{r-i-1}(r)} - \frac{\partial p_{i1}}{\partial x}.
\end{aligned}
$$

This shows that the Euler-Lagrange equations are essentially the equations of motion for the variables p_{i1}, $i = 1, 2, \ldots, r - 1$. The other equations of motion in Ostrogradskii's theorem are merely a consequence of the way the new variables in the Ostrogradskii transformation are introduced.

On the other hand, from the definition of the Hamiltonian (7.8),

$$
\begin{aligned}
\frac{\partial H}{\partial q_{i1}} &= -\frac{\partial \mathcal{L}(r, n)}{\partial q_{i1}} \\
&= -\frac{\partial \mathcal{L}(r, n)}{\partial \alpha_{r-i-1}(r)}.
\end{aligned}
$$

Combining the two results,

$$\frac{\delta \mathcal{L}(r, n)}{\delta \alpha_{r-i-1}(r)} = -\frac{\partial H}{\partial q_{i1}} - \frac{\partial p_{i1}}{\partial x}. \tag{8.14}$$

We expand $\partial T_k / \partial x$ in two different ways:

$$
\begin{aligned}
\frac{\partial T_k}{\partial x} &= \left(\frac{\delta \mathcal{L}(r, n)}{\delta \boldsymbol{\alpha}(r)} \right)^T \boldsymbol{J}(r) \left(\frac{\delta H(r, k)}{\delta \boldsymbol{\alpha}(r)} \right) \\
&= \sum_{i=1}^{r-1} \sum_{j=1}^{r-1} \frac{\delta \mathcal{L}(r, n)}{\delta \alpha_{i-1}(r)} J_{ij}(r) \frac{\delta H(r, k)}{\delta \alpha_{j-1}(r)} \\
\overset{\text{using } (8.14)}{=} &-\sum_{i=1}^{r-1} \sum_{j=1}^{r-1} \left(\frac{\partial H}{\partial q_{(r-i)1}} + \frac{\partial p_{(r-i)1}}{\partial x} \right) J_{ij}(r) \frac{\delta H(r, k)}{\delta \alpha_{j-1}(r)},
\end{aligned}
$$

29

and

$$\frac{\partial T_k}{\partial x} = \sum_{i=1}^{r-1} \sum_{j=1}^{N_{i+1}} \left(\frac{\partial T_k}{\partial q_{ij}} \frac{\partial q_{ij}}{\partial x} + \frac{\partial T_k}{\partial p_{ij}} \frac{\partial p_{ij}}{\partial x} \right).$$

As long as we do not impose the Euler-Lagrange equations, the derivatives of the phase space variables are independent variations. Hence their coefficients in both expressions for $\partial T_k/\partial x$ are equal. Expressing this equality for the coefficient of $\partial p_{i1}/\partial x$ gives

$$\frac{\partial T_k}{\partial p_{i1}} = -\sum_{j=1}^{r-1} \left(J_{(r-i)j}(r) \frac{\delta H(r,k)}{\delta \alpha_{j-1}(r)} \right)$$

$$\underset{\text{using (4.16)}}{=} -\frac{\partial \alpha_{r-i-1}(r)}{\partial t_k}$$

$$\underset{\text{using (7.1)}}{=} -\frac{\partial q_{i1}}{\partial t_k},$$

or

$$\frac{\partial q_{i1}}{\partial t_k} = \frac{\partial H_k}{\partial p_{i1}},$$

which we needed to prove. $\blacksquare$

Proof of step 2:

Assume

$$\frac{\partial q_{i\tilde{j}}}{\partial t_k} = \frac{\partial H_k}{\partial p_{i\tilde{j}}},$$

for $\tilde{j} = 1, 2, \ldots, j$. Then

$$\frac{\partial q_{i(j+1)}}{\partial t_k} \underset{\text{using (7.1)}}{=} \frac{\partial \alpha_{r-i-1}^{(j)}(r)}{\partial t_k}$$

$$\underset{\text{using (7.1)}}{=} \frac{\partial}{\partial x} \frac{\partial q_{ij}}{\partial t_k}$$

$$= \frac{\partial}{\partial x} \frac{\partial H_k}{\partial p_{ij}},$$

since the x-flow and the t_k-flow commute. In the last step the induction hypothesis is used. For any function of the variable x,

$$\frac{\partial f}{\partial x} = \sum_{i=1}^{r-1} \sum_{j=1}^{N_{i+1}} \left(\frac{\partial f}{\partial q_{ij}} \frac{\partial q_{ij}}{\partial x} + \frac{\partial f}{\partial p_{ij}} \frac{\partial p_{ij}}{\partial x} \right)$$

$$= \sum_{i=1}^{r-1} \sum_{j=1}^{N_{i+1}} \left(\frac{\partial f}{\partial q_{ij}} \frac{\partial H}{\partial p_{ij}} - \frac{\partial f}{\partial p_{ij}} \frac{\partial H}{\partial q_{ij}} \right)$$

$$= \{f, H\}, \tag{8.15}$$

by definition of the Poisson bracket. With this well-known result, the above becomes

$$\frac{\partial q_{i(j+1)}}{\partial t_k} = \left\{ \frac{\partial H_k}{\partial p_{ij}}, H \right\}$$

30

$$\overset{\text{using }(7.10)}{=} \quad \{\{q_{ij}, H_k\}, H\}$$

$$= \quad -\{\{H_k, H\}, q_{ij}\} - \{\{H, q_{ij}\}, H_k\}$$

$$\overset{\text{using }(8.15)}{=} \quad -\left\{\frac{\partial H_k}{\partial x}, q_{ij}\right\} + \{\{q_{ij}, H\}, H_k\}$$

$$= \quad \{\{q_{ij}, H\}, H_k\}$$

$$\overset{\text{using }(8.15)}{=} \quad \left\{\frac{\partial q_{ij}}{\partial x}, H_k\right\}$$

$$\overset{\text{using }(7.1)}{=} \quad \left\{q_{i(j+1)}, H_k\right\}$$

$$\overset{\text{using }(7.10)}{=} \quad \frac{\partial H_k}{\partial p_{i(j+1)}},$$

where we have used the fact that the Poisson bracket (7.10) satisfies the Jacobi identity. This is what we needed to prove. ∎

Proof of step 3:

From the commutativity of the x-flow and the t_k-flow,

$$\frac{\partial}{\partial x}\frac{\partial q_{iN_{i+1}}}{\partial t_k} \quad = \quad \frac{\partial}{\partial t_k}\frac{\partial q_{iN_{i+1}}}{\partial x}$$

$$\Rightarrow \quad \frac{\partial}{\partial x}\frac{\partial H_k}{\partial p_{iN_{i+1}}} \quad = \quad \frac{\partial}{\partial t_k}\frac{\partial H}{\partial p_{iN_{i+1}}}, \tag{8.16}$$

using steps 1 and 2 of the proof. We examine the left-hand side of this equation separately.

$$\text{Left-hand side} \quad = \quad \frac{\partial}{\partial x}\frac{\partial H_k}{\partial p_{iN_{i+1}}}$$

$$\overset{\text{using }(8.15)}{=} \quad \left\{\frac{\partial H_k}{\partial p_{iN_{i+1}}}, H\right\}$$

$$\overset{\text{using }(7.10)}{=} \quad -\left\{\{H_k, q_{iN_{i+1}}\}, H\right\}$$

$$= \quad \left\{\{q_{iN_{i+1}}, H\}, H_k\right\} + \left\{\{H, H_k\}, q_{iN_{i+1}}\right\}$$

$$\overset{\text{using }(8.15)}{=} \quad \left\{\frac{\partial q_{iN_{i+1}}}{\partial x}, H_k\right\} - \left\{\frac{\partial H_k}{\partial x}, q_{iN_{i+1}}\right\}$$

$$= \quad \left\{\frac{\partial H}{\partial p_{iN_{i+1}}}, H_k\right\},$$

again using the Jacobi identity. The factor $\partial H/\partial p_{iN_{i+1}}$ now appears both on the left and on the right of our equation. From (7.12)

$$\frac{\partial H}{\partial p_{iN_{i+1}}}$$

$$= \frac{\partial}{\partial p_{iN_{i+1}}}\left(\frac{1}{2}\left(\boldsymbol{P} - \boldsymbol{\mathcal{A}}(r, n)(\boldsymbol{q})\right)^T \boldsymbol{\mathcal{M}}(r, n)\left(\boldsymbol{P} - \boldsymbol{\mathcal{A}}(r, n)(\boldsymbol{q})\right)\right)$$

$$= \frac{\partial}{\partial p_{iN_{i+1}}}\left(\frac{1}{2}\sum_{i=1}^{r-1}\sum_{j=1}^{r-1}\mathcal{M}_{ij}(r, n)\left(p_{iN_{i+1}} - \mathcal{A}_i(r, n)(\boldsymbol{q})\right)\left(p_{jN_{j+1}} - \mathcal{A}_j(r, n)(\boldsymbol{q})\right)\right)$$

$$= \sum_{j=1}^{r-1}\mathcal{M}_{ij}(r, n)\left(p_{jN_{j+1}} - \mathcal{A}_j(r, n)(\boldsymbol{q})\right).$$

Using this result in (8.16),

$$\frac{\partial}{\partial t_k} \sum_{j=1}^{r-1} \mathcal{M}_{ij}(r,n) \left(p_{jN_{j+1}} - \mathcal{A}_j(r,n)(\boldsymbol{q}) \right)$$

$$= \left\{ \sum_{j=1}^{r-1} \mathcal{M}_{ij}(r,n) \left(p_{jN_{j+1}} - \mathcal{A}_j(r,n)(\boldsymbol{q}) \right), H_k \right\}$$

$$= \sum_{j=1}^{r-1} \mathcal{M}_{ij}(r,n) \left\{ \left(p_{jN_{j+1}} - \mathcal{A}_j(r,n)(\boldsymbol{q}) \right), H_k \right\}$$

$$= -\sum_{j=1}^{r-1} \mathcal{M}_{ij}(r,n) \left(\frac{\partial H_k}{\partial q_{jN_{j+1}}} + \{ \mathcal{A}_j(r,n)(\boldsymbol{q}), H_k \} \right).$$

Multiplying both sides of this equation by $\mathcal{G}_{si}(r,n)$ and summing over i from 1 to $r-1$, this becomes

$$\frac{\partial p_{sN_{s+1}}}{\partial t_k} - \frac{\partial \mathcal{A}_s(r,n)(\boldsymbol{q}),}{\partial t_k} = -\frac{\partial H_k}{\partial q_{sN_{s+1}}} - \{ \mathcal{A}_s(r,n)(\boldsymbol{q}), H_k \}.$$

Since $\mathcal{A}_s(r,n)(\boldsymbol{q})$ depends only on $\boldsymbol{q}$, it follows from step 2 that the second term on the left-hand side is equal to the second term on the right-hand side. Hence

$$\frac{\partial p_{sN_{s+1}}}{\partial t_k} = -\frac{\partial H_k}{\partial q_{sN_{s+1}}},$$

for $s = 1, 2, \ldots, r-1$. This terminates the proof of step 3. ∎

Note that it is necessary for the Lagrangian $\mathcal{L}(r,n)$ to be nonsingular, as we are using the matrix $\mathcal{M}(r,n)$.

Proof of step 4:

Assume

$$\frac{\partial p_{i\tilde{j}}}{\partial t_k} = -\frac{\partial H_k}{\partial q_{i\tilde{j}}},$$

for $\tilde{j} = N_{i+1}, N_{i+1} - 1, \ldots, j$. We have

$$\frac{\partial}{\partial t_k} \frac{\partial p_{ij}}{\partial x} = \frac{\partial}{\partial x} \frac{\partial p_{ij}}{\partial t_k}$$

$$\overset{\text{using (8.15)}}{=} \left\{ \frac{\partial p_{ij}}{\partial t_k}, H \right\}$$

$$= -\left\{ \frac{\partial H_k}{\partial q_{ij}}, H \right\}$$

$$\overset{\text{using (7.10)}}{=} \{\{ p_{ij}, H_k \}, H \}$$

$$= -\{\{ H_k, H \}, p_{ij} \} - \{\{ H, p_{ij} \}, H_k \}$$

$$\overset{\text{using (8.15)}}{=} -\left\{ \frac{\partial H_k}{\partial x}, H \right\} + \left\{ \frac{\partial p_{ij}}{\partial x}, H_k \right\}$$

$$= -\left\{ \frac{\partial H}{\partial q_{ij}}, H_k \right\}$$

$$\underset{\text{using (7.10)}}{=} \quad -\sum_{\gamma=1}^{r-1}\sum_{\delta=1}^{N_{\gamma+1}} \left(\frac{\partial^2 H}{\partial q_{ij}\partial q_{\gamma\delta}}\frac{\partial H_k}{\partial p_{\gamma\delta}} - \frac{\partial^2 H}{\partial q_{ij}\partial p_{\gamma\delta}}\frac{\partial H_k}{\partial q_{\gamma\delta}} \right)$$

$$= \quad -\sum_{\gamma=1}^{r-1}\sum_{\delta=1}^{N_{\gamma+1}} \frac{\partial^2 H}{\partial q_{ij}\partial q_{\gamma\delta}}\frac{\partial H_k}{\partial p_{\gamma\delta}} + \sum_{\gamma=1}^{r-1}\sum_{\delta=1}^{N_{\gamma+1}} \frac{\partial^2 H}{\partial q_{ij}\partial p_{\gamma\delta}}\frac{\partial H_k}{\partial q_{\gamma\delta}}$$

$$\underset{\text{using (8.9)}}{=} \quad -\sum_{\gamma=1}^{r-1}\sum_{\delta=1}^{N_{\gamma+1}} \frac{\partial^2 H}{\partial q_{ij}\partial q_{\gamma\delta}}\frac{\partial H_k}{\partial p_{\gamma\delta}} + \sum_{\delta=1}^{N_{i+1}} \frac{\partial^2 H}{\partial q_{ij}\partial p_{i\delta}}\frac{\partial H_k}{\partial q_{i\delta}}$$

$$\underset{\text{using (8.7)}}{=} \quad -\sum_{\gamma=1}^{r-1}\sum_{\delta=1}^{N_{\gamma+1}} \frac{\partial^2 H}{\partial q_{ij}\partial q_{\gamma\delta}}\frac{\partial H_k}{\partial p_{\gamma\delta}} + \frac{\partial^2 H}{\partial q_{ij}\partial p_{i(j-1)}}\frac{\partial H_k}{\partial q_{i(j-1)}} +$$

$$\frac{\partial^2 H}{\partial q_{ij}\partial p_{iN_{i+1}}}\frac{\partial H_k}{\partial q_{iN_{i+1}}}$$

$$\underset{\text{using (8.8)}}{=} \quad -\sum_{\gamma=1}^{r-1}\sum_{\delta=1}^{N_{\gamma+1}} \frac{\partial^2 H}{\partial q_{ij}\partial q_{\gamma\delta}}\frac{\partial H_k}{\partial p_{\gamma\delta}} + \frac{\partial H_k}{\partial q_{i(j-1)}}.$$

The left-hand side of this equation can be expressed another way as well:

$$\frac{\partial}{\partial t_k}\frac{\partial p_{ij}}{\partial x} \quad = \quad \frac{\partial}{\partial t_k}\left(-\frac{\partial H}{\partial q_{ij}} \right)$$

$$= \quad -\sum_{\gamma=1}^{r-1}\sum_{\delta=1}^{N_{\gamma+1}} \left(\frac{\partial^2 H}{\partial q_{ij}\partial q_{\gamma\delta}}\frac{\partial q_{\gamma\delta}}{\partial t_k} + \frac{\partial^2 H}{\partial q_{ij}\partial p_{\gamma\delta}}\frac{\partial p_{\gamma\delta}}{\partial t_k} \right)$$

$$= \quad -\sum_{\gamma=1}^{r-1}\sum_{\delta=1}^{N_{\gamma+1}} \left(\frac{\partial^2 H}{\partial q_{ij}\partial q_{\gamma\delta}}\frac{\partial H_k}{\partial p_{\gamma\delta}} + \frac{\partial^2 H}{\partial q_{ij}\partial p_{\gamma\delta}}\frac{\partial p_{\gamma\delta}}{\partial t_k} \right)$$

$$= \quad -\sum_{\gamma=1}^{r-1}\sum_{\delta=1}^{N_{\gamma+1}} \frac{\partial^2 H}{\partial q_{ij}\partial q_{\gamma\delta}}\frac{\partial H_k}{\partial p_{\gamma\delta}} - \frac{\partial p_{i(j-1)}}{\partial t_k},$$

where the second term has been simplified using lemma 8.1, as before. Comparing the two right-hand sides, the double-summed term drops out and one finds

$$\frac{\partial p_{i(j-1)}}{\partial t_k} = -\frac{\partial H_k}{\partial q_{i(j-1)}}.$$

This completes the proof of step 4 and hence of the Bogoyavlenskii-Novikov theorem. ∎

Let us recapitulate the most important results of the last few sections.

- The (r,n)-th stationary KP equation can be written as an N-dimensional Hamiltonian system in x, given by Ostrogradskii's theorem (7.9).

- This Hamiltonian system is completely integrable in the sense of Liouville. N independent conserved quantities in involution $T_k(\boldsymbol{q},\boldsymbol{p})$ can be constructed explicitly.

- These conserved quantities can be interpreted as Hamiltonians: The t_k-flow induces on the solution space of the (r,n)-th KP equation an evolution which is Hamiltonian with Hamiltonian $H_k = -T_k$. This Hamiltonian system shares its phase space variables, symplectic

33

structure and conserved quantities with the x-system. The t_k-evolution of the phase space variables is given by

$$\frac{\partial q_{ij}}{\partial t_k} = \frac{\partial H_k}{\partial p_{ij}}, \qquad \frac{\partial p_{ij}}{\partial t_k} = -\frac{\partial H_k}{\partial q_{ij}}, \tag{8.17}$$

for $i = 1, 2, \ldots, r-1$ and $j = 1, 2, \ldots, N_{i+1}$. The Hamiltonian is given by

$$H_k(\boldsymbol{q}, \boldsymbol{p}) = -T_k(\boldsymbol{q}, \boldsymbol{p}). \tag{8.18}$$

This only gives non-trivial results only if k is not a multiple of r and n. Strictly speaking however, this is not required for the proof of the Bogoyavlenskii-Novikov theorem.

At this point, we have established enough results to argue that N of the conserved quantities $T_k(r, n)$ are nontrivial and functionally independent:

Theorem 8.3 *In the phase space spanned by the independent coordinates $\boldsymbol{q}$ and $\boldsymbol{p}$ there are N nontrivial functionally independent conserved quantities*

Proof In the previous theorem, we have shown that the conserved quantity $T_k(\boldsymbol{q}, \boldsymbol{p})$ is minus the Hamiltonian for the evolution of the phase space variables $(\boldsymbol{q}, \boldsymbol{p})$ in the t_k-direction. So, if $T_k(\boldsymbol{q}, \boldsymbol{p})$ is trivial (*i.e.*, $T_k(\boldsymbol{q}, \boldsymbol{p})=0$ on the entire phase space), then so is the dependence of the solution of the (r, n)-th KP equation, and conversely. A typical solution of the (r, n)-th KP equation is a solution of genus N of the KP equation (see [11]) of the form

$$u = u_0 + 2\partial_x^2 \ln \Theta(\sum_{j=1}^{\infty} \boldsymbol{K}_j t_j). \tag{8.19}$$

Here all the $\boldsymbol{K}_j$ are N-dimensional vectors. If the conserved quantity $T_k(\boldsymbol{q}, \boldsymbol{p})$ is functionally dependent on any of the other $T_j(\boldsymbol{q}, \boldsymbol{p})$, $j < k$, then the vectorfield X_{H_k} is a linear combination of the vectorfields X_{H_j}, $j < k$. Hence the vector $\boldsymbol{K}_k$ is a linear combination of the vectors $\boldsymbol{K}_j$, $j < k$. If $\boldsymbol{K}_k$, is linearly dependent on the vectors with lower indices, we can use a linear transformation to obtain a solution of the form (8.19) which depends on $t_1, t_2, \ldots, t_{k-1}$, but is independent of t_k (for instance, this is possible for t_r and t_n). If $\boldsymbol{K}_k$ is independent of the vectors $\boldsymbol{K}_j$, with $j < l$, then the solution depends on t_k in a nontrivial manner. In this case, the conserved quantity $T_k(\boldsymbol{q}, \boldsymbol{p})$ has to be nontrivial and functionally independent of T_j, for $j < k$. A linear space of dimension N is spanned by N linearly independent vectors. Hence a typical solution of the (r, n)-th KP equation has N nontrivial functional independent conserved quantities $T_k(\boldsymbol{q}, \boldsymbol{p})$. ∎

Remark One often convenient way to integrate an integrable Hamiltonian system explicitly is analogous to the method of forward and inverse scattering, but restricted to systems of ordinary differential equations [45]. To invoke this method, a Lax representation [45] for the system of Hamiltonian equations is required. Such a representation was obtained in step 5 of the algorithm presented in [11].

In what follows, only the Hamiltonian system in x is considered. Any conclusions reached are however also valid for the Hamiltonian systems in any of the 'time' variables t_k.

34

9 Examples

In this section, the abstract formalism of the previous sections is illustrated using concrete examples, by assigning concrete values to r and n. The simplest cases are discussed: $r = 2$ and various values for n (the KdV hierarchy); $r = 3$ and various values for n (the Boussinesq hierarchy). A special case of $r = 3$ gives rise to stationary three-phase solutions of the KP equation, namely for $n = 4$. This case is illustrated in more detail than the other examples. It is the easiest example not covered by the work of Bogoyavlenskii and Novikov [15].

(a) The KdV hierarchy: one-dimensional solutions of the KP equation

The KdV hierarchy is obtained from the KP hierarchy by imposing the reduction $r = 2$, hence

$$L_+^2 = L^2. \tag{9.1}$$

Since $L_+^2 = \partial^2 + 2u_2$, u_2 is the only independent potential with the other potentials determined in terms of it. From $L_-^2 = 0$

$$u_3 = -\frac{u_2'}{2}, \ u_4 = \frac{u_2''}{4} - \frac{u_2^2}{2}, \ u_5 = \frac{3}{2}u_2 u_2' - \frac{u_2'''}{8}, \ etc. \tag{9.2}$$

Other ingredients needed for (4.4) are: $M(2) = M_{2,1} = \partial$ and $\beta(2, n) = \beta_{-1}(n) = \alpha_{-1}(n)$. Hence the KdV hierarchy has the form

$$\frac{\partial u_2}{\partial t_n} = \frac{\partial}{\partial x}\alpha_{-1}(n). \tag{9.3}$$

In order to write this in Hamiltonian form, we use the potential $\alpha_0(2) = 2u_2 = u$. Then $D(2) = 2$, by (4.6). Hence the Poisson structure of the KdV hierarchy is given by $J(2) = D(2)M(2) = 2\partial$. Using (4.9) with $j = 1$ and $r = 2$,

$$\alpha_{-1}(2) = \beta_{-1}(2) = \frac{2}{2+n}\frac{\delta\alpha_{-1}(2+n)}{\delta u} \tag{9.4}$$

we can recast the KdV hierarchy in its familiar Hamiltonian form [37]:

$$\frac{\partial u}{\partial t_n} = 2\frac{\partial}{\partial x}\frac{\delta H(2, n)}{\delta u}, \tag{9.5}$$

with $H(2, n) = 2\alpha_{-1}(2+n)/(2+n)$. If the factor 2 is absorbed into the definition of the Hamiltonian, then this form of the KdV hierarchy is identical to that introduced by Gardner [37]. Note that immediately all even flows (i.e., $t_n = t_{2k}$, for k a positive integer) are trivial because $2 + 2k$ is not coprime with $r = 2$, so $H(2, n) = \alpha_{-1}(2 + 2k)/(1 + k) \equiv 0$. We write out some nontrivial flows explicitly:

(i) $n = 1$: $H(2, 1) = u^2/4$ and

$$\frac{\partial u}{\partial t_1} = u_x, \tag{9.6}$$

as expected.

(ii) $n = 3$: $H(2, 3) = u^3/8 - u_x^2/16$ and

$$\frac{\partial u}{\partial t_3} = \frac{1}{4}\left(6uu_x + u_{xxx}\right), \tag{9.7}$$

the KdV equation.

(iii) $n = 5$: $H(2,5) = 5u^4/64 - 5uu_x^2/32 + u_{xx}^2/64$ and

$$\frac{\partial u}{\partial t_5} = \frac{1}{16}\left(30u^2 u_x + 20u_x u_{xx} + 10uu_{xxx} + u_{5x}\right),\tag{9.8}$$

the 5th-order KdV equation.

There is only one Casimir functional for the KdV hierarchy, namely $H(2,-1) = 2\alpha_{-1}(1) = 2u_2 = u$. It is easily verified that this is indeed a Casimir functional for the Poisson structure $J(2) = 2\partial$: $J(2)(\delta H(2,-1)/\delta u) = 2\partial(\delta u/\delta u) = 2\partial(1) = 0$.

Imposing the n-reduction, the Lagrangian $\mathcal{L}(2,n)$ has the form

$$\mathcal{L}(2,n) = H(2,n) + \sum_{k=1}^{n-2} d_k H(2,k) + h_1 u.\tag{9.9}$$

This Lagrangian was first obtained by Bogoyavlenskii and Novikov [15]. From Table 1, $N_2 = [(n-1)/2] = (n-1)/2$, since n is odd. Necessarily, the Lagrangian has the form

$$\mathcal{L}(2,n) = \frac{1}{2}a\left(u^{((n-1)/2)}\right)^2 + \hat{\mathcal{L}}(2,n),\tag{9.10}$$

for some nonzero constant a and $\hat{\mathcal{L}}(2,n)$ independent of $u^{((n-1)/2)}$. The Lagrangian is always nonsingular.

Because the case $r = 2$ was covered by Bogoyavlenskii and Novikov, no examples of the Ostrogradksii transformation and the resulting Hamiltonian system of ordinary differential equations will be given. A typical solution of the $(2,n)$-th KP equation, $i.e.$, a stationary solution of the n-KdV equation, depends on $N = N_2 = (n-1)/2$ phases. These solutions are also one-dimensional since they are independent of $y = t_2$.

(b) The Boussinesq hierarchy: stationary solutions of the KP equation

The Boussinesq hierarchy is obtained from the KP hierarchy by imposing the reduction $r = 3$, hence

$$L_+^3 = L^3.\tag{9.11}$$

Since $L_+^3 = \partial^3 + 3u_2\partial + 3u_2' + 3u_3$, u_2 and u_3 are the only independent potentials with the other potentials determined in terms of these two. From $L_-^3 = 0$

$$u_4 = -u_3' - u_2^2 - \frac{u_2''}{3}, \ u_5 = -2u_2 u_3 + 2u_2 u_2' + \frac{2}{3}u_3'' + \frac{u_2'''}{3}, \ etc.\tag{9.12}$$

Furthermore,

$$\boldsymbol{U}(3) = \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}, \ \ \boldsymbol{\beta}(3,n) = \begin{pmatrix} \beta_{-1}(n) \\ \beta_{-2}(n) \end{pmatrix}, \ \ \boldsymbol{M}(3) = \begin{pmatrix} \partial & 0 \\ -\partial^2 & \partial \end{pmatrix},\tag{9.13}$$

so that the Boussinesq hierarchy is

$$\frac{\partial \boldsymbol{U}(3)}{\partial t_n} = \boldsymbol{M}(3)\boldsymbol{\beta}(3,n) \ \Rightarrow \ \begin{cases} \dfrac{\partial u_2}{\partial t_n} = \dfrac{\partial \beta_{-1}(n)}{\partial x} \\ \dfrac{\partial u_3}{\partial t_n} = -\dfrac{\partial^2 \beta_{-1}(n)}{\partial x^2} + \dfrac{\partial \beta_{-2}(n)}{\partial x} \end{cases}.\tag{9.14}$$

In order to write this in Hamiltonian form, we use the potentials $\alpha_1(3) = 3u_2 = u$ and $\alpha_0(3) = 3u_2' + 3u_3 = v$, where u and v are introduced for notational simplicity. Then

$$\boldsymbol{\alpha}(3) = \begin{pmatrix} v \\ u \end{pmatrix}, \quad \boldsymbol{D}(3) = \begin{pmatrix} 3\partial & 3 \\ 3 & 0 \end{pmatrix}, \quad \boldsymbol{\beta}(3,n) = \frac{3}{3+n}\frac{\delta}{\delta\boldsymbol{\alpha}(3)}\alpha_{-1}(3+n). \tag{9.15}$$

Hence the Poisson structure of the Boussinesq hierarchy is

$$J(3) = D(3)M(3) = 3\begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}. \tag{9.16}$$

The Boussinesq hierarchy is written in Hamiltonian form as:

$$\frac{\partial}{\partial t_n}\begin{pmatrix} v \\ u \end{pmatrix} = 3\begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}\begin{pmatrix} \frac{\delta H(3,n)}{\delta v} \\ \frac{\delta H(3,n)}{\delta u} \end{pmatrix} = 3\frac{\partial}{\partial x}\begin{pmatrix} \frac{\delta H(3,n)}{\delta u} \\ \frac{\delta H(3,n)}{\delta v} \end{pmatrix}, \tag{9.17}$$

with $H(3,n) = 3\alpha_{-1}(3+n)/(3+n)$. Up to a factor 3, this form of the Boussinesq hierarchy is identical to the one introduced by McKean [36]. We write some flows explicitly:

(i) $n = 1$: $H(3,1) = uv/3$ and

$$\begin{cases} v_{t_1} &= v_x \\ u_{t_1} &= u_x \end{cases}, \tag{9.18}$$

as expected.

(ii) $n = 2$: $H(3,2) = -u^3/27 + u_x^2/9 + v^2/3 - vu_x/3$ and

$$\begin{cases} v_{t_2} &= -\dfrac{2}{3}uu_x - \dfrac{2}{3}u_{xxx} + v_{xx} \\ u_{t_2} &= 2v_x - u_{xx} \end{cases}. \tag{9.19}$$

Elimination of v from these two equations gives the Boussinesq equation,

$$u_{t_2 t_2} + \frac{1}{3}u_{xxxx} + \frac{2}{3}\left(u^2\right)_{xx} = 0. \tag{9.20}$$

(iii) $n = 4$: $H(3,4) = -u_{xx}^2/27 + v_x u_{xx}/9 - v_x^2/9 + uu_x^2/9 - 2uvu_x/9 - u^4/81 + 2uv^2/9$ and

$$\begin{cases} v_{t_4} &= -\dfrac{4}{9}u^2 u_x + \dfrac{4}{3}vv_x - \dfrac{4}{3}u_x u_{xx} - \dfrac{2}{3}uu_{xxx} + \dfrac{2}{3}u_x v_x + \dfrac{2}{3}uv_{xx} - \dfrac{2}{9}u_{5x} + \dfrac{2}{3}v_{xxxx} \\ u_{t_4} &= -\dfrac{2}{3}u_x^2 - \dfrac{2}{3}uu_{xx} + \dfrac{4}{3}vu_x - \dfrac{2}{3}u_{xxxx} + \dfrac{2}{3}v_{xxx} \end{cases}. \tag{9.21}$$

This is the next member of the Boussinesq hierarchy.

There are two Casimir functionals for the Boussinesq hierarchy, namely $H(3,-1) = 3\alpha_{-1}(2)/2 = 3(u_2'+u_3)/2 = v/2$ and $H(3,-2) = 3\alpha_{-1}(1) = 3u_2 = u$. For convenience u and v are used as Casimir functionals below.

Imposing the n-reduction, the Lagrangian $\mathcal{L}(3,n)$ has the form

$$\mathcal{L}(3,n) = H(3,n) + \sum_{k=1}^{n-2} d_k H(3,k) + h_1 u + h_2 v. \tag{9.22}$$

37

Theorem 10.1, given in the next section, shows that this Lagrangian is always nonsingular. A typical solution of the $(3, n)$-th KP equation, *i.e.*, a stationary solution of the n-Boussinesq equation, depends on $N = N_2 + N_3 = n - 1$ phases. These solutions are stationary solutions of the KP equation, since they are independent of $t = t_3$.

(c) Stationary 3-phase solutions of the KP equation

Consider the $r = 3$, $n = 4$ reduction. The Lagrangian is

$$
\mathcal{L}(3, 4) = -\frac{u_{xx}^2}{27} + \frac{v_x u_{xx}}{9} - \frac{v_x^2}{9} + \frac{u u_x^2}{9} - \frac{2uvu_x}{9} - \frac{u^4}{81} + \frac{2uv^2}{9} +
$$
$$
d_2 \left(-\frac{u^3}{27} + \frac{u_x^2}{9} + \frac{v^2}{3} - \frac{vu_x}{3} \right) + d_1 \frac{uv}{3} + h_1 u + h_2 v. \qquad (9.23)
$$

The Ostrogradskii transformation (7.1) for this Lagrangian is

$$
q_{11} = u, \qquad p_{11} = \frac{\delta \mathcal{L}(3, 4)}{\delta u_x} = \frac{u_{xxx}}{54} + \frac{uu_x}{9} + \frac{d_2 u_x}{18} + \frac{d_1 u}{6} + \frac{h_2}{2},
$$
$$
q_{12} = u_x, \qquad p_{12} = \frac{\delta \mathcal{L}}{\delta u_{xx}} = -\frac{2u_{xx}}{27} + \frac{v_x}{9}, \qquad (9.24)
$$
$$
q_{21} = v, \qquad p_{21} = \frac{\delta \mathcal{L}(3, 4)}{\delta v_x} = \frac{u_{xx}}{9} - \frac{2v_x}{9}.
$$

In the definition of p_{11}, the Euler-Lagrange equations have been used to eliminate v_{xx}. The Ostrogradskii transformation can be inverted:

$$
u = q_{11}, \quad u_x = q_{12}, \quad v = q_{21},
$$
$$
u_{xx} = -54 p_{12} - 27 p_{21}, \quad v_x = -27 p_{12} - 18 p_{21}, \qquad (9.25)
$$
$$
u_{xxx} = 54 p_{11} - 6 q_{11} q_{12} - 3 d_2 q_{12} - 9 d_1 q_{11} - 27 h_2.
$$

Using the inverse Ostrogradskii transformation, the Hamiltonian corresponding to the Lagrangian (9.23) is

$$
H = -27 p_{12}^2 - 27 p_{12} p_{21} - 9 p_{21}^2 + p_{11} q_{12} + \frac{q_{11}^4}{81} - \frac{q_{11} q_{12}^2}{9} - \frac{2 q_{21}^2 q_{11}}{9} + \frac{2 q_{11} q_{12} q_{21}}{9} +
$$
$$
d_2 \left(\frac{q_{11}^3}{27} - \frac{q_{12}^2}{9} - \frac{q_{21}^2}{3} + \frac{q_{12} q_{21}}{3} \right) - d_1 \frac{q_{11} q_{21}}{3} - h_1 q_{11} - h_2 q_{21}. \qquad (9.26)
$$

For simplicity the constants d_1, d_2, h_1 and h_2 are equated to zero in the remainder of this example. Using (8.5), three conserved quantities are computed for the Hamiltonian system generated by (9.26):

$$
T_1 = -H \qquad (9.27)
$$
$$
T_2 = 3 \int \left(\frac{\delta \mathcal{L}(3, 4)}{\delta v} \frac{\partial}{\partial x} \frac{\delta H(3, 2)}{\delta u} + \frac{\delta \mathcal{L}(3, 4)}{\delta u} \frac{\partial}{\partial x} \frac{\delta H(3, 2)}{\delta v} \right) dx
$$
$$
= \frac{4 q_{12} q_{11}^3}{81} - \frac{8 q_{21} q_{11}^3}{81} - 2 p_{12} q_{12} q_{11} - \frac{4 p_{21} q_{12} q_{11}}{3} + 4 p_{12} q_{11} q_{21} + 2 p_{21} q_{21} q_{11} - \frac{q_{12}^3}{27} +
$$
$$
\frac{q_{21}^3}{27} - \frac{2 q_{12} q_{21}^2}{9} + 9 p_{11} p_{21} + \frac{4 q_{12}^2 q_{21}}{27} \qquad (9.28)
$$

38

$$T_5 \;=\; 3\int\left(\frac{\delta\mathcal{L}(3,4)}{\delta v}\frac{\partial}{\partial x}\frac{\delta H(3,5)}{\delta u} + \frac{\delta\mathcal{L}(3,4)}{\delta u}\frac{\partial}{\partial x}\frac{\delta H(3,5)}{\delta v}\right)dx$$

$$\begin{aligned}
&= -\frac{2q_{11}^6}{729} + \frac{8p_{12}q_{11}^4}{27} + \frac{4p_{21}q_{11}^4}{27} - \frac{q_{12}^2 q_{11}^3}{243} + 3p_{11}p_{21}q_{21} + \frac{2q_{21}^2 q_{11}^3}{243} - \frac{2q_{12}q_{21}q_{11}^3}{243} - 9p_{12}^2 q_{11}^2 - \\
&\quad 2p_{21}^2 q_{11}^2 - 9p_{12}p_{21}q_{11}^2 + \frac{p_{11}q_{12}q_{11}^2}{3} - 3p_{11}^2 q_{11} + \frac{p_{21}q_{12}^2 q_{11}}{9} + \frac{q_{21}^4}{27} + 54p_{12}^3 - \frac{2q_{12}q_{21}^3}{27} + \\
&\quad 27p_{12}p_{21}^2 + \frac{4q_{12}^2 q_{21}^2}{81} + 81p_{12}^2 p_{21} - 3p_{11}p_{12}q_{12} - 3p_{11}p_{21}q_{12} - \frac{q_{12}^3 q_{21}}{81} - \frac{2p_{21}q_{12}q_{21}q_{11}}{9}. \quad (9.29)
\end{aligned}$$

Since $r = 3$ and $n = 4$, the conserved quantities T_3 and T_4 are trivial. It is easy to check by direct computation that these three conserved quantities are in involution. Furthermore, the dependence of the solution of the KP equation on t_2 (*i.e.*, y) is governed by the Hamiltonian system generated by $H_2 = -T_2$. The same statement holds for t_5 and $H_5 = -T_5$. We will return to this example in Section 11

10 Singular and nonsingular Lagrangians

We have shown that the Hamiltonian system in x (7.9) is completely integrable in the sense of Liouville when Ostrogradksii's theorem applies, *i.e.*, when the Lagrangian $\mathcal{L}(r,n)$ is nonsingular. This has immediate consequences for the structure of the solutions. On a compact component of phase space, almost all solutions are quasi-periodic with N phases. These phases move linearly on an N-dimensional torus [14]. Such a torus is topologically equivalent to a compact component of

$$\Lambda(r,n) = \{T_k(\boldsymbol{q},\boldsymbol{p}) = \mathcal{T}_k, k \in \Omega(r,n)\} \tag{10.1}$$

where $\Omega(r,n) = \{$first N values of k, not integer multiples of r or $n\}$. The constants $\mathcal{T}_k$ are determined by the initial conditions. The torus $\Lambda(r,n)$ is shared by all t_k-flows. The only difference between these different flows from this point of view is the linear motion on the torus. This linear motion determines the N frequencies with respect to the variable t_k.

We know from [11] that a typical solution of the (r,n)-th KP equation has $g_{max} = (r-1)(n-1)/2$ phases. In the proof of Theorem 10.1, it is shown that $N = g_{max}$ if one of r, n is even and the other one is odd. The case of both r and n even is not allowed, since r and n need to be coprime. On the other hand, if both r and n are odd, $N = g_{max}$ only if $r = 3$. Otherwise $N > g_{max}$ and the Ostrogradskii transformation introduces more variables than are needed to span the phase space. The transformation is then not invertible and the Lagrangian $\mathcal{L}(r,n)$ is nonsingular. This situation does not occur in the work of Bogoyavlenskii and Novikov [15], because for the KdV hierarchy $r = 2$. From the results in this section it follows that some rank 1, finite-genus solutions (namely $r > 3$ and odd, $n > r$ and odd), give rise to interesting examples of singular Lagrangians.

Theorem 10.1 *The Lagrangian $\mathcal{L}(r,n)$ is singular if and only if r and n are both odd and $r > 3$. For all other cases of (r,n), $N = g_{max}$.*

Proof First note that $[R/2] + [(R+1)/2] = R$, for all integers R. Using Table 6.1 we can rewrite

$$N \;=\; \sum_{j=2}^{r} N_j$$

$$\begin{aligned} &= \quad N_2 + N_3 + \sum_{j=4}^{r} N_j \\ &= \quad r + n - 4 + \sum_{j=4}^{r} \left[\frac{n + r - 2j + 2}{2} \right]. \end{aligned} \tag{10.2}$$

In the calculations below, we use that n can always be chosen to be greater than r. The calculations are slightly different depending on whether r and/or n are even or odd. Since r and n cannot both be even, there are three cases.

1. r is even and n is odd. We write $r = 2R$, $n = 2M + 1$. Use (10.2),

$$\begin{aligned} N &= \quad r + n - 4 + \sum_{j=4}^{2R} (M + R - j + 1) \\ &= \quad r + n - 4 + 2RM - 3M - 2R + 3 \\ &= \quad \frac{(r - 1)(n - 1)}{2} \\ &= \quad g_{max}. \end{aligned}$$

2. r is odd and n is even. We write $r = 2R + 1$, $n = 2M$. The calculations are similar to those in the previous case.

$$\begin{aligned} N &= \quad r + n - 4 + \sum_{j=4}^{2R+1} (M + R - j + 1) \\ &= \quad \frac{(r - 1)(n - 1)}{2} \\ &= \quad g_{max}. \end{aligned}$$

3. r is odd and n is odd. We write $r = 2R + 1$, $n = 2M + 1$. In this case, the result is quite surprising.

$$\begin{aligned} N &= \quad r + n - 4 + \sum_{j=4}^{2R+1} (R + M - j + 2) \\ &= \quad r + n - 4 + \frac{(R + M - 2)(R + M - 1)}{2} - \frac{(M - R + 1)(M - R)}{2} \\ &= \quad \frac{(r - 1)(n - 1)}{2} + \frac{r - 3}{2} \\ &= \quad g_{max} + \frac{r - 3}{2}. \end{aligned}$$

Hence, in this case, $N \neq g_{max}$. So, if r and n are both odd and $r > 3$, the dimension of the torus $\Lambda(r, n)$ in (10.1) is seemingly greater than the maximal number of phases of a solution of the (r, n)-th KP equation, according to [11]. This dimension exceeds the maximal genus by the amount of $(r - 3)/2$, which is a positive integer when r is odd and greater than three. This is an indication that the assumptions necessary for Ostrogradskii's theorem have been violated. Hence $(r, n$ both odd, $r > 3)$ is a sufficient condition for the Lagrangian $\mathcal{L}(r, n)$ to be singular.

That this condition is necessary for $\mathcal{L}(r,n)$ to be singular follows from the results of Veselov [17]. There it is demonstrated that the dimension of the phase space of the Euler-Lagrange equations (5.8) is always equal to $2g_{max}$, which is the desired dimension of the phase space of the Hamiltonian system (4.16). In such a case the Lagrangian is only singular if the Ostrogradskii transformation introduces more variables than the dimension of the phase space [28], which only happens when r, n are noth odd and $r > 3$. Hence this condition is also necessary for the singularity of the Lagrangian. ■

Note that in the generic case of [11], $r = g + 1$, $n = g + 2$, we always have $g_{max} = N$, since this automatically puts one in case 1 or case 2.

Example

The smallest odd value possible for r is $r = 3$. In this case, the count of the number of phases is still right. This corresponds to the Boussinesq hierarchy. The smallest values of r and n where the count of the number of phases is wrong occurs when $(r,n) = (5,7)$. This example is discussed below. In this example, the Lagrangian $\mathcal{L}(5,7)$ expressed in the 4 variables $\boldsymbol{\alpha}(5) = (\alpha_0(5), \alpha_1(5), \alpha_2(5), \alpha_3(5))^T$ is singular. It is shown below how one deals with the singular Lagrangian case. As it turns out, a relatively straightforward transformation reduces the Lagrangian $\mathcal{L}(5,7)$ to a nonsingular Lagrangian, expressed in the transformed variables. The Ostrogradskii transformation is applicable to this Lagrangian and it results in a completely integrable system with $N = g_{max}$.

For simplicity, denote $u = \alpha_3(5)$, $v = \alpha_2(5)$, $w = \alpha_1(5)$ and $z = \alpha_0(5)$. The Lagrangian is

$$\mathcal{L}(5,7) \;=\; -\frac{2u_{xxxx}}{25}\left(z - \frac{5}{2}w_x + \frac{9}{5}v_{xx}\right)_{xx} + \frac{w_{xx}v_{xxxx}}{5} - \frac{7z_x v_{xxxx}}{25} - \frac{7z_{xx}w_{xx}}{25} +$$
$$\tilde{\mathcal{L}}(u, u_x, u_{xx}, u_{xxx}, v, v_x, v_{xx}, v_{xxx}, w, w_x, w_{xx}, z, z_x), \tag{10.3}$$

where all dependence on the vector $\boldsymbol{X} = (u_{xxxx}, v_{xxxx}, w_{xxx}, z_{xx})^T$ is explicit. We have

$$\boldsymbol{\mathcal{G}}(5,7) = \begin{pmatrix} 0 & -18/125 & 1/5 & -2/25 \\ -18/125 & 0 & 0 & 0 \\ 1/5 & 0 & 0 & 0 \\ -2/25 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\mathcal{A}}(5,7) = \begin{pmatrix} 0 \\ w_{xx}/5 - 7z_x/25 \\ 0 \\ -7w_{xx}/25 \end{pmatrix}. \tag{10.4}$$

Clearly $\boldsymbol{\mathcal{G}}(5,7)$ is singular. If canonical variables $\boldsymbol{q}$ and $\boldsymbol{p}$ were introduced using the Ostrogradskii transformation on this Lagrangian, this would lead to $2N = 2(N_2 + N_3 + N_4 + N_5) = 26$ phase space variables. The dimension of the phase space can be obtained from the Euler-Lagrange equations [17] corresponding to $\mathcal{L}(5,7)$ and is 24. Since the corank of the matrix $\boldsymbol{\mathcal{G}}(5,7)$ is 2, it follows from the Ostrogradskii transformation (7.1) that two linear combinations between p_{14}, p_{24}, p_{33} and p_{42} exist. These are also linear in $\boldsymbol{q}$ because $\boldsymbol{\mathcal{A}}(5,7)$ is linear in $\boldsymbol{q}$:

$$p_{24} = -\frac{2}{5}p_{33} + \frac{7}{25}q_{33}, \quad p_{42} = -\frac{18}{25}p_{33} + \frac{1}{5}q_{33} - \frac{7}{25}q_{42}. \tag{10.5}$$

At this point the theory of constrained Hamiltonian systems can be used [30]. Another possibility is to use the general methods of Krupkova [28] for singular Lagrangians. However, it is possible to transform the potentials to a new set of potentials such that the Lagrangian is nonsingular when expressed in the new potentials. Motivated by the form of the Lagrangian, let

$$\hat{u} = u, \quad \hat{v} = v, \quad \hat{w} = w, \quad \hat{z} = z - \frac{5}{2}w_x + \frac{9}{5}v_{xx}. \tag{10.6}$$

This transformation is clearly invertible. In the new variables, the Lagrangian is

$$\mathcal{L}(5,7) = -\frac{2}{25}\hat{u}_{xxxx}\hat{z}_{xx} + \frac{1}{250}\hat{v}_{xxx}\hat{w}_{xx} - \frac{7}{25}\hat{v}_{xxxx}\hat{z}_x - \frac{7}{25}\hat{w}_{x\hat{x}xx} +$$
$$\hat{\mathcal{L}}(\hat{u}, \hat{u}_x, \hat{u}_{xx}, \hat{u}_{xxx}, \hat{v}, \hat{v}_x, \hat{v}_{xx}, \hat{v}_{xxx}, \hat{w}, \hat{w}_x, \hat{w}_{xx}, \hat{z}, \hat{z}_x), \tag{10.7}$$

up to total derivatives. Define a new vector $\hat{\boldsymbol{X}} = (\hat{u}_{xxxx}, \hat{v}_{xxxx}, \hat{w}_{xx}, \hat{z}_{xx})^T$. The Lagrangian is

$$\mathcal{L}(5,7) = \frac{1}{2}\hat{\boldsymbol{X}}^T \hat{\boldsymbol{\mathcal{G}}}(5,7)\hat{\boldsymbol{X}} + \hat{\boldsymbol{\mathcal{A}}}(5,7)\hat{\boldsymbol{X}} + \hat{\mathcal{L}}(5,7), \tag{10.8}$$

with

$$\hat{\boldsymbol{\mathcal{G}}}(5,7) = \begin{pmatrix} 0 & 0 & 0 & -2/25 \\ 0 & 0 & 1/250 & 0 \\ 0 & 1/250 & 0 & -7/25 \\ -2/25 & 0 & -7/25 & 0 \end{pmatrix}, \tag{10.9}$$

which is nonsingular, hence by Theorem 7.1 the Lagrangian is nonsingular. The Ostrogradskii transformation acting on the transformed Lagrangian (10.8) introduces 24 canonical variables, as many variables as the dimension of the phase space [17].

It is not clear if the approach of this example always works, *i.e.*, for other values of r and n, both odd. There is no proof and the problem of dealing with a singular Lagrangian for values of (r,n) other than $(5,7)$ describing the (r,n)-th KP equation may require the general methods alluded to above.

11 Autonomous symmetry reductions of the Hamiltonian system

As discussed in the remarks on page 6, not all solutions of the (r,n)-th KP equation have the same number of phases. A generic genus g solution of the KP equation is usually a very non-typical solution of the (r,n)-th KP equation, with $r = g + 1$ and $n = g + 2$.

A solution of the (r,n)-th KP equation is completely determined by the phase-space variables q and p. In the $2N$-dimensional phase space coordinatized by $\boldsymbol{q}$ and $\boldsymbol{p}$, for a given initial condition, the solution evolves on the torus $\Lambda(r,n)$ determined by (10.1). Usually (*i.e.*, for almost all initial data in the class of solutions of the (r,n)-th KP equation), this torus is N-dimensional [14]. However, special solutions correspond to lower-dimensional tori. For example, suppose that $N = 2$. Then almost all solutions evolve on a two-dimensional torus: for almost all values of $\mathcal{T}_1$ and $\mathcal{T}_2$, the surface Λ determined by $T_1 = \mathcal{T}_1$ and $T_2 = \mathcal{T}_2$ is topologically equivalent to a two-dimensional torus, like the one drawn in Figure 11.1. For special values of $\mathcal{T}_1$ and $\mathcal{T}_2$, however, this torus is degenerate and the resulting 'surface' is only a one-dimensional torus, *i.e.*, a circle C. This is drawn in Figure 11.1.

To a certain degree this scenario is the typical one that occurs. For almost all values of the constants $\{\mathcal{T}_k, \ k \in \Omega(r,n)\}$, the torus $\Lambda(r,n)$ in (10.1) is N-dimensional. For a special set of values $\{\mathcal{T}_k, \ k \in \Omega(r,n)\}$, the torus is $(N-1)$-dimensional, corresponding to a solution with $(N-1)$
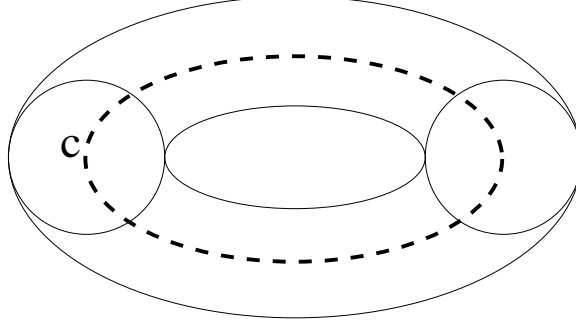
Figure 11.1: **Usually solutions evolve on 2-D tori. Special solutions correspond to lower-dimensional tori.**

phases. For an even more limited class of constants $\{\mathcal{T}_k, \; k \in \Omega(r,n)\}$, the torus $\Lambda(r,n)$ is $(N-2)$-dimensional, corresponding to solutions with $(N-2)$ phases, *etc.*

The conditions on the set of constants $\{\mathcal{T}_k, \; k \in \Omega(r,n)\}$ can be expressed in terms of the variables $\boldsymbol{q}$ and $\boldsymbol{p}$. When these variables satisfy certain constraints (to be derived below), the dimension of the torus $\Lambda(r,n)$ decreases.

We have argued above that the conserved quantities T_k for $k \in \Omega(r,n)$ are functionally independent quantities if the $\boldsymbol{q}$ and $\boldsymbol{p}$ variables are considered independent variables. The only way for the torus $\Lambda(r,n)$ to be less than N-dimensional is if the conserved quantities are *not* functionally independent. This is only possible if there are functional relationships between the variables $(\boldsymbol{q}, \boldsymbol{p})$. The constraints on the variables $\boldsymbol{q}$ and $\boldsymbol{p}$ are obtained as follows:

1. Require that the conserved quantities T_k for $k \in \Omega(r,n)$ be functionally dependent. This is expressed as

$$\text{rank} \left(\nabla T_{i_1} \; \nabla T_{i_2} \; \ldots \; \nabla T_{i_N} \right) = g < N, \qquad (11.1)$$

   where the indices i_k are the elements of $\Omega(r,n)$ and ∇T_{i_k} is defined in (7.11). In this case, $N - g$ of the conserved quantities are functionally dependent on the remaining g functionally independent conserved quantities. Without loss of generality, we assume that the first g conserved quantities remain functionally independent, whereas the last $N - g$ conserved quantities are functionally dependent:

$$T_{i_k} = F_{i_k}(T_{i_1}, T_{i_2}, \ldots, T_{i_g}), \qquad (11.2)$$

   for $k = g+1, g+2, \ldots, N$.

2. In this case, there are only g functionally independent conserved quantities T_{i_k}, $k = 1, 2, \ldots, g$. The manifold (10.1) reduces to

$$\Lambda_g(r,n) = \{T_{i_k} = \mathcal{T}_k, \; k = 1, 2, \ldots, g\}, \qquad (11.3)$$

   which is topologically equivalent to a g-dimensional torus. This g-dimensional torus is parametrizable by g phase variables moving linearly on the torus. In other words, if the evolution of a solution of the (r,n)-th KP equation is restricted to this g-dimensional torus, it has only g phases. Since this solution is also a solution of the KP equation and it has rank 1 and g phases, it must be a genus g, rank 1 solution of the KP equation.

3. Equations (11.1) results in a number of conditions on the variables q and p:

$$K_j(\boldsymbol{q}, \boldsymbol{p}) = 0, \quad j = 1, 2, \ldots, m, \qquad (11.4)$$

where m is the number of conditions. If these conditions are satisfied for the 'initial' conditions $\boldsymbol{q}(0)$ and $\boldsymbol{p}(0)$ then they are automatically satisfied for all other x-values, *i.e.*, the conditions on the variables $\boldsymbol{q}$ and $\boldsymbol{p}$ are invariant under the x-flow. This is easy to see: The conditions (11.4) are equivalent to the conditions (11.1) which are equivalent to the conditions (11.2), which only involve the conserved quantities. Clearly (11.2) are invariant conditions.

The conditions on the variables $\boldsymbol{q}$ and $\boldsymbol{p}$ (11.4) are polynomial in the $\boldsymbol{q}$ and $\boldsymbol{p}$ variables, since all the entries of the matrix on the left-hand side of (11.1) are polynomial in these variables. In practice, the conditions on the variables $\boldsymbol{q}$ and $\boldsymbol{p}$ (11.1) can be written as combinations of simpler conditions,

$$K_j = \sum_{k=1}^{m_g} Q_{j,k}(\boldsymbol{q}, \boldsymbol{p}) P_k(\boldsymbol{q}, \boldsymbol{p}), \quad j = 1, 2, \ldots, m. \qquad (11.5)$$

Here both $Q_{j,k}(\boldsymbol{q}, \boldsymbol{p})$ and $P_k(\boldsymbol{q}, \boldsymbol{p})$ are polynomials in q and p. If $P_k(\boldsymbol{q}, \boldsymbol{p}) = 0$, for $k = 1, 2, \ldots, m_g$ then clearly the conditions (11.4) are satisfied. Clearly the decomposition (11.5) is not unique. The factors P_k of a given decomposition are not necessarily invariant under the x-flow. In order to find a minimal (*i.e.*, smallest number of elements) set of conditions on the $\boldsymbol{q}$ and $\boldsymbol{p}$ variables, the invariance of such factors needs to be tested separately. Since the conditions (11.1) are invariant, as argued above, such a minimal set of invariant factors is guaranteed to exist. The existence of a *minimal* decomposition is essentially a restatement of Hilbert's Basis theorem [46]. Below, we prove that the number of elements in this set is $m_g = 2(N - g)$. Once this minimal set of conditions has been found, (11.4) can be replaced by the conditions

$$P_k(\boldsymbol{q}, \boldsymbol{p}) = 0, \quad k = 1, 2, \ldots, m_g. \qquad (11.6)$$

The invariance of the factors $P_k(\boldsymbol{q}, \boldsymbol{p})$ is easily tested. It is necessary and sufficient that [47]

$$\{P_k(\boldsymbol{q}, \boldsymbol{p}), H\} = 0, \quad \text{for } k = 1, 2, \ldots, m_g, \qquad (11.7)$$

on the solution manifold $P_k(\boldsymbol{q}, \boldsymbol{p}) = 0$, for $k = 1, 2, \ldots, m_g$.

4. The conditions on the variables $\boldsymbol{q}$ and $\boldsymbol{p}$ (11.6) are autonomous, since the conditions do not depend explicitly on x. The conditions (11.6) determine *autonomous invariant symmetry reductions* of the Hamiltonian system (7.9).

Theorem 11.1 *[48] In order for a solution of the (r, n)-th KP equation to have genus g instead of N, $2(N - g)$ conditions need to be imposed on the variables $\boldsymbol{q}$ and $\boldsymbol{p}$, i.e., $m_g = 2(N - g)$.*

Proof By Ostrogradskii's theorem, a typical solution of the (r, n)-th KP equation resides in the $2N$-dimensional phase space with coordinates $\boldsymbol{q}$ and $\boldsymbol{p}$. The existence of N conserved quantities T_{i_k}, for $k = 1, 2, \ldots, N$ guarantees that the motion starting from any initial condition evolves on a torus determined by the N conditions $T_{i_k} = \mathcal{T}_k$, for $k = 1, 2, \ldots, N$. Hence this torus is a hypersurface of codimension N, or of dimension $2N - N = N$.

44

If we impose that the rank of the matrix $(\nabla T_{i_1}, \nabla T_{i_2}, \ldots, \nabla T_{i_N})$ is $N - 1$ in order to obtain genus $N - 1$ solutions, then the motion of the solutions is restricted to an $(N - 1)$-dimensional torus. An $(N - 1)$-dimensional hypersurface in a $2N$-dimensional phase space is determined by $N + 1$ equations. $N - 1$ equations are provided by the relations $T_{i_k} = \mathcal{T}_k$, for $k = 1, 2, \ldots, N - 1$. Hence another two conditions are required on the coordinates $\boldsymbol{q}$ and $\boldsymbol{p}$.

The proof of the theorem is now easily obtained by repeating this argument $N - g$ times. ∎

Remarks

(a) The fact that $m_{N-1} = 2$ is not easily seen from (11.1). For (11.1) to be satisfied with $g = N-1$, the determinants of all $N \times N$ minors need to be zero. This seemingly results in $\begin{pmatrix} 2N \\ N \end{pmatrix}$ N-dimensional minors of which $2N - 1$ are functionally independent. The determinant of all these minors are decomposable in terms of two polynomials $P_1(\boldsymbol{q}, \boldsymbol{p})$ and $P_2(\boldsymbol{q}, \boldsymbol{p})$, which are both invariant under the x-flow. This is explicitly worked out in the example below, for $N = 3$ and $g = 2$.

(b) It should be mentioned that in [11], some ideas were given to find conditions on nontypical solutions of the (r, n)-th KP equation. Those ideas were correct, but seemingly hard to implement, as is seen in the example discussed there. This same example is discussed below. The algorithm offerered in this section to find the determining conditions on the nontypical solutions of the (r, n)-th KP equation is much more efficient.

Example: Generic genus 2 solutions of the KP equation

A generic solution of genus 2 of the KP equation is a solution of the $(3, 4)$-th KP equation. The Hamiltonian system for this case was discussed on page 38. The Hamiltonian system (7.9) corresponding to the Hamiltonian (9.26) is three-dimensional, hence a typical solution of this system executes linear motion on a 3-torus, topologically equivalent to

$$\Lambda(3, 4) = \{T_1 = \mathcal{T}_1, \ T_2 = \mathcal{T}_2, \ T_5 = \mathcal{T}_5 \} . \tag{11.8}$$

Such a solution has three phases and is hence a rank 1, genus 3 solution of the KP equation. Nevertheless, the generic rank 1, genus 2 solutions of the KP equation are obtained as solutions of the $(3, 4)$-th KP equation. These solutions are special, nontypical solutions of the $(3, 4)$-th KP equation. They are obtained by imposing an autonomous invariant symmetry reduction on the Hamiltonian system (7.9), as outlined in this section.

The condition (11.1) for a solution of the $(3, 4)$-th KP equation to have genus 2 is

$$\text{rank} \begin{pmatrix} \dfrac{\partial T_1}{\partial q_{11}} & \dfrac{\partial T_1}{\partial q_{12}} & \dfrac{\partial T_1}{\partial q_{21}} & \dfrac{\partial T_1}{\partial p_{11}} & \dfrac{\partial T_1}{\partial p_{12}} & \dfrac{\partial T_1}{\partial p_{21}} \\[2mm] \dfrac{\partial T_2}{\partial q_{11}} & \dfrac{\partial T_2}{\partial q_{12}} & \dfrac{\partial T_2}{\partial q_{21}} & \dfrac{\partial T_2}{\partial p_{11}} & \dfrac{\partial T_2}{\partial p_{12}} & \dfrac{\partial T_2}{\partial p_{21}} \\[2mm] \dfrac{\partial T_3}{\partial q_{11}} & \dfrac{\partial T_3}{\partial q_{12}} & \dfrac{\partial T_3}{\partial q_{21}} & \dfrac{\partial T_3}{\partial p_{11}} & \dfrac{\partial T_3}{\partial p_{12}} & \dfrac{\partial T_3}{\partial p_{21}} \end{pmatrix} = 2. \tag{11.9}$$

From Theorem 11.1, it follows that (11.9) is equivalent to two invariant conditions on the variables $q_{11}, q_{12}, q_{21}, p_{11}, p_{12}$ and p_{21}. These conditions are readily found in this specific case [31].

The expressions for these conditions are quite long, so we do not repeat them here. Let us denote them by

$$P_1(\boldsymbol{q}, \boldsymbol{p}) = 0, \quad p_2(\boldsymbol{q}, \boldsymbol{p}) = 0. \tag{11.10}$$

There is a more geometrical way to look at the conditions (11.9). Consider the three-dimensional space spanned by the conserved quantities T_1, T_2 and T_5. If the conditions (11.10) are satisfied, there is a functional relationship between the three conserved quantities:

$$\Omega : f(T_1, T_2, T_5) = 0, \tag{11.11}$$

which represents a surface in the space spanned by T_1, T_2 and T_5. By solving the conditions (11.10) for two of the phase space variables (p_{11} and p_{12} respectively, for instance), and substituting the result in the form of the conserved quantities (9.27), (9.28) and (9.29), a parametric representation of the surface Ω is obtained:

$$\Omega : \begin{cases} T_1 &= T_1(q_{11}, q_{12}, q_{21}, p_{21}), \\ T_2 &= T_2(q_{11}, q_{12}, q_{21}, p_{21}), \\ T_5 &= T_3(q_{11}, q_{12}, q_{21}, p_{21}). \end{cases} \tag{11.12}$$

Apparently, two too many parameters are present in this set of equations, since the parametric representation of a surface should only contain two parameters. However, the existence of a functional relationship (11.11) guarantees that these parameters appear only in two different combinations such that there are essentially two parameters present in (11.12). The most convenient way to plot the resulting surface is to equate two of the 'parameters' q_{11}, q_{12}, q_{21} and p_{21} to zero, while letting the remaining two vary. In Figure 11.2, two different views of the surface Ω are given.

Every point in the space spanned by T_1, T_2 and T_5 corresponds to a three-dimensional torus, on which the motion of the solutions of the $(3, 4)$-th KP equation takes place. A point on the surface Ω corresponds to a degenerate three-dimensional torus, on which there are in essence only two independent directions, analogous to the idea demonstrated in Figure 11.1. In other words, points on the surface Ω correspond to two-dimensional tori and to genus two solutions of the KP equation. These genus two solutions are the generic rank 1, genus 2 solutions of the KP equation, as argued above.

Note that a more drastic reduction to genus one solutions is possible. These solutions correspond to points on the singular curves on the surface Ω. As the genus one solutions are more easily obtained from the $r = 2$ case, this case is not essential.

12 Parameter count

The previous sections put us in a position to count easily the parameters that determine a solution of the (r, n)-th KP equation.

The first column of table 12.1 lists the parameters that determine a solution of the (r, n)-th KP equation. They are the 'initial' values variables $\boldsymbol{q}(0)$ and $\boldsymbol{p}(0)$ (not only for the x-flow, but any other t_k-flow), the constants h_k that are the coefficients of the Casimir functionals in (5.7), the constants d_k which are the coefficients of the lower-order flows in (5.7) and the constant u_1, discussed in the remark on page 8.

How many of each of these parameters determine a typical solution of the (r, n)-th KP equation is indicated in the second column. A typical solution of the (r, n)-th KP equation has $N = (r-1)(n-1)/2$ variables $\boldsymbol{q}$ and N variables $\boldsymbol{p}$. Each of these is determined by its initial conditions $\boldsymbol{q}(0)$ and $\boldsymbol{p}(0)$. For a typical solution of the (r, n)-th KP equation, N is also the genus of the
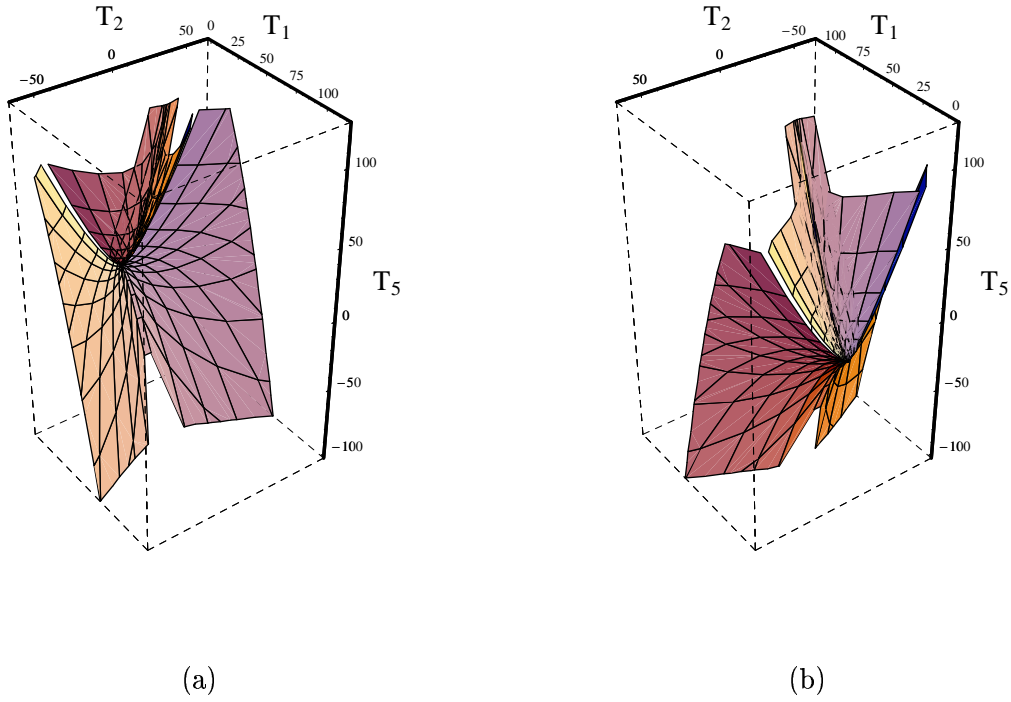
46

(a) (b)

Figure 11.2: **The surface Ω from two different view points. The cusp of the surface is at the origin. Figure (b) shows the same surface as Figure (a), but rotated around the T_5-axis by 180 degrees. This figure was obtained using (11.12) with $q_{21} = 0 = p_{21}$.**

Table 12.1: **The number of parameters determining the solutions of the (r,n)-th KP equation.**

	typical solution of the (r,n)-th KP equation	generic genus g solution of (KP)	non-typical solution of the (r,n)-th KP equation
$\boldsymbol{q}(0)$	$N = \dfrac{(r-1)(n-1)}{2}$	g	g
$\boldsymbol{p}(0)$	$N = \dfrac{(r-1)(n-1)}{2}$	g	g
h_k	$r-1$	g	$r-1$
d_k	$n-2$	g	$n-2$
u_1	1	1	1
total #	$rn-1$	$4g+1$	$2g+r+n-2$

solution. Any solution of the (r, n)-th KP equation is determined by $r - 1$ Casimir functionals (see (5.7)). Also from (5.7), it follows that $n - 2$ lower-order flows are included, accounting for the coefficients d_k. With the addition of the constant u_1, this results in a total number of $rn - 1$ parameters that determine a typical solution of the (r, n)-th KP equation.

The third column expresses the number of parameters in terms of the genus of the solution, for a generic genus g solution of the KP equation. For such a solution $r = g + 1$ and $n = g + 2$ [11]. Furthermore, the Hamiltonian system (7.9) reduces significantly such that there are exactly g variables q and p. The total number of parameters adds up to $4g + 1$. This is a known result, implied for instance in [8].

Not every nontypical solution of the (r, n)-th KP equation is generic. For such solutions, the number of variables q is equal to the genus g of the solution, as is the number of variables p. These results are given in the last column of Table 12.1.

There is an important distinction between the different types of parameters in table 12.1. The entries in the top two rows have dynamical significance: they are initial values for the variables q and p. The Hamiltonian system (7.9) is a dynamical system for the determination of the variables q and p. The other parameters merely show up as parameters in this Hamiltonian system. This distinction between two kinds of parameters, dynamical and nondynamical, is to our knowledge new.

13 Minimal Characterization of the initial data

A rank 1, finite-genus solution of the KP equation is completely determined by a solution of an (r, n)-th KP equation, for a certain r and n. The (r, n)-th KP equation is given by the Euler-Lagrange equation (5.8). This is a set of $(r - 1)$ ordinary differential equations in x. Various quantities appear in this system of ordinary differential equations: $(r - 1)$ potentials $\alpha(r)$ and their derivatives with respect to x, $(r - 1)$ constants h_k, $(n - 2)$ constants d_k and one constant potential u_1.

Next we argue that the knowledge of the initial condition for KP, $u(x, y, t = 0)$ along one direction (say the x-axis) is sufficient to determine the corresponding rank 1, finite genus solution for all x, y and t.

1. If the initial condition is specified along the x-axis for $y = 0$, all potentials and their derivatives can be found at any point on the x-axis (Note that a rank 1, finite-genus solution is analytic in all its independent variables). This is done in the following way: The Euler-Lagrange equations and their derivatives with respect to x determine *algebraic* conditions on the potentials and their derivatives, as well as the unknown parameters h_k, d_k and u_1. In these conditions, u and its derivatives with respect to x are known, by assumption. Hence taking more derivatives of the Euler-Lagrange equations (5.8) with respect to x adds more conditions than unknowns. Taking enough derivatives of the Euler-Lagrange equations, we obtain a set of polynomial equations for the unknown potentials and their derivatives, as well as the parameters h_k, d_k and u_1.

2. We have already argued that the knowledge of the x-dependence completely determines the dependence of the solution on any higher-order time variable: the Hamiltonians determining the dependence of the solution on t_k are conserved quantities for the x-evolution of the solution, $H_k = -T_k$. Furthermore, the initial conditions $q(0)$, $p(0)$ for t_k are identical to the initial conditions for the x-evolution at $x = 0$, $y = 0$.

This shows that it suffices to specify the rank 1, finite-genus initial condition along one direction: $u(x, y = 0, t = 0)$. It is clear that the above argument can be repeated if the initial condition is specified along any of the higher-order flows. This is specifically relevant for t_2 (y) and t_3 (t).

Remarks The procedure for finding the parameters h_k, d_k and u_1 given above is not very practical. A large number of derivatives of the potential are required and a large polynomial system needs to be solved to find the values of the parameters.

Acknowledgements

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