The stability analysis of the periodic traveling wave solutions of the mKdV equation

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Abstract

The stability of periodic solutions of partial differential equations has been an area of increasing interest in the last decade. In this paper, we derive all periodic traveling wave solutions of the focusing and defocusing mKdV equations. We show that in the defocusing case all such solutions are orbitally stable with respect to subharmonic perturbations: perturbations that are periodic with period equal to an integer multiple of the period of the underlying solution. We do this by explicitly computing the spectrum and the corresponding eigenfunctions associated with the linear stability problem. Next, we bring into play different members of the mKdV hierarchy. Combining this with the spectral stability results allows for the construction of a Lyapunov function for the periodic traveling waves. Using the seminal results of Grillakis, Shatah, and Strauss, we are able to conclude orbital stability. In the focusing case, we show how instabilities arise.

1 Introduction

The modified Korteweg-de Vries (mKdV) equation is given by

$$u_t + 6\delta u^2 u_x + u_{xxx} = 0, \tag{1.1}$$

where $\delta = -1$ corresponds to the defocusing case and $\delta = 1$ corresponds to the focusing case. It arises in many of the same physical contexts as the KdV equation, such as water waves and plasma physics, but in different parameter regimes [1].

It is well known (see [2], for instance) that the defocusing mKdV equation possesses the periodic traveling wave solution

$$u = k\mathrm{sn}(x - (-k^2 - 1)t, x),$$

whereas the focusing equation has the solutions

$$u = k\mathrm{cn}(x - (2k^2 - 1)t, x), \quad u = \mathrm{dn}(x - (-k^2 + 2)t, x),$$

where $\mathrm{sn}(\cdot, k)$, $\mathrm{cn}(\cdot, k)$, $\mathrm{dn}(\cdot, k)$ denote the Jacobi elliptic functions [3, 4] with elliptic modulus $k \in [0, 1)$. These do not constitute the full class of periodic traveling wave solutions of the mKdV equation, as is shown in Section 2. The orbital stability of the dn solutions was first studied in [5], where they were proven to be orbitally stable with respect to periodic perturbations of the same period. However, as noted in [2], that proof fails for the other solutions mentioned above. More recently in [2], a modified version of the Bloch decomposition and counting techniques in [6, 7] to Hamiltonian equations with a singular Poisson structure was developed. It is proven there that the sn and dn solutions are orbitally stable with respect to periodic perturbations of the same period for all values of the elliptic modulus k. The dynamics of the cn solutions changes from stable to unstable as the elliptic modulus passes through a fixed value k^*. However, the accompanying numerical investigation of the spectral stability of the cn solutions with respect to subharmonic perturbations suggests instability for *all* values of the elliptic modulus.

Stability results for other subclasses of solutions of the mKdV equation have been obtained in recent years as well. In [6], the spectral stability of small-amplitude periodic traveling wave solutions with respect to periodic perturbations of the same period was established. This result was recently extended beyond spectral stability in the work of [8]. Through the use of the periodic instability index developed in [9] in combination with a periodic version of the Evans function technique employed in [10, 11], it was proven that the small-amplitude solutions are orbitally stable with respect to periodic perturbations of the same period. The same result is also established for solutions in neighborhoods of homoclinic orbits.

There are two limitations in all the results discussed above: (i) they are restricted to special cases of traveling wave solutions. (ii) Only periodic perturbations of the same period are considered. Here we examine the spectral and (nonlinear) orbital stability of *all* periodic traveling wave solutions of the mKdV equation with respect to *subharmonic* perturbations: perturbations that are periodic with period equal to an integer multiple of the period of the underlying solution. Extension beyond periodic perturbations of the same period to subharmonic perturbations is important in that: (i) they are a significantly larger class of perturbations than the periodic ones of the same period, while retaining our ability to discuss completeness in and separability of a suitable function space. For example, this would not be the case for quasi-periodic or almost periodic perturbations [12]. (ii) There are nontrivial examples of solutions which are stable with respect to periodic perturbations of the same period, but unstable with respect to subharmonic perturbations, such as cnoidal wave solutions of the focusing nonlinear Schrödinger equation [6]. (iii) They have a greater physical relevance than periodic perturbations of the same period, since in applications one usually considers domains which are larger than the period of the solution, *e.g.*, ocean wave dynamics.

The basis of our procedure is the Lyapunov method, which was first extended to infinite-dimensional systems (partial differential equations) by V.I. Arnold [13, 14] in his study of incompressible ideal fluid flows. Since its introduction, the Lyapunov method has formed the crux of subsequent nonlinear stability techniques (see [15, 16, 17] for instance). We follow the ideas presented in [2, 18, 19], and use the algebraic connection between the eigenfunctions of the Lax pair and those of the spectral stability problem.

Due to difficulties that arise with the spectral parameter in the Lax pair for the focusing mKdV equation (see Section 8), we first restrict ourselves to the defocusing case, equating $\delta = -1$. After deriving all periodic traveling wave solutions of the defocusing mKdV equation in terms of the Weierstrass elliptic function (surprisingly, this result appears to be new), we analytically prove that all bounded periodic traveling wave solutions are spectrally and orbitally stable with respect to subharmonic perturbations. Next, we return to the focusing case. We construct all periodic traveling wave solutions, and employ a combination of analytical and numerical techniques to study their stability.

Remarks.

- Superharmonic perturbations (perturbations whose period is the base period divided by a positive integer) are covered by studying perturbations that have the same period as the underlying solution.
- Many of the details will be omitted as they are similar to what happens for the KdV equation in [18, 20].

2 Periodic traveling wave solutions

In this section we construct all periodic traveling wave solutions of the defocusing mKdV equation, see Section 8 for their construction in the focusing case. We employ a technique originally due to Poincaré, Painlevé, Briot, and Bouquet [21], though most recently reformulated in [22, 23].

Remark. A large class of solutions of the mKdV equation in terms of the Weierstrass elliptic function is derived using a different method in [24]. However, it is straightforward to check that the solutions in [24] do not constitute the full set of periodic traveling wave solutions.

To examine traveling wave solutions, we change to a moving coordinate frame

$$y = x - Vt, \quad \tau = t.$$

In the (y, τ) coordinates the mKdV equation becomes

$$u_\tau - V u_y - 6u^2 u_y + u_{yyy} = 0. \tag{2.1}$$

Stationary solutions (1.1) are time-independent solutions of (2.1). Letting $u(y, \tau) = U(y)$, stationary solutions satisfy the ordinary differential equation

$$-VU_y - 6U^2U_y + U_{yyy} = 0. \quad (2.2)$$

Integrating (2.2) gives

$$-VU - 2U^3 + U_{yy} = C, \quad (2.3)$$

for some constant C. Multiplying (2.3) by U_y and integrating a second time gives

$$-\frac{V}{2}U^2 - \frac{1}{2}U^4 + \frac{1}{2}U_y^2 - CU = E, \quad (2.4)$$

for some constant E. Thus, all stationary solutions $U(y)$ satisfy the first-order ordinary differential equation (2.4).

Defining the new variable $P = U_y$, (2.4) becomes

$$\frac{1}{2}P^2 - \frac{1}{2}U^4 - \frac{V}{2}U^2 - CU - E = 0.$$

This defines a genus one algebraic curve [25], birationally equivalent to (using the normal form algorithm found in [26], and implemented in the Maple command `algcurves[Weierstrassform]`)

$$r^2 = 4s^3 - g_2 s - g_3, \quad (2.5)$$

where

$$U = R(s, r), \quad P = S(s, r),$$

are rational functions of s and r (we omit the explicit form of R and S for brevity). The elliptic invariants g_2 and g_3 are given by

$$g_2 = \frac{4}{3}V^2 + 32E, \quad g_3 = -\frac{8}{27}V^3 + \frac{64}{3}VE - 16C^2.$$

As the curve (2.5) is in Weierstrass form, it can be parameterized in terms of the Weierstrass $\wp$-function $\wp(x, g_2, g_3)$ [27], with

$$r = \wp'(\omega x, g_2, g_3), \quad s = \wp(\omega x, g_2, g_3),$$

for some constant ω. Transforming back to our original variables gives

$$U = R(\wp(\omega x, g_2, g_3), \wp'(\omega x, g_2, g_3)), \quad P = S(\wp(\omega x, g_2, g_3), \wp'(\omega x, g_2, g_3)).$$

Imposing our original assumption, $U_y = P$, gives $\omega = \frac{1}{2}$. Thus our final solution form is

$$U(y) = \frac{\pm\sqrt{2E}\wp'(\frac{1}{2}(y + y_0), g_2, g_3) + C(2\wp(\frac{1}{2}(y + y_0), g_2, g_3) - \frac{2}{3}V)}{\left(\wp(\frac{1}{2}(y + y_0), g_2, g_3) - \frac{V}{3}\right)^2 - 8E}. \quad (2.6)$$

Here y_0 is an arbitrary shift in y determined by the initial conditions. These solutions are doubly periodic in the complex plane. When considered on the real line, they have period $2T$ determined by

$$2T = 4\int_{e_1}^{\infty} \frac{1}{\sqrt{4z^3 - g_2 z - g_3}}\, dz,$$

where e_1 is the largest root of the equation obtained by setting $r = 0$ in (2.5). This gives all periodic solutions of (2.2) due to a classic theorem by Briot and Bouquet [21].

We now determine which values of V, C, and E give rise to real bounded periodic solutions. Equating $v = U_y$ in (2.3), we have the first-order two-dimensional system

$$U_y = P, \quad P_y = VU + 2U^3 + C.$$

All fixed points (U_0, P_0) satisfy

$$P_0 = 0, \quad VU_0 + 2U_0^3 + C = 0. \tag{2.7}$$

After linearizing about $(U_0, 0)$, the resulting linear system has eigenvalues

$$\lambda = \pm\sqrt{V + 6U_0^2}. \tag{2.8}$$

We have two saddles and a center when the discriminant of the second equation in (2.7)

$$d = -8V^3 - 108C^2$$

is greater than zero, and one saddle when the discriminant is less than zero. Therefore, we can only expect periodic solutions for $V < 0$ and $d > 0$ which gives

$$|C| < \sqrt{\frac{-8V^3}{108}}.$$

Using (2.4), we see that for fixed V and C the phase space is foliated by the family of curves

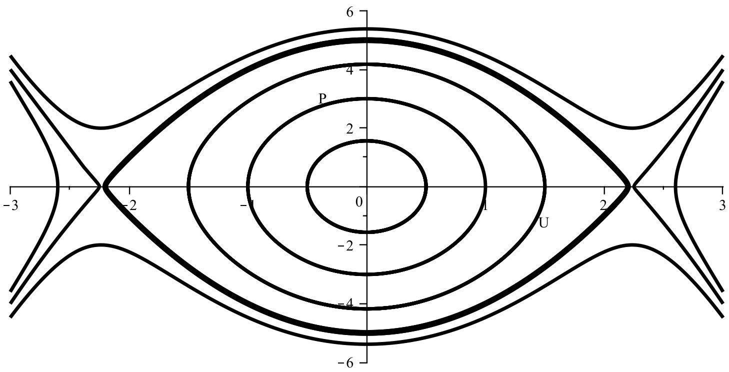
$$P^2 = U^4 + VU^2 + 2CU + 2E. \tag{2.9}$$

The parameter E is specified by the initial condition. Periodic solutions are separated from unbounded solutions by two heteroclinc orbits in the case $C = 0$, and by one homoclinc orbit in the case $C \neq 0$, see Fig. 1. All values of E which give rise to a solution lying inside the separatrix correspond to periodic solutions. Thinking of the right-hand side of (2.9) as a polynomial in U, all values of E which make its discriminant positive give rise to periodic solutions. For $C = 0$ we can write the solution in the particularly simple form

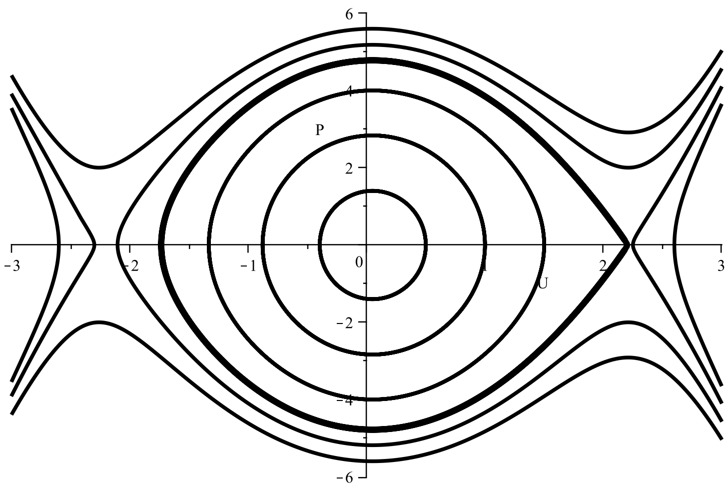
$$U(y) = \pm k\sqrt{\frac{-V}{1+k^2}}\,\mathrm{sn}\left(\sqrt{\frac{-V}{1+k^2}}\,y, k\right),$$

where E is parameterized by the elliptic modulus k

$$E = \frac{k^2 V^2}{2(k^4 + 2k^2 + 1)}.$$



(a)



(b)

Figure 1: (a) Typical (U, v) phase plane in the defocusing case for $C = 0$ (here $V = -10$). The two heteroclinic orbits are in bold. (b) For $C \neq 0$ (here $C = 0.5$, $V = -10$), the heteroclinc orbits break into a single homoclinic orbit (bold line). The homoclinic orbit exists for $0 < |C| < \sqrt{\frac{-8V}{108}}$.

3 The linear stability problem

Before we study the orbital stability of the stationary solutions, we examine their spectral and linear stability. To this end, we consider perturbations of a stationary solution

$$u(y, \tau) = U(y) + \epsilon w(y, \tau) + \mathcal{O}(\epsilon^2),$$

where ϵ is a small parameter. Substituting this in (2.1) and ignoring higher-than-first-order terms in ϵ, we find

$$w_\tau = 6U^2 w_y + 12UU_y w - w_{yyy} + V w_y, \tag{3.1}$$

at first order in ϵ. The zero order terms vanish since $U(y)$ solves the mKdV equation. By ignoring the higher-order terms in ϵ, we are restricting our attention to linear stability. The traveling wave solution is defined to be *linearly stable* if for all $\epsilon > 0$, there is a $\delta > 0$ such that if $||w(y,0)|| < \delta$ then $||w(y,\tau)|| < \epsilon$ for all $\tau > 0$. This definition depends on our choice of the norm $||\cdot||$, to be discussed later.

Next, since (3.1) is autonomous in time, we may separate variables. Let

$$w(y,\tau) = e^{\lambda\tau}W(y,\lambda). \tag{3.2}$$

Then $W(y,\lambda)$ satisfies

$$-W_{yyy} + (V + 6U^2)W_y + 12UU_yW = \lambda W, \tag{3.3}$$

or

$$J\mathcal{L}W = \lambda W, \quad J = \partial_y, \quad \mathcal{L} = -\partial_y^2 + V + 6U^2. \tag{3.4}$$

In what follows, the λ dependence of W will be suppressed. To avoid confusion with other spectra arising below, we refer to $\sigma(J\mathcal{L})$ as the *stability spectrum*.

4 Numerical Results

Before we determine the spectrum of (3.4) analytically, we compute it numerically, using Hill's method [28, 29]. Hill's method is ideally suited to a problem such as (3.4) with periodic coefficients. It allows us to compute all eigenfunctions of the form

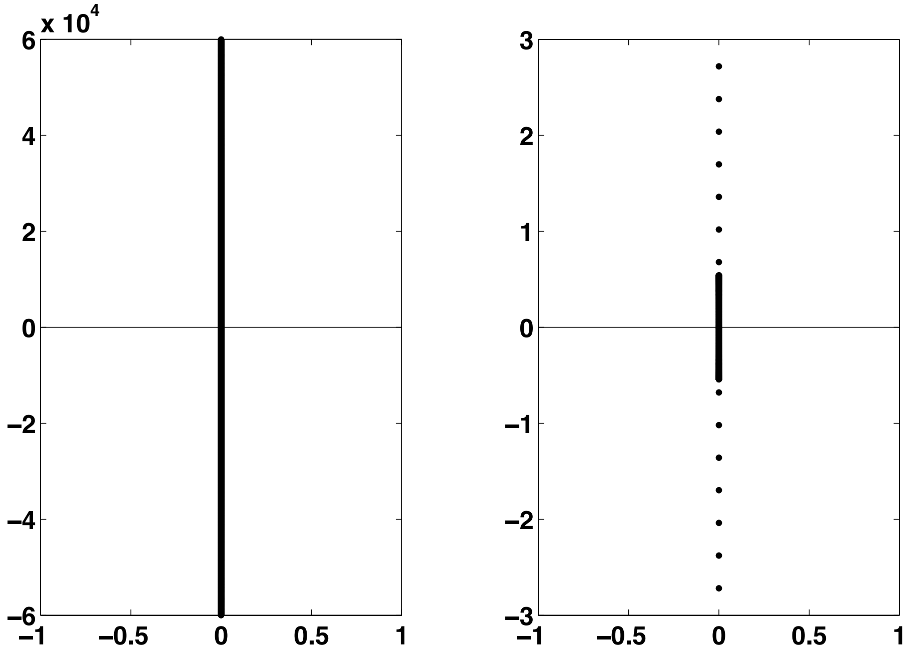
$$W = e^{i\mu y}\hat{W}(y), \quad \hat{W}(y+2T) = \hat{W}(y), \tag{4.1}$$

with $\mu \in [-\pi/4T, \pi/4T)$. It follows from Floquet's theorem that *all* bounded solutions of (3.4) are of this form. Here bounded means that $\sup_{x\in\mathbb{R}}|W(x)|$ is finite. Thus $W \in C_b^0(\mathbb{R})$. On the other hand, we also have $W \in L^2(-T,T)$ (the square-integrable functions of period $2T$) since the exponential factor in (4.1) disappears in the computation of the L^2-norm. Thus

$$W \in C_b^0(\mathbb{R}) \cap L^2(-T,T). \tag{4.2}$$

It should be noted that by this choice our investigations include perturbations of an arbitrary period that is an integer multiple of $2T$, *i.e.*, subharmonic perturbations.

Figure (2) shows discrete approximations to the spectrum of (3.4), computed using SpectrUW2.0 [28]. The solution parameters are $V = -10$, $C = 0$, and $E \approx 11.9$ ($k = 0.8$). The numerical parameters (see [28, 29]) are $N = 40$ (81 Fourier modes) and $D = 80$ (79 different Floquet exponents). The right panel (b) is a blow-up of the left panel (a) around the origin. First, it appears that the spectrum is on the imaginary axis, indicating spectral stability of the solution. Second, the numerics shows that a symmetric band around the origin has a higher spectral density than the rest of the imaginary axis. This is indeed the case, as shown in more detail in Fig. 4, where the imaginary parts $\in [-1.5, 1.5]$ of the computed eigenvalues are displayed as a function of the Floquet parameter μ (here 199 different Floquet exponents were used). This shows that λ values with $\text{Im}(\lambda) \in [-0.54, 0.54]$ (approximately) are attained for three different μ values in $[-\pi/4T, \pi/4T)$. The rest of the imaginary axis is only attained for one μ value. This picture persists if a larger portion of the imaginary λ axis is examined. These numerical results are in perfect agreement with the theoretical results below.



(a) (b)

Figure 2: (a) The numerically computed spectrum for the traveling wave solution with $V = -10$, $C = 0$, and $E = 11.9$ ($k = .8$) using Hill's method with 81 Fourier modes and 79 different Floquet exponents, see [28, 29]; (b) A blow-up of (a) around the origin, showing a band of higher spectral density;

Figure 3 shows discrete approximations to the spectrum for $C \neq 0$. The solution parameters are $V = -10$, $C = 10\sqrt{15}/9$, and $E \approx -1$. We see the same structure as for the $C = 0$ case.

5 Lax pair representation

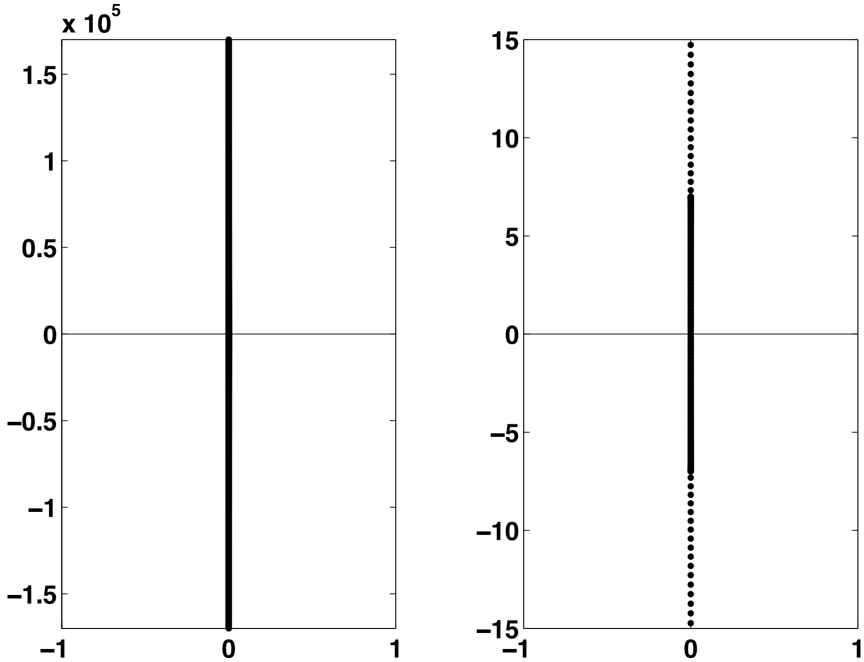
Equation (2.2) is equivalent to the compatibility of two linear ordinary differential systems:

$$\psi_y = \begin{pmatrix} -i\zeta & u \\ u & i\zeta \end{pmatrix} \psi, \tag{5.1}$$

$$\psi_\tau = \begin{pmatrix} -iV\zeta - 4i\zeta^3 - 2i\zeta u^2 & Vu + 4\zeta^2 u + 2u^3 - u_{yy} + 2i\zeta u_y \\ Vu + 4\zeta^2 u + 2u^3 - u_{yy} - 2i\zeta u_y & iV\zeta + 4i\zeta^3 + 2i\zeta u^2 \end{pmatrix} \psi. \tag{5.2}$$

In other words, the compatibility condition $\psi_{y\tau} = \psi_{\tau y}$ requires that u satisfies the defocusing mKdV equation. We can rewrite (5.1) as the spectral problem

$$\begin{pmatrix} i\partial_y & -iu \\ iu & -i\partial_y \end{pmatrix} \psi = \zeta\psi. \tag{5.3}$$



(a) (b)

Figure 3: (a) The numerically computed spectrum for the traveling wave solution with $V = -10$, $C = 10\sqrt{15}/9$, and $E \approx -1$ using Hill's method with 81 Fourier modes and 79 different Floquet exponents, see [28, 29]; (b) A blow-up of (a) around the origin, showing a band of higher spectral density;

This problem is self-adjoint, therefore $\zeta \in \mathbb{R}$. Evaluating (5.1-5.2) at the stationary solution $u(y,\tau) = U(y)$, we find

$$\psi_y = \begin{pmatrix} -i\zeta & U \\ U & i\zeta \end{pmatrix} \psi, \tag{5.4}$$

$$\psi_\tau = \begin{pmatrix} -iV\zeta - 4i\zeta^3 - 2i\zeta U^2 & VU + 4\zeta^2 U + 2U^3 - U_{yy} + 2i\zeta U_y \\ VU + 4\zeta^2 U + 2U^3 - U_{yy} - 2i\zeta U_y & iV\zeta + 4i\zeta^3 + 2i\zeta U^2 \end{pmatrix} \psi. \tag{5.5}$$

We refer to the set of all ζ values such that (5.4-5.5) has bounded solutions as the *Lax spectrum* σ_L. Since the spectral problem (5.3) is self-adjoint, the Lax spectrum is a subset of the real line: $\sigma_L \subset \mathbb{R}$. The goal of this section is to determine this subset explicitly. In the next section, we connect the Lax spectrum to the stability spectrum.

Equation (5.5) simplifies. Using (2.3) to eliminate U_{yy} gives

$$\psi_\tau = \begin{pmatrix} -iV\zeta - 4i\zeta^3 - 2i\zeta U^2 & 4\zeta^2 U + C + 2i\zeta U_y \\ 4\zeta^2 U + C - 2i\zeta U_y & iV\zeta + 4i\zeta^3 + 2i\zeta U^2 \end{pmatrix} \psi = \begin{pmatrix} A & B \\ \overline{B} & -A \end{pmatrix} \psi, \tag{5.6}$$

where $\overline{B}$ denotes the complex conjugate of B. Since A and B do not explicitly depend on τ, we separate variables

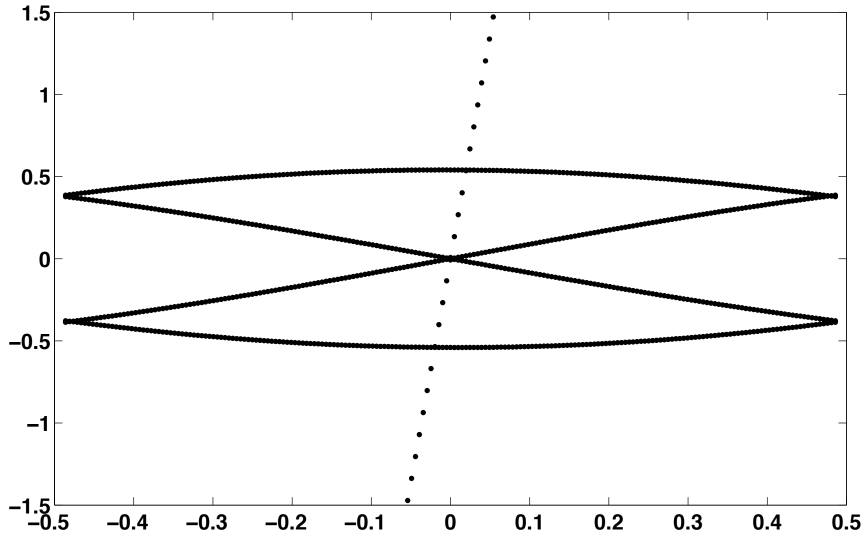


Figure 4: The imaginary part of λ as a function of μ, demonstrating the higher spectral density of the band on the imaginary axis around the origin. The parameter values are identical to those of Fig. 2, except 199 different Floquet exponents are used here.

$$\psi(y,\tau) = e^{\Omega\tau}\begin{pmatrix}\alpha(y)\\ \beta(y)\end{pmatrix}. \tag{5.7}$$

Substituting (5.7) into (5.6) and canceling the exponential, we find

$$\begin{pmatrix}A & B\\ \overline{B} & -A\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0.$$

This implies that the existence of nontrivial solutions requires

$$\Omega^2 = A^2 + |B|^2 = -16\zeta^6 - 8V\zeta^4 - (V^2+8E)\zeta^2 + C^2 = 0, \tag{5.8}$$

where we have used the explicit form of the stationary solution $U(y)$ derived earlier. This determines Ω in terms of the spectral parameter ζ. Thinking of Ω^2 as a polynomial in ζ^2, one finds that the discriminant of (5.8) has the same sign as the discriminant of the right hand side of (2.9). As discussed earlier, this discriminant is positive for periodic solutions. Also, Ω^2 is an even function of ζ. Therefore, for periodic stationary solutions, (5.8) can be written as

$$\Omega^2 = -16(\zeta-\zeta_1)(\zeta+\zeta_1)(\zeta-\zeta_2)(\zeta+\zeta_2)(\zeta-\zeta_3)(\zeta+\zeta_3)$$

for positive constants $0 \le \zeta_3 < \zeta_2 < \zeta_1$.

The eigenvector corresponding to the eigenvalue Ω is

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \gamma(y)\begin{pmatrix}-B\\ A-\Omega\end{pmatrix}, \tag{5.9}$$

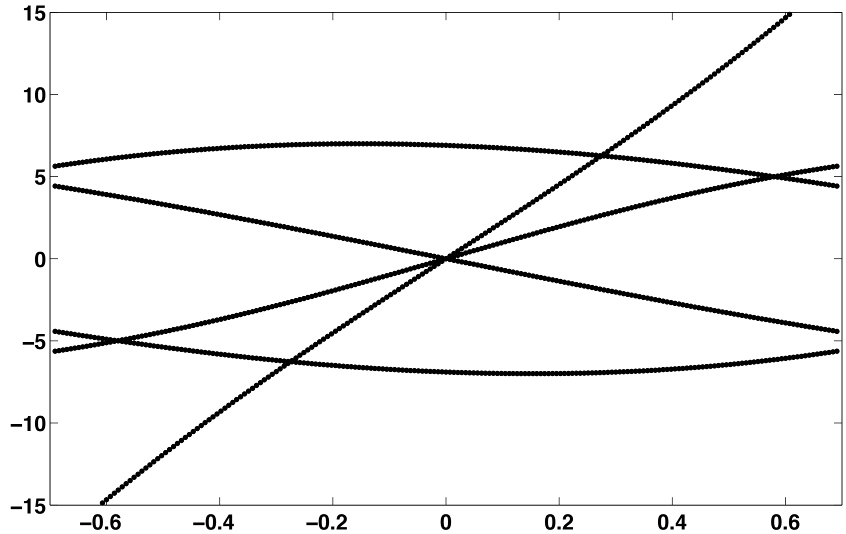


Figure 5: The imaginary part of λ as a function of μ, demonstrating the higher spectral density of the band on the imaginary axis around the origin. The parameter values are identical to those of Fig. 3, except 199 different Floquet exponents are used here.

where γ is a scalar function of x. It is determined by substitution of the above into the first equation of the Lax pair, resulting in a first-order scalar differential equation for γ. This equation may be solved explicitly giving

$$\gamma = \exp \int \left(i\zeta - \frac{A'}{A-\Omega} - \frac{UB}{A-\Omega} \right) dy,$$

up to a multiplicative constant. This simplifies to

$$\gamma = \frac{1}{A-\Omega} \exp \int \left(i\zeta - \frac{UB}{A-\Omega} \right) dy. \tag{5.10}$$

Each value of ζ results in two values of Ω (except for the six branch points $\pm\zeta_i$, $i = 1, 2, 3$, where $\Omega = 0$) and therefore (5.9) represents two eigenvectors. These solutions are clearly linearly independent. For those values of ζ for which $\Omega = 0$, only one solution is generated. A second one may be found using reduction of order, resulting in algebraically growing solutions.

To determine the Lax spectrum σ_L, we need to determine the set of all $\zeta \subset \mathbb{R}$ such that (5.9) is bounded as a function of x. Thus, we need to determine for which ζ the scalar function $\gamma(x)$ is bounded. First, one can readily check that the only values of ζ for which the denominator in (5.9) is singular are the branch points $\pm\zeta_i$, $i = 1, 2, 3$, where $\Omega = 0$. One finds that the vector part of (5.9) cancels the singularity in $\gamma(y)$. Thus, $\pm\zeta_i$, $i = 1, 2, 3$, are part of the Lax spectrum. For all other values of ζ, it is necessary and sufficient that

$$\left\langle \Re \left(-\frac{uB}{A-\Omega} \right) \right\rangle = 0. \tag{5.11}$$

Here $\langle \ \cdot \ \rangle = \frac{1}{2T}\int_{-T}^{T} \ \cdot \ dy$ denotes the average over a period. The explicit form of the above depends on whether Ω is real or imaginary. It should be noted that since $\zeta \in \mathbb{R}$, it follows from (5.8) that these are the only possibilities. Let us investigate each case separately:

- If Ω is imaginary then

$$-\frac{UB}{A-\Omega} = \frac{-U\mathrm{Re}(B)}{\mathrm{Im}(A-\Omega)}i + \frac{\partial_y\left(\frac{1}{2}\mathrm{Im}(A-\Omega)\right)}{\mathrm{Im}(A-\Omega)}, \tag{5.12}$$

  where we used that

$$\partial_y\left(\frac{1}{2}(\mathrm{Im}(A)-\Omega)\right) = -2\zeta U U_y = -U\mathrm{Im}(B).$$

  The first term in (5.12) is imaginary and the second term is a total derivative, thus giving zero average. Therefore, all ζ values for which Ω is imaginary are in the Lax spectrum.

- If Ω is real, then ignoring total derivatives and using (2.3) one finds

$$\left\langle \mathrm{Re}\left(-\frac{UB}{A-\Omega}\right)\right\rangle = \left\langle \frac{U\mathrm{Re}(B)}{\Omega^2+(\mathrm{Im}(A))^2}\right\rangle \Omega = \left\langle \frac{4\zeta^2U^2+CU}{\Omega^2+(\mathrm{Im}(A))^2}\right\rangle \Omega = 0.$$

  Obviously, the first factor in the last equality is not zero for $C=0$. A similar argument as for the case of non-trivial phase in the defocusing NLS equation gives that this average term is never zero [30]. Therefore, Ω must be identically zero, and all values of ζ for which Ω is real are not part of the Lax spectrum.

We conclude that the Lax spectrum consists of all ζ values for which $\Omega^2 \leq 0$:

$$\sigma_L = (-\infty, -\zeta_1] \cup [-\zeta_2, -\zeta_3] \cup [\zeta_3, \zeta_2] \cup [\zeta_1, \infty).$$

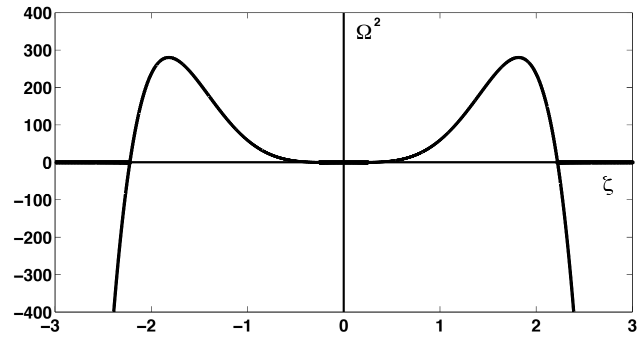
Furthermore, for all values of $\zeta \in \sigma_L$:

$$\Omega \in i\mathbb{R}.$$

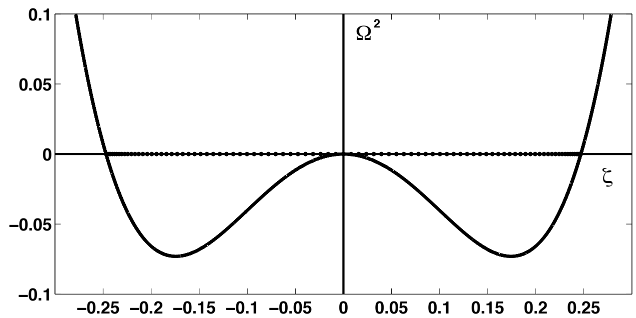
In fact, Ω^2 takes on all negative values for $\zeta \in (-\infty, -\zeta_1]$, implying that $\Omega = \pm\sqrt{|\Omega^2|}$ covers the imaginary axis. The same is true for the segment $\zeta \in [\zeta_1, \infty)$. Furthermore, for $\zeta \in [-\zeta_2, -\zeta_3]$, Ω^2 takes on all negative values in $[\Omega^2(\zeta^*), 0]$ twice, where $\Omega^2(\zeta^*)$ is the minimal value of Ω^2 attained for $\zeta \in [-\zeta_2, -\zeta_3]$. Since Ω^2 is an even function of ζ, the same is true for the segment $[\zeta_3, \zeta_2]$. Upon taking square roots, this implies that the interval on the imaginary axis $\left[-i\sqrt{|\Omega^2(\zeta^*)|}, i\sqrt{|\Omega^2(\zeta^*)|}\right]$ is covered six times, while the rest of the imaginary axis is double covered. Symbolically, we write [18]

$$\Omega \in (i\mathbb{R})^2 \cup \left[-i\sqrt{|\Omega^2(\zeta^*)|}, i\sqrt{|\Omega^2(\zeta^*)|}\right]^4, \tag{5.13}$$

where the exponents denote multiplicities (see Figs. 6 and 7).



(a)



(b)

Figure 6: (a) Ω^2 as a function of real ζ, with the same paramter values as Fig. 2. The union of the dotted line segments is the numerically computed Lax spectrum (in the complex ζ-plane) with 81 Fourier modes and 49 different Floquet exponents. (b) A blow-up of (a) around the origin;

6 Spectral stability

It is well known that there exists a connection between the eigenfunctions of the Lax pair of an integrable equation and the eigenfunctions of the linear stability problem for this integrable equation [31, 1, 32, 33, 34]. A direct calculation proves that the function

$$w(y,\tau) = \psi_1^2(y,\tau) + \psi_2^2(y,\tau) = \frac{1}{2i\zeta}\partial_y \left(\psi_2^2(y,\tau) - \psi_1^2(y,\tau)\right)$$

satisfies the linear stability problem (3.1). Here $\psi = (\psi_1, \psi_2)^T$ is any solution of (5.4-5.5) with the corresponding stationary solution $U(y)$.

In order to establish the spectral stability of equilibrium solutions of (2.2), we need to establish that *all* bounded solutions $W(y)$ of (3.3) are obtained through the squared eigenfunction connection by

$$W(y) = \alpha^2(y) + \beta^2(y).$$

If we manage to do so then we may immediately conclude that

$$\lambda = 2\Omega.$$

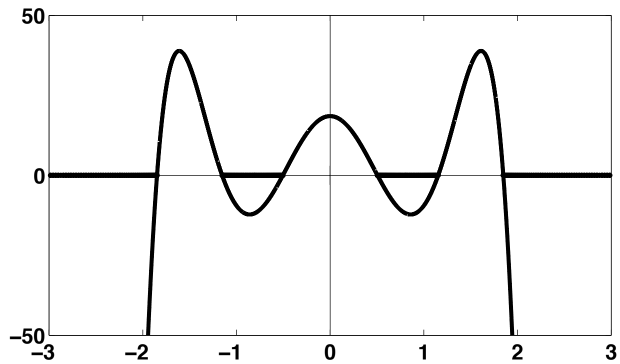


Figure 7: Ω^2 as a function of real ζ, for $C \neq 0$ (same parameter values as Fig. 3). The union of the dotted line segments is the numerically computed Lax spectrum (in the complex ζ-plane) with 81 Fourier modes and 49 different Floquet exponents.

Since $\Omega \in i\mathbb{R}$, we conclude that the stability spectrum is given by

$$\sigma(J\mathcal{L}) = i\mathbb{R}.$$

In order to obtain this conclusion, we need the following theorem.

Theorem 1 All but six *solutions of (3.3) may be written as* $W(y) = \alpha^2(y) + \beta^2(y)$, *where* $(\alpha, \beta)^T$ *solves (5.4-5.5). Specifically,* all *solutions of (3.3) bounded on the whole real line are obtained through the squared eigenfunction connection, with one exception corresponding to* $\lambda = 0$.

Proof. For any given value of $\lambda \in \mathbb{C}$, (3.3) is a third-order linear ordinary differential equation. Thus, it has three linearly independent solutions. On the other hand, we have already shown (see the previous theorem) that the formula

$$W(y) = \alpha^2(y) + \beta^2(y) \tag{6.1}$$

provides solutions of this ordinary differential equation. Let us count how many solutions are obtained this way, for a fixed value of λ. For any value of $\lambda \in \mathbb{C}$, exactly one value of $\Omega \in \mathbb{C}$ is obtained through $\Omega = \lambda/2$. Excluding the six values of λ for which the discriminant of (5.8) as a function of ζ is zero (these are the only values of λ for which Ω^2 reaches its maximum or minimum value, keeping in mind that Ω^2 is an even function of ζ), (5.8) gives rise to six values of $\zeta \in \mathbb{C}$. It should be noted that we are not restricting ourselves to $\zeta \in \sigma_L$ now, since the boundedness of the solutions is not a concern in this counting argument. Next, for a given pair $(\Omega, \zeta) \in \mathbb{C}^2$, (5.9) defines a unique solution of (5.4, 5.5). Thus, any choice of $\lambda \in \mathbb{C}$ not equal to the six values mentioned above, gives rise to exactly six solutions of (3.3), through the squared eigenfunction connection. Let us examine how many of these solutions are linearly independent.

- Since Ω^2 is an even function of ζ, the six values of ζ mentioned above come in pairs: $\pm\zeta_i$, $i = 1, 2, 3$. It can be checked that if ζ corresponds to the eigenfunction W, then $-\zeta$ corresponds to its complex conjugate $\overline{W}$. Therefore, when considering the general solution to the linear stability problem $w = a_1 e^{\Omega t} W + a_2 e^{-\Omega t} \overline{W}$, half of the ζ values provide no new solutions.

Also, if there is an exponential contribution from $\gamma(y)$ then an argument similar to that in [18] establishes the linear independence of the remaining three solutions.

- As in [18], the only possibility for the exponential factor from $\gamma(y)$ not to contribute is if $\lambda = 0 = \Omega$. Only one linearly independent solution is obtained through the squared eigenfunction connection, corresponding to translational invariance, $W = U_y$. The other two can be obtained through reduction of order. Just as for the KdV equation in [18], this allows one to construct two solutions whose amplitude grows linearly in x. A suitable linear combination of these solutions is bounded. Thus, corresponding to $\lambda = 0$ there are two eigenfunctions. One of these is obtained through the squared eigenfunction connection.

Lastly, consider the six excluded values of λ. For the two λ values where Ω^2 reaches a local minimum, the two solutions obtained through the squared eigenfunction connection are bounded, thus, these values of λ are part of the spectrum. The third solution may be constructed using reduction of order, and introduces algebraic growth. For the two values of λ where Ω^2 obtains a global maximum, three solutions are obtained through the squared eigenfunction connection, all of which are unbounded. For the other two λ values where Ω^2 reaches a local maximum, two solutions are obtained through the squared eigenfunction connection, both of which are unbounded. The third solution may be constructed using reduction of order, and introduces algebraic growth. ■

We have established the following theorem.

Theorem 2 (Spectral Stability) *The periodic traveling wave solutions of the defocusing mKdV equation are spectrally stable. The spectrum of their associated linear stability problem is explicitly given by $\sigma(J\mathcal{L}) = i\mathbb{R}$, or, accounting for multiple coverings,*

$$\sigma(J\mathcal{L}) = i\mathbb{R} \cup \left[-2i\sqrt{|\Omega^2(\zeta^*)|}, 2i\sqrt{|\Omega^2(\zeta^*)|}\right]^2, \tag{6.2}$$

where $|\Omega^2(\zeta^*)|$ is as before.

Remark. As previously mentioned, for a fixed value of Ω, only three of the six solutions (corresponding to the six different values of ζ) obtained through the squared eigenfunction connection contribute as independent solutions to the linear stability problem. Therefore, the double and sextuple coverings in the Ω representation (5.13) drop to single and triple coverings in (6.2).

It is an application of the SCS basis lemma in [2, 6] that the eigenfunctions W form a basis for $L^2_{per}([-NT, NT])$, for any integer N, when the potential U is periodic in y with period $2L$. This allows us to conclude linear stability with respect to subharmonic perturbations.

Theorem 3 (Linear Stability) *The periodic traveling wave solutions of the defocusing mKdV equation are linearly stable with respect to square-integrable subharmonic perturbations.*

In other words, all solutions of the defocusing mKdV equation with sufficiently small square integrable initial conditions remain small for all $t > 0$.

7 Nonlinear stability

7.1 Hamiltonian structure

To begin, we recall the formulation of the defocusing mKdV equation as a Hamiltonian system. We are concerned with the stability of $2T$-periodic traveling wave solutions of equation (1.1) with respect to subharmonic perturbations of period $2NT$ for any fixed positive integer N. Therefore, we naturally consider solutions u in the space of square-integrable functions of period $2NT$, $L^2_{per}[-NT, NT]$. In order to properly define the higher-order equations in the mKdV hierarchy that are necessary for our stability argument (see Section 7.2), we further require u and its derivatives of up to order three to be square-integrable as well. Therefore, we consider solutions of (2.1) defined on the function space

$$\mathbb{V} = H^3_{per}[-NT, NT],$$

equipped with natural inner product

$$\langle v, w \rangle = \int_{-NT}^{NT} \bar{v} w \, dx.$$

We write the mKdV equation in Hamiltonian form

$$u_\tau = JH'(u) \tag{7.1}$$

on $\mathbb{V}$. Here J is the skew-symmetric operator

$$J = \partial_y,$$

the Hamiltonian H is the functional

$$H(u) = \int_{-NT}^{NT} \left(\frac{1}{2} u_y^2 + \frac{1}{2} V u^2 + \frac{1}{2} u^4 \right) dy, \tag{7.2}$$

and the notation E' denotes the variational derivative of $E = \displaystyle\int_{-NL}^{NL} \mathcal{E}(u, u_x, \ldots) dx$,

$$E'(u) = \sum_{i=0}^{\infty} (-1)^i \partial_y^i \frac{\partial \mathcal{E}(u)}{\partial u_{iy}}, \tag{7.3}$$

where the sum in (7.3) terminates at the order of the highest derivatives involved. For instance, in the computation of H' the sum terminates after accounting for first derivative terms.

We allow for perturbations in a function space $\mathbb{V}_0 \subset \mathbb{V}$. In order to apply the stability result of [35], we follow [2] and restrict ourselves to the space of perturbations on which J has a well-defined and bounded inverse. This amounts to fixing the spatial average of u on $H^3_{per}[-NL, NL]$, which poses no problem since it is a Casimir of the Poisson operator J, hence, it is conserved under the mKdV flow. Therefore, we consider perturbations in $\ker(J)^\perp$, *i.e.*, zero-average subharmonic perturbations

$$\mathbb{V}_0 = \left\{ v \in H^3_{per}([-NL, NL]) : \int_{-NL}^{NL} v \; dx = 0 \right\}. \tag{7.4}$$

Remark. Physically, requiring perturbations to be zero-average makes sense. Simply stated, this means we are not allowing perturbations to add mass to or subtract mass from the system.

7.2 The mKdV hierarchy

By virtue of its integrability, the mKdV equation possesses an infinite number of conserved quantities $H_0, H_1, H_2, \ldots$, and just as the functional $H_1 = H$ defines the mKdV equation, each H_i defines a Hamiltonian system with time variable τ_i through

$$u_{\tau_i} = JH_i'(u). \tag{7.5}$$

This defines an infinite hierarchy of equations, the mKdV hierarchy. It has the following properties:

- All the functionals H_i, $i = 0, 1, \ldots$, are conserved for each member of the mKdV hierarchy (7.5).
- The flows of the mKdV hierarchy (7.5) mutually commute, and we can think of u as solving all of these equations simultaneously, *i.e.*, $u = u(\tau_0, \tau_1, \ldots)$ [36].

As all the flows in the mKdV hierarchy commute, we may take any linear combination of the above Hamiltonians to define a new Hamiltonian system. For our purposes, we define the *n-th mKdV equation* with time variable t_n as

$$u_{t_n} = J\hat{H}_n'(u), \tag{7.6}$$

where each $\hat{H}_n$ is defined as

$$\hat{H}_n := H_n + \sum_{i=0}^{n-1} c_{n,i} H_i, \quad \hat{H}_0 := H_0, \tag{7.7}$$

for constants $c_{n,i}$, $i = 0 \ldots n-1$.

Since every member of the nonlinear hierarchy (7.5) is integrable, each possesses a Lax pair. The collection of the linear equations in the Lax pairs is known as the *linear mKdV hierarchy.* We construct the Lax pair for the n-th mKdV equation (7.6) by taking the same linear combination of the lower-order flows as for the nonlinear hierarchy, and define the *n-th linear mKdV equation* as

$$\psi_{t_n} = \hat{T}_n \psi = \begin{pmatrix} \hat{A}_n & \hat{B}_n \\ \hat{C}_n & -\hat{A}_n \end{pmatrix} \psi, \tag{7.8}$$

$$\hat{T}_n := T_n + \sum_{i=0}^{n-1} c_{n,i} T_i, \quad \hat{T}_0 := T_0.$$

7.3 Stationary solutions

Stationary solutions of the mKdV hierarchy are defined as solutions such that

$$u_{t_n} = 0$$

for some integer n and constants $c_{n,0}, \ldots, c_{n,n-1}$ in (7.6-7.7). Thus, a stationary solution of the n-th mKdV equation satisfies the ordinary differential equation

$$J\hat{H}_n'(u) = 0$$

with independent variable y.
The stationary solutions have the following properties:

- Since all the flows commute, the set of stationary solutions is invariant under any of the mKdV equations, *i.e.*, a stationary solution of the n-th equation remains a stationary solution after evolving under any of the other flows.
- Any stationary solution of the n-th mKdV equation is also stationary with respect to all of the higher-order time variables t_m, $m > n$. In such cases, the constants $c_{m,i}$, $i \geq n$ are undetermined coefficients. We make use of this fact when constructing a Lyapunov function later.

The traveling wave solution U is a stationary solution of the first mKdV equation with $c_{1,0} = V$. It fact, it is stationary with respect to all the higher-order flows. For example, it is a stationary solution of the second mKdV equation with

$$c_{2,0} = c_{2,1}(V^2 - 4E) + V^2 - 4E \tag{7.9}$$

for any value of $c_{2,1}$.

7.4 Stability

Consider the problem of nonlinear stability. The invariance of the mKdV equation under translation is represented by the Lie group

$$G = \mathbb{R},$$

which acts on $u(y, \tau)$ according to

$$T(g)u(y, \tau) = u(y + y_0, \tau), \quad g = y_0 \in G.$$

Stability is considered modulo this symmetry. We use the following definition.

Definition 1 *The equilibrium solution U is orbitally stable with respect to perturbations in $\mathbb{V}_0$ if for a given $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$||u(y, 0) - U(y)|| < \delta \Rightarrow \inf_{g \in G} ||u(y, \tau) - T(g)U(y)|| < \epsilon,$$

where $|| \cdot ||$ denotes the norm obtained through $\langle \cdot, \cdot \rangle$ on $\mathbb{V}_0$.

As a first step to prove orbital stability, we search for a Lyapunov function. For Hamiltonian systems, this is a constant of the motion $E(u)$ for which U is an unconstrained minimum:

$$\frac{\partial}{\partial \tau}E(u) = 0, \quad E'(U) = 0, \quad \langle v, E''(U)v \rangle > 0, \quad \forall v \in \mathbb{V}_0,\ v \neq 0.$$

We obtain an infinite number of candidate Lyapunov functions through the mKdV hierarchy.

Linearizing (7.6) about the equilibrium solution U gives

$$w_{t_n} = J\mathcal{L}_n w,$$

where $\mathcal{L}_n$ is the Hessian of $\hat{H}_n$ evaluated at the stationary solution. Through the same squared eigenfunction connection we have

$$2\Omega_n W(x) = J\mathcal{L}_n W(x), \tag{7.10}$$

where Ω_n is defined through

$$\psi(x, t_n) = e^{\Omega_n t_n} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \tag{7.11}$$

and due to the commuting property of the flows, the Lax hierarchy shares the common (complete) set of eigenfunctions $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ from before (still assuming the solution is stationary with respect to the first flow). Substituting (7.11) into the second equation in (7.8) determines a relationship between Ω_n and ζ, and in general, Ω_n^2 is a polynomial of degree $2n+1$ in ζ^2. When evaluated at a stationary solution of the mKdV equation, Ω_n^2 takes a degenerate form.

Theorem 4 *Let U be a stationary solution of the first mKdV equation. Then for all $n > 1$, the equation for the n-th algebraic curve becomes*

$$\Omega_n^2(\zeta) = p_n^2(\zeta)\Omega^2(\zeta),$$

where $p_n(\zeta)$ is a polynomial of degree $n-1$ in ζ^2. Furthermore, $p_n(\zeta)$ depends on the free parameters $c_{n,1}, \ldots, c_{n,n-1}$ in such a way that we have total control over the roots when considered as a function of ζ^2.

Proof. When evaluated at a stationary solution of the mKdV equation, all the higher-order flows become linearly dependent. The theorem is a direct consequence of this linear dependence and the functional form the Lax operators take as polynomials in ζ. The proof follows the same outline as in [20].

With the above facts established, we return to nonlinear stability. Just as we considered the norm of a solution modulo symmetries, we shall in effect do the same when considering a Lyapunov function. We have the following theorem due to [37, 38]:

Theorem 5 *Let U be a spectrally stable equilibrium solution of equation (7.1) such that the eigenfunctions W of the linear stability problem (3.3) form a basis for the space of allowed perturbations $\mathbb{V}_0$, on which the operator J has bounded inverse. Furthermore, suppose there exists an integer $n \geq 1$ and constants $c_{n,0}, \ldots, c_{n,n-1}$ such that the Hamiltonian for the n-th equation in the nonlinear hierarchy satisfies the following:*

1. *The kernel of* $\hat{H}''_n(U)$ *on* $\mathbb{V}_0$ *is spanned by the infinitesimal generators of the symmetry group* G *acting on* U*.*

2. *For all eigenfunctions not in the kernel of* $\hat{H}''_n(U)$

$$K_n(W) := \langle W, \hat{H}''_n(U)W \rangle > 0.$$

Then U *is orbitally stable with respect to perturbations in* $\mathbb{V}_0$*.*

Let us consider the implications of this theorem for the problem at hand:

- An application of the SCS basis lemma in [6] establishes that the eigenfunctions W form a basis for $L^2_{per}([-NT, NT])$, for any integer N, when the potential U is periodic in y with period $2T$.
- Due to translation invariance, we know that U_y is in the kernel of $\hat{H}''_1(U)$. It is well established [2, 35] that the kernel of $\hat{H}''_1(U)$ is spanned by U_y when considered on $\mathbb{V}_0$. This is the infinitesimal generator of G, ∂_y, acting on U. Furthermore, it is a direct consequence of the Riemann surface relations that the kernel of $\hat{H}''_1(U)$ is equal to the kernel of $\hat{H}''_n(U)$, for all $n \geq 1$.

What is left to verify is condition 2 in the nonlinear stability theorem, *i.e.*, to prove orbital stability we need to find an n such that

$$K_n = \langle W, \mathcal{L}_n W \rangle = \int_{-NT}^{NT} \overline{W} \mathcal{L}_n W \, dy \geq 0,$$

with equality obtained only on the kernel of $\mathcal{L}_n$, *i.e.*, only for $\Omega = 0$.

To calculate the higher-order K_n, we make use of the following. Assume our solution is an equilibrium solution of the n-th flow. Then from equation (7.10) we have

$$\mathcal{L}_n W = 2\Omega_n J^{-1} W.$$

This gives

$$K_n = \int_{-NT}^{NT} \overline{W} \mathcal{L}_n W \, dx = 2\Omega_n \int_{-NT}^{NT} \overline{W} J^{-1} W \, dy.$$

Using that U is a stationary solution of the second flow and substituting for Ω_n in the above gives

$$K_n(\zeta) = \Omega_n(\zeta) \frac{K_1(\zeta)}{\Omega(\zeta)} = p_n(\zeta) K_1(\zeta). \tag{7.12}$$

Therefore, when considering stationary solutions of the defocusing mKdV equation, one simply needs to calculate K_1 in order to calculate any of the higher order K_i. Let us do so. From (3.3) and the squared eigenfunction connection we have

$$\mathcal{L} W = 2\Omega J^{-1} W = \frac{\Omega}{i\zeta} \left(\beta^2 - \alpha^2\right).$$

This gives

$$\overline{W}\mathcal{L}W = \frac{\Omega}{i\zeta}\left(|\beta|^4 - |\alpha|^4 + (\overline{\alpha})^2\beta^2 - (\overline{\beta})^2\alpha^2\right).$$

Now

$$\alpha = -\gamma B, \quad \beta = \gamma(A - \Omega).$$

Using (5.12), we have (up to a multiplicative constant)

$$\gamma = \frac{1}{\sqrt{(\mathrm{Im}(A-\Omega))}} \exp\left(i\int \frac{u\mathrm{Re}(B)}{\mathrm{Im}(A-\Omega)}dy\right)\exp\left(\int i\zeta dy\right).$$

Therefore,

$$|\gamma|^2 = \frac{1}{\mathrm{Im}(A-\Omega)}.$$

Along with $\Omega^2 - A^2 - |B|^2 = 0$, the above implies

$$|\beta|^2 = \mathrm{Im}(A-\Omega), \quad |\alpha|^2 = \mathrm{Im}(A+\Omega), \quad \overline{\alpha}^2\beta^2 = -\overline{B}^2, \quad \overline{\beta}^2\alpha^2 = -B^2.$$

Therefore,

$$\int_{-NT}^{NT} \overline{W}\mathcal{L}_1 W dy = \int_{-NT}^{NT} \frac{\Omega}{i\zeta}\left(4A\Omega + 4\mathrm{Re}(B)\mathrm{Im}(B)i\right) dy.$$

Using that $\mathrm{Re}(B)\mathrm{Im}(B)$ is a total derivative gives

$$K_1 = 4\Omega^2 \int_{-NT}^{NT} \frac{A}{i\zeta}\, dy = 4\Omega^2 \int_{-NT}^{NT} \left(-V - 4\zeta^2 - 2u^2\right)\, dy.$$

Let us revisit the second condition of the nonlinear stability theorem. Using $\Omega^2 = A^2 + |B|^2$ we see that K_1 can be zero only if $\Omega \geq 0$. We also see that K_1 is linear in ζ^2, thus, it may change sign at some point ζ_0^2. Therefore, no stability conclusion can be drawn from K_1. However, let us go one flow higher and calculate K_2. A direct calculation gives

$$\Omega_2^2 = (-4\zeta^2 + V + c_{2,1})^2\Omega^2,$$

with $c_{2,0}$ as in (7.9). Therefore, choosing $c_{2,1} = 4\zeta_0^2 - V$ makes K_2 of definite sign.

We have proved the following theorem:

Theorem 6 (Orbital Stability) *There exists a constant $c_{2,1}$ such that K_2 is positive on the Lax spectrum. Therefore, all traveling wave solutions U of the defocusing mKdV equation are orbitally stable with respect to zero-average subharmonic perturbations,* i.e., *perturbations in the function space*

$$\mathbb{V}_0 = \left\{v \in H^3_{per}([-NT, NT]) : \int_{-NT}^{NT} v\, dx = 0\right\},$$

where $2T$ is the period of the initial condition and N is any integer.

Remark. There is no restriction on the spatial average of the traveling wave solution, only on the spatial average of the perturbation.

8 Focusing case

We now examine the focusing mKdV equation

$$u_t + 6u^2 u_x + u_{xxx} = 0. \tag{8.1}$$

8.1 Traveling wave solutions

We change to a moving coordinate frame

$$y = x - Vt, \quad \tau = t.$$

In the (y, τ) coordinates the focusing mKdV equation becomes

$$u_\tau - Vu_y + 6u^2 u_y + u_{yyy} = 0. \tag{8.2}$$

We look for stationary solutions $u_\tau = 0$. Letting $u(y, \tau) = U(y)$, stationary solutions satisfy the ordinary differential equation

$$-VU_y + 6U^2 U_y + U_{yyy} = 0. \tag{8.3}$$

Integrating (8.3) gives

$$-VU + 2U^3 + U_{yy} = C, \tag{8.4}$$

for some constant C. Multiplying (8.4) by U_y and integrating a second time gives

$$-\frac{V}{2}U^2 + \frac{1}{2}U^4 + \frac{1}{2}U_y^2 - CU = E, \tag{8.5}$$

for some constant E. Therefore, all stationary solutions $U(y)$ satisfy the first-order ordinary differential equation (8.5). Following the same procedure as for the defocusing case, we find that all periodic solutions are given by

$$U(y) = \frac{\pm\sqrt{2E}\wp'(\frac{1}{2}(y+y_0), g_2, g_3) + C(2\wp(\frac{1}{2}(y+y_0), g_2, g_3) - \frac{2}{3}V)}{\left(\wp(\frac{1}{2}(y+y_0), g_2, g_3) - \frac{V}{3} - 2\sqrt{-2E}\right)\left(\wp(\frac{1}{2}(y+y_0), g_2, g_3) - \frac{V}{3} + 2\sqrt{-2E}\right)}. \tag{8.6}$$

Here y_0 is an arbitrary shift in y determined by the initial conditions.

We now determine which values of V, C, and E give rise to bounded periodic solutions. Letting $P = U_y$ in (2.3), we have the first-order two-dimensional system

$$U_y = P, \quad P_y = VU - 2U^3 + C.$$

All fixed points (U_0, P_0) satisfy

$$P_0 = 0, \quad VU_0 - 2U_0^3 + C = 0. \tag{8.7}$$

After linearizing about $(U_0, 0)$, the resulting linear system has eigenvalues

$$\lambda = \pm\sqrt{V - 6U_0^2}. \tag{8.8}$$

We have two centers and a saddle when the discriminant

$$d = 8V^3 - 108C^2$$

is greater than zero, and one center when the discriminant is less than zero. Consider the following cases

- $V < 0$. This implies $d < 0$, giving one center and periodic solutions for all values of C. For $C = 0$ the solution reduces to $U(y) = \pm k\sqrt{\frac{V}{2k^2-1}}\mathrm{cn}\left(\sqrt{\frac{V}{2k^2-1}}y, k\right)$. This solution is imaginary for $k > \frac{1}{\sqrt{2}}$ and $\lim\limits_{k\to\frac{1}{\sqrt{2}}^-} U = \pm\infty$.

- $V > 0$, $|C| < \sqrt{\frac{8V^3}{108}}$. There are two centers and one saddle. Periodic solutions exist for all values of E, except for one value giving rise to two homoclinic orbits, corresponding to the saddle, see Fig. 8. For $C = 0$ the solution reduces to $U(y) = \pm k\sqrt{\frac{V}{2k^2-1}}\mathrm{cn}\left(\sqrt{\frac{V}{2k^2-1}}y, k\right)$. This solution is inside the homoclinic orbits for $|k| > 1$, and goes to zero as $k \to \infty$, using $\mathrm{cn}(y, k) = \mathrm{dn}(ky, \sqrt{\frac{1}{k^2}})$ [27]. It is outside the homoclinic orbits for $\frac{1}{\sqrt{2}} < k < 1$. For $k = 1$ it gives the soliton solution, which corresponds to the homoclinic orbit.

- $V > 0$, $|C| > \sqrt{\frac{8V^3}{108}}$. There is one center and the solutions are periodic solutions for all values of E.

8.2 Stability

The linear stability problem for the focusing mKdV equation takes the form

$$w_\tau = J\mathcal{L}w, \qquad J\mathcal{L}W = \lambda W, \qquad J = \partial_y, \qquad \mathcal{L} = -\partial_{yy} + V - 6U^2. \tag{8.9}$$

The squared eigenfunction connection is given by

$$w(y, \tau) = \psi_1(y, \tau)^2 - \psi_2(y, \tau)^2 = -\frac{1}{2i\zeta}\partial_y\left(\psi_2(y, \tau)^2 + \psi_1(y, \tau)^2\right)$$

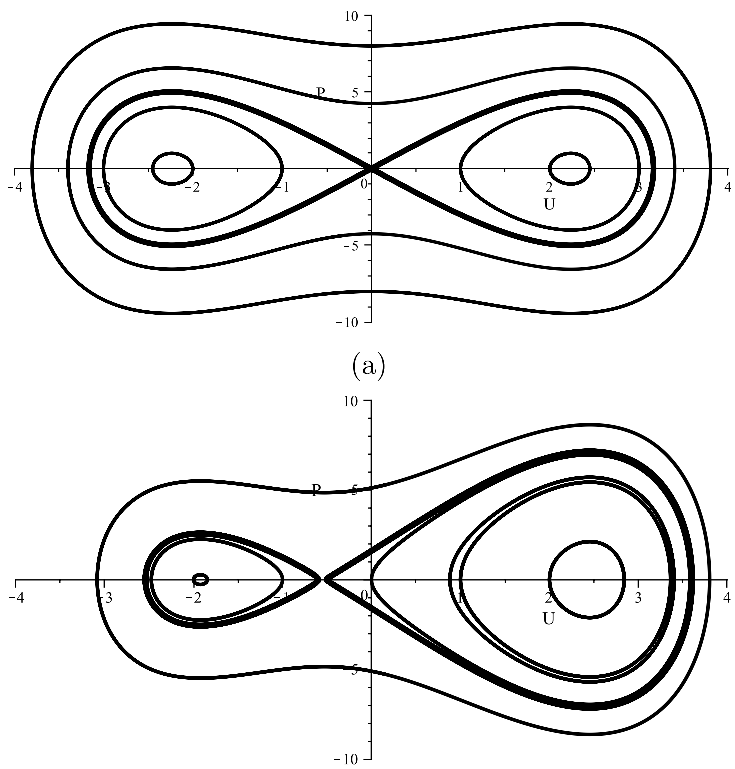
where ψ_1 and ψ_2 are obtained from the Lax pair representation

$$\psi_y = \begin{pmatrix} -i\zeta & u \\ -u & i\zeta \end{pmatrix}\psi, \qquad \psi_\tau = \begin{pmatrix} -iV\zeta - 4i\zeta^3 + 2i\zeta u^2 & 4\zeta^2 u + C + 2i\zeta u_y \\ -4\zeta^2 u - C + 2i\zeta u_y & iV\zeta + 4i\zeta^3 - 2i\zeta u^2 \end{pmatrix}\psi. \tag{8.10}$$

However, unlike the defocusing case, the associated spectral problem for ζ,

$$\begin{pmatrix} i\partial_y & -iu \\ -iu & -i\partial_y \end{pmatrix}\psi = \zeta\psi,$$

is not self-adjoint. Therefore, ζ is not necessarily restricted to the real axis. Several difficulties arise as a result:



(b)

Figure 8: (a) Typical (U, P) phase plane in the focusing case for $C = 0$. The two homoclinic orbits are in bold. (b) For $C \neq 0$, the homoclinic orbits change size relative to each other. Both homoclinic orbits persist until $|C| = \sqrt{\frac{8V^3}{108}}$, at which point only one center remains, surrounded by periodic orbits.

- As in the defocusing case, we separate variables and find a relationship between Ω and ζ:

$$\Omega^2 = -16\zeta^6 - 8V\zeta^4 + (-V^2 + 8E)\zeta^2 - C^2.$$

However, since ζ is not confined to the real axis, Ω is no longer restricted to $\mathbb{R} \cup i\mathbb{R}$.

- Looking for bounded eigenfunctions, one arrives at the necessary and sufficient condition

$$\left\langle \mathrm{Re}\left(i\zeta - \frac{A'}{A - \Omega} + \frac{uB}{A - \Omega} \right) \right\rangle = 0, \tag{8.11}$$

which is nearly identical to the condition obtained in the defocusing case. However, since ζ is no longer confined to the real axis, explicit analysis of (8.11) is much more difficult. This is the main stumbling block to examining stability in the focusing case. It should be noted that (8.11) still lends itself to numerical computation, which is simpler than numerically tackling the original spectral problem since it does not involve solving any differential equations.

We make several observations. For real ζ, the spectral problem for Ω is skew-adjoint. Therefore, for all real ζ in the Lax spectrum $\Omega(\zeta)$ is imaginary. For such ζ values the squared eigenfunction connection immediately implies the corresponding solution to the linear stability problem is bounded. However, numerical results suggest that ζ is not confined to the real axis (see Figs. 9 and 13).

For solutions lying within the homoclinic orbits in Fig. 8 (*i.e.*, the dn solutions when $C = 0$), it appears from the numerical results that the Lax spectrum is confined to the union of the real and imaginary axis (see Fig. 9). We hypothesize that this is due to an underlying symmetry of the spectral problem

$$\begin{pmatrix} i\partial_y & -iu \\ -iu & -i\partial_y \end{pmatrix}^2 \psi = \zeta^2 \psi.$$

Although this problem is not self-adjoint, it may have PT-symmetry (for instance, see [39]), which would confine ζ^2 to the real axis, and hence $\zeta \in \mathbb{R} \cup i\mathbb{R}$. Also, we see numerically that the dn solution appears to be stable with respect to subharmonic perturbations. Furthermore, under such assumptions the analytic formula for $\Omega(\zeta)$ predicts the band of higher spectral density on the imaginary axis seen in the numerically computed spectrum, see Figs. 10-12. In addition, if one assumes that $\zeta \in \mathbb{R} \cup i\mathbb{R}$ then the essential parts of the nonlinear stability calculations in the defocusing case carry over to the focusing case. In fact, one finds $c_{1,0} = V^2 + 4E$, $c_{2,0} = (V^2 + 4E)c_{2,1} + V^2 + 4E$, and

$$K_2 = (-4\zeta^2 + V + c_{2,1})K_1 = (-4\zeta^2 + V + c_{2,1}) \int_{-NT}^{NT} \left(-V - 4\zeta^2 + 2u^2\right) dy$$

Therefore, we should expect spectrally stable solutions to also be nonlinearly (orbitally) stable.

For the solutions outside the homoclinic orbits in Fig. 8, (*i.e.*, the cn solutions for $C = 0$), the Lax spectrum appears much more complicated (see Fig. 13), and numerical studies of the stability spectrum point to spectral instability, see Fig. 14 and Figure 1 in [2]. It is interesting to note that in numerical investigations the cn solutions appear stable with respect to periodic perturbations of the same period, but unstable with respect to subharmonic perturbations [2].

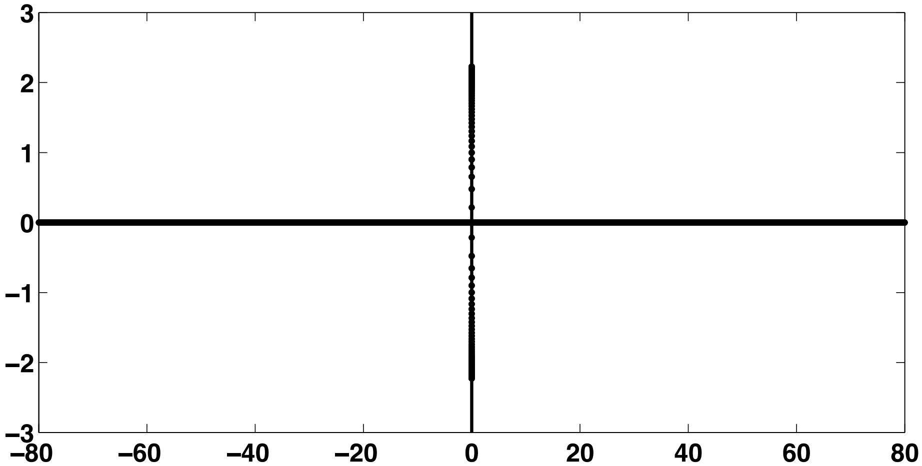


Figure 9: Numerically computed Lax spectrum for the traveling wave solution (8.6) with $V = 10$, $C = 0$, and $k = 1.8$ using Hill's method with 81 Fourier modes and 49 different Floquet exponents.

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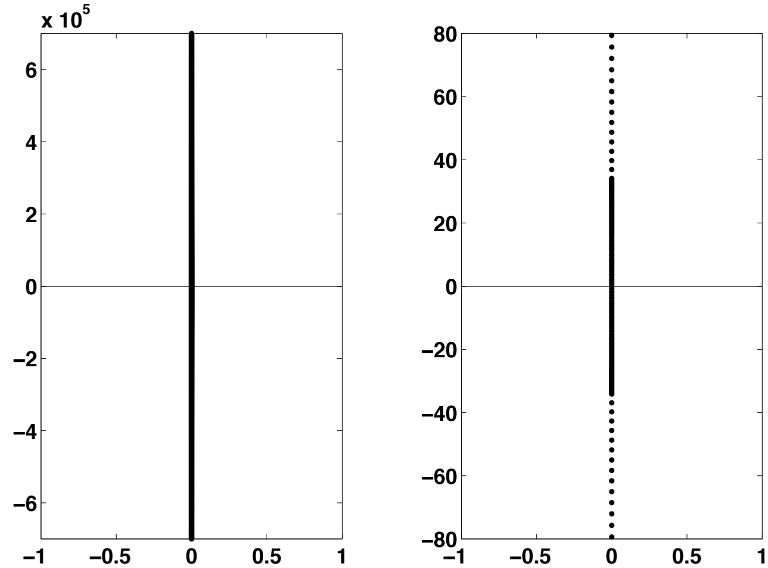

(a) (b)

Figure 10: (a) The numerically computed spectrum for the traveling wave solution (8.6). The parameter values are identical to those of Fig. 9. (b) A blow-up of (a) around the origin, showing a band of higher spectral density.

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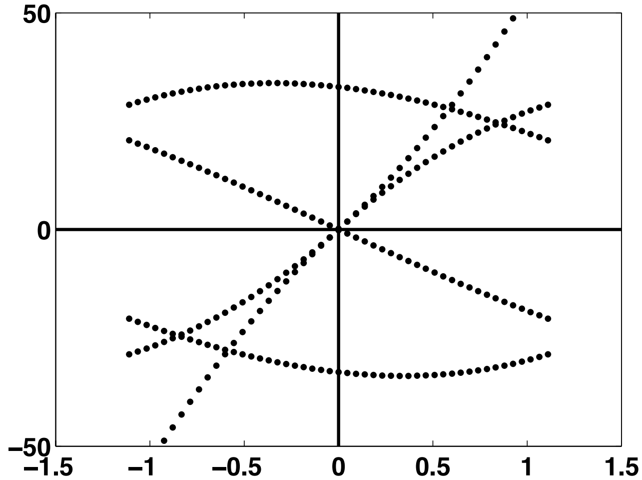

Figure 11: The imaginary part of λ as a function of μ, demonstrating the higher spectral density, for the traveling wave solution (8.6). The parameter values are identical to those of Fig. 9.

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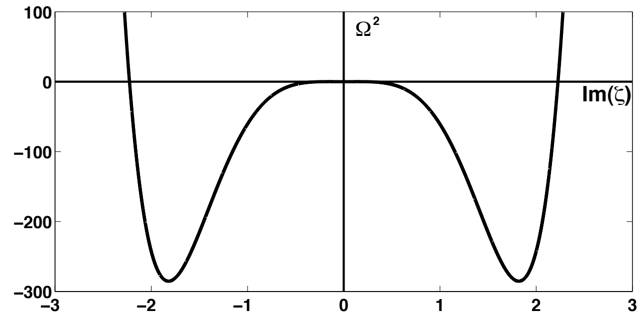
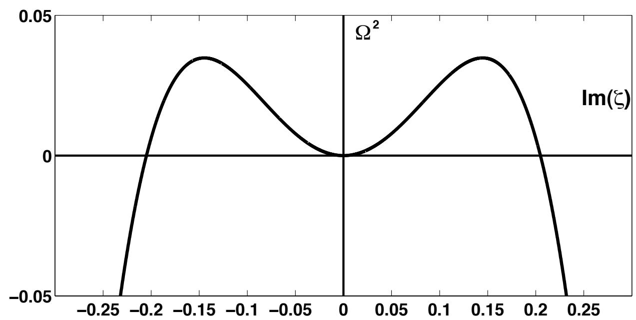


(a)



(b)

Figure 12: (a) Ω^2 as a function of $\text{Im}(\zeta)$ for the traveling wave solution (8.6) with the same parameter values as Fig. 9. (b) A blow-up of (a) around the origin.

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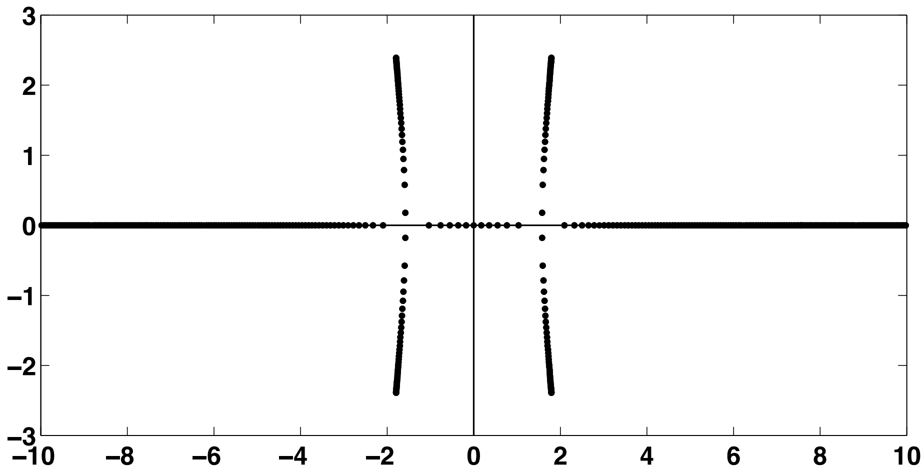

Figure 13: The imaginary part of λ as a function of μ, demonstrating the higher spectral density, for the traveling wave solution (8.6). for the traveling wave solution with $V = 10$, $C = 0$, and $k = .8$ using Hill's method with 81 Fourier modes and 49 different Floquet exponents.

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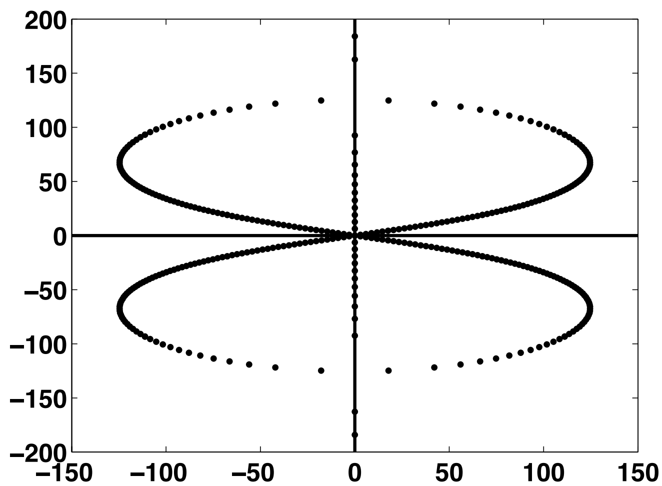

Figure 14: The numerically computed spectrum for the traveling wave solution (8.6). The parameter values are identical to those of Fig. 13.