On the spectral and orbital stability of spatially periodic stationary solutions of generalized Korteweg-de Vries equations

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This paper is dedicated to Walter Craig on the occasion of his 60^{th} *birthday.*

Abstract In this paper we generalize previous work on the spectral and orbital stability of waves for infinite-dimensional Hamiltonian systems to include those cases for which the skew-symmetric operator $\mathscr{J}$ is singular. We assume that $\mathscr{J}$ restricted to the orthogonal complement of its kernel has a bounded inverse. With this assumption and some further genericity conditions we (a) derive an unstable eigenvalue count for the appropriate linearized operator, and (b) show that the spectral stability of the wave implies its orbital (nonlinear) stability, provided there are no purely imaginary eigenvalues with negative Krein signature. We use our theory to investigate the (in)stability of spatially periodic waves to the generalized KdV equation for various power nonlinearities when the perturbation has the same period as that of the wave. Solutions of the integrable modified KdV equation are studied analytically in detail, as well as solutions with small amplitudes for higher-order pure power nonlinearities. We conclude by studying the transverse stability of these solutions when they are considered as planar solutions of the generalized KP-I equation.

1 Introduction

The study of the stability of spatially periodic stationary solutions of nonlinear wave equations has seen different advances the past few years. There are advances both in the numerical investigation of spectral stability [15, 43], as well as in the analytical

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study of spectral and orbital stability (see [6, 8, 19, 20, 25, 41] and the references therein). We focus specifically on the study of the stability of periodic solutions of Hamiltonian partial differential equations, as in [25]. However, the results also apply to the study of localized waves for systems in which the appropriate linearized system has a compact resolvent, e.g., the Gross-Pitaevski equation (see [37] and the references therein). The equations of interest are written abstractly as

$$u_t = \mathscr{J}\mathscr{E}'(u), \qquad u(0) = u_0, \tag{1}$$

on a Hilbert space X, where $\mathscr{J} : X \to \text{range}(\mathscr{J}) \subset X$ is skew symmetric, and $\mathscr{E} : X \to \mathbb{R}$ is a C^2-functional. In previous works (see [25, 35] and the references therein) it was assumed that $\mathscr{J}$ is nonsingular with bounded inverse. We do not make that assumption here. We allow $\ker(\mathscr{J})$ to be nontrivial; however, we do assume that $\mathscr{J}|_{\ker(J)^\perp}$ has a *bounded* inverse.

We are interested in the spectral and orbital stability of spatially periodic waves to (1). The waves are realized as critical points of a constrained energy, and the stability of the waves is determined by closely examining the Hessian, say $\mathscr{L}$, of the constrained energy. It will henceforth be assumed that $\mathrm{n}(\mathscr{L}) < \infty$, where the notation $\mathrm{n}(\mathscr{S})$ is used to denote the number of negative eigenvalues (counting multiplicities) of the self-adjoint operator $\mathscr{S}$. If $\mathscr{J}$ is nonsingular with bounded inverse, then it was seen in [21] that there is a symmetric matrix D such that if $\mathrm{n}(\mathscr{L}) - \mathrm{n}(D) = 0$, then the wave is orbitally stable. The matrix D is intimately related to the conserved quantities of (1) which are generated by its group invariances. It was shown in [25, 36] and the references therein that if this difference is positive, then there exists a close relationship between this difference and the structure of $\sigma(\mathscr{J}\mathscr{L})$, where $\mathscr{J}\mathscr{L}$ is the linearization of (1) about the critical point. As we demonstrate, this formula must be modified if $\mathscr{J}$ is singular. In particular, the formula must take into account the fact that the only nontrivial flow of (1) occurs on $\ker(\mathscr{J})^\perp$ (see Theorem 1 for a precise statement).

A concrete example to which the theory is applicable is the determination of the orbital (in)stability of spatially periodic stationary solutions of the generalized Korteweg-de Vries (gKdV) equations ($p \in \mathbb{N}_0$)

$$u_t = \partial_x\left(-u_{xx} \pm u^{p+1}\right). \tag{2}$$

Here $\mathscr{J} = \partial_x$, and $u \in \ker(\mathscr{J})^\perp$ if $\overline{u} = 0$, where $\overline{u}$ represents the spatial average. Note that the $\pm$ sign is irrelevant when p is odd, but not so when p is even. With $p = 1$ equation (2) is the integrable KdV equation, and with $p = 2$ it is the integrable modified KdV (mKdV) equation. For $p \geq 3$ the equation is not integrable by any definition. If $p \in \mathbb{R}^+$ is not an integer in the gKdV, then unless we know that solutions are always positive the nonlinear term must be replaced via $u^{p+1} \mapsto |u|^p u$. When we consider the integrable cases we can do all calculations explicitly for even large amplitude waves, providing examples that are far more robust than for the non-integrable cases. The stability theory we develop is applicable for *all* cases, as long as there is a local well-posedness theory for the initial value problem.

In this paper we restrict ourselves to proving the orbital stability of spatially periodic stationary solutions of gKdV for $p \geq 2$ with respect to perturbations of the same period. This corresponds to the case of zero Floquet exponent in [25], for which the determination of $\sigma(\mathscr{J}\mathscr{L})$ was left open there because it results in $\mathscr{J}$ being singular. For the mKdV equation the results depend on the period of the wave as well as on which sign is considered. A few non-integrable examples are discussed as well in the regime of small amplitude waves.

The paper is organized as follows. In Section 2 we discuss the spectral theory and the orbital stability theory for relative equilibria to (1) under the assumption that $\ker(\mathscr{J})$ is nontrivial. In Section 3 we apply the results of the theory to (2) in the case of periodic perturbations of the same period as the underlying solution. We consider the case of the mKdV equation ($p = 2$) in detail for three of its periodic solutions. We find that two of the solutions are orbitally stable, whereas the stability of the third solution depends upon its period. We finish the section by considering the case of small solutions for any $p \in \mathbb{N}$, and find that all of the solutions under consideration are orbitally stable. Finally, in Section 4 we consider the transverse stability of the gKdV solutions when they are considered as solutions of the generalized KP-I equation.

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Addendum. This paper has a long history. The original work was completed in 2009, and a major revision was done in 2010. In its original form this paper was joined with the companion paper by Deconinck and Kapitula [14]. At the suggestion of an editor the two papers were severed, and the second was soon thereafter published. This paper was submitted elsewhere, where after some time and consideration the editors decided that it was not appropriate for the journal. We subsequently submitted this paper to another journal in the fall of 2011. As far as we can tell, the paper then fell through the cracks in the editor/referee system of that journal (we joked that it had fallen into a “Refereeing Purgatory”). While the paper as of Fall 2014 is still unpublished, the work has been noticed in the community. From the time of original submission to Fall 2014 the paper has been referenced to at least 31 times (according to Google Scholar). Some of the referencing papers are Benzoni-Gavage [4], Bottman et al. [7], Bronski et al. [9, 10], Chen et al. [12], Farah and Scialom [17], Hakkaev et al. [23], Johnson [28], Johnson and Zumbrun [29, 30], Johnson et al. [31], Kapitula and Promislow [32], Kapitula and Stefanov [34], Nivala and Deconinck [39], Pava and Natali [40], Pelinovsky [42], Stanislavova and Stefanov [45]. The results and ideas presented herein are also a highlighted example in the recent book on stability theory by Kapitula and Promislow [33, Chapter 6.1.2].

2 Theoretical results

Much of the following discussion can be found in [35, Section 2]. It is included here for the sake of completeness. Let U, V, X denote three real Hilbert spaces with $U \subset X \subset V$ being dense and continuous embeddings. Throughout this paper, we use only the scalar product $\langle \cdot, \cdot \rangle$ on the space V. In particular, we have $U \subset X \subset V \subset X^*$ where X^* denotes the dual space of X. Adjoint operators are always taken with respect to the scalar product on V.

We are interested in relative equilibria of (1). These are solutions of (1) whose functional form is in some sense invariant under the dynamics. Typical examples include traveling wave solutions which are invariant under translation in x (as for (2)), or which are invariant under multiplication by a unitary scalar (as for nonlinear Schrödinger-type equations). In order to make this notion of invariance precise so as to formally define what is meant by relative equilibria, we need a few elements from the theory of Lie groups and Lie algebras.

Let G be a finite-dimensional abelian Lie group with Lie algebra $\mathfrak{g}$. Denote by $\exp(\omega) = \mathrm{e}^{\omega}$ for $\omega \in \mathfrak{g}$ the exponential map from $\mathfrak{g}$ into G. Next, assume that $T : G \to L(V)$ is a unitary representation of G on V so that $T'(e)$ maps $\mathfrak{g}$ into the space of closed skew-symmetric operators on V with domain X. Since $\mathscr{J}$ is nonsingular with bounded inverse on $\ker(\mathscr{J})^{\perp}$, we make the additional assumption that:

Assumption 1. *The derivative of the group action $T(\cdot)$ satisfies $T_\omega : X \mapsto \ker(\mathscr{J})^{\perp}$.*

In Assumption 1 the notation $T_\omega := T'(e)\omega$ is used for the linear skew-symmetric operator which is the generator of the semigroup $T(\mathrm{e}^{\omega t})$. Note that T_ω is also linear in $\omega \in \mathfrak{g}$. We assume that U is contained in the domain of T_ω^2. The group orbit Gu of an element $u \in X$ is defined by $Gu := \{T(g)u;\ g \in G\}$.

2.1 Existence of relative equilibria

We need two compatibility assumptions. First, we assume that $\mathscr{E}$ is invariant under G so that

$$\mathscr{E}(T(g)u) = \mathscr{E}(u)$$

for all $u \in X$ and all $g \in G$. Second, there is a type of commutation between the group action and the skew operator,

$$T(g)\mathscr{J} = \mathscr{J}T(g^{-1})^*, \quad \text{all } g \in G. \tag{3}$$

As a consequence of Assumption 1 we can define the bounded functional $M_\omega : X \mapsto \mathbb{R}$ by

$$M_\omega(u) := \frac{1}{2}\langle \mathscr{J}^{-1} T_\omega u, u\rangle, \quad \omega \in \mathfrak{g}.$$

Its second derivative $M''_\omega(u) = \mathscr{J}^{-1}T_\omega : X \mapsto \ker(\mathscr{J})^\perp$ is a bounded symmetric linear operator by (3). Note that $M_\omega(T(g)u) = M_\omega(u)$, i.e., that M_ω is a conserved functional under the group action.

We are now positioned to define what we mean by relative equilibria of (1). These are solutions $u(t)$ whose time orbit is contained in the group orbit Gu_0 so that $u(t) \in Gu(0)$ for all t. Thus, $\phi \in X$ is a relative equilibrium of (1) if and only if there is an $\omega \in \mathfrak{g}$ so that $u(t) = T(\mathrm{e}^{\omega t})\phi$ satisfies (1). Substituting the ansatz $u(t) = T(\mathrm{e}^{\omega t})\phi$ into (1) we get

$$T_\omega \phi = \mathscr{J}\mathscr{E}'(\phi), \quad \omega \in \mathfrak{g}. \tag{4}$$

As a consequence of Assumption 1 both sides are in $\ker(\mathscr{J})^\perp$. Let the operator $P_\mathscr{J} : X \mapsto \ker(\mathscr{J})^\perp$ be the orthogonal projection onto the range of $\mathscr{J}$. Since $P_\mathscr{J} T_\omega = T_\omega P_\mathscr{J}$, (4) is equivalent to

$$P_\mathscr{J}\left[\mathscr{E}'(\phi) - \mathscr{J}^{-1}T_\omega\phi\right] = 0. \tag{5}$$

Note that (5) implies that

$$\mathscr{E}'(\phi) - \mathscr{J}^{-1}T_\omega\phi \in \ker(\mathscr{J}).$$

In conclusion, we see from (5) that $\phi \in X$ is a relative equilibrium if and only if $P_\mathscr{J}\mathscr{H}'_\omega(\phi) = 0$, where

$$\mathscr{H}_\omega := \mathscr{E} - M_\omega, \quad \omega \in \mathfrak{g}.$$

Note that this does not necessarily imply that ϕ is a critical point of $\mathscr{H}_\omega$. We assume throughout that there exists a smooth family of bound states:

Assumption 2 (Relative equilibria). *There exists a non-empty open set $\Omega \subset \mathfrak{g}$ and a $\mathscr{C}^1$ function $\phi : \Omega \to U$, $\omega \mapsto \phi_\omega$ such that ϕ_ω is a relative equilibrium of (1), i.e., $P_\mathscr{J}\mathscr{H}'_\omega(\phi_\omega) = 0$ for each $\omega \in \Omega$. We assume that the isotropy subgroups $\{g \in G;\ T(g)\phi_\omega = \phi_\omega\}$ are discrete for all ω so that the group orbits $G\phi_\omega$ satisfy* $\dim(G\phi_\omega) = \dim(G)$.

Remark 1. Since G is abelian, the entire group orbit $T(g)\phi_\omega$ with $g \in G$ consists of relative equilibria with time evolution $T(\mathrm{e}^{\omega t})$.

2.2 Formulation of the evolution equation

Without loss of generality we will henceforth assume that the relative equilibrium is a critical point of $\mathscr{H}_\omega$. Indeed, if the relative equilibrium satisfies

$$\mathscr{H}'_\omega(\phi_\omega) = z, \quad z \in \ker(\mathscr{J}),$$

then the mapping

$$\widetilde{\mathscr{H}_\omega}(u) = \mathscr{H}_\omega(u) - \langle z, u\rangle$$

yields that the relative equilibrium is a critical point of $\widetilde{\mathscr{H}_\omega}(u)$. The self-adjoint Hessian of the energy at the relative equilibrium ϕ_ω is defined as

$$\mathscr{L} := \mathscr{H}_\omega''(\phi_\omega) : U \mapsto V. \tag{6}$$

Note that the linearization of (1) around the relative equilibrium ϕ_ω in the "co-moving" frame is given by $\mathscr{J}\mathscr{L}$. Due to the invariance of H_ω under the abelian group G, one has that the tangent space of the group orbit $G\phi_\omega$ at ϕ_ω is contained in $\ker(\mathscr{L})$. As a consequence of Assumption 2 it follows from [**21**, p. 314] that

$$\ker(\mathscr{L}) := \operatorname{span}\{T_\alpha\phi_\omega;\ \alpha \in \mathfrak{g}\}, \tag{7}$$

Upon writing $X = \ker(\mathscr{J}) \oplus H_1$, where $H_1 := \ker(\mathscr{J})^\perp$, let $Q_{\mathscr{J}} := \mathbb{1} - P_{\mathscr{J}} : X \mapsto \ker(\mathscr{J})$ be the orthogonal projection onto $\ker(\mathscr{J})$, where $P_{\mathscr{J}}$ was defined in the previous subsection. One may rewrite the system

$$u_t = \mathscr{J}\mathscr{H}_\omega'(\phi_\omega + u), \quad u(0) = u_0, \tag{8}$$

for which the relative equilibrium ϕ_ω satisfies $P_{\mathscr{J}}\mathscr{H}_\omega'(\phi_\omega) = 0$, as the system,

$$\begin{aligned} &\partial_t P_{\mathscr{J}} u = \mathscr{J} P_{\mathscr{J}} \mathscr{H}_\omega'(\phi_\omega + P_{\mathscr{J}} u + Q_{\mathscr{J}} u), \quad P_{\mathscr{J}} u(0) = P_{\mathscr{J}} u_0 \\ &\partial_t Q_{\mathscr{J}} u = 0, \quad Q_{\mathscr{J}} u(0) = Q_{\mathscr{J}} u_0. \end{aligned} \tag{9}$$

From (9) it is seen that $Q_{\mathscr{J}} u(t) = Q_{\mathscr{J}} u(0)$ for all $t \geq 0$; in other words, nontrivial evolution of the initial data only occurs in H_1. Consider an initial condition for (9) which satisfies $Q_{\mathscr{J}} u_0 = 0$. One sees from (9) that $Q_{\mathscr{J}} u(t) = 0$ for all $t > 0$. Using $P_{\mathscr{J}} u = u$ for all $t \geq 0$, the evolution equation of interest is given by

$$u_t = \mathscr{J} P_{\mathscr{J}} \mathscr{H}_\omega'(\phi_\omega + u), \quad u(0) = u_0. \tag{10}$$

Since the evolution occurs on H_1, $\mathscr{J}$ is now skew symmetric with *bounded* inverse.

2.3 The eigenvalue count

The spectral problem associated with the stability problem for relative equilibria of (10) is

$$\mathscr{J}\mathscr{L}|_{H_1} u = \lambda u, \quad \mathscr{L}|_{H_1} := P_{\mathscr{J}} \mathscr{L} P_{\mathscr{J}}. \tag{11}$$

By assumption $\mathscr{J}$ is skew-symmetric, and it has bounded inverse on H_1. It is clear that $\mathscr{L}$, and hence $\mathscr{L}|_{H_1}$, are self-adjoint. In what follows, the following assumptions are used:

Assumption 3. *It is assumed that:*

1. $\mathscr{L}$ *has a compact resolvent, and* $\sigma(\mathscr{L}) \cap \mathbb{R}^-$ *is a finite set,*
2. *There is a self-adjoint operator* $\mathscr{L}_0$ *with compact resolvent such that:*
   a. $\mathscr{L} = \mathscr{L}_0 + \mathscr{A}$, *where* $\mathscr{A}$ *is* $\mathscr{L}_0$*-compact and satisfies*

$$\|\mathscr{A} u\| \leq a\|u\| + b\| \, |\mathscr{L}_0|^r u\|,$$

   for some positive constants a, b *and* $r \in [0, 1)$,

   b. *The increasing sequence of nonzero eigenvalues* ω_j *of* $\mathscr{L}_0$ *satisfies*

$$\sum_{j=1}^{\infty} |\omega_j|^{-s} < \infty,$$

   for some $s \geq 1$,

   c. *There exists a subsequence of eigenvalues* $\{\omega_{n_k}\}_{k\in\mathbb{N}}$ *and constants* $c > 0$ *and* $r' > r$ *such that*

$$\omega_{n_k+1} - \omega_{n_k} \geq c\, \omega_{n_k+1}^{r'}.$$

3. $\mathrm{Im}(\mathscr{J}) = \mathrm{Im}(\mathscr{L}|_{H_1}) = 0$, *where* Im *denotes the imaginary part.*

Remark 2. Assumption 3(a)-(b) are also assumed in [**25**]. It is known that these assumptions are not absolutely necessary (see [**35**] where the assumptions are removed), but they are satisfied for the applications we have in mind. It is clear that Assumption 3(a)-(b) for $\mathscr{L}$ imply that $\mathscr{L}|_{H_1}$ has the same properties. As seen in [**25**], Assumption 3(c) is not necessary, and it is assumed here only for the sake of simplicity. As a consequence of this assumption, eigenvalues for (11) come in quartets $\{\pm\lambda, \pm\lambda^*\}$.

In contrast to [**25**] we do not assume that $\mathscr{L}|_{H_1}$ is nonsingular. In fact, as a consequence of Assumption 1 one has $\ker(\mathscr{L}) \subset H_1$; consequently, $\ker(\mathscr{L}) \subset \ker(\mathscr{L}|_{H_1})$. One has that $\ker(\mathscr{L}) \subset H_1$ and $\ker(\mathscr{J}) \subset \ker(\mathscr{L})^\perp$. Upon defining

$$\ker_{\mathrm{a}}(\mathscr{L}) := \{z \in \ker(\mathscr{J}) : \mathscr{L}^{-1} z \in H_1\},$$

one has

$$\ker(\mathscr{L}|_{H_1}) = \ker(\mathscr{L}) \oplus \ker_{\mathrm{a}}(\mathscr{L}).$$

The definition of $\ker_{\mathrm{a}}(\mathscr{L})$ makes sense because $\ker(\mathscr{J}) \subset \ker(\mathscr{L})^\perp$. Let $\ker(\mathscr{J}) = \mathrm{Span}(z_1, \ldots, z_m)$, where $\{z_1, \ldots, z_m\}$ are orthonormal. Let $J \in \mathbb{C}^{m\times m}$ be the Hermitian matrix whose entries are given by

$$J_{ij} = \langle z_i, \mathscr{L}^{-1} z_j \rangle, \tag{12}$$

i.e., J is a matrix representation for the quadratic form $\langle u, \mathscr{L}^{-1}|_{\ker(\mathscr{J})} u\rangle$. Note that

$$\dim[\ker(J)] = \dim[\ker_{\mathrm{a}}(\mathscr{L})];$$

hence, $\ker(\mathscr{L}) = \ker(\mathscr{L}|_{H_1})$ if and only if J is nonsingular. This is henceforth assumed.

With J nonsingular, one has

$$\ker(\mathscr{L}|_{H_1})^{\perp} = \ker(\mathscr{L})^{\perp} \cap H_1.$$

Using the notation

$$\mathscr{A}(S) := \{y : y = \mathscr{A}s \text{ for some } s \in S\},$$

one has that as a consequence of [35, equation (3.2)],

$$\mathscr{J}^{-1}(\ker(\mathscr{L}|_{H_1})) \subset \ker(\mathscr{L}|_{H_1})^{\perp}. \tag{13}$$

Since $\ker(\mathscr{L}|_{H_1}) \subset H_1$ one can refine (13) to say that $\mathscr{J}^{-1}(\ker(\mathscr{L}|_{H_1})) \subset \ker(\mathscr{L}|_{H_1})^{\perp} \cap H_1$; hence, there is a generalized eigenspace $X_{\mathscr{L}} \subset H_1$ such that

$$\mathscr{L}|_{H_1} X_{\mathscr{L}} = \mathscr{J}^{-1}(\ker(\mathscr{L}|_{H_1})).$$

Define

$$D_{ij} := \langle y_i, \mathscr{L}|_{H_1} y_j \rangle, \tag{14}$$

where $\{y_i\} \subset X_{\mathscr{L}}$ is any basis for $X_{\mathscr{L}}$. If D is nonsingular, then by the Fredholm alternative

$$\mathrm{gker}(\mathscr{J}\mathscr{L}|_{H_1}) = \ker(\mathscr{L}|_{H_1}) \oplus X_{\mathscr{L}}, \tag{15}$$

with $\mathrm{m_g}(0) = \dim(\ker(\mathscr{L}|_{H_1}))$ and $\mathrm{m_a}(0) = 2\mathrm{m_g}(0)$. Here $\mathrm{m_g}(\lambda)$ is the geometric multiplicity of the eigenvalue λ, $\mathrm{m_a}(\lambda) \geq \mathrm{m_g}(\lambda)$ is the algebraic multiplicity, and $\mathrm{gker}(\mathscr{A})$ refers to the generalized kernel of the operator $\mathscr{A}$.

In order to apply the results of [25] we must recast the eigenvalue problem in the appropriate subspace so that the operator $\mathscr{L}|_{H_1}$ no longer has a nontrivial kernel. Let $P_1 : H_1 \mapsto \ker(\mathscr{L}|_{H_1})^{\perp}$ be the orthogonal projection, and set $Q_1 := \mathbb{1} - P_1$. For $\lambda \neq 0$ rewrite (11) as

$$\mathscr{L}|_{H_1} u = \lambda \mathscr{J}^{-1} u. \tag{16}$$

Upon using the projections P_1, Q_1 one sees that (16) is equivalent to the system,

$$\begin{aligned} P_1 \mathscr{L}|_{H_1} P_1 \cdot P_1 u &= \lambda P_1 \mathscr{J}^{-1} P_1 u + \lambda P_1 \mathscr{J}^{-1} Q_1 u \\ 0 &= \lambda Q_1 \mathscr{J}^{-1} P_1 u + \lambda Q_1 \mathscr{J}^{-1} Q_1 u. \end{aligned} \tag{17}$$

As a consequence of (13) one has that $Q_1 \mathscr{J}^{-1} Q_1 u = 0$; thus, from the second line of (17) one has for $\lambda \neq 0$ the identities,

$$\begin{aligned} P_1 \mathscr{J}^{-1} P_1 u &= (P_1 + Q_1) \mathscr{J}^{-1} P_1 u = \mathscr{J}^{-1} P_1 u \\ P_1 \mathscr{J}^{-1} Q_1 u &= (P_1 + Q_1) \mathscr{J}^{-1} Q_1 u = \mathscr{J}^{-1} Q_1 u. \end{aligned}$$

The first line of (17) can be rewritten as

$$\mathscr{J}P_1\mathscr{L}|_{H_1}P_1\cdot P_1u=\lambda P_1u+\lambda Q_1u, \tag{18}$$

which, upon using $P_1Q_1=0$, becomes

$$P_1\mathscr{J}P_1\cdot P_1\mathscr{L}|_{H_1}P_1\cdot P_1u=\lambda P_1u. \tag{19}$$

In other words, when looking for nonzero eigenvalues (11) is equivalent to (19). Once (19) is solved, then Q_1u is uniquely determined by (18). It is an exercise to show that the same conclusion holds when considering generalized eigenfunctions.

By the Fredholm alternative, the solvability of (19) requires that

$$P_1u\in\ker(P_1\mathscr{L}|_{H_1}P_1\cdot P_1\mathscr{J}P_1)^{\perp}.$$

Since $P_1\mathscr{L}|_{H_1}P_1$ is nonsingular on $\ker(\mathscr{L}|_{H_1})^{\perp}$, the above is equivalent to requiring that

$$P_1u\in\ker(P_1\mathscr{J}P_1)^{\perp}=[\mathscr{J}^{-1}(\ker(\mathscr{L}|_{H_1}))]^{\perp}.$$

Next, define the orthogonal projection $P_2:H_1\mapsto[\mathscr{J}^{-1}(\ker(\mathscr{L}|_{H_1}))]^{\perp}$ and set $Q_2=\mathbb{1}-P_2$. Note that as a consequence of (13), $P_1P_2=P_2P_1$, and that $P_2(P_1\mathscr{J}P_1)=(P_1\mathscr{J}P_1)P_2$. Applying the operator P_2 to (19) yields

$$\Pi\mathscr{J}\Pi\cdot\Pi\mathscr{L}|_{H_1}\Pi\cdot\Pi u+\Pi\mathscr{J}\Pi\cdot\Pi\mathscr{L}|_{H_1}\Pi\cdot Q_2P_1u=\lambda\Pi u,\quad \Pi:=P_1P_2. \tag{20}$$

Applying the operator Q_2 to (19) and using the fact that $Q_2:H_1\mapsto\ker(P_1\mathscr{J}P_1)$ one gets

$$0=\lambda Q_2P_1u. \tag{21}$$

Thus, by (21) for nonzero λ (20) becomes

$$\Pi\mathscr{J}\Pi\cdot\Pi\mathscr{L}|_{H_1}\Pi\cdot\Pi u=\lambda\Pi u, \tag{22}$$

which is equivalent to (19) and hence (11).

Set

$$R_{\mathscr{L}}:=[\ker(\mathscr{L})\oplus\mathscr{J}^{-1}(\ker(\mathscr{L}))]^{\perp},$$

and note that $\Pi:R_{\mathscr{L}}\mapsto R_{\mathscr{L}}$ and $\Pi\mathscr{J}\Pi\cdot\Pi\mathscr{L}|_{H_1}\Pi:R_{\mathscr{L}}\mapsto R_{\mathscr{L}}$. By construction it is clear that $\Pi\mathscr{J}\Pi:R_{\mathscr{L}}\mapsto R_{\mathscr{L}}$ is nonsingular. In addition,

$$\ker(\Pi\mathscr{L}|_{H_1}\Pi)=\ker(\mathscr{L}|_{H_1})\oplus X_{\mathscr{L}},$$

and since D is nonsingular one has that

$$\ker(\Pi\mathscr{L}|_{H_1}\Pi)\cap R_{\mathscr{L}}=\{0\}.$$

This follows from $P_1X_{\mathscr{L}}=X_{\mathscr{L}}$ and the fact that for any $y\in X_{\mathscr{L}}$,

$$P_2y=y-\sum\langle y,\mathscr{J}^{-1}\ell_i\rangle\mathscr{J}^{-1}\ell_i=y-\sum\langle y,\mathscr{L}|_{H_1}y_i\rangle\mathscr{J}^{-1}\ell_i,$$

where $\{\ell_i\} \subset \ker(\mathscr{L}|_{H_1})$ are chosen so that $\{\mathscr{J}^{-1}\ell_i\}$ is an orthonormal basis for $\mathscr{J}^{-1}(\ker(\mathscr{L}|_{H_1}))$, and $\mathscr{J}\mathscr{L}|_{H_1}y_i = \ell_i$. In conclusion, both of the operators $\Pi\mathscr{J}\Pi$ and $\Pi\mathscr{L}|_{H_1}\Pi$ are nonsingular when acting on $R_{\mathscr{L}}$. Furthermore, these operators satisfy Assumption 3.

Before continuing our study of the eigenvalue problem in (22), we need to briefly discuss the notion of the Krein signature of purely imaginary eigenvalues. For a nonzero purely imaginary eigenvalue λ let $E_\lambda \subset R_{\mathscr{L}}$ be its associated eigenspace. The Krein signature of λ is determined via the nonsingular quadratic form $\langle w, (\mathscr{L}|_{H_1})|_{E_\lambda} w\rangle$. For a self-adjoint operator $\mathscr{A}$, let $\mathrm{n}(\langle w, \mathscr{A}w\rangle)$ denote the dimension of the maximal subspace for which $\langle w, \mathscr{A}w\rangle < 0$. The eigenvalue is said to have negative Krein signature if

$$k_{\mathrm{i}}^-(\lambda) := \mathrm{n}(\langle w, (\mathscr{L}|_{H_1})|_{E_\lambda} w\rangle) \geq 1;$$

otherwise, if $k_{\mathrm{i}}^-(\lambda) = 0$, then the eigenvalue is said to have positive Krein signature. If the eigenvalue λ is geometrically and algebraically simple with eigenfunction u_λ, then

$$k_{\mathrm{i}}^-(\lambda) = \begin{cases} 0, & \langle u_\lambda, \mathscr{L}|_{H_1} u_\lambda\rangle > 0 \\ 1, & \langle u_\lambda, \mathscr{L}|_{H_1} u_\lambda\rangle < 0 \end{cases}.$$

We set the total Krein signature to be

$$k_{\mathrm{i}}^- := \sum_{\lambda \in \mathrm{i}\mathbb{R}\setminus\{0\}} k_{\mathrm{i}}^-(\lambda).$$

Since Assumption 3(c) implies that $k_{\mathrm{i}}^-(\lambda) = k_{\mathrm{i}}^-(\overline{\lambda})$, one has that k_{i}^- is necessarily even.

The theoretical ideas and results in [**25**] can now be applied to (22). For a given $w \in H_1$ one has

$$w = w_{\mathscr{L}} + w_X + w_R; \quad w_{\mathscr{L}} \in \ker(\mathscr{L}),\ w_X \in \mathscr{J}^{-1}(\ker(\mathscr{L}|_{H_1})),\ w_R \in R_{\mathscr{L}}.$$

Since $\mathscr{L}|_{H_1} : X_{\mathscr{L}} \mapsto \mathscr{J}^{-1}(\ker(\mathscr{L}|_{H_1}))$, and since D is nonsingular, one may write alternatively

$$w = w_{\mathscr{L}} + y + w_R; \quad w_{\mathscr{L}} \in \ker(\mathscr{L}),\ y \in X_{\mathscr{L}},\ w_R \in R_{\mathscr{L}}. \tag{23}$$

A simple modification of the proof leading to [**25**, Proposition 2.8] yields

$$\begin{aligned} \langle w, \mathscr{L}|_{H_1} w\rangle &= \langle w_{\mathscr{L}}, \mathscr{L}|_{H_1} w_{\mathscr{L}}\rangle + \langle y, \mathscr{L}|_{H_1} y\rangle + \langle w_R, \mathscr{L}|_{H_1} w_R\rangle \\ &= \langle y, \mathscr{L}|_{H_1} y\rangle + \langle w_R, \mathscr{L}|_{H_1} w_R\rangle. \end{aligned} \tag{24}$$

By [**25**, Theorem 2.13] applied to (22) one has that

$$k_{\mathrm{r}} + k_{\mathrm{c}} + k_{\mathrm{i}}^- = \mathrm{n}(\langle w_R, \mathscr{L}|_{H_1} w_R\rangle). \tag{25}$$

Here k_r refers to the number of real eigenvalues of $\mathscr{J}\mathscr{L}|_{H_1}$ in the open right-half plane, k_c (even) is its number of complex-valued eigenvalues in the open right-half plane, and k_i^- (even) is the total negative Krein signature. In conclusion, upon noting that $\mathrm{n}(\mathscr{A}) = \mathrm{n}(\langle w, \mathscr{A}w\rangle)$ for self-adjoint operators $\mathscr{A}$, one has from (24) and (25) that

$$\mathrm{n}(\mathscr{L}|_{H_1}) = \mathrm{n}(D) + k_\mathrm{r} + k_\mathrm{c} + k_\mathrm{i}^-. \tag{26}$$

We wish to further refine (26). For $x \in X$ write $x = z_1 + w_1$, where $z_1 \in \ker(\mathscr{J})$ and $w_1 \in H_1$. Since J is nonsingular one can alternatively write $x = \mathscr{L}^{-1}z + w$, where $z \in \ker(\mathscr{J})$ and $w \in H_1$. Since

$$\langle x, \mathscr{L}x\rangle = \langle z, \mathscr{L}^{-1}z\rangle + \langle w, \mathscr{L}w\rangle,$$

and since $\langle w, \mathscr{L}w\rangle = \langle w, \mathscr{L}|_{H_1}w\rangle$, one has from the above that

$$\mathrm{n}(\mathscr{L}) = \mathrm{n}(J) + \mathrm{n}(\mathscr{L}|_{H_1}). \tag{27}$$

Combining (26) and (27) leads to the following theorem:

Theorem 1. *Suppose that Assumption 1 and Assumption 3 hold, and that J given in (12) and D given in (14) are nonsingular. Then for the eigenvalue problem of (11)*

$$k_\mathrm{r} + k_\mathrm{c} + k_\mathrm{i}^- = \mathrm{n}(\mathscr{L}) - \mathrm{n}(J) - \mathrm{n}(D).$$

2.4 *Orbital stability*

Assume that there is a local well-posedness theory for (10); in other words, the initial value problem has a unique solution for at least some time. The key condition that must be verified in order to demonstrate orbital stability is that $\mathrm{n}(\mathscr{L}|_{H_1}) = \mathrm{n}(D)$ [**21**, Theorem 4.1]. By (27) one has that

$$\mathrm{n}(\mathscr{L}|_{H_1}) = \mathrm{n}(\mathscr{L}) - \mathrm{n}(J);$$

thus, by applying Theorem 1 one has:

Theorem 2. *Under the assumptions of Theorem 1, if $k_\mathrm{r} = k_\mathrm{c} = k_\mathrm{i}^- = 0$, then the relative equilibria of (4) are orbitally stable. In other words, when considering solutions of (8) it is true that for each $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$Q_{\mathscr{J}}u_0 = 0,\ \|u_0 - \phi_\omega\| < \delta \quad \Longrightarrow \quad \inf_{g\in G}\|u(t) - T(g)\phi_\omega\| < \varepsilon.$$

Here $Q_{\mathscr{J}}$ is the orthogonal projection onto $\ker(\mathscr{J})$.

Remark 3. Theorem 2 can be considered as a natural generalization of the results of [**5**], as well as the related works of [**3**, **41**]. Unfortunately, the result cannot be used to furnish an alternate proof of the results in [**5**], for in this work the operator $P_{\mathscr{J}}\mathscr{J}P_{\mathscr{J}}$ does not have a bounded inverse.

3 Application: generalized KdV

Consider the generalized KdV equation with power nonlinearity,

$$u_t + (u_{xx} \pm u^{p+1})_x = 0, \quad p \geq 1. \tag{28}$$

The equation with the plus sign is referred to as the focusing gKdV, whereas that with the minus sign is the defocusing gKdV. As discussed in [13], if p is odd then the sign is irrelevant, but if p is even then the two cases are genuinely distinct. Our interest in this section is in the orbital stability of $2L$-periodic solutions of (28) with respect to perturbations of period $2L$, i.e., harmonic perturbations, using the terminology of [6]. It is seen in [13] that global solutions exist to (28) for integer $1 \leq p \leq 3$ for initial data of any size in the appropriate space, whereas solutions are known to exist globally for integer $p \geq 4$ for sufficiently small initial data in the appropriate space. For explicit calculations we focus most of our attention on the cases $p = 1$ (KdV) and $p = 2$ (mKdV). As stated in Section 1, both of these cases are completely integrable.

Following the notation of the previous section one has that $\mathscr{J} = \partial_x$. The space H_1 is given by

$$H_1 = \{u \in L^2_{\mathrm{per}}[-L, L] : \overline{u} = 0\}, \quad \overline{u} := \frac{1}{2L}\int_{-L}^{L} u(x)\,\mathrm{d}x.$$

The projection operator is $P_{\mathscr{J}}u = u - \overline{u}$, so $Q_{\mathscr{J}}U$ has zero mean. The relevant group action is $T(\omega)u(x,t) = u(x+\omega, t)$. Since $T_\omega = \omega\partial_x = \omega\mathscr{J}$, Assumption 1 holds. The Hamiltonian associated with (28) is

$$\mathscr{E}(u) = \int_{-L}^{L}\left(\frac{1}{2}u_x^2 \mp \frac{1}{p+2}u^{p+2}\right)\mathrm{d}x, \tag{29}$$

while the functional M_ω is given by

$$M_\omega = \frac{\omega}{2}\int_{-L}^{L} u^2\,\mathrm{d}x,$$

leading to

$$\mathscr{H}_\omega(u) = \mathscr{E}(u) - M_\omega(u) = \int_{-L}^{L}\left(\frac{1}{2}u_x^2 \mp \frac{1}{p+2}u^{p+2} - \frac{\omega}{2}u^2\right)\mathrm{d}x.$$

We find that the relative equilibria of interest satisfy

$$u_{xx} = cu \mp u^{p+1}, \tag{30}$$

with $c = -\omega$. This familiar result is usually obtained by writing (28) in a moving frame via $x \mapsto x - ct$, and looking for stationary solutions. The point of introducing the functional M_ω is that our approach works equally systematic for other equations where different symmetry reductions lead to the relative equilibria, such as the nonlinear Schrödinger equation.

Remark 4. Note that the solutions we use by no means exhaust the stationary periodic solutions of the gKdV equation; in particular, the full class of solutions can be found only by considering the ODE

$$u_{xx} = cu \mp u^{p+1} + a_0, \quad a_0 \in \mathbb{R} \tag{31}$$

(for instance, see [**26**], where a larger class of stationary solutions is constructed). We restrict ourselves to the case $a_0 = 0$ because the functional form of the solutions is the simplest, enabling explicit calculations. The case of $a_0 \neq 0$ was carried out in [**9**, **27**].

Let a $2L$-periodic solution to (30) be denoted by $U(x)$. Upon linearizing, we obtain a linear eigenvalue problem of the form (11) with

$$\mathscr{J} = \partial_x, \quad \mathscr{L} = -\partial_x^2 + c \mp (p+1)U^p(x). \tag{32}$$

Note that $\mathscr{L}$ is a Hill operator [**38**]. In the space $L^2_{\text{per}}[-L,L]$ it is known that the countable set of eigenvalues for $\mathscr{L}$ can be ordered as $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ with $\lim_{n\to\infty} \lambda_n = +\infty$, and that the associated normalized eigenfunctions ϕ_j form an orthonormal basis.

Before continuing, we need to verify Assumption 3. Since $\mathscr{L}$ is a Hill operator, it has compact resolvent, and the number of negative eigenvalues is finite. Moreover, using $\mathscr{L}_0 = -\partial_x^2$ the Hill operator satisfies the compactness condition of (b)-(1) with $b = r = 0$. Since the eigenvalues are explicitly given by $(j\pi/L)^2$ (double eigenvalues for $j \geq 1$), the growth conditions on the eigenvalues, (b)-(2) and (b)-(3), are also satisfied. The final condition (c) clearly holds.

As a consequence of the spatial translation invariance associated with (28) one knows that $\mathscr{L}U_x = 0$. Using the notation from Section 2,

$$\begin{aligned} &\ker(\mathscr{J}) = \text{Span}\{1\}, \quad \ker(\mathscr{L}) = \text{Span}\{U_x\} \\ &\mathscr{J}^{-1}(\ker(\mathscr{L})) = \text{Span}\{U - \overline{U}\}, \quad \mathscr{L}^{-1}(\ker(\mathscr{J})) = \text{Span}\{\mathscr{L}^{-1}(1)\}. \end{aligned}$$

Note that the assumption that J is nonsingular in Theorem 1 is equivalent to $\langle \mathscr{L}^{-1}(1), 1\rangle \neq 0$.

In order to construct the one-by-one dimensional matrix D one must first find a basis for the one-dimensional generalized eigenspace $X_{\mathscr{L}}$. Assume that J is nonsingular. Let

$$u_C = \mathscr{L}^{-1}(U) - C\mathscr{L}^{-1}(1), \quad C = \frac{\langle \mathscr{L}^{-1}(U), 1\rangle}{\langle \mathscr{L}^{-1}(1), 1\rangle}. \tag{33}$$

It is clear that

$$\mathscr{L}u_C = U - C,$$

and since $u_C \in H_1$ one has that

$$\mathscr{L}|_{H_1} u_C = U - \overline{U}.$$

It follows that u_C provides a basis for $X_{\mathscr{L}}$, where C is given in (33). Since $u_C \in H_1$ one has that

$$\langle u_C, \mathscr{L}|_{H_1} u_C \rangle = \langle u_C, \mathscr{L} u_C \rangle,$$

which upon using (33) and simplifying finally yields

$$D = \frac{\begin{vmatrix} \langle \mathscr{L}^{-1}(U), U \rangle & \langle \mathscr{L}^{-1}(U), 1 \rangle \\ \langle \mathscr{L}^{-1}(U), 1 \rangle & \langle \mathscr{L}^{-1}(1), 1 \rangle \end{vmatrix}}{\langle \mathscr{L}^{-1}(1), 1 \rangle}, \tag{34}$$

where we have used that $\mathscr{L}^{-1}$ is self adjoint.

Remark 5. Assuming that J is nonsingular, one sees from (34) that $D = 0$ if and only u_C satisfies $\langle u_C, U \rangle = 0$.

The expression in (34) is difficult to calculate in general, hence we wish to simplify it. Assume that $U(x)$ is even, which in particular implies that $\mathscr{L}$ maps even (odd) functions to even (odd) functions. As stated, one solution of $\mathscr{L}\phi = 0$ is $\phi = U'(x)$. Reduction of order provides a second solution Ψ, which satisfies

$$\mathscr{L}\Psi = 0, \quad \begin{vmatrix} U' & \Psi \\ U'' & \Psi' \end{vmatrix} = 1. \tag{35}$$

Formally,

$$\Psi(x) = U'(x) \int^x \frac{1}{[U'(s)]^2} \, \mathrm{d}s.$$

The above formulation is problematic because U' will generally have at least one zero. For any even f which is $2L$-periodic one has by variation of parameters that

$$\mathscr{L}^{-1}(f) = U'(x) \int_0^x \Psi(s) f(s) \, \mathrm{d}s - \Psi(x) \int_0^x U'(s) f(s) \, \mathrm{d}s + c_f \Psi(x), \tag{36}$$

where

$$c_f := \int_0^L U'(s) f(s) \, \mathrm{d}s - \frac{1}{2} U''(L) \frac{\langle \Psi, f \rangle}{\Psi'(L)}$$

is chosen so that

$$\frac{\mathrm{d}}{\mathrm{d}x} \mathscr{L}^{-1}(f)|_{x=L} = 0.$$

This final condition guarantees that $\mathscr{L}^{-1}(f)$ is $2L$-periodic. Using (36) it is straightforward to check that

$$\begin{aligned}
\langle \mathscr{L}^{-1}(1), 1\rangle &= \left(2U(L) - \frac{1}{2}U''(L)\frac{\langle \Psi, 1\rangle}{\Psi'(L)}\right)\langle \Psi, 1\rangle - 2\langle \Psi, U\rangle \\
\langle \mathscr{L}^{-1}(U), 1\rangle &= \frac{1}{2}U^2(L)\langle \Psi, 1\rangle - \frac{3}{2}\langle \Psi, U^2\rangle + \left(U(L) - \frac{1}{2}U''(L)\frac{\langle \Psi, 1\rangle}{\Psi'(L)}\right)\langle \Psi, U\rangle \\
\langle \mathscr{L}^{-1}(U), U\rangle &= -\langle \Psi, U^3\rangle + \left(U^2(L) - \frac{1}{2}U''(L)\frac{\langle \Psi, U\rangle}{\Psi'(L)}\right)\langle \Psi, U\rangle.
\end{aligned} \tag{37}$$

Further simplification of these expressions is possible for specific values of p, and for a choice of focusing or defocusing. An example of such simplifications is found below, where we consider specific instances of the generalized KdV equation.

3.1 Modified KdV: $p = 2$

With $p = 2$ and for the focusing case, we simplify the expressions in (37) even more. Integration by parts and use of the fact that U and Ψ are even yields

$$\langle \Psi'', U\rangle = 2U(L)\Psi'(L) + \langle \Psi, U''\rangle.$$

Consequently,

$$0 = \langle \mathscr{L}\Psi, U\rangle = -2U(L)\Psi'(L) + \langle \Psi, \mathscr{L}U\rangle,$$

and since $\mathscr{L}U = -2U^3$

$$0 = -2U(L)\Psi'(L) - 2\langle \Psi, U^3\rangle \quad \Longrightarrow \quad \langle \Psi, U^3\rangle = -U(L)\Psi'(L).$$

Similarly,

$$0 = \langle \mathscr{L}\Psi, 1\rangle \quad \Longrightarrow \quad \langle \Psi, U^2\rangle = \frac{1}{3}c\langle \Psi, 1\rangle - \frac{2}{3}\Psi'(L).$$

Substitution of the above into (37) gives

$$\begin{aligned}
\langle \mathscr{L}^{-1}(1), 1\rangle &= \left(2U(L) - \frac{1}{2}U''(L)\frac{\langle \Psi, 1\rangle}{\Psi'(L)}\right)\langle \Psi, 1\rangle - 2\langle \Psi, U\rangle \\
\langle \mathscr{L}^{-1}(U), 1\rangle &= \frac{1}{2}(U^2(L) - c)\langle \Psi, 1\rangle + \Psi'(L) + \left(U(L) - \frac{1}{2}U''(L)\frac{\langle \Psi, 1\rangle}{\Psi'(L)}\right)\langle \Psi, U\rangle \\
\langle \mathscr{L}^{-1}(U), U\rangle &= U(L)\Psi'(L) + \left(U^2(L) - \frac{1}{2}U''(L)\frac{\langle \Psi, U\rangle}{\Psi'(L)}\right)\langle \Psi, U\rangle.
\end{aligned} \tag{38}$$

In order to evaluate (34) one must compute the expressions in (38). This is done in subsequent sections for two explicit cases.

For the focusing case with $p = 2$ we work with the two families of stationary periodic solutions given by

$$U = \sqrt{2}\,\mu\,\mathrm{dn}(\mu x, k), \quad c = \mu^2(2-k^2),$$
$$U = \sqrt{2}\,\mu k\,\mathrm{cn}(\mu x, k), \quad c = \mu^2(-1+2k^2).$$

Here $\mu > 0$, $0 \le k < 1$, and $\mathrm{dn}(y,k)$ and $\mathrm{cn}(y,k)$ are Jacobi elliptic functions. For the defocusing case we use

$$U = \sqrt{2}\,\mu k\,\mathrm{sn}(\mu x, k), \quad c = -\mu^2(1+k^2).$$

The waves proportional to $\mathrm{cn}(y,k)$ and $\mathrm{sn}(y,k)$ have period $4K(k)/\mu$, where $K(k)$ is the complete elliptic integral of the first kind. The wave proportional to $\mathrm{dn}(y,k)$ has period $2K(k)/\mu$. The function $K(k)$ is smooth, strictly increasing, and satisfies the limits

$$\lim_{k\to 0^+} K(k) = \frac{\pi}{2}, \quad \lim_{k\to 1^-} K(k) = +\infty.$$

Thus, the inclusion of the parameter μ allows us to consider the entire family of dnoidal solutions for fixed L. In all that follows we set $\mu = 1$, as the effect of including μ is simply an overall eigenvalue scaling in all of the calculations listed below. In particular, the parameter μ is needed only to consider the entire family of cnoidal waves for a fixed period.

Define

$$\mathscr{L}_0 := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 6k^2\,\mathrm{sn}^2(x,k).$$

Since $\mathscr{L}_0$ has a two-gap potential, when considering $\sigma(\mathscr{L}_0)$ on $L^2_{\mathrm{per}}([-2K(k), 2K(k)]; \mathbb{C})$ it is known that the first five eigenvalues are simple, and that all other eigenvalues have multiplicity two. In particular, the first five eigenvalues, as well as the associated eigenfunctions, are given by

$$\begin{aligned}
&\lambda_0 = 2(1+k^2-a(k));\ \phi_0(x) = k^2\,\mathrm{sn}^2(x,k) - \frac{1}{3}(1+k^2+a(k)),\\
&\lambda_1 = 1+k^2;\ \phi_1(x) = \partial_x\,\mathrm{sn}(x,k),\\
&\lambda_2 = 1+4k^2;\ \phi_2(x) = \partial_x\,\mathrm{cn}(x,k),\\
&\lambda_3 = 4+k^2;\ \phi_3(x) = \partial_x\,\mathrm{dn}(x,k,)\\
&\lambda_4 = 2(1+k^2+a(k));\ \phi_4(x) = k^2\,\mathrm{sn}^2(x,k) - \frac{1}{3}(1+k^2-a(k)),
\end{aligned} \tag{39}$$

where $a(k) := \sqrt{1-k^2+k^4}$ [**18**]. The following proposition is useful for all subsequent calculations. As a consequence of Proposition 1 the evaluation of $\langle \mathscr{L}^{-1}(1), 1\rangle$ will be straightforward.

Proposition 1. *For* $j \notin \{0,4\}$, $\langle\phi_j, 1\rangle = 0$.

Proof. An observation of (39) reveals that the result holds for $j = 1,2,3$. Now consider $j \geq 5$. Upon using the fact that $\mathscr{L}_0\phi_j = \lambda_j\phi_j$ and integrating both sides over one period one sees that

$$\lambda_j\langle\phi_j, 1\rangle = 6\langle\phi_j, k^2\,\mathrm{sn}^2(x,k)\rangle.$$

Set $b(k) := (1+k^2-a(k))/3$. Using the representation of $\phi_4(x)$ given in (39) allows one to rewrite the above as

$$(\lambda_j - 6b(k))\langle\phi_j, 1\rangle = 6\langle\phi_j, \phi_4\rangle = 0,$$

where the second inequality follows from the orthogonality of the eigenfunctions. Since $\lambda_j - 6b(k) > 0$ for $j \geq 5$, the desired conclusion follows.

3.1.1 Focusing mKdV: solution $\sqrt{2}\,\mathrm{dn}(x,k)$

The wave is given by

$$U(x;k) := \sqrt{2}\,\mathrm{dn}(x,k), \quad c(k) := 2-k^2. \tag{40}$$

The fundamental period of $U(x;k)$ is $2K(k)$, so that $L = K(k)$. Upon using the identity $1-k^2\,\mathrm{dn}^2(x,k) = k^2\,\mathrm{sn}^2(x,k)$ one finds that the linearization around U yields the operator

$$\mathscr{L} := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + c - 3U^2 = \mathscr{L}_0 - (4+k^2). \tag{41}$$

The spectrum of $\mathscr{L}$ is derived from that of $\mathscr{L}_0$ via $\lambda_j \mapsto \lambda_j - (4+k^2)$ in (39). Set $\widetilde{\lambda}_j := \lambda_j - (4+k^2)$. Since the fundamental period is $2K(k)$, the eigenvalues associated with λ_1, λ_2 are not relevant, as the associated eigenfunctions have fundamental period $4K(k)$. Consequently, $\mathrm{n}(\mathscr{L}) = 1$ for all $k \in [0,1)$.

As a consequence of Proposition 1 one has that for any $0 \leq k < 1$,

$$\langle\mathscr{L}^{-1}(1), 1\rangle = \frac{\langle\phi_0, 1\rangle^2}{\widetilde{\lambda}_0} + \frac{\langle\phi_4, 1\rangle^2}{\widetilde{\lambda}_4}. \tag{42}$$

Using the explicit expressions given in (39), and using the identities

$$\begin{aligned}
\int_{-K(k)}^{K(k)} k^2\,\mathrm{sn}^2(x,k)\,\mathrm{d}x &= 2(K(k)-E(k)) \\
\int_{-K(k)}^{K(k)} k^4\,\mathrm{sn}^4(x,k)\,\mathrm{d}x &= \frac{2}{3}[(2+k^2)K(k) - 2(1+k^2)E(k)],
\end{aligned} \tag{43}$$

we have explicitly

$$\langle \mathscr{L}^{-1}(1), 1\rangle = \frac{2K(k) - (2+k^2)E(k)}{k^4} > 0, \tag{44}$$

for $k > 0$. This inequality is easily verified using the series expansions of $K(k)$ and $E(k)$ [**1**]. As a consequence, it is now known that $\mathrm{n}(J) = 0$.

In order to complete the calculation, we compute D. Recall that formally

$$\Psi(x) = U'(x) \int^x \frac{1}{[U'(s)]^2}\,\mathrm{d}s.$$

Upon using the identities

$$\begin{aligned}
\frac{1}{[\partial_x \mathrm{dn}(x;k)]^2} &= \frac{1}{k^4}\left(\frac{1}{\mathrm{sn}^2(x;k)} + \frac{1}{\mathrm{cn}^2(x;k)}\right),\\
\frac{\partial}{\partial x}\frac{\mathrm{sn}(x;k)}{\mathrm{cn}(x;k)} &= \frac{\mathrm{dn}(x;k)}{\mathrm{cn}^2(x;k)}, \quad \frac{\partial}{\partial x}\frac{\mathrm{cn}(x;k)}{\mathrm{sn}(x;k)} = -\frac{\mathrm{dn}(x;k)}{\mathrm{sn}^2(x;k)},
\end{aligned}$$

integrating by parts, and normalizing with (35) to get $\Psi(0) = 1/(\sqrt{2}k^2)$, one eventually gets

$$\Psi(x) = \frac{1}{\sqrt{2}k^2}\left(\frac{1-2\,\mathrm{sn}^2(x;k)}{\mathrm{dn}(x;k)} - k^2\,\mathrm{sn}(x;k)\,\mathrm{cn}(x;k)\int_0^x \frac{1-2\,\mathrm{sn}^2(t;k)}{\mathrm{dn}^2(t;k)}\,\mathrm{d}t\right).$$

Differentiating and evaluating at $x = L = K(k)$ yields

$$\Psi'(L) = \frac{1}{2}U(L)\int_0^L \frac{1-2\,\mathrm{sn}^2(t;k)}{\mathrm{dn}^2(t;k)}\,\mathrm{d}t.$$

Upon integrating by parts one has that

$$\langle \Psi, 1\rangle = \frac{1}{k^2}U(L)\int_0^L \frac{1-2\,\mathrm{sn}^2(t;k)}{\mathrm{dn}^2(t;k)}\,\mathrm{d}t \quad \Longrightarrow \quad \Psi'(L) = \frac{1}{2}k^2\langle \Psi, 1\rangle.$$

Since $U''(L) = k^2 U(L)$, one can now rewrite (38) as

$$\begin{aligned}
\langle \mathscr{L}^{-1}(1), 1\rangle &= U(L)\langle \Psi, 1\rangle - 2\langle \Psi, U\rangle,\\
\langle \mathscr{L}^{-1}(U), 1\rangle &= \frac{1}{2}(U^2(L) - c + k^2)\langle \Psi, 1\rangle = 0,\\
\langle \mathscr{L}^{-1}(U), U\rangle &= U^2(L)\langle \Psi, U\rangle + \frac{1}{2}k^2 U(L)\langle \Psi, 1\rangle - U(L)\frac{\langle \Psi, U\rangle^2}{\langle \Psi, 1\rangle}.
\end{aligned} \tag{45}$$

Using the results of (45) in (34) one sees that

$$D = \langle \mathscr{L}^{-1}(U), U\rangle.$$

An expression for $\langle \Psi, 1\rangle$ is given above. Upon integrating by parts and simplifying with (43) one sees that

$$\langle \Psi, U\rangle = \frac{1}{k^4}\left(2E(k) - (2-k^2)K(k) + k^2(1-k^2)\int_0^L \frac{1-2\,\mathrm{sn}^2(t;k)}{\mathrm{dn}^2(t;k)}\,\mathrm{d}t\right).$$

In order to complete the calculation the integral must be computed. Using the fact that the integrand is even in t and $2L$-periodic one has that

$$\int_0^L \frac{1-2\,\mathrm{sn}^2(t;k)}{\mathrm{dn}^2(t;k)}\,\mathrm{d}t = \int_L^{2L} \frac{1-2\,\mathrm{sn}^2(t;k)}{\mathrm{dn}^2(t;k)}\,\mathrm{d}t.$$

Since

$$\mathrm{sn}(t+L;k) = \frac{\mathrm{sn}(t;k)}{\mathrm{dn}(t;k)}, \quad \mathrm{dn}(t+L;k) = \frac{\sqrt{1-k^2}}{\mathrm{dn}(t;k)},$$

one has

$$\int_L^{2L} \frac{1-2\,\mathrm{sn}^2(t;k)}{\mathrm{dn}^2(t;k)}\,\mathrm{d}t = \frac{1}{1-k^2}\int_0^L [-1+(2-k^2)\,\mathrm{sn}^2(t;k)]\,\mathrm{d}t.$$

The latter integral can be computed with (43) to finally get

$$\int_0^L \frac{1-2\,\mathrm{sn}^2(t;k)}{\mathrm{dn}^2(t;k)}\,\mathrm{d}t = \frac{1}{k^2(1-k^2)}(2(1-k^2)K(k) - (2-k^2)E(k)).$$

Consequently, one concludes that

$$\langle \Psi, 1\rangle = \frac{U(L)}{k^4(1-k^2)}(2(1-k^2)K(k) - (2-k^2)E(k)), \quad \langle \Psi, U\rangle = -\frac{K(k)-E(k)}{k^2}.$$

Plugging these expressions into (45) and evaluating the resulting expression yields

$$\langle \mathscr{L}^{-1}(U), U\rangle = -\frac{(1-k^2)K^2(k) - E^2(k)}{2(1-k^2)K(k) - (2-k^2)E(k)} < 0,$$

for $k > 0$. Here the inequality follows as before, using the series expansions of $E(k)$ and $K(k)$ [**1**] to establish that the denominator has a definite sign. The definite sign of the numerator follows similarly from $E(k) > \sqrt{1-k^2}K(k)$, for $k > 0$.

In conclusion, from Theorem 1 it is seen that $k_\mathrm{r} = k_\mathrm{c} = k_\mathrm{i}^- = 0$. Recalling that the scaling μ was unimportant in the above calculations, and applying Theorem 2, the following result is obtained.

Theorem 3. *Consider the solution* $U_\mu(x) = \sqrt{2}\,\mu\,\mathrm{dn}(\mu x, k)$ *on* $L^2_{\mathrm{per}}([-K(k)/\mu, K(k)/\mu]; \mathbb{R})$ *endowed with the natural inner-product. For a given* $\varepsilon > 0$ *sufficiently small there is a* $\delta > 0$ *such that if* $\|u(0) - U_\mu\| < \delta$ *with* $\overline{u(0)} = \overline{U_\mu}$*, then*

$$\inf_{\omega\in\mathbb{R}} \|u(t) - U_\mu(\cdot+\omega)\| < \varepsilon.$$

Thus the dn solution of the focusing mKdV equation is orbitally stable with respect to periodic perturbations of the same period for all values of the elliptic modulus.

Remark 6. The result of Theorem 3 was recently established in [**2**]; however, the proof there is different than that presented here. In particular, the proof in [**2**] fails when $\mathrm{n}(\mathscr{L}) = 2$, which will be the case in next problem.

3.1.2 Focusing mKdV: solution $\sqrt{2}k\,\mathrm{cn}(x,k)$

Next set

$$U(x;k) := \sqrt{2}k\,\mathrm{cn}(x,k), \quad c(k) := -1+2k^2. \tag{46}$$

For this second case, the fundamental period of $U(x;k)$ is $4K(k)$, so that now $L = 2K(k)$. The linearization around U gives the operator

$$\mathscr{L} := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + c - 3U^2 = \mathscr{L}_0 - (1+4k^2). \tag{47}$$

The spectrum of $\mathscr{L}$ is derived from that of $\mathscr{L}_0$ via $\lambda_j \mapsto \lambda_j - (1+4k^2)$ in (39). Set $\widetilde{\lambda}_j := \lambda_j - (1+4k^2)$. Since the fundamental period is $4K(k)$, all of the eigenvalues in (39) are relevant. Consequently, $\mathrm{n}(\mathscr{L}) = 2$ for all k.

The result of Proposition 1 still holds. Thus, $\langle \mathscr{L}^{-1}(1), 1\rangle$ is still given by (42) with the appropriate substitution. Using the explicit expressions given in (39), and using (43), allows one to explicitly compute

$$\langle \mathscr{L}^{-1}(1), 1\rangle = -4(2E(k) - K(k)) = -4\frac{\mathrm{d}}{\mathrm{d}k}kE(k). \tag{48}$$

Since

$$\lim_{k\to 0^+} \frac{\mathrm{d}}{\mathrm{d}k}kE(k) = E(0) > 0, \quad \lim_{k\to 1^-} \frac{\mathrm{d}}{\mathrm{d}k}kE(k) = -\infty, \quad \frac{\mathrm{d}^2}{\mathrm{d}k^2}kE(k) < 0,$$

there is a unique $k^* \sim 0.909$ such that $\langle \mathscr{L}^{-1}(1), 1\rangle = 0$ for $0 \le k < k^*$. In conclusion,

$$\mathrm{n}(\langle \mathscr{L}^{-1}(1), 1\rangle) = \begin{cases} 1, & 0 \le k < k^*, \\ 0, & k^* < k < 1. \end{cases} \tag{49}$$

Now we compute D in order to complete the calculation. The calculation is similar to that presented in the previous subsection, and hence only the highlights will be given. One has

$$\Psi(x) = \frac{1}{\sqrt{2}k}\left(\mathrm{cn}(x;k) - k^2\,\mathrm{sn}(x;k)\,\mathrm{dn}(x;k)\int_0^x \frac{2-\mathrm{sn}^2(t;k)}{\mathrm{dn}^2(t;k)}\,\mathrm{d}t\right),$$

from which one gets that

$$\Psi'(L) = -\frac{1}{2}U(L)\int_0^L \frac{2-\mathrm{sn}^2(t;k)}{\mathrm{dn}^2(t;k)}\,\mathrm{d}t = -\frac{1}{2}\langle \Psi, 1\rangle.$$

The second equality is again found by integrating by parts. The analogue of (45) with $U''(L) = -U(L)$ is now

$$\begin{aligned}\langle \mathscr{L}^{-1}(1), 1\rangle &= U(L)\langle \Psi, 1\rangle - 2\langle \Psi, U\rangle,\\ \langle \mathscr{L}^{-1}(U), 1\rangle &= 0,\\ \langle \mathscr{L}^{-1}(U), U\rangle &= U^2(L)\langle \Psi, U\rangle - \frac{1}{2}U(L)\langle \Psi, 1\rangle - U(L)\frac{\langle \Psi, U\rangle^2}{\langle \Psi, 1\rangle};\end{aligned} \tag{50}$$

hence, as in the previous subsection we conclude with

$$D = \langle \mathscr{L}^{-1}(U), U\rangle.$$

Note that the potential singularity at $k = k^*$ has been removed.

Calculating as before one has that

$$\begin{aligned}\int_0^L \frac{2-\mathrm{sn}^2(t;k)}{\mathrm{dn}^2(t;k)}\,\mathrm{d}t &= 2\int_{L/2}^L \frac{2-\mathrm{sn}^2(t;k)}{\mathrm{dn}^2(t;k)}\,\mathrm{d}t\\ &= \frac{2}{k^2(1-k^2)}((1-k^2)K(k) - (1-2k^2)E(k)),\end{aligned}$$

from which one eventually gets that

$$\langle \Psi, 1\rangle = \frac{2U(L)}{k^2(1-k^2)}((1-k^2)K(k) - (1-2k^2)E(k)), \quad \langle \Psi, U\rangle = \frac{2}{1-k^2}E(k).$$

The second equality requires the use of (43) and the identity

$$\frac{\partial}{\partial x}\frac{\mathrm{cn}(x;k)}{\mathrm{dn}(x;k)} = -(1-k^2)\frac{\mathrm{sn}(x;k)}{\mathrm{dn}^2(x;k)}.$$

Substituting the above into the expression for $\langle \mathscr{L}^{-1}(U), U\rangle$ yields a negative sign for $k > 0$, as before. Thus $\mathrm{n}(D) = 1$.

Upon using the result of Theorem 1 one has that

$$k_\mathrm{r} + k_\mathrm{c} + k_\mathrm{i}^- = \begin{cases} 0, & 0 \le k < k^*,\\ 1, & k^* < k < 1.\end{cases}$$

Since k_i^- and k_c are even, it then follows that $k_\mathrm{c} = k_\mathrm{i}^- = 0$, but

$$k_\mathrm{r} = \begin{cases} 0, & 0 \le k < k^*,\\ 1, & k^* < k < 1.\end{cases} \tag{51}$$

Hence, for $k < k^*$ the cn wave of period $2L$ is a constrained minimizer and thus stable with respect to periodic perturbations of period $2L$. On the other hand, the cn wave is unstable for $k > k^*$.

Theorem 4. *Consider the solution* $U_\mu(x) = \sqrt{2}\,\mu k\,\mathrm{cn}(\mu x, k)$ *on* $L^2_{\mathrm{per}}([-2K(k)/\mu, 2K(k)/\mu];\mathbb{R})$ *endowed with the natural inner-product. Define* k^* *as the unique value satisfying* $K(k^*) = 2E(k*)$*. If* $k < k^* \sim 0.909$*, then for a given* $\varepsilon > 0$ *sufficiently small there is a* $\delta > 0$ *such that if* $\|u(0) - U_\mu\| < \delta$ *with* $\overline{u(0)} = \overline{U_\mu}$*, then*

$$\inf_{\omega \in \mathbb{R}} \|u(t) - U_\mu(\cdot + \omega)\| < \varepsilon.$$

Thus, the cn solution of the focusing mKdV equation is orbitally stable with respect to periodic perturbations of the same period, provided the elliptic modulus $k < k^*$*. If* $k > k^*$*, then the wave is unstable.*

Remark 7. Since $\mathrm{n}(D) = 1$ for all k, the spectral structure at $k = k^*$ is such that the origin is an eigenvalue of $\mathscr{J}\mathscr{L}|_{H_1}$ of algebraic multiplicity four and geometric multiplicity two; furthermore, there are two nontrivial Jordan blocks.

Remark 8. If the operator $\mathscr{J}$ were nonsingular, then $\mathrm{n}(\mathscr{L}) = 2$ with $\mathrm{n}(D) = 1$ would imply that $k_{\mathrm{r}} = 1$ for all values of k. This example shows that the modification of the result of [**36**] presented in Theorem 1 is indeed necessary, and is not simply a technical detail.

Remark 9. Without loss of generality assume that $\mu = 1$. A numerical calculation of $\sigma(\mathscr{J}\mathscr{L})$ shows that the cnoidal wave for focusing mKdV is unstable for any value of k in the space $L^2_{\mathrm{per}}([-2nK(k), 2nK(k)];\mathbb{R})$ for any integral $n \geq 2$. This is illustrated in Figure 1. The spectra illustrated there were computed using SpectrUW2.0, using 20 Fourier modes and 400 equally-spaced Floquet exponents, with $P = 1$ [**11**]. Note the different scalings of the different figures. Since the density of the eigenvalues computed is not uniform, there are some parts of the spectrum where we have less information than elsewhere. Nevertheless, between what is known theoretically about the spectra and what we observe numerically, we feel the statements made below are safe inferences based on the numerical results plotted. For all plots in Figure 1, the entire imaginary axis is part of the spectrum, whereas the real axis is not (except for the origin, and perhaps two other points, see below). It should also be pointed out that all plots are consistent with the results of Bronski and Johnson [**8**]: at the origin, the spectrum should generically consist of either the imaginary axis (with multiplicity three), or of three distinct components, all intersecting at the origin. In our case, the second scenario unfolds for all but one value of k, see below. From the numerical results, it appears that all self-intersection points of the spectrum occur at eigenvalues corresponding to eigenfunctions with period $4K(k)$.

- For $k < k^*$, in addition to the imaginary axis, the spectrum consists of the boundary of two lobes, each cut in half by the imaginary axis. Of course, the lobes are symmetric with respect to the real and imaginary axes. The lobes touch at the origin. The boundary of the upper lobe has a second intersection point with the imaginary axis which approaches the origin as $k \to k^{*-}$. One of the eigenfunctions corresponding to this point is periodic with period $4K(k)$. It is the first (in terms of distance to the origin) non-zero eigenvalue on the imaginary axis corresponding to a period $4K(k)$ eigenfunction. This case is illustrated in Figure 1a-c.

For the last panel $k = 0.908$, very close to $k^* \sim 0.909$. For $k < k^*$, the origin is the only point on the real axis that is in the spectrum.

- It appears that three eigenvalues with periodic eigenfunctions collide at the origin for $k = k^*$. For $k < k^*$ all three eigenvalues are on the imaginary axis. For $k > k^*$ all three are on the real axis, giving rise to unstable and stable directions. For $k = k^*$, it appears the spectrum near the origin appears to consists of more than three components (counting multiplicities), leading to the conclusion that for $k = k^*$ the discriminant of Bronski and Johnson [**8**] is zero. The case of a zero discriminant is not discussed in [**8**].
- For $k > k^*$, in addition to the imaginary axis, the spectrum consists of additional curves, bounding a total of six regions in $\mathbb{C}$, three in the right-half plane, three in the left-half plane. We describe the ones in the right-half plane. Using the left-right symmetry of the spectrum completes the picture. There is a region touching the origin, which has a point furthest from the origin, which is a self-intersection point of the spectrum. This point corresponds to an eigenfunction with period $4K(k)$. It is the unstable eigenvalue given by (48). To the right of this point are two more lobes, one above the real axis, one below. For k greater than but close to k^*, the region touching the origin is small, and the two remaining lobes are relatively large. This is illustrated in Figure 1d-e. Figure 1e is a zoom-in of Figure 1d near the origin. Due to the low density of computed eigenvalues the left (right) side of the right (left) upper and lower lobes is not visible in Figure 1e. For Figure 1f, $k = 0.95$, and the outer lobes have decreased in size, whereas the regions touching the origin have grown.

In summary, the numerical results show that the cn solution of the mKdV equation is unstable with respect to perturbations of period $4nK(k)$ for any $n \geq 2$, even if it is stable with respect to perturbations of period $4K(k)$.

3.1.3 Defocusing mKdV: solution $\sqrt{2}k\,\mathrm{sn}(x,k)$

In the defocusing regime there exists a branch of solutions

$$U(x;k) := \sqrt{2}k\,\mathrm{sn}(x,k), \quad c(k) := -(1+k^2). \tag{52}$$

The linearization around U yields the operator

$$\mathscr{L} := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + c + 3U^2 = \mathscr{L}_0 - (1+k^2). \tag{53}$$

The spectrum of $\mathscr{L}$ is derived from that of $\mathscr{L}_0$ via $\lambda_j \mapsto \lambda_j - (1+k^2)$ in (39). Setting $\widetilde{\lambda}_j := \lambda_j - (1+k^2)$, one sees that $\mathrm{n}(\mathscr{L}) = 1$. Arguing as in the previous cases gives that $\langle \mathscr{L}^{-1}(1), 1\rangle < 0$ (the explicit calculations are left for the interested reader). Since $\mathrm{n}(\mathscr{L}) = \mathrm{n}(J) = 1$, by applying Theorem 1 one has the following:

Theorem 5. *Consider the solution* $U_\mu(x) = \sqrt{2}\mu k \operatorname{sn}(\mu x, k)$ *on* $L^2_{\mathrm{per}}([-2K(k)/\mu, 2K(k)/\mu]; \mathbb{R})$ *endowed with the natural inner-product. For a given* $\varepsilon > 0$ *sufficiently small there is a* $\delta > 0$ *such that if* $\|u(0) - U_\mu\| < \delta$ *with* $\overline{u(0)} = \overline{U_\mu}$*, then*

$$\inf_{\omega\in\mathbb{R}} \|u(t) - U_\mu(\cdot + \omega)\| < \varepsilon.$$

Thus the sn solution of the defocusing mKdV equation is orbitally stable with respect to perturbations of the same period.

Remark 10. The comments of Remark 8 also apply here, except now $\mathrm{n}(\mathscr{L}) = 1$ with $\mathrm{n}(D) = 0$.

Remark 11. Using the integrability of the mKdV equation extensively, more explicit statements are possible, see [**16**], where the stability and instability of all stationary solutions of the mKdV equation is discussed. For instance, it is shown there that the stationary solutions of the defocusing equation are orbitally stable with respect to subharmonic perturbations. For the focusing equation, the stability with respect to these perturbations depends on whether the solutions have higher (dn-like solutions; stable) or lower (cn-like solutions; unstable) energy than the soliton solutions which act as a separatrix between these two classes in the phase plane of the stationary equation (31).

3.2 *Perturbative results:* $p \geq 3$

As a consequence of [**27**, Theorem 5.6] it is known that for any $p \neq 2$ the periodic waves which are analogous to the dn-wave when $p = 2$ are orbitally stable if they are sufficiently small perturbations of the appropriate nonzero constant state. For this reason, these waves are not discussed here. Instead, we focus on the orbital stability of the small waves analogous to the cn-wave (focusing) and sn-wave (defocusing) when $p \in \mathbb{N}_0$.

After rescaling x, t, u (28) can be rewritten as

$$u_t + (\omega u_{xx} + u + \varepsilon\delta u^{p+1})_x = 0, \quad \delta \in \{-1, +1\}, \tag{54}$$

where the interest will be on the orbital stability of 2π-periodic solutions for $0 < \varepsilon \ll 1$. Note that the wave speed has been fixed. The free parameter ω allows us to perturbatively construct the desired steady-state solutions via the Poincaré-Lindstedt method.

First consider the existence problem for the steady-state solutions, i.e.,

$$\omega u'' + u + \varepsilon\delta u^{p+1} = a_0, \quad a_0 \in \mathbb{R}. \tag{55}$$

Looking for even solutions yields the expansions

$$u(x) = U(x) \sim \underbrace{a_0 + b\cos x}_{U_0} + \varepsilon U_1(x), \quad \omega \sim 1 + \varepsilon\omega_1.$$

Setting $\mathscr{L}_0 := -(\partial_x^2 + 1)$, one sees that at $\mathscr{O}(\varepsilon)$,

$$\mathscr{L}_0 U_1 = 2\omega_1 \partial_x^2 U_0 + \delta U_0^{p+1}. \tag{56}$$

The standard solvability condition for (56) that removes the secular terms is

$$\omega_1 = \frac{\delta}{2\pi b^2}\langle U_0^{p+2} - a_0 U_0^{p+1}, 1\rangle. \tag{57}$$

Furthermore, using a finite cosine-series representation,

$$U_1(x) = -\frac{\delta}{2\pi}\langle U_0^{p+1}, 1\rangle + \sum_{j=2}^{p+1} c_j \cos(jx), \tag{58}$$

for suitably chosen constants c_j.

The linearization about the wave U gives the linear operator

$$\mathscr{L} \sim \mathscr{L}_0 + \varepsilon\mathscr{L}_\varepsilon; \quad \mathscr{L}_\varepsilon := -\omega_1 \frac{\mathrm{d}^2}{\mathrm{d}x^2} - \delta(p+1)U_0^p(x). \tag{59}$$

The principal eigenvalue is given by $\lambda_0 = -1 + \mathscr{O}(\varepsilon)$, and the associated eigenfunction $\phi_0 = 1/\sqrt{2\pi} + \mathscr{O}(\varepsilon)$ satisfies $\langle \phi_0, 1\rangle = 1 + \mathscr{O}(\varepsilon)$. The next nonzero eigenvalue is given by $\lambda_1 = \varepsilon\lambda_\varepsilon + \mathscr{O}(\varepsilon^2)$, and the associated eigenfunction is

$$\phi_1 = \phi_1^0 + \varepsilon\phi_\varepsilon + \mathscr{O}(\varepsilon^2), \quad \phi_1^0 := \frac{1}{\sqrt{\pi}}\cos x.$$

Using regular perturbation theory results in

$$\begin{aligned} \lambda_\varepsilon &= \frac{1}{\pi}\langle \cos x, \mathscr{L}_\varepsilon(\cos x)\rangle \\ &= -\frac{\delta}{\pi b^2}\left(p\langle U_0^{p+2} - a_0 U_0^{p+1}, 1\rangle - a_0(p+1)\langle U_0^{p+1} - a_0 U_0^p, 1\rangle\right). \end{aligned} \tag{60}$$

A Maple-assisted calculation reveals that when $p = 1$, $\lambda_\varepsilon = 0$; otherwise, for p odd one has $\lambda_\varepsilon = -\delta a_0 f(a_0, b, p)$, where $f > 0$, while for p even $\lambda_\varepsilon = -\delta g(a_0, b, p)$, where again $g > 0$. The above calculations characterize $\sigma(\mathscr{L})$ in the following manner: if $\lambda_1 < 0$, then a left band edge of $\sigma(\mathscr{L})$ is at $\lambda = 0$; otherwise, the right band edge is at the origin (see Figure 2). In conclusion, for $0 < \varepsilon \ll 1$,

$$\mathrm{n}(\mathscr{L}) = \begin{cases} 1, & \lambda_1 > 0 \\ 2, & \lambda_1 < 0. \end{cases} \tag{61}$$

Remark 12. For $p = 1$ one finds by continuing the perturbation expansion that

$$\lambda_1 \sim \left(2a_0^2 + \frac{11}{6}b^2 + b^3\right)\varepsilon^2 > 0.$$

Furthermore, note that $\lambda_\varepsilon \neq 0$ for all (a_0, b) if p is even, and for p odd it is true as long as $a_0 \neq 0$.

Next, we compute J and D. Using the expansion of (59),

$$\mathscr{L}^{-1}(1) = -1 + \mathscr{O}(\varepsilon);$$

hence,

$$\langle \mathscr{L}^{-1}(1), 1\rangle = -2\pi + \mathscr{O}(\varepsilon) \quad \Longrightarrow \quad \mathrm{n}(J) = 1.$$

Since

$$\langle \mathscr{L}^{-1}(U_0), 1\rangle = \langle \mathscr{L}^{-1}(1), U_0\rangle = -2\pi a_0 + \mathscr{O}(\varepsilon),$$

an examination of (34) reveals that the dominant term for D follows from the calculation of $\langle \mathscr{L}^{-1}(U_0), U_0\rangle$. This in turn requires an expansion for Ψ. First write

$$\widetilde{\Psi} = \cos x + \varepsilon \Psi_\varepsilon + \mathscr{O}(\varepsilon^2).$$

Then

$$\mathscr{L}_0 \Psi_\varepsilon = -\mathscr{L}_\varepsilon \cos x,$$

which upon solving yields the existence of constants e_j such that

$$\Psi_\varepsilon = \frac{1}{2}\lambda_\varepsilon x \sin x + e_0 + \sum_{j=2}^{p+1} e_j \cos(jx).$$

Setting

$$\Psi = \frac{1}{b}\left(\cos x + \varepsilon\left([U_1''(0) - \Psi_\varepsilon(0)]\cos x + \Psi_\varepsilon\right) + \mathscr{O}(\varepsilon^2)\right)$$

yields the normalization

$$U_0''(0)\Psi(0) = -1 + \mathscr{O}(\varepsilon^2).$$

Since

$$\Psi'(\pi) = -\frac{\pi}{2b}\lambda_\varepsilon \varepsilon + \mathscr{O}(\varepsilon^2),$$

an examination of the last line of (37) yields that

$$\langle \mathscr{L}^{-1}(U_0), U_0\rangle = \frac{b^2\pi}{\lambda_1} + \mathscr{O}(1).$$

In conclusion, for $p \geq 2$ and $\varepsilon > 0$ sufficiently small one has

$$\mathrm{n}(D) = \begin{cases} 0, & \lambda_1 > 0, \\ 1, & \lambda_1 < 0. \end{cases} \tag{62}$$

Upon using (61) and (62), along with the fact that $\mathrm{n}(J) = 1$, one can conclude via Theorem 1 that $k_\mathrm{r} = k_\mathrm{c} = k_\mathrm{i}^- = 0$. By Theorem 2 this yields:

Theorem 6. *Let $p \geq 2$, and let (a_0, b) be such that λ_ε given in (60) is nonzero. Consider the solution $U_0(x) = a_0 + b\cos x + \mathscr{O}(\varepsilon)$ to (54) on the space $L^2_{\mathrm{per}}([-\pi,\pi];\mathbb{R})$ endowed with the natural inner product. If $\varepsilon > 0$ is sufficiently small, then U_0 is orbitally stable.*

4 Transverse instabilities of gKP-I

Recently Rousset and Tzvetkov [**44**] considered the parameter-dependent eigenvalue problem

$$\mathscr{J}(\ell)\mathscr{L}(\ell)u = \lambda u, \quad \ell \in [0, +\infty). \tag{63}$$

They considered the situation where $\mathscr{J}(\ell)$ is skew symmetric and invertible for all ℓ, and where the self-adjoint operator $\mathscr{L}(\ell)$ satisfies the assumptions:

1. there is an $L > 0$ and $\alpha > 0$ such that $\mathscr{L}(\ell) \geq \alpha\mathbb{1}$ for $\ell \geq L$,
2. if $\ell_1 > \ell_2$, then $\mathscr{L}(\ell_1) > \mathscr{L}(\ell_2)$; furthermore, if for some $\ell > 0$ the operator $\mathscr{L}(\ell)$ has a nontrivial kernel, then $\langle \mathscr{L}'(\ell)\phi, \phi\rangle > 0$ for any $\phi \in \ker(\mathscr{L}(\ell))$ (here $\mathscr{L}'(\ell)$ is the derivative of $\mathscr{L}(\ell)$ with respect to ℓ),
3. $\mathrm{n}(\mathscr{L}(0)) = 1$.

There is also an assumption on the essential spectrum; however, we do not state it here, since it is not relevant for our considerations. Under these assumptions Rousset and Tzvetkov [**44**] showed there is an $\ell > 0$ such that (63) has a bounded solution for $\mathrm{Re}\,\lambda > 0$. In other words, they developed an instability criterion. The goal in this section is to restate the instability result in terms of the index theory from our previous results. The theoretical result will be applied to the spectral stability study of periodic waves of the generalized Kadomtsev-Petviashvili equation with strong surface tension (gKP-I).

Consider the theory and arguments leading to the result of Theorem 1. They are clearly independent of any parameter dependence of the operators. For the parameter-dependent problem it is often the case that the operator $\mathscr{L}(\ell)$ is invertible except for a finite number of values ℓ; hence, without loss of generality, we assume that $\mathscr{L}(\ell)$ is nonsingular. We further assume that $\mathscr{J}(\ell)$ is nonsingular with bounded inverse. The theorem can be restated to say:

Theorem 7. *Consider the eigenvalue problem (63). Suppose that $\mathscr{L}(\ell)$ satisfies Assumption 3 for each $\ell \geq 0$, and further assume that $\mathscr{J}(\ell)$ has bounded inverse for each $\ell \geq 0$. If $\mathscr{L}(\ell)$ is nonsingular, then for each $\ell \geq 0$,*

$$k_\mathrm{r} + k_\mathrm{c} + k_\mathrm{i}^- = \mathrm{n}(\mathscr{L}(\ell)).$$

In particular, if $\mathrm{n}(\mathscr{L})$ is odd, then $k_\mathrm{r} \geq 1$.

Remark 13. The instability proof used in [44] required that $\mathrm{n}(\mathscr{L}(0)) = 1$. The other assumptions are clearly not necessary for the statement of Theorem 7; however, if it is the case that $\mathrm{n}(\mathscr{L}(\ell)) = 0$ for $\ell \geq L$, then there will be no unstable spectrum for $\ell \geq L$.

Let us apply the result of Theorem 7 to the study of spatially periodic waves of the generalized KP-I equation (gKP-I), which is a two-dimensional version of the gKdV given by

$$u_t = \partial_x(-u_{xx} + cu \mp u^{p+1}) + \partial_x^{-1} u_{yy}. \tag{64}$$

This problem was recently studied in [22] as an application of the results of [44] (also see [30] for an Evans function analysis). We consider solutions to (64) that are spatially periodic in both x and y; in particular, we assume that

$$u(x, y + 2\pi/\ell) = u(x,y), \quad \ell > 0. \tag{65}$$

The period in x, say $2L_x$, will be determined by one of the y-independent solutions considered in the previous sections. In order for the operator ∂_x^{-1} to make sense, it must be assumed that

$$\int_{-L_x}^{L_x} u_{yy}(x,y)\,\mathrm{d}x = 0, \tag{66}$$

i.e., in the language of the previous sections, it must be the case that

$$u_{yy} \in H_{1,x},$$

where the notation $H_{1,x}$ denotes the fact that the spatial average must be zero in the x-direction only.

Let $U(x)$ represent a y-independent spatially periodic steady-state solution, i.e.,

$$-U'' + cU \mp U^{p+1} = 0, \quad U(x + 2L_x) = U(x),$$

and set the linearization about this wave to be

$$\mathscr{L}_0 = -\partial_x^2 + c \mp (p+1)U^p.$$

Writing the perturbation as

$$u(x,y) = U(x) + v(x,y),$$

the linearized eigenvalue problem is

$$\partial_x \mathscr{L}_0 v + \partial_x^{-1} v_{yy} = \lambda v, \quad v(x + 2L_x, y) = v(x,y). \tag{67}$$

Using the fact that the perturbation is $2\pi/\ell$-periodic in y implies that we can write

$$v(x,y) = \sum_{\ell=-\infty}^{+\infty} v_\ell(x) \mathrm{e}^{\mathrm{i}\ell y}.$$

Plugging this expansion into (67) yields the parameter dependent eigenvalue problem

$$\partial_x \mathscr{L}_0 v_\ell - \ell^2 \partial_x^{-1} v_\ell = \lambda v_\ell, \quad v_\ell(x+2L_x) = v_\ell(x). \tag{68}$$

The system (68) is not yet in the desired form of (63). Write

$$u = \partial_x v_\ell, \tag{69}$$

so that (68) becomes

$$-\partial_x \mathscr{L}_0 \partial_x u + \ell^2 u = -\lambda \partial_x u, \quad u(x+2L_x) = u(x). \tag{70}$$

Equating $\mathscr{J} = -\partial_x^{-1}$ yields

$$\mathscr{J} \underbrace{\left(-\partial_x \mathscr{L}_0 \partial_x + \ell^2\right)}_{\mathscr{L}(\ell)} u = \lambda u, \quad u(x+2L_x) = u(x). \tag{71}$$

This is the desired Hamiltonian form for the eigenvalue equation.

As for (71), some care must be taken regarding the space in which it is being considered. The transformation (69), and the periodicity condition of (68), require that $u \in H_{1,x}$. As we have already seen, in this space ∂_x has bounded inverse; hence, for (71) $\mathscr{J}$ is a skew-symmetric operator with bounded inverse. The operator $\mathscr{L}(\ell)$ on the space of $2L_x$-periodic functions is self adjoint and has a compact resolvent. Hence, the same is true for the operator $\mathscr{L}(\ell)|_{H_{1,x}}$. Assuming that $\mathscr{L}(\ell)|_{H_{1,x}}$ is nonsingular, by Theorem 7 we have for the eigenvalue problem (71)

$$k_\mathrm{r} + k_\mathrm{c} + k_\mathrm{i}^- = \mathrm{n}\left(\mathscr{L}(\ell)|_{H_{1,x}}\right). \tag{72}$$

We wish to make (72) more definitive. Let us first consider the problem (71) for $\ell \ll 1$. With $\ell = 0$ the problem is

$$\mathscr{L}_0 \partial_x u = \lambda u \quad \Rightarrow \quad (\partial_x \mathscr{L}_0)^\mathrm{a} u = (-\lambda) u,$$

where $\mathscr{T}^\mathrm{a}$ is used to denote the adjoint of the operator $\mathscr{T}$. This is precisely the adjoint problem for the problem studied in the previous sections; hence, the spectrum is completely known for the worked examples. If $\langle \mathscr{L}_0^{-1}(1), 1\rangle \neq 0$, then $\lambda = 0$ is an eigenvalue with $\mathrm{m_g}(0) = 1$ and $\mathrm{m_a}(0) = 2$. For $\ell > 0$ but small the double eigenvalue at zero splits into a pair of eigenvalues, each of which is $\mathscr{O}(\ell)$ (see [**35**, Theorem 4.1]). Since eigenvalues come in quartets, it must be the case that the pair is either purely real, and thus contributes to a linear instability, or purely imaginary. Furthermore, if there are any eigenvalues with $\mathrm{Re}\,\lambda > 0$ when $\ell = 0$, these continue to have positive real part for small ℓ.

In order to compute the right-hand side of (72) for $\ell = \mathscr{O}(1)$, we need to consider $\sigma\left(\mathscr{L}(\ell)|_{H_{1,x}}\right)$. The eigenvalue problem

$$\mathscr{L}(\ell) u = \lambda u$$

is equivalent to

$$-\partial_x\mathscr{L}_0\partial_x u=\gamma u,\quad \gamma=\lambda-\ell^2. \tag{73}$$

Thus, if one can compute $\mathrm{n}(-\partial_x\mathscr{L}_0\partial_x|_{H_{1,x}})$, one can readily compute the desired quantity $\mathrm{n}(\mathscr{L}(\ell)|_{H_{1,x}})$. In particular, if the negative eigenvalues of $-\partial_x\mathscr{L}_0\partial_x|_{H_{1,x}}$ are ordered as $\lambda_0<\lambda_1\le\cdots\le\lambda_N<0$, then the (potential) negative eigenvalues of $\mathscr{L}(\ell)|_{H_{1,x}}$ are given by $\lambda_0+\ell^2<\cdots<\lambda_N+\ell^2$. It is clearly the case that if $\ell^2>-\lambda_0$, then $\mathscr{L}(\ell)|_{H_{1,x}}$ is a positive operator; otherwise, the operator will have a finite number of negative directions, and this number will decrease as increasing ℓ moves a negative eigenvalue λ_j across the origin.

In conclusion, in order to refine (72) we must compute $\mathrm{n}(-\partial_x\mathscr{L}_0\partial_x|_{H_{1,x}})$. We will show that it will be enough to consider $\mathrm{n}(\mathscr{L}_0|_{H_{1,x}})$. Integrating (73) yields that for $\gamma\neq 0$,

$$0=\langle(-\partial_x\mathscr{L}_0\partial_x)u,1\rangle=\gamma\langle u,1\rangle.$$

This implies that all eigenfunctions associated with nonzero eigenvalues are in $H_{1,x}$, so that

$$\mathrm{n}\left(-\partial_x\mathscr{L}_0\partial_x|_{H_{1,x}}\right)=\mathrm{n}(-\partial_x\mathscr{L}_0\partial_x).$$

Regarding the computation of $\mathrm{n}(-\partial_x\mathscr{L}_0\partial_x)$, let us consider the quadratic form

$$\langle-\partial_x\mathscr{L}_0\partial_x v,v\rangle=\langle\mathscr{L}_0 u,u\rangle,\quad u=\partial_x v.$$

In order for the two quadratic forms to be equivalent it must be the case that both u and v are $2L_x$-periodic, which in turn implies that $u\in H_{1,x}$. In other words, we have

$$\langle-\partial_x\mathscr{L}_0\partial_x v,v\rangle=\langle\mathscr{L}_0|_{H_{1,x}}u,u\rangle\quad\Rightarrow\quad \mathrm{n}(-\partial_x\mathscr{L}_0\partial_x)=\mathrm{n}\left(\mathscr{L}_0|_{H_{1,x}}\right).$$

But, from (27) we have

$$\mathrm{n}\left(\mathscr{L}_0|_{H_{1,x}}\right)=\mathrm{n}(\mathscr{L}_0)-\mathrm{n}(\langle\mathscr{L}_0^{-1}(1),1\rangle);$$

thus, we conclude that

$$\mathrm{n}\left(-\partial_x\mathscr{L}_0\partial_x|_{H_{1,x}}\right)=\mathrm{n}(\mathscr{L}_0)-\mathrm{n}(\langle\mathscr{L}_0^{-1}(1),1\rangle). \tag{74}$$

The computation of the right-hand side of (74) has been done in the previous sections for specific examples. Summarizing, we have that for $p=2$,

$$\begin{aligned}
U(x)&=\sqrt{2}\,\mathrm{dn}(x,k):\quad \mathrm{n}(\mathscr{L}_0)-\mathrm{n}(\langle\mathscr{L}_0^{-1}(1),1\rangle)=1,\\
U(x)&=\sqrt{2}k\,\mathrm{cn}(x,k):\quad \mathrm{n}(\mathscr{L}_0)-\mathrm{n}(\langle\mathscr{L}_0^{-1}(1),1\rangle)=\begin{cases}1, & 0\le k<k^*,\\ 2, & k^*<k<1,\end{cases}\\
U(x)&=\sqrt{2}k\,\mathrm{sn}(x,k):\quad \mathrm{n}(\mathscr{L}_0)-\mathrm{n}(\langle\mathscr{L}_0^{-1}(1),1\rangle)=0.
\end{aligned}$$

While we do not prove it here, it is not difficult to show in the second case, i.e., $U(x)\propto\mathrm{cn}(x,k)$, that the two negative eigenvalues are simple. For the perturbative results proved for $p\ge 3$, we have that

$$\mathrm{n}(\mathscr{L}_0) - \mathrm{n}(\langle \mathscr{L}_0^{-1}(1), 1\rangle) = \begin{cases} 0, & \lambda_1 > 0, \\ 1, & \lambda_1 < 0. \end{cases}$$

In other words, the count is zero when the right band edge is at the origin (left panel of Figure 2), and is one when the left band is at the origin (right panel of Figure 2). In all of the above cases except when $U(x) \propto \mathrm{cn}(x,k)$ for $k > k^*$ we saw that the waves were orbitally stable; otherwise, we saw that $k_\mathrm{r} = 1$, and that the unstable eigenvalue is $\mathscr{O}(1)$. We can now apply the result of Theorem 7 to say the following about transverse instabilities of these waves:

Theorem 8. *Suppose that for a gKdV spatially periodic wave* $\mathrm{n}(\mathscr{L}_0) = 1$ *(left panel of Figure 2). If* $\langle \mathscr{L}_0^{-1}(1), 1\rangle < 0$*, then the wave is spectrally transversely stable. Otherwise, the wave is spectrally unstable transversely with* $k_\mathrm{r} = 1$ *to perturbations of period* $2\pi/\ell$ *for* $0 < \ell < \sqrt{-\lambda_0}$*, where* $0 > \lambda_0 \in \sigma(\mathscr{L}_0|_{H_{1,x}})$ *is the ground state eigenvalue. All waves are spectrally stable for* $\ell > \sqrt{-\lambda_0}$*. If* $\mathrm{n}(\mathscr{L}_0) = 2$ *(right panel of Figure 2), the wave is spectrally transversely unstable.*

Remark 14. For all of the unstable waves considered in this paper we have that $k_\mathrm{r} = 1$ expect when $p = 2$ with $U(x) \propto \mathrm{cn}(x,k)$. In this case, if $0 < k < k^*$, then $k_\mathrm{r} = 1$, whereas if $k^* < k < 1$, we have that $k_\mathrm{r} = 1$ for $\sqrt{-\lambda_0} < \ell < \sqrt{-\lambda_1}$, and $k_\mathrm{r} = 2$ for $0 < \ell < \sqrt{-\lambda_1}$. Here $0 > \lambda_1 > \lambda_0 \in \sigma(\mathscr{L}_0|_{H_{1,x}})$. It should be noted that these cnoidal waves were not studied in [**22**]; furthermore, the methods of [**22**] fail for $k > k^*$.

Remark 15. If $p = 1$, then we saw in Remark 12 that for small waves, $\mathrm{n}(\mathscr{L}_0) = 1$. The following calculation which led to $\langle \mathscr{L}_0^{-1}(1), 1\rangle = -2\pi + \mathscr{O}(\varepsilon)$ was independent of p; hence, by Theorem 8 we can conclude the wave is spectrally transversely stable. This is precisely the result of Hărăguş [**24**], which was derived by a careful perturbation calculation for the entire spectrum for the problem (68).

Remark 16. When $p = 2$, the wave proportional to $\mathrm{sn}(x,k)$ is spectrally stable for all $\ell > 0$, as $\sigma(\mathscr{L}_0) \subset \mathbb{R}^+$. This answers an open question posed in [**22**, Section 5].

Remark 17. If $p \geq 3$, then if the right band edge is at the origin, the wave is spectrally stable for all $\ell > 0$, while if the left band edge is at the origin, then $k_\mathrm{r} = 1$ for $0 < \ell < \sqrt{-\lambda_0} = \mathscr{O}(1)$.

Remark 18. While stated in a different way, the result of Theorem 8 regarding transverse instability to long wavelengths ($0 < \ell \ll 1$) is precisely that given by Johnson and Zumbrun [**30**, Theorem 1]. In that paper the transverse instability criteria for gKP-I required a calculation of the quantity $\{T, M\}_{E,a}$; in particular,

$$\{T, M\}_{E,a} > 0 \quad \Rightarrow \quad \text{transversely unstable.}$$

Now, Bronski et al. [**9**] show that

$$\{T, M\}_{E,a} = T_E \langle \mathscr{L}_0^{-1}(1), 1\rangle,$$

where for the problems at hand,

$$T_E \begin{cases} < 0, & \mathrm{n}(\mathscr{L}_0) = 2, \\ > 0, & \mathrm{n}(\mathscr{L}_0) = 1. \end{cases}$$

Thus, if $T_E > 0\,(\mathrm{n}(\mathscr{L}_0) = 1)$, the instability criteria becomes $\langle \mathscr{L}_0^{-1}(1), 1\rangle > 0$, which is precisely what we have. On the other hand, if $T_E < 0\,(\mathrm{n}(\mathscr{L}_0) = 2)$, then [30] provide only a partial instability result. In particular, they show that the wave is transversely unstable if $\langle \mathscr{L}_0^{-1}(1), 1\rangle < 0$, which implies $k_\mathrm{r} = 1$. The reason for the lack of a complete description on their part is that their results depend upon the calculation of a parity (orientation) index, which yields definitive results only if k_r is odd for small ℓ.

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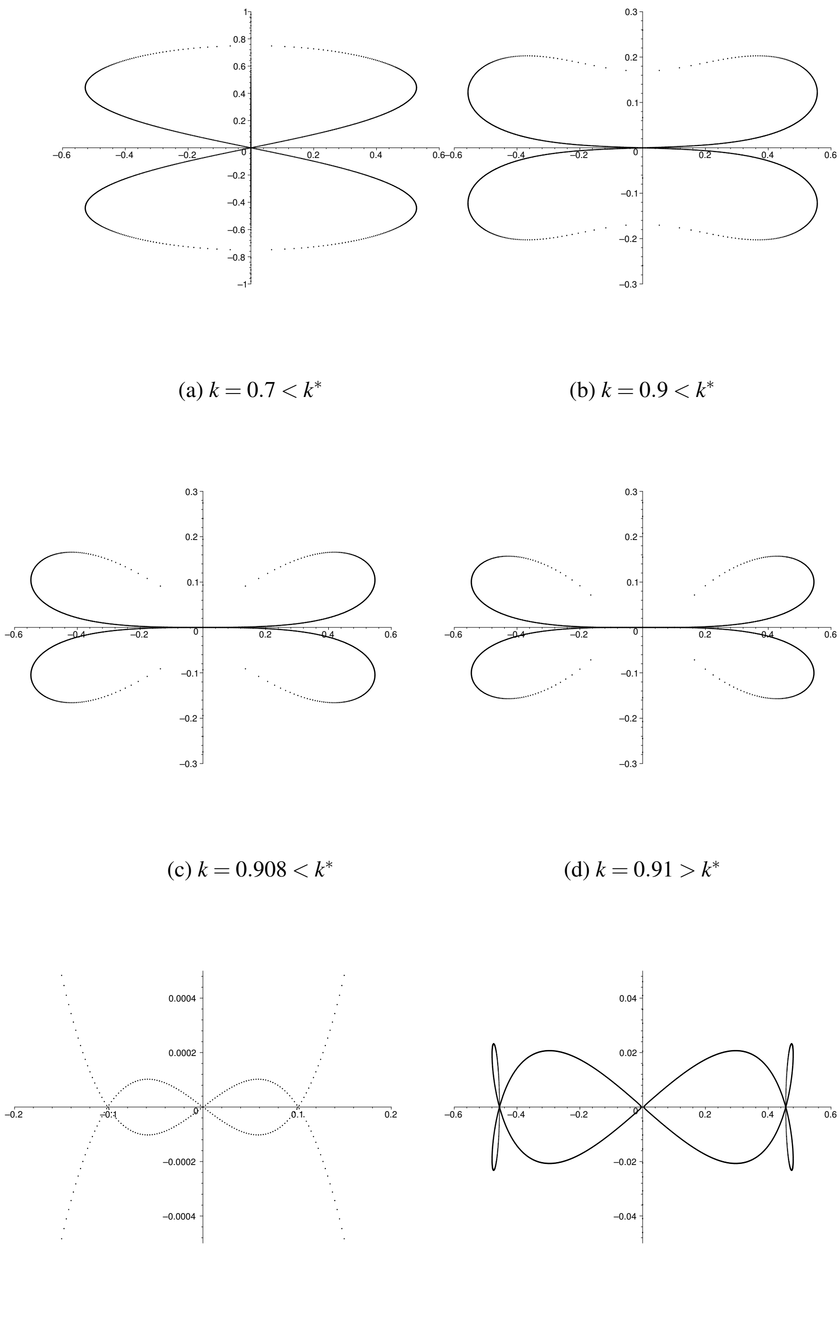

(a) $k = 0.7 < k^*$ (b) $k = 0.9 < k^*$

(c) $k = 0.908 < k^*$ (d) $k = 0.91 > k^*$

(e) $k = 0.91 > k^*$ (zoom) (f) $k = 0.95 > k^*$

Fig. 1 Numerical computations of $\sigma(\mathscr{J}\mathscr{L})$ for the cnoidal wave of focusing mKdV using the space $L^2_{\mathrm{per}}([-2nK(k)/\mu, 2nK(k)/\mu]; \mathbb{R})$ with $n = 400$. For a detailed explanation, see the main text.

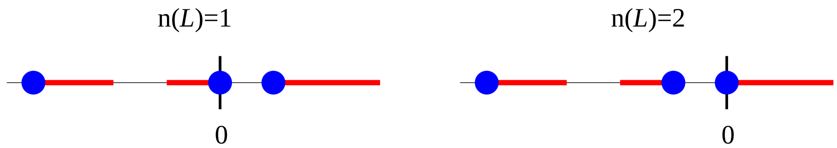

Fig. 2 The spectrum of the operator $\mathscr{L}$ for the perturbative solutions is marked with thick (red) lines. The eigenvalues associated with eigenfunctions which are 2π-periodic are marked with filled (blue) circles.