On the nonintegrability of equations for long- and short-wave interactions

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Abstract

We examine the integrability of two models used for the interaction of long and short waves in dispersive media. One is more classical but arguably cannot be derived from the underlying water wave equations, while the other one was recently derived. We use the method of Zakharov and Schulman to attempt to construct conserved quantities for these systems at different orders in the magnitude of the solutions. The coupled KdV-NLS model is shown to be nonintegrable, due to the presence of fourth-order resonances. A coupled real KdV - complex KdV system is shown to suffer the same fate, except for three special choices of the coefficients, where higher-order calculations or a different approach are necessary to conclude integrability or the absence thereof.

1 Introduction

Systems that couple long and short waves have generated significant interest recently (e.g. [3, 4, 9, 11, 13]). Much attention in this area has been devoted to the following system, known as the cubic nonlinear Schrödinger-Korteweg-deVries (NLS-KdV) system:

$$\begin{aligned} iu_t + u_{xx} + \alpha\left|u\right|^2 u &= -\beta uv, \\ v_t + \gamma v v_x + v_{xxx} &= -\beta(\left|u\right|^2)_x, \end{aligned} \tag{1}$$

where α, β and γ are real constants, $x \in \mathbb{R}$, v is a real-valued function, and u is a complex-valued function. Recently, it was shown that (1) cannot be consistently derived starting from the underlying water wave equations [12]. The following coupled KdV-CKdV (Complex KdV) model was suggested as an alternative with a consistent derivation:

$$\begin{aligned} u_t + 2\beta u_x + \alpha u_{xxx} &= -2\beta(uv)_x, \\ v_t + \beta v_x + \beta v v_x + \gamma v_{xxx} &= -\beta(\left|u\right|^2)_x. \end{aligned} \tag{2}$$

As above, v (u) is a real- (complex-) valued function and α, β and γ are real constants. We examine whether or not the two systems (1) and (2) are integrable in a sense detailed below.

A method for showing the nonintegrability of a system developed by Zakharov and Schulman [18, 19] distinguishes between *completely integrable systems* and *solvable systems*. Completely integrable systems are those for which we can find action-angle variables and solvable equations are those which can be solved by the inverse scattering transform (IST) [1]. Since integrability is a feature of the equations and not of a particular solution, we may always assume that we are working in a neighborhood of a solution with a nondegenerate linearization.

The test for complete integrability has the following steps:

1. Any completely integrable Hamiltonian system may be written locally in action-angle variables.
2. A system in action-angle variables is equivalent to a collection of uncoupled harmonic oscillators, so its Hamiltonian is quadratic.
3. Near-identity normal-form transformations [16] can be used to reduce any Hamiltonian to quadratic as long as there are no obstructions from resonances.
4. Any obstruction in the above steps due to resonances implies the system is not completely integrable.

The normal-form transformation that removes n-th order terms from the Hamiltonian gives rise to a resonance manifold which describes the process of scattering p waves ($p \in \mathbb{N}$) into $n-p$ waves. For example, if a system admits two dispersion laws $\omega^{(1)}$ and $\omega^{(2)}$, an n-th order resonance manifold is defined by

$$M = \left\{ (k_1, \ldots, k_n) \in \mathbb{R}^n \middle| \sum_{j=1}^{n} \sigma_j k_j = 0 \text{ and } \sum_{j=1}^{n} \sigma_j \omega^{(\ell)}(k_j) = 0 \right\}, \tag{3}$$

with any combination of $\sigma_j \in \{-1, 1\}$ and $\ell \in \{1, 2\}$. Associated with each resonance manifold is an amplitude function. If the amplitude vanishes on the resonance manifold then the singularity of the normal form transformation is removable and the transformation is valid. If the amplitude does not vanish on the resonance manifold, complete integrability is not possible but solvability may be.

The test for solvability has the following steps:

1. Every system solvable by the IST has an infinite hierarchy of equations solvable by the IST. The members of the hierarchy share conserved quantities.
2. By assumption, any equation solvable by the IST is linearizable with nondegenerate linearization, so each member of the hierarchy has quadratic terms in the Hamiltonian, at least in the small amplitude limit.
3. Every member of the hierarchy has a linearly independent Hamiltonian, so the original system has infinitely many conserved quantities with linearly independent quadratic terms (see e.g. [14]).
4. If there exist only finitely many conserved quantities with quadratic terms for our PDE equation, it is not solvable by the IST.

The method of Zakharov and Schulman begins by removing all higher-order nonresonant terms as above. Next an ansatz is made about the existence of an additional conserved quantity in a power series in terms of unknown amplitudes. Upon enforcing that the quantity is independent of t, resonance manifolds appear as above. However, in this case, the resonance manifold amplitude is multiplied by another quantity:

$$\sum_{j=1}^{n} \sigma_j \Phi^{(\ell)}(k_j),$$

where σ_j and ℓ are the same as in (3) and $\Phi^{(\ell)}$ are the unknown quadratic amplitudes in the power series. If the functions $\Phi^{(\ell)}$ can be found to satisfy this relationship and if they are linearly independent from the two relations defining the resonance manifold, then the manifold is called degenerate [15]. If any of the n-th order resonance manifolds are nondegenerate and have nonzero amplitude, the constructed quantity is not conserved. The fact that another conserved quantity with linearly independent quadratic terms cannot be constructed implies that the system must not be solvable by the IST.

Determining whether or not a resonance manifold is degenerate poses challenges. We use the theory of web geometry [6] to check degeneracy as described in Appendix A. In Sections 2 and 3 we examine the integrability of (1) and (2).

2 Coupled NLS & KdV Model

The Hamiltonian for (1) on the whole line is

$$H = \int \left(|u_x|^2 + \frac{1}{2}v_x^2 - \frac{\alpha}{2}|u|^4 - \frac{\gamma}{6}v^3 - \beta|u|^2 v \right) \mathrm{d}x,$$

for the variables $z = (u, iu^*, v)$ with non-canonical Poisson structure

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \partial_x \end{pmatrix},$$

so that (1) is equivalent to $z_t = J\delta H/\delta z$, where $\delta/\delta z$ denotes the variational gradient with respect to the components of z [5]. Here and throughout integrals without bounds are to be interpreted as whole line integrals. This system admits two types of waves with dispersion relations $\omega_k = k^2$ and $\Omega_k = -k^3$. Here and throughout, k subscripts are indices, not partial derivatives. We introduce the Fourier transform,

$$u(x) = \frac{1}{\sqrt{2\pi}} \int u(k) e^{ikx} \mathrm{d}k = \frac{1}{\sqrt{2\pi}} \int u_k e^{ikx} \mathrm{d}k. \tag{4}$$

Applying the Fourier transform to u and v results in a Hamiltonian system for (u_k, v_k) with Hamiltonian

$$\begin{aligned} H(u_k, v_k) = \int k^2 u_k u_k^* \mathrm{d}k + \int_0^\infty k^2 v_k v_k^* \mathrm{d}k - \frac{\beta}{\sqrt{2\pi}} \int u_1^* v_2 u_3 \delta_{1-2-3} \mathrm{d}_{123} \\ - \frac{\gamma}{6\sqrt{2\pi}} \int v_1 v_2 v_3 \delta_{123} \mathrm{d}_{123} - \frac{\alpha}{2(2\pi)} \int u_1 u_2 u_3^* u_4^* \delta_{12-3-4} \mathrm{d}_{1234}, \end{aligned} \tag{5}$$

where we use the notation $u_j = u_{k_j}, \mathrm{d}_{123} = \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3$, u_k^* denotes the complex conjugate of u_k, and $\delta_{12-3} = \delta(k_1 + k_2 - k_3)$ where $\delta(\cdot)$ is the Dirac-delta function. The integral with quadratic integrand in v_k found in (5) is reduced to an integral on the half-line using the fact that $v_k^* = v_{-k}$ since $v(x)$ is real. In Fourier variables, the dynamics are

$$i\dot{u}_k = \frac{\delta H}{\delta u_k^*}, \qquad \dot{v}_k = ik\frac{\delta H}{\delta v_k^*}.$$

We introduce a_k by

$$v_k = |k|^{1/2}(a_k \theta_{-k} + a_{-k}^* \theta_k),$$

where

$$\theta_k = \theta(k) = \begin{cases} 0, & k < 0, \\ 1, & k \geq 0, \end{cases}$$

is the Heaviside-function. The dynamics are

$$i\dot{u}_k = \frac{\delta H}{\delta u_k^*}, \qquad i\dot{a}_k = \frac{\delta H}{\delta a_k^*},$$

with

$$\begin{aligned} H(u_k, a_k) &= H_2(u_k, a_k) + H_3(u_k, a_k) + H_4(u_k, a_k), \\ H_2(u_k, a_k) &= \int \omega_k u_k u_k^* \mathrm{d}k + \int_{-\infty}^0 \Omega_k a_k a_k^* \mathrm{d}k, \\ H_3(u_k, a_k) &= \int U_{123}(a_1^* a_2 a_3 + a_1 a_2^* a_3^*) \delta_{1-2-3} \mathrm{d}_{123} + \int V_{123}(u_1^* a_2 u_3 + u_1 a_2^* u_3^*) \delta_{1-2-3} \mathrm{d}_{123}, \\ H_4(u_k, a_k) &= \int W_{1234} u_1 u_2 u_3^* u_4^* \delta_{12-3-4} \mathrm{d}_{1234}, \\ U_{123} = -\frac{\gamma}{2\sqrt{2\pi}} &\left|k_1 k_2 k_3\right|^{1/2} \theta_{-1}\theta_{-2}\theta_{-3}, \quad V_{123} = -\frac{\beta}{\sqrt{2\pi}} \left|k_2\right|^{1/2} \theta_{-2}, \quad W_{1234} = -\frac{\alpha}{2(2\pi)}. \end{aligned} \tag{6}$$

This system is identical to that used to study the integrability of Langmuir Waves [8] up to third-order and with $k \to -k$. The canonical near-identity transformation,

$$
\begin{aligned}
a_k &= \tilde{a}_k + \int \left(U^{(1)}_{012} \tilde{a}_1 \tilde{a}_2 - 2U^{(1)}_{120} \tilde{a}_1 \tilde{a}^*_2 - U^{(2)}_{102} \tilde{u}_1 \tilde{u}^*_2 \right) \mathrm{d}_{12}, \\
u_k &= \tilde{u}_k + \int \left(U^{(2)}_{012} \tilde{a}_1 \tilde{u}_2 - U^{(2)}_{210} \tilde{a}^*_1 \tilde{u}_2 \right) \mathrm{d}_{12}, \\
U^{(1)}_{\ell mn} &= -\frac{U_{\ell mn}}{\Omega_\ell - \Omega_m - \Omega_n} \delta_{\ell-m-n}, \qquad U^{(2)}_{\ell mn} = -\frac{V_{\ell mn}}{\omega_\ell - \Omega_m - \omega_n} \delta_{\ell-m-n},
\end{aligned}
\tag{7}
$$

removes third-order terms from the Hamiltonian so that $H(\tilde{u}_k, \tilde{a}_k) = H_2(\tilde{u}_k, \tilde{a}_k) + H_4(\tilde{u}_k, \tilde{a}_k) + \tilde{H}_4(\tilde{u}_k, \tilde{a}_k)$ where H_2 is unchanged from (6) and $\tilde{H}_4$ are the quartic terms which arise from H_3 under (7).

The transformation (7) gives rise to two resonance manifolds,

$$
\begin{aligned}
M_1 &= \left\{ (k_1, k_2, k_3) \in \mathbb{R}^3 : k_1 - k_2 - k_3 = 0 \text{ and } \Omega(k_1) - \Omega(k_2) - \Omega(k_3) = 0 \right\}, \\
M_2 &= \left\{ (k_1, k_2, k_3) \in \mathbb{R}^3 : k_1 - k_2 - k_3 = 0 \text{ and } \omega(k_1) - \Omega(k_2) - \omega(k_3) = 0 \right\}.
\end{aligned}
$$

The amplitudes $U_{123} = 0$ on M_1 and $V_{123} = 0$ on M_2 so the singularities in $U^{(1)}_{123}$ and $U^{(2)}_{123}$ are removable. Next we seek to remove the fourth-order terms from the Hamiltonian using a near-identity transformation. One resonance manifold appearing in such a transformation is defined by

$$
M_3 = \left\{ (k_1, k_2, k_3, k_4) \in \mathbb{R}^4 : k_1 + k_2 - k_3 - k_4 = 0 \text{ and } \omega(k_1) + \Omega(k_2) - \omega(k_3) - \Omega(k_4) = 0 \right\}, \tag{8}
$$

corresponding to the process of converting two waves k_1 and k_2 with frequency $\omega(k_1)$ and $\Omega(k_2)$ respectively to two with frequency $\omega(k_3)$ and $\Omega(k_4)$. The amplitude of this process is found by collecting the fourth-order terms which multiply the quantity $\delta_{12-3-4}/(\omega_1 + \Omega_2 - \omega_3 - \Omega_4)$:

$$
\begin{aligned}
T_{k_1,k_2,k_3,k_4} &= T^{(1)}_{k_1,k_2,k_3,k_4} + T^{(2)}_{k_1,k_2,k_3,k_4}, \\
T^{(1)}_{k_1,k_2,k_3,k_4} &= 2 \left(\frac{V_{k_3+k_4,k_4,k_3} V_{k_1+k_2,k_2,k_1}}{\omega_{k_1} + \Omega_{k_2} - \omega_{k_1+k_2}} + \frac{V_{k_1,k_4,k_1-k_4} V_{k_3,k_2,k_3-k_2}}{\omega_{k_1} - \Omega_{k_4} - \omega_{k_1-k_4}} \right) \\
&\quad + 4 \left(\frac{V_{k_1,k_1-k_3,k_3} U_{k_4,k_2,k_4-k_2}}{\Omega_{k_4} - \Omega_{k_2} - \Omega_{k_4-k_2}} + \frac{V_{k_3,k_3-k_1,k_1} U_{k_4,k_2,k_2-k_4}}{\Omega_{k_2} - \Omega_{k_4} - \Omega_{k_2-k_4}} \right), \\
T^{(2)}_{k_1,k_2,k_3,k_4} &= \frac{\omega_{k_1+k_2} V_{k_4+k_3,k_4,k_3} V_{k_1+k_2,k_2,k_1}}{(\omega_{k_4+k_3} - \Omega_{k_4} - \omega_{k_3})(\omega_{k_2+k_1} - \Omega_2 - \omega_1)} + \frac{\omega_{k_3-k_2} V_{k_3,k_2,k_3-k_2} V_{k_1,k_4,k_1-k_4}}{(\omega_{k_3} - \Omega_{k_2} - \omega_{k_3-k_2})(\omega_{k_1} - \Omega_{k_4} - \omega_{k_1-k_4})} \\
&\quad + 2 \frac{\Omega_{k_4-k_2} U_{k_4,k_2,k_4-k_2} V_{k_1,k_1-k_3,k_3}}{(\Omega_{k_4} - \Omega_{k_2} - \Omega_{k_4-k_2})(\omega_{k_1} - \Omega_{k_1-k_3} - \omega_{k_3})} + 2 \frac{\Omega_{k_2-k_4} U_{k_2,k_4,k_2-k_4} V_{k_3,k_3-k_1,k_1}}{(\Omega_{k_2} - \Omega_{k_4} - \Omega_{k_2-k_4})(\omega_3 - \Omega_{k_3-k_1} - \omega_{k_1})}.
\end{aligned}
$$

The quantity $T^{(1)}_{k_1,k_2,k_3,k_4}$ is (10) in [8] but $T^{(2)}_{k_1,k_2,k_3,k_4}$ is mistakenly omitted from the full expression for T_{k_1,k_2,k_3,k_4} [7]. The interaction coefficient $T^{(2)}_{k_1,k_2,k_3,k_4}$ originates from the product of the two quadratic terms of the transformation (7) when applied to the quadratic part of the Hamiltonian, H_2. The result of [8] remains unchanged, T_{k_1,k_2,k_3,k_4} is not identically zero on M_3.

The scattering process defined by the resonance manifold M_3 is proven to be nondegenerate in [8] using elementary methods. Here we use web geometry (Appendix A) to show nondegeneracy since this technique generalizes in a much more straightforward manner and will be used later for studying (2). We define a family of foliations of M_3 by

$$
k_j = \text{constant}, \quad j = 1, 2, 3, 4, \tag{9}
$$

which is a 4-web of M_3. The process (8) does not correspond to billiard scattering since M_3 can be parameterized by

$$
k_1 = \frac{1}{2} \left(-k_2^2 - k_2 k_4 - k_4^2 + k_4 - k_2 \right), \quad k_3 = \frac{1}{2} \left(-k_2^2 - k_2 k_4 - k_4^2 + k_2 - k_4 \right).
$$

We use Mathematica to calculate the invariants introduced in [2] to show that this 4-web is linearizable only if $\beta = 0$. However $\beta = 0$ corresponds to an uncoupled system of KdV and NLS equations and is known to be integrable. Since the web is not linearizable it must have rank 2, hence this process is not degenerate.

Since there exists a fourth-order resonance manifold with nonzero amplitude, fourth-order terms cannot be removed from the Hamiltonian, thus the system is not completely integrable. Since this resonance manifold is also nondegenerate, a new conserved quantity cannot be constructed with linearly independent quadratic part, so the system must not be solvable by the IST. Equation (1) is nonintegrable in either sense defined in Section 1.

3 Coupled KdV-CKdV Model

The Hamiltonian for (2) on the whole line is

$$H = \int \left(\frac{\alpha}{2}|u_x|^2 + \frac{\gamma}{2}v_x^2 - \frac{\beta}{6}v^3 - \beta|u|^2 v - \beta|u|^2 - \frac{\beta}{2}v^2 \right) \mathrm{d}x,$$

for the variables (u, iu^*, v). The dynamics are

$$u_t = 2\partial_x \frac{\delta H}{\delta u^*}, \qquad v_t = \partial_x \frac{\delta H}{\delta v}.$$

Equation (2) admits two types of waves with frequencies $\omega_k = 2\beta k - \alpha k^3$ and $\Omega_k = \beta k - \gamma k^3$. Applying the Fourier transform (4) to u and v results in a Hamiltonian system for (u_k, v_k) with Hamiltonian

$$\begin{aligned} H(u_k, v_k) = {} & \frac{1}{2}\int (\alpha k^2 - 2\beta) u_k u_k^* \mathrm{d}k + \int_0^\infty (\gamma k^2 - \beta) v_k v_k^* \mathrm{d}k - \frac{\beta}{6\sqrt{2\pi}} \int v_1 v_2 v_3 \delta_{123} \mathrm{d}_{123} \\ & - \frac{\beta}{\sqrt{2\pi}} \int u_1^* v_2 u_3 \delta_{1-2-3} \mathrm{d}_{123}, \end{aligned}$$

and dynamics

$$\dot{u}_k = 2ik \frac{\delta H}{\delta u_k^*}, \qquad \dot{v}_k = ik \frac{\delta H}{\delta v_k^*}.$$

Here as in (5), the integral with quadratic integrand in v_k is reduced to a half-line integral using the reality of $v(x)$.

Introducing the variables a_k and b_k by

$$v_k = |k|^{1/2}(a_k \theta_{-k} + a_{-k}^* \theta_k), \qquad u_k = |2k|^{1/2} b_{-k}^*,$$

the dynamical equations are rewritten as

$$i\dot{a}_k = \frac{\delta H}{\delta a_k^*}, \qquad i\dot{b}_k = \frac{\delta H}{\delta b_k^*}, \qquad -i\dot{b}_{-k} = \frac{\delta H}{\delta b_{-k}^*}, \tag{10}$$

with

$$\begin{aligned} H(a_k, b_k, b_{-k}) &= H_2(a_k, b_k, b_{-k}) + H_3(a_k, b_k, b_{-k}), \\ H_2(a_k, b_k, b_{-k}) &= \int_{-\infty}^0 \Omega_k a_k a_k^* \mathrm{d}k + \int_{-\infty}^0 \omega_k b_k b_k^* dk + \int_{-\infty}^0 \omega_k b_{-k} b_{-k}^* \mathrm{d}k, \\ H_3(a_k, b_k, b_{-k}) &= \int U_{123} \left(b_{-1} a_2 b_{-3}^* \delta_{1-2-3} + b_1^* a_2 b_{-3} \delta_{1-23} + b_1^* a_2 b_3 \delta_{1-2-3} \right) \mathrm{d}_{123} + c.c. \\ & \quad + \int V_{123} a_1^* a_2 a_3 \delta_{1-2-3} \mathrm{d}_{123} + c.c., \\ V_{123} &= -\frac{\beta}{2\sqrt{2\pi}} |k_1 k_2 k_3|^{1/2} \theta_{-1}\theta_{-2}\theta_{-3}, \qquad U_{123} = 4V_{123}, \end{aligned}$$

where *c.c.* is the complex conjugate of the preceding terms. The dynamical equations (10) are to be interpreted for $k<0$ only. Half line integrals are used so that our system is in normal Hamiltonian variables with the quadratic terms of the Hamiltonian being multiplied by the frequencies [17].

This system is Hamiltonian with canonical variables $(ia_k, a_k^*, ib_k, b_k^*, ib_{-k}^*, b_{-k})$ and canonical Poisson structure

$$J=\begin{pmatrix} J_1 & 0 & 0\\ 0 & J_1 & 0\\ 0 & 0 & J_1\end{pmatrix}, \qquad J_1=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}.$$

The canonical near-identity transformation to the variables $(i\tilde{a}_k, \tilde{a}_k^*, i\tilde{b}_k, \tilde{b}_k^*, i\tilde{b}_{-k}^*, \tilde{b}_{-k})$ given by

$$\begin{aligned}
a_k &= \tilde{a}_k + \int\left(\int A^{(1)}_{012}\tilde{a}_1\tilde{a}_2 - 2A^{(1)}_{120}\tilde{a}_1\tilde{a}_2^* - B^{(2)}_{102}\tilde{b}_{-1}^*\tilde{b}_{-2} - B^{(1)}_{102}\tilde{b}_1\tilde{b}_{-2}^* - B^{(2)}_{102}\tilde{b}_1\tilde{b}_2^*\right)\mathrm{d}_{12},\\
b_k &= \tilde{b}_k + \int\left(B^{(1)}_{012}\tilde{a}_1\tilde{b}_{-2} + B^{(2)}_{012}\tilde{a}_1\tilde{b}_2 - B^{(2)}_{120}\tilde{b}_1\tilde{a}_2^*\right)\mathrm{d}_{12},\\
b_{-k} &= \tilde{b}_{-k} + \int\left(-B^{(2)}_{120}\tilde{b}_{-1}\tilde{a}_2 + B^{(2)}_{012}\tilde{a}_1^*\tilde{b}_{-2} + B^{(1)}_{120}\tilde{b}_1\tilde{a}_2^*\right)\mathrm{d}_{12},\\
A^{(1)}_{\ell mn} &= -\frac{V_{\ell mn}}{\Omega_\ell-\Omega_m-\Omega_n}\delta_{\ell-m-n}, \quad B^{(1)}_{\ell mn} = -\frac{U_{\ell mn}}{\omega_\ell-\Omega_m+\omega_n}\delta_{\ell-mn}, \quad B^{(2)}_{\ell mn} = -\frac{U_{\ell mn}}{\omega_\ell-\Omega_m-\omega_n}\delta_{\ell-m-n},
\end{aligned} \tag{11}$$

removes third-order terms from the Hamiltonian. The transformation (11) gives rise to three separate three-wave resonance manifolds:

$$\begin{aligned}
M_1 &= \left\{(k_1,k_2,k_3)\in\mathbb{R}^3 : k_1-k_2-k_3=0 \text{ and } \Omega(k_1)-\Omega(k_2)-\Omega(k_3)=0\right\},\\
M_2 &= \left\{(k_1,k_2,k_3)\in\mathbb{R}^3 : k_1-k_2+k_3=0 \text{ and } \omega(k_1)-\Omega(k_2)+\omega(k_3)=0\right\},\\
M_3 &= \left\{(k_1,k_2,k_3)\in\mathbb{R}^3 : k_1-k_2-k_3=0 \text{ and } \omega(k_1)-\Omega(k_2)-\omega(k_3)=0\right\}.
\end{aligned}$$

Since the amplitude U_{123} vanishes on M_2 and M_3, $B^{(1)}_{123}$ and $B^{(2)}_{123}$ have removable singularities only. Further, the amplitude $V_{123}=0$ on M_1 unless $\gamma=0$. However, the process defining M_1 is degenerate since all three-wave interaction processes in one-dimension are degenerate [18]. Thus we must try to remove fourth-order terms.

One resonance manifold which appears when attempting to remove fourth-order terms from H is defined by

$$\mathcal{M}_1=\{(k_1,k_2,k_3,k_4): k_1=k_2+k_3+k_4 \text{ and } \omega_1=\Omega_2+\omega_3+\Omega_4\}.$$

This manifold splits into two components with local coordinates,

$$\begin{aligned}
k_1 &= \frac{1}{2}(k_2+k_4)\pm\frac{1}{2\sqrt{3\alpha}}\left[4\beta-(k_2^2+k_4^2)(\alpha-4\gamma)-2k_2k_4(\alpha+2\gamma)\right]^{1/2},\\
k_3 &= -\frac{1}{2}(k_2+k_4)\pm\frac{1}{2\sqrt{3\alpha}}\left[4\beta-(k_2^2+k_4^2)(\alpha-4\gamma)-2k_2k_4(\alpha+2\gamma)\right]^{1/2},
\end{aligned} \tag{12}$$

where the plus/minus in k_1 and k_3 are to be taken the same on each part of the manifold which we label $\mathcal{M}_1^+$ and $\mathcal{M}_1^-$. Defining a family of foliations of $\mathcal{M}_1^+$ and $\mathcal{M}_1^-$ as in (9), we find the 4-web is linearizable only in three cases: (i) $\alpha=0$ (for which a parameterization different from (12) must be used), (ii) $\gamma=0$, and (iii) $\alpha=\gamma$. We ignore the case $\beta=0$ since this corresponds to two uncoupled linear PDEs which are integrable. In any of the above three cases, the process defining $\mathcal{M}_1$ is degenerate. The amplitude of this process is given by

$$\begin{aligned}
T^{(1)}_{k_1,k_2,k_3,k_4} &= P^{(1)}_{k_1,k_2,k_3,k_4} + S^{(1)}_{k_1,k_2,k_3,k_4},\\
P^{(1)}_{k_1,k_2,k_3,k_4} &= -\frac{\Omega_{k_1-k_3}U_{k_1,k_1-k_3,k_3}V_{k_2+k_4,k_2,k_4}}{(\omega_{k_1}-\Omega_{k_1-k_3}-\omega_{k_3})(\Omega_{k_2+k_4}-\Omega_{k_2}-\Omega_{k_4})} - \frac{\omega_{k_1-k_2}U_{k_1,k_2,k_1-k_2}U_{k_3+k_4,k_3,k_4}}{(\omega_{k_1}-\Omega_{k_2}-\omega_{k_1-k_2})(\omega_{k_3+k_4}-\Omega_{k_4}-\omega_{k_3})},\\
S^{(1)}_{k_1,k_2,k_3,k_4} &= 2\frac{U_{k_2+k_3,k_2,k_3}U_{k_1,k_4,k_1-k_4}}{\omega_{k_1}-\Omega_{k_4}-\omega_{k_1-k_4}} + 2\frac{V_{k_2+k_4,k_2,k_4}U_{k_1,k_1-k_3,k_3}}{\omega_{k_1}-\Omega_{k_1-k_3}-\omega_{k_3}},
\end{aligned}$$

defined on $\mathcal{M}_1$. We restrict attention to $k_j < 0, j = 1,2,3,4$, and find that both $P^{(1)}$ and $S^{(1)}$ are strictly negative for $\mathcal{A}_1 = \{\alpha < 0, \beta > 0, \alpha < \gamma < 0\}$ and strictly positive for $\mathcal{A}_2 = \{\alpha > 0, \beta < 0, \alpha > \gamma > 0\}$. It follows that $T^{(1)} \neq 0$ on both $\mathcal{A}_1$ and $\mathcal{A}_2$. The complement of $\mathcal{A}_1 \cup \mathcal{A}_2$ gives exactly the three cases mentioned above: (i) $\alpha = 0$, (ii) $\gamma = 0$, and (iii) $\alpha = \gamma$, (again ignoring $\beta = 0$). It follows that fourth-order terms cannot be removed from the Hamiltonian using a normal form transformation and thus the system (2) cannot be integrable except possibly in these three cases.

Other resonance manifolds appear when attempting to remove fourth-order terms from H. They are defined by

$$\begin{aligned}
\mathcal{M}_2 &= \{(k_1,k_2,k_3,k_4) : k_1 + k_2 = k_3 + k_4 \text{ and } \omega_1 + \Omega_2 = \omega_3 + \Omega_4\},\\
\mathcal{M}_3 &= \{(k_1,k_2,k_3,k_4) : k_1 + k_2 + k_3 = k_4 \text{ and } \omega_1 + \Omega_2 + \omega_3 = \Omega_4\},\\
\mathcal{M}_4 &= \{(k_1,k_2,k_3,k_4) : k_1 + k_2 + k_3 + k_4 = 0 \text{ and } \omega_1 + \Omega_2 + \omega_3 + \Omega_4 = 0\},\\
\mathcal{M}_5 &= \{(k_1,k_2,k_3,k_4) : k_1 + k_3 = k_2 + k_4 \text{ and } \omega_1 + \omega_3 = \Omega_2 + \Omega_4\}.
\end{aligned}$$

The investigation of each manifold results in resonances except in the three cases (i) $\alpha = 0$, (ii) $\gamma = 0$, and (iii) $\alpha = \gamma$.

The three singled-out systems are:

$$\begin{aligned}
&\text{(i)} \quad \alpha = 0: \begin{cases} u_t + 2\beta u_x = -2\beta(uv)_x, \\ v_t + \beta v_x + \beta v v_x + \gamma v_{xxx} = -\beta(|u|^2)_x. \end{cases}\\
&\text{(ii)} \quad \gamma = 0: \begin{cases} u_t + 2\beta u_x + \alpha u_{xxx} = -2\beta(uv)_x, \\ v_t + \beta v_x + \beta v v_x = -\beta(|u|^2)_x. \end{cases}\\
&\text{(iii)} \quad \alpha = \gamma: \begin{cases} u_t + 2\beta u_x + \gamma u_{xxx} = -2\beta(uv)_x, \\ v_t + \beta v_x + \beta v v_x + \gamma v_{xxx} = -\beta(|u|^2)_x. \end{cases}
\end{aligned} \tag{13}$$

In order to determine if these systems are integrable, one must look to remove fourth-order terms from the Hamiltonian. This is not pursued here. The further investigation of these singled-out systems and their potential physical relevance is an interesting topic for future study.

4 Conclusion

Using normal-form theory, we find that the coupled NLS-KdV system (1) is not integrable by either definition in Section 1 since it has a nondegenerate fourth-order resonance manifold with nonzero amplitude. We find that the coupled KdV-CKdV system (2) is not integrable for the same reasons except potentially for three choices of parameters: (i) $\alpha = 0$, (ii) $\alpha = \gamma$, or (iii) $\gamma = 0$. We cannot verify the integrability of equations (13) using the methods described in this paper. In particular, the tools borrowed from the theory of web geometry cannot be used when looking at fifth-order resonances and higher since the results on linearizability and rank are unique to 4-webs. Our methods do however provide a way to isolate potentially interesting problems and can be used to show the nonintegrability of other systems of equations, particularly those with complicated four-wave interactions.

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Appendix A Web geometry background

It is sufficient for our purposes to define web geometry [10] for 2-dimensional manifolds.

Definition Let (x, y) be local coordinates for a 2D (real) manifold. Then a *d-web* is the local foliation of the manifold by d curves defined by

$$u_j(x, y) = \text{const}, \quad 1 \leq j \leq d,$$

where $u_j(x, y)$ are smooth functions.

We need a definition regarding the geometry of the webs.

Definition A d-web is *linearizable* if it is diffeomorphic to a d-web formed by d one-parameter foliations of straight lines on the plane [2].

Example For a 2D manifold with local coordinates (x, y), the two families of curves

$$x = c_1, \quad y = c_2, \quad \frac{x}{y} = c_3, \quad x + y = c_4,$$

for $c_1, c_2, c_3, c_4 \in \mathbb{R}$ define a 4-web. The web is linear since the defining curves are lines. Since the web is linear, it is trivially linearizable.

For our purposes d will always equal 4. An important invariant of a given web is the *rank* of the web.

Definition The *rank of a d-web* is equal to the number of linearly independent relations of the form

$$\sum_{j=1}^{d} f_j(x, y) = 0.$$

The 4-webs we work with are always defined on the resonance manifold M (3) and hence always have rank at least equal to 2 since

$$\sum_{j=1}^{4} \sigma_j k_j = 0, \quad \text{and} \quad \sum_{j=1}^{4} \sigma_j \omega^{(\ell)}(k_j) = 0. \tag{14}$$

The rank of a 4-web is equal to $2, 3$ or ∞ [10]. Generally infinite-rank webs are disregarded in the theory of web geometry, but in our application they are possible. The web has infinite rank when the only solution to (14) is of the form $k_\alpha = k_\beta$ and $k_\gamma = k_\delta$ for $(\alpha, \beta, \gamma, \delta) \in \{1, 2, 3, 4\}$. This corresponds to so-called billiard scattering [6]. Therefore as long as the resonance manifold does not correspond to billiard scattering, the rank of the web is either 2 or 3. Poincaré's Theorem of web geometry states that a planar 4-web is of rank three if it is linearizable. Since our resonance manifolds are degenerate if there exists another linearly independent relation on the manifold, it is sufficient to determine whether or not a 4-web defined on the manifold is linearizable to determine if it is degenerate. To determine if a 4-web is linearizable, we use the algorithm and Mathematica code developed in [2].

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