The orbital stability of the cnoidal waves of the Korteweg–de Vries equation

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A R T I C L E I N F O	A B S T R A C T
	The cnoidal wave solution of the integrable Korteweg–de Vries equation is the most basic of its periodic solutions. Following earlier work where the linear stability of these solutions was established, we prove in this Letter that cnoidal waves are (nonlinearly) orbitally stable with respect to so-called subharmonic perturbations: perturbations that are periodic with period any integer multiple of the cnoidal-wave period. Our method of proof combines the construction of an appropriate Lyapunov function with the seminal results of Grillakis, Shatah and Strauss (1987, 1990) [17,18]. The integrability of the Korteweg–de Vries equation is used in that we need the presence of at least one extra conserved quantity in addition to those expected from the Lie point symmetries of the equation. © 2010 Elsevier B.V. All rights reserved.

1. Introduction

The stability of spatially periodic stationary solutions of nonlinear wave equations has seen different advances the past few years. There are advances both in the numerical investigation of spectral stability [12,26], as well as in the analytical study of spectral and orbital stability (see [4,6,20,14,15,8,3,19] and the references therein). For integrable equations such as the Korteweg–de Vries (KdV) equation,

$$u_t + \left(\frac{u^2}{2} + u_{xx} \right)_x = 0, \tag{1.1}$$

the investigations can be made more concrete, leading to an explicit determination of the spectrum of the operator for the spectral stability problem. We show in this Letter that explicit orbital stability results can also be obtained.

We consider the cnoidal wave solutions of (1.1):

$$u(x,t) = 12k^2 \operatorname{cn}^2(x - ct, k), \quad c = 8k^2 - 4. \tag{1.2}$$

Here $\operatorname{cn}(x,k)$ is the Jacobi elliptic cosine function with elliptic modulus $k \in [0, 1)$ [2,9]. It generalizes the cosine function to which it is equal for $k = 0$, and limits to a hyperbolic secant as $k \to 1^-$. The cnoidal wave is periodic with period $T(k) = 2K(k)$, where $K(k)$

is the complete elliptic integral of the first kind [2,9] given by

$$K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 z}} \, dz. \tag{1.3}$$

The cnoidal waves are the simplest of the periodic solutions of the KdV equation, represent the most general stationary (in a frame translating with speed c) solution of the equation, and contain the famous soliton solution as a limit case. The solutions are of practical importance in many applications such as, for instance, near-shore ocean waves [27]. As such the issue of their stability is important. Further, it is clear from this particular application that it is insufficient to consider the stability of the cnoidal waves with respect to perturbations of period $T(k)$. Indeed, physically the only restriction on the perturbations is that they be continuous and bounded on $x \in \mathbb{R}$. In order to work in a more friendly setting in terms of the function space we can consider, we restrict ourselves to so-called *subharmonic* perturbations[1]: perturbations that are periodic with period $nT(k)$, where n is any nonzero positive integer. Note that the class of subharmonic perturbations is larger than that of perturbations of the same period, and is a significant step closer to the desired class of perturbations.

The spectral and linear stability of the cnoidal waves were the subject of [6], and their orbital stability with respect to harmonic (same period) perturbations was covered in [25] (using integrability) and [4] (not using integrability). Leaving the realm of harmonic

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[1] Note that superharmonic perturbations (perturbations whose period is the base period divided by a positive (> 1) integer) are covered by studying perturbations that have the same period as the underlying solution.

perturbations leaves us with the question posed by Benjamin in 1972 [5, p. 12]:

"The question thus arises whether, when released for the present sufficient condition for stability (i.e. *restriction to perturbations with the same period*), *periodic waves become unstable."*

In other words, Benjamin was asking whether waves which are energetically unstable, but are spectrally stable, can be orbitally stable. Benjamin was exploring the possibility of extending the methods he used to establish the orbital stability of the soliton solution of the KdV equation to the stability analysis of the cnoidal wave solution. His considerations indicated that such should be possible when considering harmonic perturbations. In that case it appeared a Lyapunov function could be used which was in essence the energy function for the KdV equation, see below. This indication was proven to be correct more than 30 years later [4]. Next, Benjamin showed that these methods would fail, and the energy functional would not act as a Lyapunov functional for the cnoidal waves, when considering subharmonic perturbations (even when their period was only twice the period of the cnoidal wave). He examined this problem in some detail and provided some indications that point to the instability of the cnoidal waves with respect to this class of perturbations. Ultimately, no rigorous results were found, and the problem remained open. Bottman and Deconinck [6] were able to confirm the spectral and linear stability of the cnoidal waves with respect to bounded perturbations.

Up until now the question of the orbital stability of these waves with respect to subharmonic perturbations was open, primarily because in this case the wave is not a local minimizer of a constrained energy. By using the integrable structure associated with KdV, the results of [6] on the nature of the spectrum for the linearized operator, and the Lyapunov function construction ideas presented in [24], we are able to show that these energetically unstable waves are indeed orbitally stable.

Remark. The method used in this Letter can be easily generalized to study the orbital stability of periodic traveling wave solutions of integrable equations with respect to subharmonic perturbations. In fact, manuscripts discussing the stability of the periodic traveling wave solutions for the defocusing Nonlinear Schrödinger equation and the modified KdV equation have been submitted [7,13].

2. Linear stability results for the cnoidal wave solutions

We rewrite the KdV equation in a frame moving with constant velocity c, so that the cnoidal wave is a stationary (i.e., time-independent) solution of the equation. Let

$$y = x - ct, \qquad \tau = t. \tag{2.1}$$

The KdV equation becomes

$$u_\tau + \left(\frac{u^2}{2} + u_{yy} - cu\right)_y = 0, \tag{2.2}$$

with stationary solution

$$U(y) = 12k^2 \operatorname{cn}^2(y, k). \tag{2.3}$$

The linear stability problem for this solution is given by

$$v_\tau + (Uv + v_{yy} - cv)_y = 0, \tag{2.4}$$

obtained by discarding terms of order two or higher in (2.2), when we let $u(y, \tau) = U(y) + \epsilon v(y, \tau) + \mathcal{O}(\epsilon^2)$. The study of this linear stability problem is the focus of [6]. In this section, we review those results from [6] that are necessary for our considerations below.

The linear stability problem (2.4) is autonomous in τ. A separation of variables $v(y, \tau) = e^{\lambda \tau} V(y)$ yields the spectral problem

$$\lambda V = \partial_y \mathcal{L} V, \quad \mathcal{L} = -\partial_y^2 + c - U. \tag{2.5}$$

Spectral stability of the cnoidal wave (2.3) with respect to perturbations that are bounded on the whole line is established by demonstrating that the spectrum $\sigma(\partial_y \mathcal{L})$ of the operator $\partial_y \mathcal{L}$ does not intersect the right-half complex λ plane. Since (2.2) is a Hamiltonian partial differential equation [16], the spectrum $\sigma(\partial_y \mathcal{L})$ is symmetric under reflection with respect to both the real and imaginary axes. As a consequence, spectral stability requires proving that $\sigma(\partial_y \mathcal{L})$ is confined to the imaginary axis. In order to do so, Bottman and Deconinck [6] exploit the squared-eigenfunction connection [1] between solutions of (2.4) and the components of the eigenfunction of the Lax pair for the KdV equation. We review the steps of this process, omitting their calculational details, which can be found in [6].

The KdV equation (2.2) is an integrable equation [23,16,6] with Lax pair given by

$$\psi_y = \begin{pmatrix} 0 & 1 \\ \zeta - u/6 & 0 \end{pmatrix} \psi, \tag{2.6}$$

and

$$\psi_\tau = \begin{pmatrix} u_y/6 & -4\zeta + c - u/3 \\ -4\zeta^2 + c\zeta + (u^2 - 3cu + 6\zeta u + 3u_{yy})/18 & -u_y/6 \end{pmatrix} \psi. \tag{2.7}$$

Since (2.6) is equivalent to the linear Schrödinger equation, the spectral parameter ζ belongs to a subset of the real line. By restricting the Lax pair (2.6)–(2.7) to the stationary solution $u(y, \tau) = U(y)$, the spectrum σ_L of the Lax problem (2.6) can be determined explicitly. First, (2.7) with $u(y, \tau) = U(y)$ is autonomous in τ, from which it follows that

$$\psi(y, \tau) = e^{\Omega \tau} \begin{pmatrix} c\alpha(y) \\ \beta(y) \end{pmatrix}, \tag{2.8}$$

where $\beta = \alpha_y$, using (2.6). A priori, the separation constant Ω may depend on both y and ζ. Using the commutativity of (2.6) and (2.7), it follows that Ω does not depend on y; in fact, one finds explicitly

$$\Omega^2 = 16(\zeta - k^2 + 1)(\zeta - 2k^2 + 1)(\zeta - k^2). \tag{2.9}$$

The vector $(\alpha, \beta)^T$ is easily determined from (2.7) and subsequent use of (2.6). One finds

$$\begin{pmatrix} \alpha(y) \\ \beta(y) \end{pmatrix} = \gamma_0 \exp\left(\int \frac{-U_y/6 - \Omega}{4\zeta - c + U/3} \, dy\right) \begin{pmatrix} c4\zeta - c + U/3 \\ U_y/6 - \Omega \end{pmatrix}, \tag{2.10}$$

where γ_0 is a real constant. Here $\int \cdot dy$ denotes the antiderivative with respect to y. The constant of integration affects only the value of γ_0. Using the explicit form of the eigenfunction ψ, the statement $\zeta \in \sigma_L$ is found to be equivalent with

$$\operatorname{Re}\left\langle \frac{-U_y/6 - \Omega}{4\zeta - c + U/3} \right\rangle = 0, \tag{2.11}$$

where $\langle \cdot \rangle$ denotes the spatial average of its argument. From this, it follows easily that

$$\sigma_L = \left\{ \zeta \in \mathbb{R} \colon \zeta \in \left(-\infty, k^2 - 1\right] \cup \left[2k^2 - 1, k^2\right] \right\}. \tag{2.12}$$

Having determined the solution of the Lax pair equations (2.6)–(2.7), we turn to the squared eigenfunction connection, from which it follows that $\psi_1(y, \tau)\psi_2(y, \tau)$ solves (2.4), see [6]. The conclusion that $\alpha(y)\beta(y)$ solves (2.5) with $\lambda = 2\Omega$ is immediate. In fact, it is shown in [6] that all bounded solutions of (2.5) are obtained

this way. Since for the values of $\zeta \in \sigma_L$, $\lambda = 2\Omega$ is imaginary, the spectral stability of the cnoidal wave solutions follows. An application of the SCS Basis Lemma (with parameters $s = 1$, $r = 3/5$, $r' = 4/5$, using the notation of [20]) allows for the conclusion of linear stability.

3. Orbital stability of the cnoidal wave solutions

Using the results reviewed in the previous section, we now show that the cnoidal waves (2.3) are stable when considered on the space $L^2_{\mathrm{per}}([-nK(k), nK(k)])$ for any $n \in \mathbb{N}$, i.e., with respect to subharmonic perturbations. In essence our approach follows the algorithm outlined in [21]. Using a sufficiently large number of conserved quantities of the KdV equation, we are able to construct a Lyapunov functional for the cnoidal waves, hence establishing so-called *formal stability*. Next, we verify that the additional assumptions of Theorem 1 in [17] are satisfied, from which orbital stability follows.

We first show that the cnoidal waves are not constrained minimizers of the "natural" energy for the KdV equation. This problem is overcome by considering the extra conserved quantities of the KdV equation, as shown explicitly below. The fact that we are able to construct a Lyapunov functional when it is clear the energy cannot play this role, is a consequence of the integrability of the KdV equation, which provides us with an infinite number of candidate Lyapunov functionals. This same idea was used recently to establish the orbital stability of the multi-soliton solutions of equations in the AKNS hierarchy [22]. Particularly relevant for us is the seminal work of [24], as it is particularly devoted to the case of the KdV equation.

Note that the operator $\mathcal{L}$ is the Hessian (second variation) of the Hamiltonian

$$H = \frac{1}{2} \int_{-nK(k)}^{nK(k)} \left(u_y^2 + cu^2 - \frac{1}{3}u^3 \right) dy \qquad (3.1)$$

for the KdV equation (2.2), evaluated at the cnoidal wave. We begin by calculating the Krein signature associated with the operator $\mathcal{L}$, parametrized by the Lax spectral parameter ζ. The Krein signature is defined as the sign of

$$\mathcal{K}_1 = \langle V, \mathcal{L}V \rangle_n := \int_{-nK(k)}^{nK(k)} V^* \mathcal{L}V \, dy, \qquad (3.2)$$

where $n \in \mathbb{N}$ and V is a solution of the spectral problem (2.5). We wish to calculate $\mathcal{K}_1$ for all bounded V. As was shown in [6] and stated in the previous section, all such V are of the form $V = \alpha\beta = \alpha\alpha_y$ for all $\zeta \in \sigma_L$, and we may consider $\mathcal{K}_1$ as being parameterized by the Lax spectral parameter $\zeta \in \sigma_L$. We have

$$V = \gamma^2 \left(4\zeta - c + \frac{U}{3} \right) \left(\frac{U_y}{6} - \Omega(\zeta) \right), \qquad (3.3)$$

for $\zeta \in (-\infty, k^2 - 1] \cup [2k^2 - 1, k^2]$. Further,

$$\begin{aligned} \gamma(y) &= \gamma_0 \exp\left(\int \frac{-U'/6 - \Omega(\zeta)}{4\zeta - c + U/3} \, dy \right) \\ &= \frac{\gamma_0}{2(\zeta - k^2 - k^2 \mathrm{sn}^2(y,k) + 1)^{1/2}} e^{i \arg \gamma(y)}, \end{aligned} \qquad (3.4)$$

where $\arg \gamma(x)$ is the argument of $\gamma(x)$. Note that (3.3) represents two eigenfunctions, one for each branch of the Riemann surface defined by $\Omega(\zeta)$.

A direct calculation shows that the integrand of $\mathcal{K}_1(\zeta)$ is given by

$$\begin{aligned} V^* \mathcal{L}V &= -64 |\gamma(y)|^4 \Omega(\zeta) \left(\zeta - k^2 - k^2 \mathrm{sn}^2(y,k) + 1 \right)^3 \\ &\quad \times \left(4k^2 \mathrm{cn}(y,k) \mathrm{dn}(y,k) \mathrm{sn}(y,k) - \Omega(\zeta) \right) \\ &= -4\Omega(\zeta) \left(\zeta - k^2 - k^2 \mathrm{sn}^2(y,k) + 1 \right) \\ &\quad \times \left(4k^2 \mathrm{cn}(y,k) \mathrm{dn}(y,k) \mathrm{sn}(y,k) - \Omega(\zeta) \right). \end{aligned}$$

Since

$$\mathrm{cn}(y,k) \mathrm{dn}(y,k) \mathrm{sn}(y,k) \sim \frac{\partial}{\partial y} \left(\zeta - k^2 - k^2 \mathrm{sn}^2(y,k) + 1 \right),$$

(where $\sim$ denotes proportionality up to a constant factor) the term

$$-\Omega(\zeta) \left(\zeta - k^2 - k^2 \mathrm{sn}^2(y,k) + 1 \right) \mathrm{cn}(y,k) \mathrm{dn}(y,k) \mathrm{sn}(y,k)$$

does not contribute to $\mathcal{K}_1(\lambda)$, as the integral is over a number of periods of the periodic integrand. Thus, we can conclude that

$$\mathcal{K}_1(\zeta) = 4\Omega^2(\zeta) \int_{-nK(k)}^{nK(k)} \left(\zeta - k^2 - k^2 \mathrm{sn}^2(y,k) + 1 \right) dy. \qquad (3.5)$$

Instead of using the above direct calculations, the reader may note that, up to a y derivative, $V^* \mathcal{L}V = \Omega\alpha^2 \partial_y (\alpha^*)^2$, leading to the same result.

There are two parts to the Lax spectrum σ_L, as shown in [6, Figure 4]. As reiterated above, $\zeta \in (-\infty, k^2 - 1]$ or $\zeta \in [2k^2 - 1, k^2]$. This gives rise to two parts of the essential spectrum of $\partial_y \mathcal{L}$. The semi-infinite component of σ_L gives rise to a single covering of the imaginary axis, while an additional double covering of a symmetric interval on the imaginary axis around the origin corresponds to $\zeta \in [2k^2 - 1, k^2]$. Restricting ourselves to eigenfunctions with period $2nK(k)$ results in a discrete subset of these spectral sets. Considering the above, we observe that for $\zeta \leqslant k^2 - 1$ the integrand in (3.5) is negative, resulting in $\mathcal{K}_1(\zeta) \geqslant 0$, with equality attained only for $\zeta = k^2 - 1$. This corresponds to the single covering of the imaginary axis having positive Krein signature. Next, when $\zeta \in [2k^2 - 1, k^2]$ the integrand in (3.5) is positive so that $\mathcal{K}_1(\zeta) \leqslant 0$, with equality attained again at the endpoints only. Thus the double covering of the imaginary interval around the origin has negative Krein signature. Although informative, no orbital stability conclusion about the KdV cnoidal wave can be reached from these considerations since $\mathcal{K}_1(\zeta)$ does not have definite sign.

At this point, we invoke more tools arising from the integrability of KdV. The KdV equation is the first nontrivial equation of an infinite hierarchy of partial differential equations that have mutually commuting flows. The equations in the hierarchy dictate how the function u changes under the flow of these different dynamical systems, parameterized by their respective time variable. The first three members of this hierarchy [1] are

$$u_{\tau_0} = u_y, \qquad (3.6)$$

$$u_{\tau_1} = -uu_y - u_{yyy} + c_{10} u_y, \qquad (3.7)$$

$$\begin{aligned} u_{\tau_2} &= \frac{5}{6} u^2 u_y + \frac{10}{3} u_y u_{yy} + \frac{5}{3} u u_{yyy} \\ &\quad + u_{yyyyy} + c_{21}(-uu_y - u_{yyy}) + c_{20} u_y. \end{aligned} \qquad (3.8)$$

It is clear that $\tau_1 = \tau$, provided $c_{10} = c$. It is common in the literature to equate all constants c_{jk} to zero, but the inclusion of the lower flows with each given flow is important for our purposes, as may already be obvious from the fact that (2.2) only appears for nonzero choice of c_{10}. Clearly the inclusion of the lower equations does not affect the commutativity of the different flows.

Each equation in the KdV hierarchy is Hamiltonian,

$$u_{\tau_k} = \partial_y H'_k(u), \qquad (3.9)$$

where the prime denotes the variational derivative of the Hamiltonian H with respect to u. For the k-th KdV equation, the Hamiltonian $H_k(u)$ depends on u and its first k derivatives with respect to y. For instance,

$$H_2(u) = \int_{-nK(k)}^{nK(k)} \left(\frac{1}{2} u_{yy}^2 - \frac{5}{6} u u_y^2 + \frac{5}{72} u^4 \right.$$
$$\left. + c_{21} \left(\frac{1}{2} u_y^2 - \frac{1}{6} u^3 \right) + c_{20} \frac{1}{2} u^2 \right) dy. \tag{3.10}$$

Since all the flows in the KdV hierarchy commute, the cnoidal wave solution (2.3) of KdV (2.2) is a stationary (with respect to τ_2) solution of the second KdV equation for a suitable choice of the constants c_{21} and c_{20}. This fixes c_{20} in terms of c_{21}, but it imposes no constraint on c_{21}, since the cnoidal wave (2.3) is a stationary solution of (2.2). Thus, c_{21} is a free parameter: for *any* choice of c_{21} the cnoidal wave solution (2.3) is a stationary periodic solution of the second KdV equation (3.8). Indeed, direct substitution of the cnoidal wave solution shows that

$$c_{20} = c_{21}(8k^2 - 4) - 56k^4 + 56k^2 - 16. \tag{3.11}$$

We now consider the stability problem with respect to the dynamical variable τ_2 for the cnoidal wave solution. The linear stability problem is given by

$$v_{\tau_2} = \partial_y \mathcal{L}_2 v, \tag{3.12}$$

with associated spectral stability problem

$$\lambda_2 V = \partial_y \mathcal{L}_2 V, \qquad v = e^{\lambda_2 \tau_2} V, \tag{3.13}$$

where

$$\mathcal{L}_2 = \partial_y^4 + \frac{5}{3} U(y) \partial_y^2 + \frac{5}{3} U'(y) \partial_y$$
$$+ \frac{5}{3} U''(y) + \frac{5}{6} U^2(y) + c_{21}\left(-\partial_y^2 + c - U(y) \right) - 16$$
$$+ 56k^2(1 - k^2).$$

The freedom in choosing c_{21} gives rise to a one-parameter family of linear stability problems. The Krein signature $\mathcal{K}_2$ associated with the eigenvalues of the operator $\mathcal{J}\mathcal{L}_2$ is the sign of

$$\mathcal{K}_2 := \langle V, \mathcal{L}_2 V \rangle = \int_{-nK(k)}^{nK(k)} V^* \mathcal{L}_2 V \, dy. \tag{3.14}$$

The eigenfunction $V(y)$ is the same as before, due to the commuting property of the flows. This is also easily verified using a direct calculation. All eigenfunctions of $\mathcal{J}\mathcal{L}_2$ are obtained this way, since the eigenfunctions V form a complete set in $L^2_{\text{per}}([-nL, nL])$, see [6]. We remind the reader that this statement requires the invertibility of the Poisson operator ∂_y of the KdV hierarchy, requiring us to fix the average of the solutions of the hierarchy, see also [11]. In other words, we are considering perturbations that do not alter the average of the cnoidal wave. Since $\mathcal{K}_2$ is computed for the same set of eigenfunctions as $\mathcal{K}_1$, we may think of $\mathcal{K}_2$ as being parameterized by the Lax spectral parameter ζ in the same way that $\mathcal{K}_1$ is.

A simple linear algebra calculation along the lines of that done in [6, Section 5] using the Lax pair of the second KdV equation demonstrates that

$$\Omega_2^2(\zeta) = \Omega^2(\zeta)\left(4\zeta - c_{21} + 8k^2 - 4\right)^2. \tag{3.15}$$

Here Ω_2 is the separation constant determining the exponential dependence of ψ on τ_2. Note that ψ is the simultaneous solution

of all Lax equations corresponding to the KdV hierarchy. Another direct calculation gives an expression for the integrand of (3.14) as

$$V^* \mathcal{L}_2 V = 64|\gamma(x)|^4 \Omega(\zeta)\left(4\zeta - c_{21} + 8k^2 - 4\right)$$
$$\times \left(\zeta - k^2 - k^2 \operatorname{sn}^2(y, k) + 1\right)^3$$
$$\times \left(4k^2 \operatorname{cn}(y, k) \operatorname{dn}(y, k) \operatorname{sn}(y, k) - \Omega(\zeta)\right).$$

Using the same simplifications and substitutions as above, we find the remarkable result that $\mathcal{K}_2(\zeta)$ is linearly related to $\mathcal{K}_1(\zeta)$ via the relationship

$$\mathcal{K}_2(\zeta) = -\left(4\zeta - c_{21} + 8k^2 - 4\right)\mathcal{K}_1(\zeta).$$

The factor $4\zeta - c_{21} + 8k^2 - 4$ changes sign when ζ crosses $1 - 2k^2 + c_{21}/4$. If we choose c_{21} so that its product with $\mathcal{K}_1(\zeta)$ is always positive (or zero, where $\mathcal{K}_1(\zeta) = 0$) for all values of $\zeta \in \sigma_L$, then $\mathcal{K}_2(\zeta)$ does not change sign. To this end we need

$$k^2 - 1 < 1 - 2k^2 + c_{21}/4 < 2k^2 - 1,$$

which is easily satisfied by choosing

$$4(3k^2 - 2) < c_{21} < 4(4k^2 - 2). \tag{3.16}$$

In conclusion, we see it is possible to choose the free parameter c_{21} so that $\mathcal{K}_2(\zeta)$ is positive definite for any choice of the elliptic modulus k. Note that $\mathcal{K}_2(\zeta) = 0$ only when ζ equals one of the three roots of the right-hand side of (2.9). For these values $\lambda = 2\Omega = 0$, which using (3.15) implies $\lambda_2 = 2\Omega_2 = 0$. Thus $\mathcal{K}_2 \geqslant 0$ as a function of λ_2, with equality attained only when $\lambda_2 = 0$. This is an important observation. As shown in [6], the values of ζ for which $\Omega(\zeta) = 0$ all correspond to a single eigenfunction $V \sim \operatorname{sn}(y, k) \operatorname{cn}(y, k) \operatorname{dn}(y, k) \sim \partial_y \operatorname{cn}^2(y, k)$, which is the sole generator of the Lie point symmetry group of the cnoidal wave, consisting of spatial translation. We also have that $\mathcal{K}_2(\zeta) > 0$ implies that $\mathcal{L}_2$ is positive definite in the orthogonal complement of its kernel. This is an easy consequence of working with periodic boundary conditions (so that the zero eigenvalue is isolated) and the use of Theorem 2.6 from [11]. This is one of the assumptions we had to verify was satisfied in order to invoke Theorem 1 of [17]. The other assumptions (existence of solutions and existence of the so-called bound states, i.e., the traveling waves) are immediate for the KdV equation, see [10] for the former and for instance [4] for the latter. In [10], it is shown that solutions to the KdV equations exist globally for initial data in $H^s(\mathbb{T})$, for $s \geqslant -1/2$. Here $\mathbb{T}$ denotes the torus, i.e., periodic boundary conditions are used. Note that the period used is not necessarily the minimal period of the cnoidal wave. From this result, it is clear global solutions to the τ_2 flow exist in $H^2(\mathbb{T})$ (and perhaps even in $H^{1/2}(\mathbb{T})$, but this is not necessary for our results). Indeed, for $s = 2$, the quantity $H_2(u)$ is well-defined, for all time. Since the τ_1 flow is globally defined and the flows commute, the τ_2 flow is globally defined as well, see [24] where this argument originated. Thus, in the results below, we assume all dynamics takes place in $H^2([-nK(k), nK(k)])$. It follows that all conditions for Theorem 1 of [17] are satisfied, finishing the proof of orbital stability of the cnoidal wave solutions of the second KdV equation (3.8) with respect to perturbations in $H^2([-nK(k), nK(k)])$, modulo the translational symmetry. Thus we have proved the following theorem.

Theorem 3.1. *Consider the solution $U(y) = 12k^2 \operatorname{cn}^2(y, k)$ on the space $H^2([-nK(k), nK(k)])$ endowed with the natural inner-product, where $n \in \mathbb{N}_0$ is given. For a given $\epsilon > 0$ sufficiently small there is a $\delta > 0$ such that if $\|u - U\| < \delta$ with $\langle u \rangle = \langle U \rangle$, then*

$$\inf_{\omega \in \mathbb{R}} \left\| v(\tau_2) - U(\cdot + \omega) \right\| < \epsilon.$$

Now let us consider the implications of this calculation. This is where we invoke the same argument that was used in [24] to establish the orbital stability of the N-soliton solutions of the KdV equation. Our calculation of the Krein signature implies that the quantity (3.10), with c_{20} and c_{21} satisfying (3.11) and (3.16) respectively, acts as a Lyapunov functional for the cnoidal wave solutions of the KdV equation with respect to the τ_2 dynamics. This implies that all perturbations that are not mere shifts of the cnoidal wave in $H^2([-nK(k), nK(k)])$ give rise to an increase in the value of $H_2(u)$. But all flows of the KdV hierarchy commute, and $H_2(u)$ is a conserved quantity with respect to the τ dynamics as well. Hence, (3.10) also allows us to establish formal stability of the cnoidal waves under the KdV dynamics. Finally, since the eigenfunctions of the linear stability problems for the τ and τ_2 flows are identical, the Krein signature calculation necessary to invoke Theorem 1 of [17] does not need to be repeated as it remains the same. In short, orbital stability of the cnoidal waves with respect to perturbations in $H^2([-nK(k), nK(k)])$ under the τ_2 dynamics establishes orbital stability of these cnoidal waves with respect to the same class of perturbations under the τ dynamics. Thus, the cnoidal wave solutions of the KdV equation are orbitally stable with respect to subharmonic perturbations which respect the average value of the solution.

Theorem 3.2. *Consider the solution $U(y) = 12k^2 \operatorname{cn}^2(y, k)$ on the space $H^2([-nK(k), nK(k)])$ endowed with the natural inner-product, where $n \in \mathbb{N}$ is given. For a given $\epsilon > 0$ sufficiently small there is a $\delta > 0$ such that if $\|u - U\| < \delta$ with $\langle u \rangle = \langle U \rangle$, then*

$$\inf_{\omega \in \mathbb{R}} \left\| u(t) - U(\cdot + \omega) \right\| < \epsilon.$$

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