KDV CNOIDAL WAVES ARE SPECTRALLY STABLE

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ABSTRACT. Going back to considerations of Benjamin (1974), there has been significant interest in the question of stability for the stationary periodic solutions of the Korteweg-deVries equation, the so-called cnoidal waves. In this paper, we exploit the squared-eigenfunction connection between the linear stability problem and the Lax pair for the Korteweg-deVries equation to completely determine the spectrum of the linear stability problem for perturbations that are bounded on the real line. We find that this spectrum is confined to the imaginary axis, leading to the conclusion of spectral stability. An additional argument allows us to conclude the completeness of the associated eigenfunctions.

1. **Introduction.** The Korteweg-deVries equation (KdV) for the dependent variable $u(x, t)$

$$u_t + uu_x + u_{xxx} = 0 \tag{1}$$

(indices denote partial differentiation; x and t are real independent variables) is one of the canonical integrable equations of mathematical physics and applied mathematics [2, 38]. It describes the weakly nonlinear dynamics of long one-dimensional waves propagating in a dispersive medium. For instance, KdV has been used to model long waves in shallow water [2] and ion-acoustic waves in plasmas [10].

Beyond doubt, the most famous solution of KdV is its soliton solution:

$$u = u_0 + 12\kappa^2 \text{sech}^2\left(\kappa\left(x - x_0 - (4\kappa^2 + u_0)t\right)\right), \tag{2}$$

where u_0, κ and x_0 are arbitrary parameters. For physically relevant solutions, these parameters should be chosen to be real. Note that the presence of the parameter u_0 implies that the soliton does not necessarily approach 0 as $x \to \pm\infty$. The stability of this solution was first studied by Benjamin [6], and later by Bona [8], who sharpened Benjamin's methods. Neither made use of the integrability of KdV, beyond its first few conserved quantities. Benjamin's methods use ideas from Lyapunov theory, and as such his results establish nonlinear orbital stability of the KdV soliton in a suitable function space.

KdV also possesses a family of periodic time-translating solutions expressed in terms of elliptic functions, referred to as the cnoidal waves. These were first written down by Korteweg and deVries [25]:

$$u = u_0 + 12k^2\kappa^2\text{cn}^2(\kappa(x - x_0 - (8\kappa^2k^2 - 4\kappa^2 + u_0)t), k), \qquad (3)$$

where, as for the soliton solution, u_0, κ and x_0 are arbitrary parameters that should be chosen real, for physically relevant solutions. Here $\text{cn}(\cdot, k)$ denotes the Jacobi elliptic cosine function [29, 9] with elliptic modulus $k \in [0, 1)$. A more detailed discussion of the solutions (3) is presented in the next section.

The stability of the cnoidal waves has received far less attention than that of the soliton. Benjamin contemplated various aspects of this problem [7], suggesting that it is likely that the cnoidal waves (3) are stable with respect to perturbations of the same period, but unstable with respect to perturbations of larger periods. The first conjecture was verified by McKean [33], who considered the orbital stability of all periodic finite-genus solutions of KdV with respect to perturbations of the same period. McKean relied heavily on the integrable structure of KdV. Earlier, Kuznetsov and Mikhaĭlov [27] demonstrated stability of (3) with respect to localized perturbations, by examining the KdV initial-value problem for such perturbations, using the inverse scattering method. In the 1980's, Ercolani, Flaschka, Forest and McLaughlin considered both the modulational stability of KdV wavetrains (containing (3) as a special case) [16, 35] and the stability of wave trains of perturbed KdV equations [17, 15], with results varying, depending on which perturbations are considered. Around the same time, Kuznetsov, Spector and Fal'kovich [28, 46] approached the cnoidal-wave stability problem using integrability by way of the dressing method. They focused to a large extent on the stability of the cnoidal waves with respect to transverse perturbations.

Much more recently, Angulo, Bona and Scialom [3] returned to Benjamin's original question of the stability of (3) within the context of KdV. Extending Benjamin's Lyapunov methods to periodic problems, they proved that (3) is nonlinearly orbitally stable with respect to perturbations of the same period. This reconfirms Benjamin's first conjecture, without relying on the integrability of KdV more than Benjamin [6] or Bona [8] did.

Currently under review is a manuscript by Angulo and Muñoz Grajales [4], where the spectral instability of (3) with respect to perturbations of twice the period of (3) is claimed. This result, if correct, would prove the second of Benjamin's conjectures. As in [3], the integrability of KdV is not used in an essential way to obtain this result.

In the current manuscript, we use the integrability of KdV to great advantage. We are able to prove the spectral stability of (3) with respect to perturbations of arbitrary period, or even perturbations that are quasi-periodic. Using the integrable structure of KdV, we are able to explicitly determine the essential spectrum[1] of the linear stability problem of (3). We show that this spectrum is confined to the imaginary axis, leading to the conclusion of spectral stability. In addition, we show that the spectrum covers the whole imaginary axis once, except for a region around the origin, which is covered thrice. An additional completeness argument allows us to generalize our result to obtain linear stability. Our results agree completely with numerical results obtained using Hill's method [13]. These numerical results are

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[1]In the sense of "spectrum independent of the boundary conditions", see [41], pp244

shown in Section [4]. As an interesting side note, we are able to recover the known results for the soliton case, as a limit of our results. Our results overlap with those obtained by Spector [46] and Kuznetsov, Spector and Fal'kovich [28], although our methods are more systematic. Our results contradict those of Angulo [4], mentioned above. The rest of the paper deals with the different steps of our method.

Remark 1. Since the completion of this paper Deconinck and Kapitula [11] have completed a follow-up manuscript where the nonlinear stability of the cnoidal wave is proven with respect to perturbations that are periodic with period NT, where N is any positive integer and T is the period of the cnoidal wave. That work relies on the results presented here. The integrability of the KdV equation is necessary to prove the nonlinear stability. The results of [11] conclusively answer Benjamin's question about the stability of the cnoidal wave solution of the KdV equation.

2. **The cnoidal wave solution of the KdV equation.** In this section, we examine the class of cnoidal-wave solutions in more detail. The results mentioned here are not new. Most of these considerations may be found, for instance, in [3] or [45]. Overviews of the applicability of cnoidal waves in coastal engineering are found in [44, 48]. Lastly, all facts about elliptic functions stated here are well-known. They are found in all standard works, such as [29, 9].

As stated above, the cnoidal waves are given by (3). Here the elliptic modulus k of the Jacobian cosine function $\operatorname{cn}(x,k)$ is constrained to the interval $[0,1)$. The solutions (3) are periodic in x, with period given by

$$T(k) = \frac{2K(k)}{\kappa} = \frac{2}{\kappa} \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 s}} ds, \tag{4}$$

where $K(k)$ is the complete elliptic integral of the first kind. Further, $\operatorname{cn}(x,0) = \cos x$ and $K(0) = \pi/2$, but the cnoidal wave (3) reduces to a trivial (*i.e.*, constant) solution. On the other hand, as $k \to 1$, $\operatorname{cn}(x,k) \to \operatorname{sech} x$ and $K(k) \to \infty$ and (3) reduces to (2).

The solution (3) depends on four parameters. All but the elliptic modulus have direct analogues in (2). The elliptic modulus stands out for another reason. As shown above it determines the period of the solution, and in that sense the space of functions we choose to work with. The three remaining parameters are reflections of the classical Lie-point symmetries of the KdV equation. The parameter x_0 is inherited from the translational invariance of the equation; κ reflects the scaling symmetry; and u_0 represents the effect of a Galilean boost.

The cnoidal-wave solutions are the simplest nontrivial examples of the large class of so-called finite-genus solutions of the KdV equation [5, 38]. The genus g solutions may be characterized as stationary solutions of the g-th member of the KdV hierarchy [31, 37]. This fact is used below, although restricted to the $g = 1$ case for the cnoidal waves, where it is easily explicitly verified.

3. **The linear stability problem.** To examine the stability of the cnoidal waves (3), we change to a moving coordinate frame by introducing the coordinates

$$y = x - Vt, \ \tau = t, \tag{5}$$

with $V = 8\kappa^2 k^2 - 4\kappa^2 + u_0$. In the (y, τ) coordinates the KdV equation becomes

$$u_\tau - V u_y + u u_y + u_{yyy} = 0. \tag{6}$$

The cnoidal wave is rewritten as

$$u = U(y) = u_0 + 12k^2\kappa^2 \text{cn}^2(\kappa(y - x_0), k), \tag{7}$$

and is a stationary (*i.e.* time-independent) solution of (6). Note that (7) retains only the single symmetry parameter x_0, as the other two appear in the definition of the velocity V: moving to a translating frame of reference with velocity V broke two of the three Lie symmetries: the parameters κ and u_0 (and k) define the equation (6) and the "stationary point" of the equation (6) depends on only one free parameter.

Next we consider perturbations of this stationary cnoidal wave:

$$u(y, \tau) = U(y) + \epsilon w(y, \tau) + \theta(\epsilon^2), \tag{8}$$

where ϵ is a small parameter. Substituting this in (6) and ignoring higher-than-first-order terms in ϵ, we find

$$w_\tau - V w_y + w U_y + U w_y + w_{yyy} = 0, \tag{9}$$

at first order in ϵ. The zeroth order terms vanish since $U(y)$ solves KdV. By ignoring the higher-order terms in ϵ, we are restricting our attention to examining linear stability. This is a restriction of our approach. The cnoidal-wave solution is defined to be *linearly stable* if for all $\epsilon > 0$, there is a $\delta > 0$ such that if $||w(y, 0)|| < \delta$ then $||w(y, \tau)|| < \epsilon$ for all $\tau > 0$. This definition depends on our choice of the norm $|| \cdot ||$, to be determined in the next section.

Next, since (9) is autonomous in time, we may separate variables. Let

$$w(y, \tau) = e^{\lambda\tau} W(y, \lambda), \tag{10}$$

then $W(y, \lambda)$ satisfies

$$-W_{yyy} + (V - U)W_y - U_y W = \lambda W, \tag{11}$$

or

$$\mathcal{L}W = \lambda W, \ \mathcal{L} = -\partial_y^3 + (V - U)\partial_y - U_y. \tag{12}$$

In what follows, the λ dependence of W will be suppressed. Thus, we address the stability problem for the cnoidal wave by examining the spectrum $\sigma(\mathcal{L})$ of the differential operator $\mathcal{L}$. To avoid confusion with other spectra arising below, we refer to $\sigma(\mathcal{L})$ as the *stability spectrum* of the cnoidal waves of KdV. Of course, the definition of $\sigma(\mathcal{L})$ depends on the space of functions we are considering [22, 41, 42], which goes back to the question of which function norm we elect to use. The cnoidal-wave solution of KdV is defined to be *spectrally stable* if $\sigma(\mathcal{L})$ does not intersect the right half of the complex λ plane. Since KdV is a Hamiltonian equation [18], $\sigma(\mathcal{L})$ is symmetric under reflection with respect to both the real and imaginary axis. Thus, the cnoidal wave is spectrally stable, if $\sigma(\mathcal{L})$ is contained on the imaginary axis. Spectral stability of the cnoidal wave will imply its linear stability if the set of eigenfunctions corresponding to $\sigma(\mathcal{L})$ can be shown to be complete in the space defined by $|| \cdot ||$, so that all solutions of (9) in that space can be obtained as linear combinations of solutions of (11).

Benjamin [7] pointed out several stumbling blocks in the road to proving nonlinear stability of the cnoidal-wave solution of KdV with respect to perturbations

of period equal to an integer (> 1) multiple of the cnoidal wave. If one tries to generalize the nonlinear stability proof for the soliton (2) [6], different obstacles are encountered. Some of these may be interpreted at hinting at the possibility of instability of the cnoidal wave with respect to these long period perturbations. Nevertheless, none of this is conclusive. Benjamin concluded his ruminations by briefly discussing the linear stability problem and how the conclusions reached from its study are indicators as to the stronger results desired. The question about the stability of the cnoidal waves remained open.

Remark 2. Since KdV is an integrable equation, we might investigate the stability of the cnoidal waves by solving the initial-value problem for initial conditions that start near the cnoidal wave (this point of view was put forward by Harvey Segur at the Workshop on Stability and Instability of Nonlinear Waves at the University of Washington in September of 2006). This is in essence what is done by Kuznetsov and Mikhaĭlov [27], for a restricted class of perturbations. We agree with this point of view if one is only interested in the stability of the exact solution of the integrable equation. On the other hand, if one wants to consider the stability of stationary solutions of near-integrable equations (equations that in some limit reduce to an integrable one), it might be beneficial to know as much as possible about the stability spectrum of the solution of the integrable equation. This is likely to be the starting point for a perturbative approach to examine the spectral stability of the non-integrable case. This is what is accomplished here: we will present a completely explicit analytical determination of the stability spectrum of the cnoidal waves.

4. **Numerical Results.** Before we determine the spectrum of (12) analytically, we compute it numerically, using Hill's method [13]. Hill's method is ideally suited to a problem such as (12) with periodic coefficients. It allows us to compute all eigenfunctions of the form

$$W = e^{i\mu y}\hat{W}(y), \ \hat{W}(y + T(k)) = \hat{W}(y), \tag{13}$$

with $\mu \in [-\pi/2T(k), \pi/2T(k))$. It follows from Floquet's theorem that *all* bounded solutions of (12) are of this form. Here bounded means that $\max_{x \in \mathbb{R}} |W(x)|$ is finite. Thus $W \in C_b^0(\mathbb{R})$. On the other hand, we also have $W \in L_{per}^2(-T/2, T/2)$ (the square-integrable functions of period T) since the exponential factor in (13) disappears in the computation of the L^2-norm. Thus

$$W \in C_b^0(\mathbb{R}) \cap L_{per}^2(-T/2, T/2). \tag{14}$$

It should be noted that by this choice our investigations include perturbations of an arbitrary period that is an integer multiple of $T(k)$, specifically those of period $2T(k)$ remarked on by Benjamin [7] and examined in [4]. Indeed, periodic perturbations are obtained for those μ-values for which $\mu T(k)/\pi$ is rational.

Figure 1a shows a discrete approximation to the spectrum of (12), computed using SpectrUW 2.0 [12]. The parameters are $k = 0.8$, $\kappa = 1$, $x_0 = 0$ and $u_0 = 0$. The numerical parameters (see [13, 12]) are $N = 40$ (81 Fourier modes) and $D = 50$ (49 different Floquet exponents). Figure 1b is a blow-up of Fig. 1a around the origin. First, it appears that the spectrum is on the imaginary axis[2], indicating

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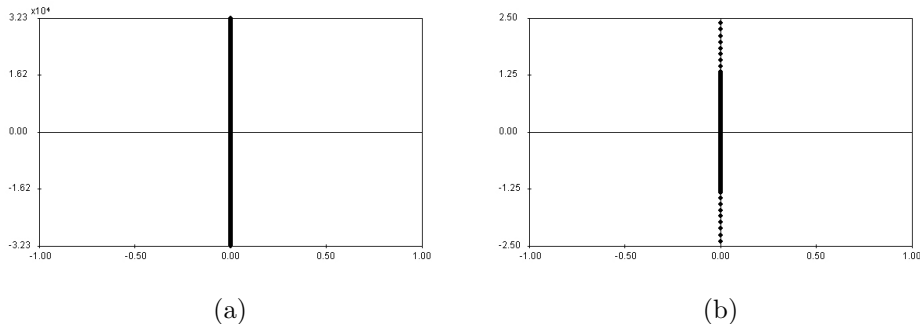[2]The largest real part computed (in absolute value) is 1.363353910^{-11}.

FIGURE 1.   (a) The numerically computed spectrum of (12) with $k = 0.8$, $\kappa = 1$, $x_0 = 0$ and $u_0 = 0$, using Hill's method with $N = 40$ (81 Fourier modes) and $D = 50$ (49 different Floquet exponents), see [13, 12]; (b) A blow-up of (a) around the origin, showing a band of higher spectral density.

spectral stability of the cnoidal wave. Second, the numerics shows that a symmetric band around the origin has a higher spectral density than does the rest of the imaginary axis. This is examined in more detail in Fig. 2, where the imaginary parts $\in [-2.5, 2.5]$ of the computed eigenvalues are displayed as a function of μ. This shows that λ values with $\text{Im}(\lambda) \in [-1.32, 1.32]$ (approximately) are attained for three different μ values in $[-\pi/2T(k), \pi/2T(k))$. The rest of the imaginary axis is only attained for a single μ value. This picture persists if a larger portion of the imaginary λ axis is examined. Also indicated in Fig. 2 is the analytical result obtained in Section 7. Note that the analytical and numerical results appear to agree perfectly.

The above considerations remain true for different values of the elliptic modulus $k \in [0, 1)$, although the spectrum does depend on k. Note that the parameters (x_0, κ, u_0) may be eliminated by simple transformations and they do not affect the spectra. Thus, for all values of $k \in [0, 1)$, the spectrum of the cnoidal-wave solution appears to be confined to the imaginary axis, indicating the spectral stability of the cnoidal waves. Also, for all these k values, the spectrum $\sigma(\mathcal{L})$ covers a symmetric interval around the origin three times, whereas the rest of the imaginary axis is single covered. The edge point on the imaginary axis where the transition from spectral density three to one occurs depends on k and is denoted $\lambda_c(k)$. The k-dependence of $\lambda_c(k)$ is shown in Fig. 3. Again, both numerical and analytical results (see Section 7) are displayed. For these numerical results, Hill's method with $N = 50$ was used.

5. **The Lax pair restricted to the cnoidal wave.** It is well known [19, 30] that KdV is equivalent to the compatibility condition of two linear ordinary differential systems:

$$\hat{\psi}_x = \begin{pmatrix} 0 & 1 \\ \zeta - u/6 & 0 \end{pmatrix} \hat{\psi}, \ \hat{\psi}_t = \begin{pmatrix} u_x/6 & -4\zeta - u/3 \\ -4\zeta^2 + (u^2 + 6\zeta u + 3u_{xx})/18 & -u_x/6 \end{pmatrix} \hat{\psi}.$$
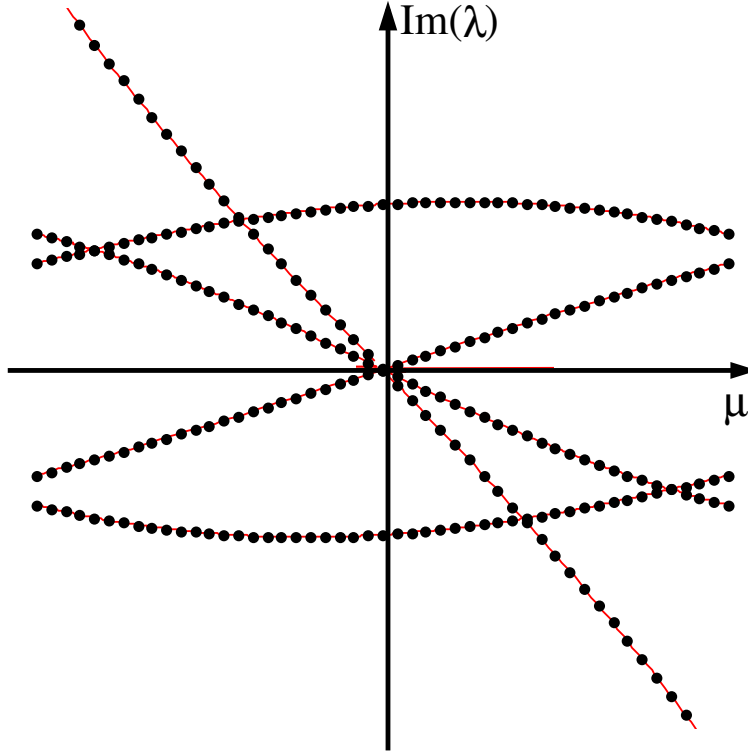(15)

FIGURE 2. The imaginary part of λ as a function of μ, demonstrating the higher spectral density (three *vs.* one) in Fig. 1b. The small circles indicate the numerical data, whereas the solid curve corresponds to the analytical result (44).

In other words, the compatibility condition $\hat{\psi}_{xt} = \hat{\psi}_{tx}$ requires that u satisfies KdV. For our purposes, it is more convenient to work with (6) instead. This equation differs from KdV only by an additional u_y term. Since the equation $u_\tau = u_y$ is the zeroth member of the KdV hierarchy [2], the Lax pair (15) is easily modified by including the zeroth member of the linear Lax pair hierarchy [30]. We obtain

$$\psi_y = \begin{pmatrix} 0 & 1 \\ \zeta - u/6 & 0 \end{pmatrix} \psi, \tag{16}$$

$$\psi_\tau = \begin{pmatrix} u_y/6 & -4\zeta + V - u/3 \\ -4\zeta^2 + \zeta V + (u^2 - 3Vu + 6\zeta u + 3u_{yy})/18 & -u_y/6 \end{pmatrix} \psi \tag{17}$$

Indeed, the condition $\psi_{y\tau} = \psi_{\tau y}$ results in (6). It is more common to use a scalar form for the Lax pair of KdV [30], which involves the stationary Schrödinger equation. We prefer to use the equivalent systems form (16–17), since it suits our methods better. Nevertheless, since ζ is the spectral parameter for the stationary Schrödinger equation, we refer to the set of all ζ values such that (16-17) has bounded (in the same sense as above) solutions as the *Lax spectrum* σ_L. Note that because ζ is a spectral parameter for the stationary Schrödinger equation, the Lax
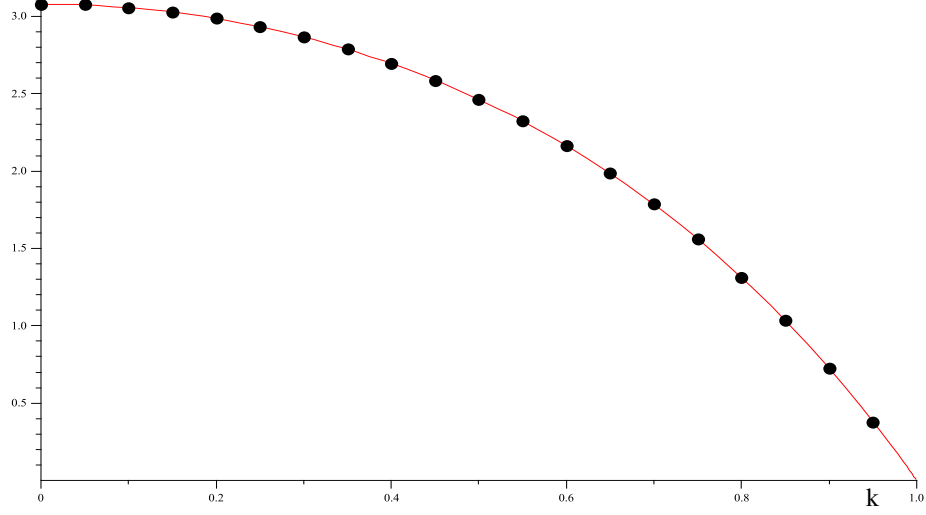
FIGURE 3. Numerical and analytical results for the imaginary part of the edge point $\lambda_c(k)$ of the triple-covered region as a function of the elliptic modulus k. The solid curve displays the analytical result, the small circles are obtained numerically.

spectrum of the cnoidal wave (7) is a subset of the real line: $\sigma_L \subset \mathbb{R}$. The goal of this section is to determine this subset explicitly. In the next section, we connect the Lax spectrum σ_L of the KdV cnoidal wave to its stability spectrum $\sigma(\mathcal{L})$.

Thus we examine the problem

$$\psi_y = \begin{pmatrix} 0 & 1 \\ \zeta - U/6 & 0 \end{pmatrix} \psi, \tag{18}$$

$$\psi_\tau = \begin{pmatrix} U_y/6 & -4\zeta + V - U/3 \\ -4\zeta^2 + \zeta V + (U^2 - 3VU + 6\zeta U + 3U_{yy})/18 & -U_y/6 \end{pmatrix} \psi \tag{19}$$

Many solution methods for integrable systems use the first Lax equation (here (18)) to great extent, while the second equation (here (19)) plays only a minor role. Such is true in the method of inverse scattering [19, 2], the use of Darboux transformations [32], *etc.* In contrast, we obtain great benefits from using (19) first. This is in the spirit of work of Krichever [26, 5]. Indeed, notice that the right-hand side of (19) does not explicitly depend on τ, allowing us to separate variables: let

$$\psi(y, \tau) = e^{\Omega \tau} \begin{pmatrix} \alpha(y) \\ \beta(y) \end{pmatrix}. \tag{20}$$

We refer to the collection of all Ω values such that (20) is bounded as a function of x as the τ-spectrum σ_τ of the cnoidal wave. Then

$$\begin{pmatrix} U_y/6 - \Omega & -4\zeta + V - U/3 \\ -4\zeta^2 + \zeta V + (U^2 - 3VU + 6\zeta U + 3U_{yy})/18 & -U_y/6 - \Omega \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0. \tag{21}$$

It follows that Ω is determined by the eigenvalue equation

$$\Omega^2 = 16\zeta^3 - 8V\zeta^2 + \left(V^2 - \frac{2C_1}{3}\right)\zeta + \frac{C_2}{36} + \frac{C_1 V}{6}, \tag{22}$$

where the constants C_1 and C_2 are given by

$$C_1 = U_{yy} + \frac{1}{2}U^2 - UV, \ C_2 = U_y^2 + \frac{1}{3}U^3 - VU^2 - 2C_1 U. \tag{23}$$

That C_1 and C_2 are indeed independent of y follows immediately from the ordinary differential equation $-VU_y + UU_y + U_{yyy} = 0$, which the cnoidal wave (7) satisfies. Alternatively, one might use the different commutation relations (see [14], using that the trace of a commutator vanishes) between the coefficient matrices in (16-17) to prove the y-independence of Ω. In our case, a more explicit calculation is possible, since we know the functional form (7) of $U(y)$ explicitly. We find that

$$C_1 = 24k^2 k'^2, \ C_2 = 0, \tag{24}$$

where $k' = \sqrt{1-k^2}$ is the complimentary elliptic modulus. Note that in the above, and from here on out, we have equated $x_0 = 0$, $u_0 = 0$, $\kappa = 1$. These choices do not affect our results. Our equation (22) for Ω^2 becomes

$$\Omega^2 = 16(\zeta - \zeta_1)(\zeta - \zeta_2)(\zeta - \zeta_3), \tag{25}$$

with

$$\zeta_1 = k^2 - 1 \leq \zeta_2 = 2k^2 - 1 \leq \zeta_3 = k^2. \tag{26}$$

The inequalities above are easily verified. The first one becomes an equality for $k = 0$, the last one for $k \to 1$.

Having determined Ω, we turn to the vector $(\alpha, \beta)^T$. Clearly, it is the eigenvector corresponding to the eigenvalue Ω:

$$\begin{aligned}
\begin{pmatrix} \alpha \\ \beta \end{pmatrix} &= \gamma(y) \begin{pmatrix} 4\zeta - V + U/3 \\ U_y/6 - \Omega \end{pmatrix} \\
&= \delta(y) \begin{pmatrix} U_y/6 + \Omega \\ -4\zeta^2 + \zeta V + (U^2 - 3VU + 6\zeta U + 3U_{yy})/18 \end{pmatrix}.
\end{aligned} \tag{27}$$

Of course, there are two eigenvectors to be considered. But, (22) is an algebraic curve representation for a genus one (elliptic, if $k \neq 0$, 1) Riemann surface [47, 5], and we may think of $(\alpha, \beta)^T$ as being a single-valued vector on this double-sheeted surface. This interpretation is valid for all ζ values, except $\zeta = \zeta_i$, $i = 1, 2, 3$, which are branch points of (25) with $\Omega = 0$, and only one eigenvector is obtained. We return to this issue below. Equation (27) gives two representations of this eigenvector, referred to as the γ- and δ-representations. Which one is used is largely a matter of convenience. Both representations depend on a scalar function of y, which is undetermined for now. It cannot be determined by using (19). Instead we have to resort to (18). This equation is a two-dimensional linear differential system, but substitution of either the γ- or δ-representation collapses it to a scalar linear first-order differential equation for either γ or δ. Thus, this equation may be solved explicitly. Using either representation, we find

$$\gamma = \gamma_0 \exp\left(\int \frac{-U_y/6 - \Omega}{4\zeta - V + U/3} dy\right), \tag{28}$$

or

$$\delta = \delta_0 \exp\left(\int \frac{\zeta U/3 - 4\zeta^2 + \zeta V + U^2/18 - UV/6}{U_y/6 + \Omega} dy\right). \qquad (29)$$

Combined with (27), these equations completely determine *all* two-dimensional vector functions that solve both (18) and (19). Since we are interested in the Lax spectrum of the cnoidal wave, we need to determine for which values of ζ (28), (29) are bounded as functions of y.

Since the vector parts (given in (27)) of the eigenvectors are bounded for all choices of ζ, we need the exponential contributions to be bounded. To this end, it is necessary and sufficient that

$$\left\langle \Re\left(\frac{-U_y/6 - \Omega}{4\zeta - V + U/3}\right)\right\rangle = 0, \qquad (30)$$

or a similar condition using (29). Here $\langle \cdot \rangle = \frac{1}{T(k)}\int_0^{T(k)} \cdot \, dy$ denotes the average over a period and $\Re$ denotes the real part. One easily checks that this average is well defined for all real $\zeta < k^2 - 1$ or $\zeta > 2k^2 - 1$, as the denominator of (30) has no zeros then. For $\zeta \in (\zeta_1 = k^2 - 1, \zeta_2 = 2k^2 - 1)$ this results in both eigenvector solutions being unbounded, thus these ζ values do not belong to the Lax spectrum σ_L. For $\zeta = \zeta_1 = k^2 - 1$, both representations result in the same bounded eigenfunction: $\alpha \sim \mathrm{sn}(x, k)$, $\beta \sim \mathrm{cn}(x, k)\mathrm{dn}(x, k)$. Thus $\zeta = \zeta_1 \in \sigma_L$. Similarly, for $\zeta = \zeta_2 = 2k^2 - 1$, we obtain one bounded eigenfunction: $\alpha \sim \mathrm{cn}(x, k)$, $\beta \sim -\mathrm{sn}(x, k)\mathrm{dn}(x, k)$, thus $\zeta = \zeta_1 \in \sigma_L$. For all other values of ζ, we examine the average defined above. The condition (30) easily simplifies. First, it is equivalent to

$$\left\langle \frac{1}{4\zeta - V + U/3}\right\rangle \Re(\Omega) = 0, \qquad (31)$$

since U_y is odd over a period and ζ is real. It follows from (25) that Ω is either real or imaginary. The condition (31) implies that all $\zeta \in \mathbb{R}$ for which Ω^2 is negative or zero are part of the Lax spectrum σ_L. This requires $\zeta \in (-\infty, k^2 - 1] \cup [2k^2 - 1, k^2]$. Next, it is easy to show that if ζ is outside of this set, then the average term is not zero. We conclude that the Lax spectrum of the cnoidal wave is given by

$$\sigma_L = (-\infty, k^2 - 1] \cup [2k^2 - 1, k^2]. \qquad (32)$$

Also, we have established that the allowed Ω values are imaginary, covering the entire imaginary axis, thus

$$\sigma_\tau = i\mathbb{R}. \qquad (33)$$

Indeed, Ω^2 takes on all negative values for $\zeta \in (-\infty, k^2 - 1]$, implying that $\Omega = \pm\sqrt{\Omega^2}$ covers the imaginary axis. Furthermore, for $\zeta \in [2k^2 - 1, k^2]$, Ω^2 takes on all negative values in $[\Omega_{\min}^2, 0]$ twice, where $\Omega_{\min}^2$ is the minimal value of Ω^2 attained for ζ in this interval. Upon taking square roots, this implies that the interval on the imaginary axis $\left[-i\sqrt{|\Omega_{\min}^2|}, i\sqrt{|\Omega_{\min}^2|}\right]$ is double covered in addition to the single covering already mentioned. Thus this symmetric interval around the origin is triple covered, whereas the rest of the imaginary axis is single covered. Symbolically, we might write
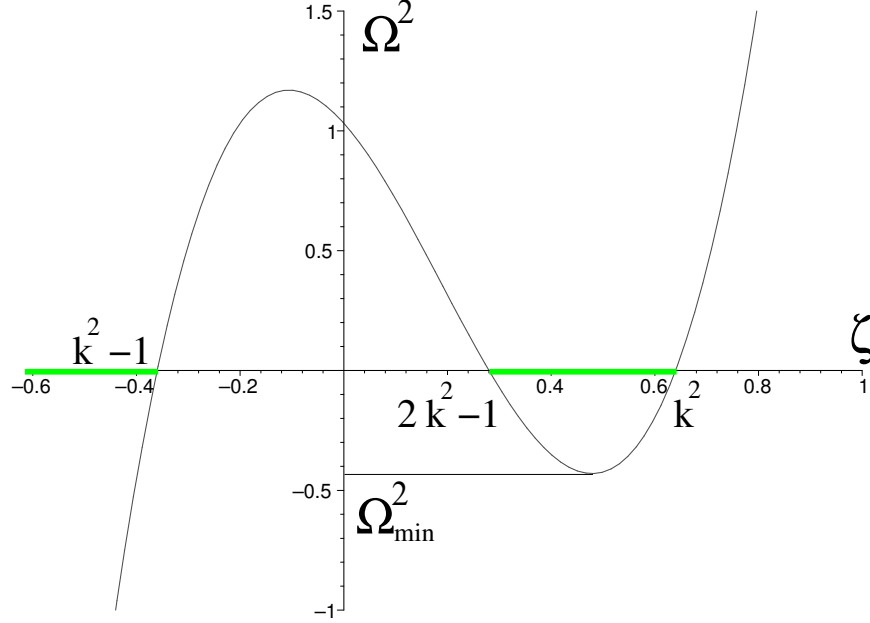
FIGURE 4. Ω^2 as a function of real ζ, for $k = 0.8$. The union of the thick line segments is the Lax spectrum σ_L of the cnoidal wave.

$$\sigma_\tau = i\mathbb{R} \cup \left[-i\sqrt{|\Omega^2_{\min}|}, i\sqrt{|\Omega^2_{\min}|} \right]^2, \tag{34}$$

instead of (33). Here the square is used to denote multiplicity. The expressions used above can be calculated explicitly in terms of the elliptic modulus. We have

$$|\Omega^2_{\min}| = \frac{16}{27}(k^2 + 1 + \Delta)(1 - 2k^2 + \Delta)(k^2 - 2 + \Delta), \tag{35}$$

where $\Delta^2 = 1 - k^2 k'^2 \geq 0$. This value is attained for

$$\zeta = \frac{4k^2 - 2 + \Delta}{3}. \tag{36}$$

Remark 3. • The condition (30) may be obtained differently. It is possible (see *e.g.* [5]) to express the solutions of the Lax pair (18-19) using Riemann theta functions of an argument that is linear in x. The condition determining the spectrum is that the coefficient of x is so that the theta function is bounded for real x. Of course, the condition obtained is equivalent to (30).
• For $\zeta = \zeta_i$, $i = 1, 2, 3$, the second linearly independent solution of the differential equation (18) is found using reduction of order, resulting in algebraically growing solutions as $y \to \pm\infty$. It should be remarked that these algebraically growing solutions solve (18), but they do not provide solutions of (19). It is remarkable that *all* solutions of (18) give rise to simultaneous solutions of

(19), *except* these three. Indeed, for all values of $\zeta \in \mathbb{C}$, (27) gives rise to a simultaneous solution of both (18) and (19), even if $\zeta \notin \sigma_L$. This is used in Section 7.

6. **The squared-eigenfunction connection.** It is well known that there exists a connection between the eigenfunctions of the Lax pair of an integrable equation and the eigenfunctions of the linear stability problem for this integrable equation [1, 2, 20, 36, 43]. This connection has been used extensively to study the soliton solutions and their stability, but to our knowledge, it has not been exploited to do the same for the finite-gap solutions of integrable equations. This is surprising, as the same connection exists. After all, establishing the squared-eigenfunction connection is an algebraic statement, which does not involve boundary conditions. The essence of this calculation is included below.

Theorem 6.1. *The product* $w(y,\tau) = \psi_1(y,\tau)\psi_2(y,\tau) = \partial_y\psi_1^2(y,\tau)/2$ *satisfies the linear stability problem (9) for* $U(y)$*. Here* $\psi = (\psi_1, \psi_2)^T$ *is any solution of (18-19) with the corresponding coefficient function* $U(y)$*.*

Proof. The proof is a straightforward calculation: calculate $w_\tau = (\psi_1\psi_2)_\tau$ using the product rule and (19). Alternatively, calculate w_τ using (9), substituting $w = \psi_1\psi_2$. In both expressions so obtained, eliminate y-derivatives of ψ_1 and ψ_2 (up to order 3) using (18). The resulting expressions are equal, finishing the proof. $\square$

Remark 4.   • Of course, boundary conditions do come into play when one worries about the completeness of these squared eigenfunctions. Again, much work has been done for the case of the whole line problem, see for instance [23, 24]. For the case of periodic solutions of KdV, the completeness of periodic eigenfunctions (with period equal to or double that of the KdV solution) was proven by McKean and Trubowitz [34], but we are not aware of any results about the full set of eigenfunctions, including those of arbitrary large periods, and those that are not periodic.
   • It is possible to repeat this calculation for any solution $u(y,\tau)$ of (6). It is not necessary that the solution is stationary in some appropriately translating frame.
   • More often, the above argument for KdV uses the eigenfunction of the stationary Schrödinger equation $\psi_1(x,t)$ (t playing a parametric role), and one shows that ψ_1^2 solves the formal adjoint equation to the linear stability problem. This extra complication is avoided by working with a first-order system instead.

7. **Spectral and linear stability.** In order to establish the spectral stability of the cnoidal wave solutions (3) of (1), we need to establish that *all* bounded solutions $W(y)$ of (11) are obtained through the squared-eigenfunction connection by

$$w(y,t) = e^{2\Omega\tau}\alpha(y)\beta(y). \tag{37}$$

If we manage to do so then we may immediately conclude that

$$\lambda = 2\Omega. \tag{38}$$

Since $\sigma_\tau = i\mathbb{R}$, we conclude that the stability spectrum is given by

$$\sigma(\mathcal{L}) = i\mathbb{R}. \tag{39}$$

In order to obtain this conclusion, we need the following theorem.

Theorem 7.1. *All but six solutions of (11) may be written as $W(y) = \alpha(y)\beta(y) = \partial_y \alpha^2(y)/2$, where $(\alpha, \beta)^T$ solves (18,21). More specifically,* all *but one solution of (11) bounded on the whole real line are obtained through the squared eigenfunction connection.*

Proof. For any given value of $\lambda \in \mathbb{C}$, (11) is a third-order linear ordinary differential equation. Thus, it has three linearly independent solutions. On the other hand, we have already shown (see Theorem 1) that the formula

$$W(y) = \alpha(y)\beta(y) = \partial_y \alpha^2(y)/2 \tag{40}$$

provides solutions of this ordinary differential equation. Let us count how many solutions are obtained this way, for a fixed value of λ. For any value of $\lambda \in \mathbb{C}$, exactly one value of $\Omega \in \mathbb{C}$ is obtained through $\Omega = \lambda/2$. Excluding the four values of λ for which the discriminant of (25) as a function of ζ is zero (these turn out to be only the values of λ for which Ω^2 reaches its maximum or minimum value in Fig. 4), (25) gives rise to three values of $\zeta \in \mathbb{C}$. It should be noted that we are not restricting ourselves to $\zeta \in \sigma_L$ now, since the boundedness of the solutions is not a concern in this counting argument. Next, for a given pair $(\Omega, \zeta) \in \mathbb{C}^2$, (27) defines a unique solution of (18,21). Thus, any choice of $\lambda \in \mathbb{C}$ not equal to the four values mentioned above, gives rise to exactly three solutions of (11), through the squared eigenfunction connection of Theorem 1. Before we consider the excluded values separately, we need to show that the three functions $W(y)$ obtained as described above are indeed linearly independent. This argument proceeds as follows:

- Provided there is an exponential contribution from $\gamma(y)$ in the solutions $W(y)$, the solutions $W(y)$ are linearly independent: indeed for fixed λ (*i.e.*, for fixed Ω) the exponents are so that for the three different ζ values, the three corresponding $W(y)$ are linearly independent. This is easily seen by using the second representation in (27) with $\delta(y)$ given in (29). Using this representation the ζ dependence is confined to the numerator of the integrand in the exponent. For different ζ different terms with singularities of different order in the complex y-plane are present with different coefficients.
- The only possibility for the exponential $\gamma(y)$ to not contribute to the solution $W(y)$ is if the integrand in (28) is proportional to a logarithmic derivative. A quick calculation confirms that this happens only for $\lambda = 0$ (thus $\Omega = 0$). For this value, it is easily checked that the three resulting $W(y)$ are all proportional to $\text{sn}(y, k)\text{cn}(y, k)\text{dn}(y, k)$, resulting in a single eigenfunction for $\lambda = 0$. This is not a surprise: the solution $U(y)$ (7) has only one invariance (translational invariance, marked by the presence of x_0) resulting in a single eigenfunction proportional to $\partial U/\partial x_0 \sim \text{sn}(y, k)\text{cn}(y, k)\text{dn}(y, k)$. The other solutions of (11) are obtained through reduction of order. This allows one to construct two solutions whose amplitude grows linearly in x. A suitable linear combination of these solutions is bounded. Thus, corresponding to

$\lambda = 0$ there are two eigenfunctions. One of these is not obtained through the squared-eigenfunction connection.

For the four excluded values, two linearly independent solutions of (11) are found. The third one may be constructed using reduction of order, and introduces algebraic growth. For the two λ values for which Ω^2 reaches its minimum value, the two solutions obtained from (40) are bounded, thus these values of λ are part of the spectrum. The two values of λ for which Ω^2 reaches its maximum value only give rise to unbounded solutions and are not part of the spectrum.

We conclude that *all* but one bounded solution of (11) are obtained through the squared eigenfunction connection. $\qquad \square$

The above considerations are summarized in the following theorem.

Theorem 7.2 (Spectral stability)**.** *The cnoidal wave solutions (3) of the KdV equation (1) are spectrally stable. The spectrum of their associated linear stability problem (11) is explicitly given by $\sigma(\mathcal{L}) = i\mathbb{R}$, or, accounting for multiple coverings,*

$$\sigma(\mathcal{L}) = i\mathbb{R}_0 \cup \left[-2i\sqrt{|\Omega^2_{\min}|}, 2i\sqrt{|\Omega^2_{\min}|} \right]^2, \tag{41}$$

where $\lambda_c(k) = 2i|\Omega_{\min}|$, and $|\Omega^2_{\min}|$ is defined by (35).

At this point, we are also able to compare our analytical results with the numerical results shown in Fig. 2. To this end, we obtain parametric expressions for the Floquet parameter μ and the spectral elements. Comparing the expression (13) with our analytical expression for the eigenfunctions, we find that

$$\mu = \frac{N\pi}{T(k)} - i \left\langle \frac{-U_y/6 - \Omega}{4\zeta - V + U/3} \right\rangle, \tag{42}$$

where N is an integer. This may be simplified to

$$\mu = \frac{N\pi}{T(k)} \pm \left\langle \frac{\Im\sqrt{16(\zeta - \zeta_1)(\zeta - \zeta_2)(\zeta - \zeta_3)}}{4\zeta - V + U/3} \right\rangle, \tag{43}$$

using (30) and (25). Here $\Im$ denotes the imaginary part. With the explicit expressions for $U(y)$ and V, we find a parametric form for the curves in Fig. 2:

$$\begin{cases} \mu & = & \dfrac{N\pi}{2K(k)} \pm \dfrac{\sqrt{|\zeta - \zeta_1||\zeta - \zeta_2||\zeta - \zeta_3|}}{K(k)} \displaystyle\int_0^{K(k)} \dfrac{dy}{\zeta - k^2 + \mathrm{dn}^2(y, k)} \\[2ex] \Im\lambda & = & \pm 8\sqrt{|\zeta - \zeta_1||\zeta - \zeta_2||\zeta - \zeta_3|} \end{cases}, \tag{44}$$

for $\zeta < k^2 - 1$ and $\zeta \in [2k^2 - 1, k^2]$. The integral on the right-hand side can be rewritten using elliptic integrals of the third kind. The solid curves in Fig. 2 were obtained using a direct numerical evaluation of (44) instead.

In order to work towards nonlinear stability of the solution (3) of (1) (see [11]), we need to show that all solutions (in a suitable space) of the linearized KdV equation (9) can be expanded in terms of the eigenfunction solutions of (11). In order to accomplish this, it suffices to show that this set of eigenfunctions is complete in the same space. Here, we argue that the set of eigenfunctions of (11) of period

NT $(0 < N \in \mathbb{N})$ is complete in $L^2(-NT/2, NT/2)$, which allows us to establish the linear stability of the cnoidal wave solution (3) of the KdV equation (1) with respect to perturbations of period NT, for any positive integer N.

Indeed, for the case of periodic perturbations of period T, this was shown by McKean and Trubowitz [34]. For $N > 1$, the result follows from the work of Haragus and Kapitula [21]. They prove spectral stability of the cnoidal wave solution with respect to perturbations of the form (13). Further, for any Floquet parameter $\mu \neq 0$, they obtain L^2-completess of the eigenfunctions from the general considerations of the SCS Basis Lemma [21]. By grouping solution sets with common period (corresponding to Floquet exponents that are commensurate with T with common denominators) the result follows. The result of Haragus and Kapitula applies to cnoidal wave solutions of small amplitude (in essence, small values of k). However that restriction was necessary for their spectral stability result. Once that result is established, their completeness argument carries through without obstructions.

This allows us to formulate the following theorem.

Theorem 7.3 (Completeness). *The eigenfunctions with period an integer multiple of the period of the cnoidal wave solutions (3) of the KdV equation (1) are complete in L^2.*

8. **The soliton limit.** It is tempting to push our arguments to the soliton limit $k \to 1$ and see to what extent the known results in that case are recovered here. To the extent we do this here, this is no more than an entertaining observation, and more work is required if one desires to turn this observation into rigorous mathematics. Recall [6, 8] that the soliton solution is known to be nonlinearly (orbitally) stable.

In the limit $k \to 1$, the cnoidal wave solution (3) reduces to the soliton solution (2). The spectral stability problem (11) maintains its formal appearance, but is now defined on the entire real line. Thus, the spectrum consists of a collection of discrete eigenvalues and the essential spectrum. Using a standard asymptotic constant-coefficient argument, one finds the essential spectrum to be the entire imaginary axis. Finding the discrete spectrum is more involved.

Proceeding with the study of the Lax pair restricted to the soliton, one finds that the Riemann surface (25) degenerates to the rational (*i.e.*, genus 0) algebraic curve

$$\Omega^2 = 16\zeta(\zeta - 1)^2. \tag{45}$$

From this, one might want to conclude that the Lax spectrum σ_L is given by the negative real line (essential spectrum), and an eigenvalue at $\zeta = 1$ (discrete spectrum). This is indeed the result obtained using the forward scattering method [19], using a one-soliton initial condition. Pushing this further, one sees that the τ-spectrum σ_τ is the imaginary line, with an added embedded eigenvalue at the origin. An irrelevant scaling factor of 2 results in a conjectured stability spectrum $\sigma(\mathcal{L}) = i\mathbb{R}$, with an eigenvalue at the origin. These conclusions, obtained rather carelessly here, are indeed correct [39, 40].

9. **Conclusion.** Using explicit constructions, we have shown that the cnoidal wave solutions of the integrable KdV equation are spectrally stable. Specifically, we have constructed the spectrum and the set of eigenfunctions for the linear stability problem analytically. To prepare for the proof of nonlinear stability in [11], we

have also provided a completeness argument of the eigenfunctions with period any integer multiple of the cnoidal wave period. To obtain our results we have used the well-known squared eigenfunction connection and extended their use to a problem with periodic coefficients, but not restricted to periodic boundary conditions.

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