NL3395 **Kadomtsev-Petviashvili equation**

The Kadomtsev-Petviashvili (KP) equation was derived by Kadomtsev & Petviashvili (1970) to examine the stability of the one-soliton solution of the Korteweg–de Vries (KdV) equation under transverse perturbations. As such, it is relevant for almost all applications where the KdV equation arises. After rescaling of its coefficients, the equation takes the form

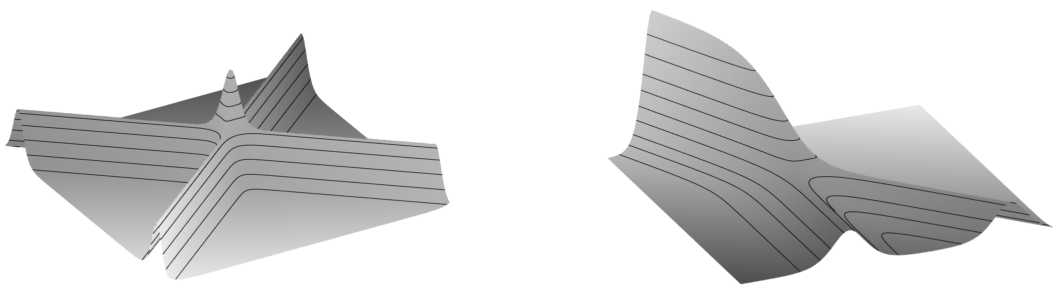
$$(-4u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0, \tag{1}$$

where indices denote differentiation, and σ is a constant parameter. If $\sigma^2 = -1$ $(+1)$, the equation is referred to as the KP1 (KP2) equation. All other real values of σ^2 can be rescaled to one of these two cases. In what follows, a reference to KP as opposed to KP1 or KP2 implies that the result in question is independent of the sign of σ^2.

Depending on the physical context, an asymptotic derivation can result in *either* the KP1 or the KP2 equation. In all such derivations, the equation describes the dynamics of weakly dispersive, nonlinear waves whose wavelength is long compared to its amplitude, and whose variations in the second space dimension (rescaled y) are slow compared to their variations in the main direction of propagation (rescaled x). Two examples are

- Surface waves in shallow water: in this case u is a rescaled wave amplitude, and a rescaled velocity. The wavelength is long compared to the depth of the water h, which is large compared to the wave amplitude. The sign of σ^2 is determined by the magnitude of the coefficient of surface tension τ. KP1 results for large surface tension $\tau/(gh^2) > 1/3$, *i.e.*, thin films. Otherwise KP2 results. Here g is the acceleration of gravity. For most applications in shallow water surface tension plays a sufficiently unimportant role and KP2 is the relevant equation (Ablowitz & Segur, 1979).
- Magneto-elastic waves in antiferromagnetic materials: here u is a rescaled strain tensor and a rescaled velocity. The sign of σ^2 is determined by the difference between the linear velocities of the magnons and phonons, and the strength and direction of the external magnetic field. (Turitsyn & Fal'kovich, 1979)

The KP equation has different classes of soliton solutions. A first class is a generalization of the solitons of the KdV equation. These solutions decay exponentially as $x, y \to \pm\infty$, in all but a finite number of directions along which they limit to a constant. For this reason these solutions are referred to as line solitons. By appropriately choosing their parameters the direction of propagation of each line soliton can be chosen to be anything but the y-direction. In the simplest case the solitons all propagate in the x-direction, adding a second dimension to the KdV solitons. Many other scenarios are possible. Two line solitons can interact with different types of interaction regions to produce two line solitons, but two line solitons can also merge to produce a single line soliton. Alternatively, a single line soliton can disintegrate in two line solitons. The production or annihilation of a line soliton is sometimes referred to as soliton resonance. Although both KP1 and KP2 have line soliton solutions, soliton resonance only occurs for the KP2 equation. Line soliton solutions of the KP2 equation are stable, whereas line soliton solutions of the KP1 equation are unstable. More possibilities exist when more than two line solitons are involved. Two distinct line soliton interactions are illustrated in Fig. 1.



(a) (b)

Figure 1. Two types of spatial ($t = 0$) line soliton interactions for the KP2 equation: (a) Two identical line solitons with an interaction that does not change their characterists; (b) Two line solitons merge to produce one line soliton.

Another class of soliton solutions exists only for the KP1 equation and decays algebraically in all directions as $\sqrt{x^2+y^2} \to \infty$. These soliton solutions are referred to as lumps and are unstable. Individual lumps in multi-lump solutions do interact with each other, but leave no trace of this interaction. A lump soliton is shown in Fig. 2 (Ablowitz & Segur, 1981).

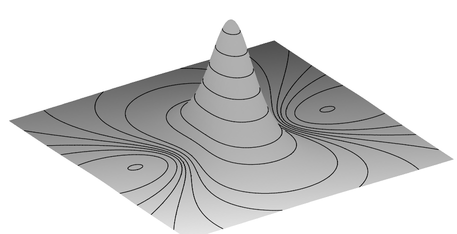


Figure 2. A lump soliton ($t = 0$) solution of the KP1 equation

Another important class of solutions of the KP equation generalizes the exact periodic and quasiperiodic solutions of the KdV equation. A (quasi)periodic solution with g phases is expressed in terms of the Riemann theta function $\theta(\boldsymbol{z}|\boldsymbol{B})$ by

$$u(x,y,t) = c + 2\frac{\partial^2}{\partial x^2} \ln \theta(\boldsymbol{k}x + \boldsymbol{l}y + \boldsymbol{\omega}t + \boldsymbol{\phi}|\boldsymbol{B}). \tag{2}$$

Here c is a constant, $\boldsymbol{k}$, $\boldsymbol{l}$, $\boldsymbol{\omega}$ and $\boldsymbol{\phi}$ are g-dimensional vectors that are interpreted as wave vectors ($\boldsymbol{k}$, $\boldsymbol{l}$), frequencies ($\boldsymbol{\omega}$) and phases ($\boldsymbol{\phi}$). These parameters and the $g \times g$ Riemann matrix $\boldsymbol{B}$ are determined by a genus g compact connected Riemann surface, and a set of g points on it.

For $g = 1$, the solution (2) generalizes the cnoidal-wave solution of the KdV equation to two spatial dimensions. For $g = 2$, the solution is still periodic in space. Its basic period cell is a hexagon, which tiles the (x, y)-plane. These solutions translate along a direction in the (x, y)-plane. For $g \geq 3$ the solution (2) is typically no longer periodic or translating in time. For some values of their parameters, these (quasi)periodic solutions can be interpreted as infinite nonlinear superpositions of line solitons. Solutions with $g \leq 2$ have been compared to experiments in shallow water, with agreement being more than satisfactory (Hammack, *et al* , 1995). A two-phase solution of the KP2 equation is shown in Fig. 3. A good review of the finite-phase solutions of the KP equation is given by Dubrovin (1981).

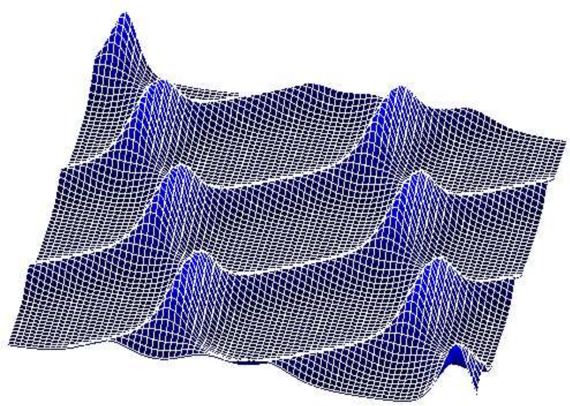


Figure 3. A two-phase periodic solution of the KP2 equation

Unlike for the KdV equation, where only a restricted class of Riemann surfaces arises, any compact connected Riemann surface gives rise to a set of solutions of the KP equation. The reverse statement is also true: if (2) is a solution of the KP equation, then the matrix $\boldsymbol{B}$ is the normalized period matrix of a genus g Riemann surface. This statement is due to Novikov. It provides a solution to the century-old Schottky problem. Its proof is due to Shiota (1986).

The KP equation is the compatibility condition $\Psi_{yt} = \Psi_{ty}$ of the two linear equations

$$\sigma\Psi_y = \Psi_{xx} + u\Psi, \qquad \Psi_t = \Psi_{xxx} + \frac{3}{2}u\Psi_x + \frac{3}{4}(u_x + w)\Psi, \tag{3}$$

with $w_x = \sigma u_y$. These equations constitute the Lax Pair of the KP equation. Using the inverse scattering method, it is the starting point for the solution of the initial-value problem for the KP equation on the whole (x, y)-plane with initial conditions that decay at infinity. The inclusion of line solitons is possible as well (Ablowitz & Clarkson, 1981). Although the initial-value problem with periodic boundary conditions for KP2 was solved by Krichever (1989), this approach was unable to solve the same problem for the KP1 equation. In this context, the solutions (2) are referred to as finite-gap solutions, as they give rise to operators (3) with spectra that have a finite-number of forbidden gaps in them.

More details and different aspects of the theory of the KP equation are found in Ablowitz & Segur (1981); Ablowitz & Clarkson (1981); Dubrovin (1981); Krichever (1989); Shiota (1986), and references therein.

Bernard Deconinck

See also Inverse scattering method; Korteweg–de Vries equation; Multidimensional solitons; Theta functions; Water waves

Further Reading

Ablowitz, M. J. & Segur, H. 1979. On the evolution of packets of water waves. *Journal of Fluid Mechanics*, 92: 691–715

Ablowitz, M. J. & Segur, H. 1981. *Solitons and the inverse scattering transform*. SIAM Studies in Applied Mathematics, Philadelphia.

Ablowitz, M. J. & Clarkson, P. A. 1991. *Solitons, nonlinear evolution equations and inverse scattering*. Cambridge University Press

Dubrovin, B. A. 1981. Theta functions and non–linear equations. *Russian Mathematical Surveys*, 36: 11-92

Hammack, J., McCallister, D., Scheffner, N. & Segur, H. 1995. Two-dimensional periodic waves in shallow water. Part 2. Asymmetric waves. *Journal of Fluid Mechanics*, 285: 95–122

Kadomtsev, B. B. & Petviashvili, V. I. 1970. On the stability of solitary waves in weakly dispersive media. *Soviet Physics Doklady*, 15: 539–541

Krichever, I. M. 1989. Spectral theory of two-dimensional periodic operators and its applications. *Russian Mathematical Surveys*, 44: 145–225

Shiota, T. 1986. Characterization of Jacobian varieties in terms of soliton equations. *Inventiones Mathematicae*, 83: 333–382

Turitsyn, S. K. & Fal'kovich, G. E. 1986. Stability of magnetoelastic solitons and self-focusing of sound in antiferromagnets. *Soviet Physics JETP*, 62: 146–152