The KP equation with quasiperiodic initial data

Bernard Deconinck* and Harvey Segur†
Department of Applied Mathematics
University of Colorado, Boulder, USA 80309-0526
Fax: (303) 492-4066

Accepted for publication.

November 16, 1997

Abstract

The Kadomtsev-Petviashvili (KP) equation is known to admit exact, quasiperiodic solutions that can be written in terms of Riemann theta functions, with a finite number of phases in each solution. In this paper, we propose a method to solve the initial-value problem for the KP equation, for initial data taken from this class of quasiperiodic functions.

Keywords:
Kadomtsev-Petviashvili Equation, Initial-Value Problem, Quasiperiodic, Riemann Surface

*Tel.: (303) 492-3462, e-mail: deconinc@newton.colorado.edu
†Tel.: (303) 492-0592, e-mail: segur@newton.colorado.edu

1 Introduction

The study of solitons and of completely integrable partial differential equations began with the discovery of solitons by Zabusky & Kruskal [1] in the Korteweg-de Vries (KdV) equation, which we write as

$$-4u_t + 6uu_x + u_{xxx} = 0. \qquad \text{(KdV)}$$

The KdV equation arises as an approximate model in many physical systems, and it continues to be the best known and most studied partial differential equation that is completely integrable. It involves one space-like (x) and one time-like (t) variable, so it is said to be in (1+1) dimensions.

Based on physical considerations, Kadomtsev & Petviashvili [2] generalized the KdV equation from (1+1) to (2+1) dimensions when they introduced a pair of equations equivalent to

$$\begin{aligned} \left(-4u_t + 6uu_x + u_{xxx}\right)_x + 3\sigma^2 u_{yy} &= 0, \quad \sigma^2 = -1, & \text{(KP1)} \\ \left(-4u_t + 6uu_x + u_{xxx}\right)_x + 3\sigma^2 u_{yy} &= 0, \quad \sigma^2 = +1. & \text{(KP2)} \end{aligned}$$

From a given physical problem with fixed parameters, one might derive *either* (KP1) *or* (KP2). For real-valued solutions, the two equations have different physical meanings and different properties. For example, Kadomtsev & Petviashvili [2] showed that a (one-dimensional) KdV soliton is unstable to transverse perturbations in (KP1), but not in (KP2). On the other hand, some statements are valid for either sign of σ^2. In what follows, we refer simply to 'the KP equation' or to '(KP)' if the sign of σ^2 is not important.

The KP equation acquired added interest when it was discovered to be completely integrable. This point is controversial [3, 4], but the following discoveries all suggested both that the KP equation is integrable, and that it has a mathematical structure closely related to the structure of the KdV equation.

- A Lax pair for the KP equation was found by Dryuma [5] and by Zakharov & Shabat [6]. For KP solutions that are y-independent, the Lax pair of (KP) reduces to that of (KdV). Explicit N-soliton solutions of the KP equation were found as a consequence of this work.

- Krichever [7, 8] showed that the KP equation, like the KdV equation, admits a huge family of quasiperiodic solutions in the form

$$u = 2\partial_x^2 \ln \Theta_g \left(\phi_1, \phi_2, \ldots, \phi_g\right), \qquad \text{(1a)}$$

  where Θ_g is a Riemann theta function with g phases, and each phase is linear in x, y, t; *i.e.,*

$$\phi_j = k_j x + l_j y + \omega_j t + \phi_{0j}, \qquad j = 1, 2, \ldots, g. \qquad \text{(1b)}$$

- Manakov and others [9] generalized the method of inverse scattering, which had been developed for one-dimensional problems like (KdV), to solve (KP1) for initial data that decay rapidly as $(x^2 + y^2) \to \infty$. This method fails for (KP2), but Ablowitz, Bar Yaacov & Fokas [10] introduced another generalization to solve (KP2) for rapidly decaying initial data.

These facts show that the theory for the KP equation parallels the theory for the KdV equation in many respects. However, the two problems differ in one important way: the KdV equation with periodic boundary conditions can be solved as an initial-value problem; the corresponding statement for the KP equation is currently false. The objective of this paper is to begin to fill in this gap in KP theory.

Important work on (KdV) with periodic boundary conditions was done by Novikov [11] and Lax [12]. For a well written overview of the history of the periodic problem, see the introduction of [13]. Broadly speaking, the solution of the initial-value problem for the KdV equation with periodic boundary conditions has two steps. Step one is to solve the initial-value problem for initial data that are periodic in x, and are of the form (1) with some finite number of phases (g). The theories presented in both [11] and [12] are restricted in this way. Step two is to show that this restricted family of initial conditions is dense in the set of all smooth, periodic functions of x. For KdV, step two was proved in [14]. This pattern of solving the periodic problem in two steps seems to be standard for integrable problems in (1+1) dimensions, although step two has been carried to completion only for some of these problems [15, 16].

The objective of this paper is to outline a method to solve step one for the KP equation. Specifically, suppose u(x, y, 0) has the form (1) with some finite number (g) of phases. If u(x, y, 0) represents initial data for the KP equation, then the method presented here finds u(x, y, t) with three properties: (a) it solves (KP); (b) it has the form (1) with g phases; and (c) it coincides with the given initial data at t = 0.

We now identify some features of this method of solution.

- In the form given above, the KP equation is not well-posed: If $u(x, y, 0) \equiv 0$ initially, then $\{u(x, y, t) \equiv 0\}$ and $\{u(x, y, t) = t\}$ both satisfy (KP), and they coincide at $t = 0$. Requiring that the solution has the form (1) removes this ambiguity.

- The method proposed here does not require that the initial data be spatially periodic; if the initial data happen to be spatially periodic, then the KP solution retains that spatial periodicity. Another method to solve the initial-value problem for (KP), proposed in [17], is restricted to initial data that are periodic in x and periodic in y. The two methods are very different. On the other hand, our method is closely related to earlier work by Previato [18].

- It has been asserted [17, 3] that (KP1) and (KP2) are quite different. For the problem posed here, we see no difference between the two equations. Our method of solution works equally well for both.

- Whether the family of solutions defined by (1) is dense in some more general class of initial data is an important question, but we do not address it in this paper. See [17, 19] for discussions of this issue.

The KP equation admits solutions in the form (1) because of its intimate connection with the theory of Riemann surfaces. Using this connection, Krichever [7, 8] showed how to construct a KP solution of the form (1), starting with two ingredients: (i) a compact, connected Riemann surface, and (ii) a set of points on that surface, called a divisor. Therefore, to solve the KP equation with quasiperiodic initial data, it is sufficient to construct from the initial data: (i) the appropriate Riemann surface, and (ii) its divisor. The method of solution proposed here does exactly this.

The connection between the KP equation and the theory of Riemann surfaces plays a crucial role in all of this work, so we review this connection in §2. Our method is in spirit similar to the method presented in [20] so we give a brief overview of this method in §3. §4 contains our

main result: a method to construct the Riemann surface and its divisor from quasiperiodic initial data, $u(x, y, 0)$, that are of the form (1) with a finite number of phases. In order to simplify our presentation as much as possible, we defer nonessential proofs of the main points of the method to the Appendix. Some concrete examples are worked out in §5. These examples include: (i) all KP solutions with exactly 1 phase; (ii) all KP solutions with exactly 2 phases; (iii) all one-dimensional solutions of the form (1); (iv) the connection between our method and the traditional monodromy approach.

2 Relation between the KP equation and Riemann surfaces

We begin by recalling the basic ingredients of Riemann surface theory [21, 22]. Consider a polynomial relation between two complex variables, w, z:

$$R(w, z) = 0. \tag{2}$$

$R(w, z)$ is polynomial in w, with each coefficient polynomial in z; the polynomials are of finite degree. Such a relation is said to define an *algebraic curve*. More precisely, (2) defines the *affine part* of the curve, *i.e.,* all finite points on the curve. If $R(w, z)$ is a polynomial of degree 1 (*i.e,* linear) in w, then (2) defines $w(z)$ uniquely. If $R(w, z)$ is polynomial in w of degree p ($p > 1$), then (2) defines a p-valued function, $w(z)$. A standard way to handle this multi-valuedness is to define p *branches* of $w(z)$: a branch is a copy of the complex $z-$plane, with a boundary (*i.e.,* a number of cuts), and $w(z)$ is defined to be single-valued and continuous on each branch. Viewed in this way, $w(z)$ has p unconnected branches lying over the complex $z-$plane.

Alternatively, one can connect the separate branches, essentially by gluing one branch to another at their edges, in such a way that $w(z)$ remains single-valued and continuous as one passes from one branch to the next. This gluing necessarily starts and stops at a *branch point*, *i.e.,* a point where $w(z)$ has fewer than p distinct values and where $w(z)$ does **not** return to its original value when one follows $w(z)$ around the point on a small circle that encloses the point on a particular branch. In addition, if one compactifies the complex z-plane in the usual way, by adding a point at infinity, then the z-plane can be identified with a Riemann sphere, and the separate branches of $w(z)$ glue together to form a compact, connected *Riemann surface*. Thus, a Riemann surface represents a multi-sheeted covering of the extended z-plane, required because $w(z)$ is multi-valued. (In what follows, we omit 'compact, connected', which will always be understood.)

In the simplest cases, the surface obtained in this way is simply connected, so it is topologically equivalent to a sphere. However, most Riemann surfaces are not simply connected, and the number of holes in a surface is called the *genus* of the surface.

The most studied curves are those in which $R(w, z)$ is a quadratic monic polynomial in w. Then perhaps after a change of variables, (2) can be written in the form

$$w^2 = P_n(z), \tag{3}$$

where $P_n(z)$ is a polynomial of degree n in z. One can show the following [23].

1. If n is 1 or 2, then the Riemann surface has genus 0, and is topologically equivalent to a sphere. These curves are called *rational*.

2. If n is 3 or 4, then the Riemann surface has genus 1, and is topologically equivalent to a torus. These curves are called *elliptic*.

3. If n is either $(2g+1)$ or $(2g+2)$, then the Riemann surface has genus g. These curves are called *hyperelliptic* if $g > 1$. Figure 1 shows a Riemann surface for a hyperelliptic curve of genus 2. In all of these cases, the point at infinity is a branch point if n is odd, and it is not a branch point if n is even.

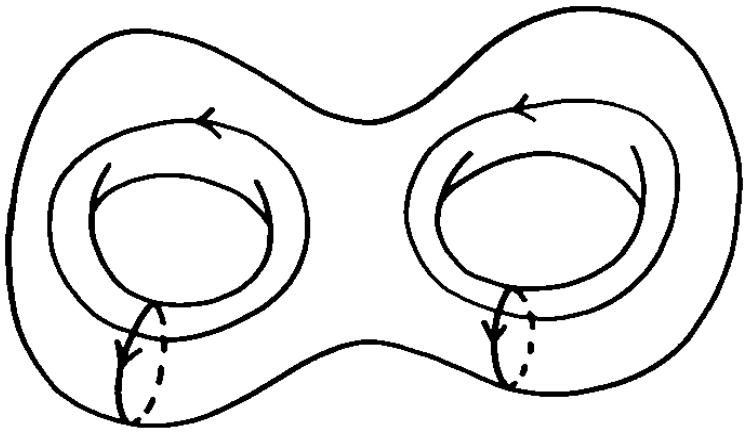

Figure 1: **A hyperelliptic surface of genus 2, with a basis of 4 cycles.**

Riemann surfaces are related to the KP equation because a Riemann surface of genus g generates a Riemann theta function with g phases, like those in (1). To see this, we need two features of a Riemann surface. The first feature is a *cycle*: a closed, oriented, piecewise smooth curve on the Riemann surface [24]. One difference between a Riemann surface of finite genus and the complex plane is that the Riemann surface admits cycles that cannot be continuously deformed to a point; no such curves exist on the complex plane. (In what follows, we use 'cycle' to mean only those cycles that cannot be shrunk to a point.) One can show that on a Riemann surface of genus g, the cycles have a natural group structure with $2g$ independent elements. One chooses a basis of $2g$ cycles to represent the entire group. A possible choice of basis cycles is drawn in Figure 1.

A second natural feature of a Riemann surface is a *holomorphic differential* (also called an *Abelian differential*) ω, which is regular everywhere on the surface and can be written locally as $\omega = f(z)dz$, where $f(z)$ is analytic in the local coordinate z. On a Riemann surface of genus g, one shows that g linearly independent holomorphic differentials exist [21]. The set of holomorphic differentials forms a g-dimensional linear vector space.

A Riemann surface of genus g has g holomorphic differentials, and we have chosen a basis of $2g$ cycles. Integrating the g differentials around the $2g$ cycles provides $2g^2$ complex numbers. Following a standard procedure [22], one uses g of the cycles to normalize the differentials, then integrates the normalized differentials around the other g cycles to obtain a *Riemann matrix*: a $g \times g$, symmetric matrix whose real part is negative definite.

This Riemann matrix, constructed from intrinsic features of the Riemann surface, represents the surface in a natural way. It is independent of the coordinates on the surface, and of the coordinates on the cycles [23]. The Riemann matrix depends on the choice of the basis of cycles, but all Riemann matrices obtained from a particular Riemann surface can be transformed into one another in a well-established way [22].

Thus every Riemann surface generates a Riemann matrix, but the converse is false: not every Riemann matrix can be associated with a Riemann surface. The *Schottky problem* in algebraic

geometry can be stated as follows: Out of all Riemann matrices, determine which ones can be associated with some Riemann surface [25]. We will see that the KP equation not only solves this problem, it also identifies the Riemann surface in question.

Now let $\boldsymbol{B}$ be a $g \times g$ Riemann matrix, which may or may not be associated with a Riemann surface. It induces a Riemann theta function with g phases, according to

$$\Theta(\boldsymbol{\Phi}; \boldsymbol{B}) = \sum_{\boldsymbol{m} \in \boldsymbol{Z}^g} \exp\left(\frac{1}{2}\boldsymbol{m} \cdot \boldsymbol{B} \cdot \boldsymbol{m} + i\boldsymbol{m} \cdot \boldsymbol{\Phi}\right); \tag{4}$$

here $\boldsymbol{\Phi} = (\phi_1, \phi_2, \ldots, \phi_g)$ is a g-component vector of phase variables, $\boldsymbol{m} = (m_1, m_2, \ldots, m_g)$ is a g-component vector of integers in which each component runs over all positive and negative integers separately, and the sum is a g-fold sum over each component of $\boldsymbol{m}$. Thus, a theta function is defined by a g-fold Fourier series. Because the real part of $\boldsymbol{B}$ is negative definite, the Fourier coefficients in (4) decay faster than exponentially, so the series converges uniformly and absolutely for any choice of components of $\boldsymbol{\Phi}$. By construction, $\Theta(\boldsymbol{\Phi}; \boldsymbol{B})$ is periodic in each component of $\boldsymbol{\Phi}$, and this is the definition of a quasiperiodic function.

After this rather long introduction, we come at last to Krichever's important work [7, 8]. Krichever starts with a Riemann surface of some finite genus, constructs from it a Riemann matrix and from that a theta function, according to (4). For each of the phase variables in (4), he uses an expression of the form (1b), and obtains the parameters (k_j, l_j, w_j) in each phase by integrating particular nonholomorphic differentials around the basis cycles on the Riemann surface. Then he proves that this theta function, when substituted into (1), provides a solution to the KP equation. In this way every Riemann surface, of any finite genus, generates solutions of the KP equation. This is a remarkable fact.

In (1), the arbitrary phase constants, (ϕ_{0j}), are not determined by the Riemann surface. To completely specify the KP solution, one can either specify these g numbers, or one can specify a set of g points on the Riemann surface. Then the location of these points determines the phase constants. This set of g points is called the *divisor* for the KP solution.

Krichever's construction starts with a theta function associated with some Riemann surface, rather than with an arbitrary theta function. This fact led S.P. Novikov to conjecture that (4) with (1) generates a KP solution only if the theta function can be associated with some Riemann surface [1] [22]. In other words, Novikov conjectured that a way to solve the Schottky problem, and to determine whether a given Riemann matrix can be associated with some Riemann surface, is to use the Riemann matrix to generate a theta function, according to (4), then use that theta function to construct $u(x, y, t)$, according to (1), then check to see whether $u(x, y, t)$ solves (KP). Novikov conjectured that the function constructed in this way would solve KP if and only if the Riemann matrix was associated with some Riemann surface.

Like Krichever's construction, Novikov's conjecture is remarkable. It is perhaps even more remarkable that the conjecture is correct. It was proved by Shiota [26], following earlier work by Mulase [27]. To summarize the results of this section, there is a close connection between the KP equation and Riemann surfaces: every Riemann surface, of any finite genus, generates KP solutions of the form (1). Conversely, every KP solution of the form (1) corresponds to some Riemann surface. The number of phases in the KP solution equals the genus of the surface. Such a solution is called a finite-genus solution.

Krichever's discovery allows one to construct large families of special solutions of the KP equation. This construction has been made effective for low genera by several people [22, 28, 29, 30, 31,

[1] This is called "Novikov's conjecture", but we are not aware that the problem was ever stated in print by him. It seems to have propagated by word of mouth.

32], and efficient computer programs to produce and to display these solutions are now available. Necessarily, these methods start either with a Riemann surface or with some version of a Riemann matrix. They do not start with initial data for the KP equation. Moreover, the methods do not suggest how to obtain a Riemann matrix from initial data. This is the problem addressed in §4.

3 Finite-genus solutions of (1+1)-dimensional integrable equations

The main tool for solving the periodic problem for (1+1)-dimensional equations has been the monodromy matrix (e.g. [14, 15, 23, 33]). There is however a method that allows one to solve the initial-value problem for finite-genus solutions in a purely local way [20], [34]. Since our method to solve the KP equation is very similar to that in [20], we now review its main results.

We want to solve a system of (1+1)-dimensional equations for N functions $u_k(x,t)$, $k = 1,\ldots,N$. Let $\boldsymbol{u} = (u_1,\ldots,u_N)$. We call $\boldsymbol{u}$ the potential. The system of equations is completely integrable, so it has an associated linear problem for an r-dimensional vector-valued wave function $\boldsymbol{\Psi}$

$$\boldsymbol{\Psi}_x = \boldsymbol{U}(\boldsymbol{u},\lambda)\boldsymbol{\Psi}, \tag{5a}$$
$$\boldsymbol{\Psi}_t = \boldsymbol{V}(\boldsymbol{u},\lambda)\boldsymbol{\Psi}. \tag{5b}$$

As examples, we mention: (a) KdV, for which $r = 2$, $\boldsymbol{u} = u$ (*i.e.*, a scalar), and (5a) can be written in scalar second-order form; (b) any equation from the AKNS hierarchy [35], for which $r = 2$, $\boldsymbol{u} = (q,r)$; (c)Boussinesq, for which $r = 3$ [36], $\boldsymbol{u} = (p,q)$ with $q_t = p_x$, and (5a) can be written in scalar form. The $r \times r$ matrices $\boldsymbol{U}(\boldsymbol{u},\lambda)$ and $\boldsymbol{V}(\boldsymbol{u},\lambda)$ depend on x and t through the potential and a finite number of its derivatives with respect to x. They also depend in a nontrivial way on a *spectral parameter* λ. The compatibility condition of (5a) and (5b) is

$$\frac{\partial \boldsymbol{U}}{\partial t}(\boldsymbol{u},\lambda) - \frac{\partial \boldsymbol{V}}{\partial x}(\boldsymbol{u},\lambda) = \left[\boldsymbol{V}(\boldsymbol{u},\lambda), \boldsymbol{U}(\boldsymbol{u},\lambda)\right]. \tag{6}$$

This equation is equivalent to the original set of integrable equations for $\boldsymbol{u}$. For scalar equations ($\boldsymbol{u} = u$), (6) is the matrix form of the familiar Lax equation.

Every equation of the form (6) is a member of an infinite hierarchy. If $\boldsymbol{U}(\boldsymbol{u},\lambda)$ and $\boldsymbol{V}(\boldsymbol{u},\lambda)$ are polynomial in λ, this hierarchy can be obtained [35] by finding all matrices $\boldsymbol{W}_k(\boldsymbol{u},\lambda)$ that satisfy

$$\frac{\partial \boldsymbol{U}}{\partial t_k}(\boldsymbol{u},\lambda) - \frac{\partial \boldsymbol{W}_k}{\partial x}(\boldsymbol{u},\lambda) = \left[\boldsymbol{W}_k(\boldsymbol{u},\lambda), \boldsymbol{U}(\boldsymbol{u},\lambda)\right], \tag{7}$$

where $\boldsymbol{W}_k$ is again polynomial in λ; k is an index we use to label the members of the hierarchy. All members of the hierarchy automatically commute with each other. It is then possible [37] to find a wave function $\boldsymbol{\Psi}(x,t,t_3,t_4,\ldots)$ that simultaneously satisfies all equations in the corresponding linear hierarchy

$$\boldsymbol{\Psi}_{t_k} = \boldsymbol{W}_k(\boldsymbol{u},\lambda)\boldsymbol{\Psi}, \qquad k = 1,2,3,\ldots \tag{8}$$

with $t_1 = x, t_2 = t$ and $\boldsymbol{W}_1 = \boldsymbol{U}, \boldsymbol{W}_2 = \boldsymbol{V}$. Note that the potentials can depend on all 'time' variables.

Consider a potential $\boldsymbol{u}$ that makes the t_k-flow (7) stationary, $\boldsymbol{u}_{t_k} = 0$. As a consequence we can separate variables in the k-th equation of (8): let

$$\boldsymbol{\Psi} = e^{\mu t_k} \widetilde{\boldsymbol{\Psi}} \left(x, t, t_2, \ldots, t_{k-1}, t_{k+1} \ldots\right). \tag{9}$$

The k-th equation of (8) now reduces to an algebraic eigenvalue problem for $(\widetilde{\boldsymbol{\Psi}}, \mu)$:

$$\mu \widetilde{\boldsymbol{\Psi}} = \boldsymbol{W}_k(\boldsymbol{u}, \lambda) \widetilde{\boldsymbol{\Psi}}. \tag{10}$$

The eigenvalue equation is

$$\Gamma : \det\left(\boldsymbol{W}_k(\boldsymbol{u}, \lambda) - \mu \boldsymbol{I}_r\right) = 0, \tag{11}$$

with $\boldsymbol{I}_r$ the r-dimensional identity matrix. This is an algebraic relationship between the two parameters μ and λ (as discussed in §2). It determines the affine part of a Riemann surface Γ. Clearly the entries of $\boldsymbol{W}_k(\boldsymbol{u}, \lambda)$ depend on x and t. Nevertheless, because of the particular combinations of the matrix entries that appear in (11), one shows that the coefficients of different powers of λ and μ are $x-$ and $t-$independent [20]. They are conserved quantities for the evolution (in both x and t) of the potential. Hence Γ is independent of the initial time and the spatial point where the calculations are done. Thus the Riemann surface is determined in a purely local way, at fixed (x, t), but it is valid for all values of (x, t).

The Riemann surface Γ as a covering of the λ-plane has r sheets. If no two of these sheets are identical, the corresponding potential is said to have *rank 1* [38, 39]. In other words, the potential has rank 1 if the equation of the Riemann surface as a polynomial in μ has no double roots, except at isolated values of λ (*i.e.,* at discriminant points). The equation (11) for the surface Γ was found earlier in [33]. There it is shown that for solutions of the form (1) the Riemann surface can be obtained from $\boldsymbol{W}_k(\boldsymbol{u}, \lambda)$ instead of from the monodromy matrix.

The eigenvector $\widetilde{\boldsymbol{\Psi}}$ of (10) defines the divisor, which is required for the reconstruction of the potentials. The divisor is the set of finite points (μ, λ) where $\widetilde{\boldsymbol{\Psi}}$ has a pole singularity at $(x = 0, t = 0)$. These points are solutions of the equations

$$\det\left(\boldsymbol{W}_k^{(i1)}(\boldsymbol{u}, \lambda) - \mu \boldsymbol{I}_r^{(i1)}\right)\Big|_{x=0, t=0} = 0, \qquad i = 1, 2, \ldots, r. \tag{12}$$

Here $\boldsymbol{W}^{(i1)}$ is the minor matrix of element w_{i1} in the matrix $\boldsymbol{W}$. We thus require all cofactors of the first column of $(\boldsymbol{W}_k(\boldsymbol{u}, \lambda) - \mu \boldsymbol{I}_r)$ to vanish. This results in a set of points that constitute the divisor. The vanishing of the cofactors of the first column at $(x = 0, t = 0)$ can be regarded as the defining relations for other Riemann surfaces. The divisor points are defined by the intersection of all these surfaces. Since these surfaces are evaluated at $(x = 0, t = 0)$, they do not depend on x or t. Hence the divisor is independent of x and t by construction. The above construction of the divisor is justified, since the wave function $\boldsymbol{\Psi}$ corresponds to the (possibly vector-valued) *Baker-Akhiezer function* on the Riemann surface [20].

We now have a Riemann surface and a divisor. Krichever's construction allows us to reconstruct the potential $\boldsymbol{u}(x, t)$ in the form of Riemann theta functions. This shows that the solutions obtained this way are indeed finite-genus solutions. Novikov [11, 40] and Lax [12] were the first to notice the important connection between higher order stationary flows and finite-genus solutions for (KdV).

4 Method to solve the forward problem for the KP equation

This section contains our main results. Starting from initial conditions for (KP) of the form (1), we derive a Riemann surface and a divisor on it. This constitutes the forward problem for the inverse scattering scheme for (KP) with these initial data. It is justified to talk of a forward problem, since Krichever's construction [7, 8] can be regarded as the solution of an inverse problem.

The Riemann surface and the divisor are the spectral data for the finite genus solutions. Since Krichever's construction allows one to reconstruct the potential from exactly these data, the spectral data are complete for the problem posed. Since the Riemann surface and the divisor show up through an eigenvalue problem, they do indeed constitute spectral data.

Our method to solve the KP equation is analogous to that outlined in §3 for $(1+1)$-dimensional equations. For those equations, one stationary flow from the hierarchy of commuting flows reduces the problem to an algebraic eigenvalue problem. For the KP equation, we will show that two stationary flows from the KP hierarchy reduce the problem to an algebraic eigenvalue problem. Following Burchnall & Chaundy [41, 42] and Previato [18], we use the two stationary flows to construct the so-called Burchnall-Chaundy (BC) matrix, which leads naturally to an algebraic eigenvalue problem. The eigenvalue equation defines the Riemann surface; the eigenvector determines the divisor.

It follows from [26, 27] that any finite-genus solution of (KP) makes stationary all but a finite number of flows in the KP hierarchy. This means that every KP solution of finite genus and rank 1 (*i.e.,* every solution of the form (1)) can be obtained by the method presented here.

One can divide this method into two parts. The objective of the first part (in steps 1, 2 and 3 of what follows) is to identify the pair of stationary flows that corresponds to the initial data in question. Depending on details of the initial data, these steps might select a unique pair of stationary flows, or they might narrow the choices to a finite list of pairs of flows. For each pair of stationary flows, the second part of the method (in steps 4 and 6) constructs an algebraic eigenvalue problem algorithmically. Step 5 provides the set of differential equations the initial condition needs to satisfy for a given pair of stationary flows, so at this point one can check for compatibility with the given initial data. Incompatibility means that one has chosen the wrong pair of stationary flows for the given initial data. For every set of initial data of the form (1), there is at least one appropriate pair of stationary flows in the hierarchy.

Step 1. Determine the genus of the initial condition

Rewrite (1b) in the form

$$\phi_j = \boldsymbol{\kappa}_j.\boldsymbol{x} + \omega_j t + \phi_{0j}, \qquad j = 1, 2, \ldots, g, \tag{1b'}$$

with $\boldsymbol{\kappa}_j = (k_j, l_j)$ and $\boldsymbol{x} = (x, y)$. A two-dimensional Fourier transform of the initial data resolves all the vectors $\boldsymbol{\kappa}_j, j = 1, 2, \ldots, g$, unless there is a relationship of the form

$$\sum_{i=1}^{g} n_i \boldsymbol{\kappa}_i = 0, \tag{13}$$

with the n_i integers, not all zero. In this case the wave vectors $\boldsymbol{\kappa}_i$ are commensurable. Even if (13) holds, the Fourier transform allows one to find a lower bound on the genus. Because the initial data contains only a finite number of phases, the Fourier transform is necessarily discrete; *i.e.,* it consists of isolated spikes. Generically (*i.e.,* when (13) does not hold), this procedure allows us to find the genus of the initial condition by counting the spikes in the Fourier transform, modulo harmonics.

Finding the genus of the initial data from a Fourier transform is simple, but it fails completely in an important special case: when the initial data are spatially periodic. In this case (13) is automatically satisfied, as any two wave vectors κ_i are commensurable. For spatially periodic initial data, one must find the genus by some other means. One possibility is given in step 5, but it is cumbersome.

Either the genus of the intial data or a lower bound on the genus can also be found from a one-dimensional Fourier transform of the initial data, along a certain direction in the (x, y)-plane. One may choose to do the one -dimensional procedure along a number of different directions, to avoid incomplete resolution of the spikes.

Not only the number of spikes is important. Their locations matter as well: they give the actual value of the wavenumbers. One can use this information to check the values of the wavenumbers obtained from the reconstruction procedure [8]. From here on, we assume that the genus of the initial data is known and denote it by g.

Step 2. Check for one-dimensional solutions

The KP equation is invariant under transformations of the form

$$\begin{aligned} \bar{x} &= x + \alpha y + \frac{3}{4}\sigma^2\alpha^2 t, \\ \bar{y} &= y + \frac{3}{2}\sigma^2\alpha t, \\ \bar{t} &= t, \end{aligned} \tag{14}$$

corresponding to

$$\partial_x = \partial_{\bar{x}}, \quad \partial_y = \partial_{\bar{y}} + \alpha\partial_{\bar{x}}, \quad \partial_t = \partial_{\bar{t}} + \frac{3}{2}\sigma^2\alpha\partial_{\bar{y}} + \frac{3}{4}\sigma^2\alpha^2\partial_{\bar{x}}. \tag{15}$$

Therefore, if the initial data satisfy

$$u_y(x, y, 0) = \beta u_x(x, y, 0) \tag{16}$$

for all (x, y) and for some fixed β, then there is a transformation of the form (14) under which the initial data become $y-$independent. These initial data are one-dimensional. Perhaps after such a transformation, they are initial data for the KdV equation. This problem can be solved by the method described in §3 and [20].

Condition (16) is unusual, but the problem becomes so simple that one should always check for it first. This special case is worked out in detail in §5.

Remark: If a solution is one-dimensional, the spikes in the two-dimensional Fourier transform (step 1) are collinear in the (k, l)-plane.

Step 3. Select two stationary flows (r, n)

The KP equation can be written as the compatibility condition of the following linear system:

$$\sigma\psi_y = \psi_{xx} + u\psi, \tag{17a}$$

$$\psi_t = \psi_{xxx} + \frac{3}{2}u\psi_x + \frac{3}{4}\left(u_x + w\right)\psi, \tag{17b}$$

with $w_x = \sigma u_y$. In what follows we absorb σ into the y-derivative. Just as in the (1+1)-dimensional case, the KP equation is a member of an infinite hierarchy. This hierarchy is the set of compatibility conditions of an infinite number of linear equations

$$\psi_{t_k} = A_k(u)\psi, \qquad k = 1, 2, 3, \ldots. \tag{18}$$

Let us compare this hierarchy to that in (8). In (8), $\boldsymbol{W}_k(\boldsymbol{u}, \lambda)$ is an $r \times r$ matrix, and the $k-$th flow is defined by a system of linear ordinary differential equations. In (18), $A_k(u)$ is a kth-order differential operator (in x), and the kth linear flow is defined by a single partial differential equation. The operator $A_k(u)$ depends on u and on a finite number of its derivatives and antiderivatives. The first four equations of this hierarchy are:

$$\begin{aligned}
\psi_{t_1} &= \psi_x + c_{10}\psi, && \text{(19a)}\\
\psi_{t_2} &= \psi_{xx} + u\psi + c_{21}\psi_x + c_{20}\psi, && \text{(19b)}\\
\psi_{t_3} &= \psi_{xxx} + \frac{3}{2}u\psi_x + \frac{3}{4}\left(u_x + w_1\right)\psi + c_{32}\left(\psi_{xx} + u\psi\right) + \\
&\quad c_{31}\psi_x + c_{30}\psi, && \text{(19c)}\\
\psi_{t_4} &= \psi_{4x} + 2u\psi_{xx} + \left(w_1 + 2u_x\right)\psi_x + \left(u^2 + \frac{1}{2}w_{1,x} + \frac{1}{2}w_2 + u_{xx}\right)\psi + \\
&\quad c_{43}\left(\psi_{xxx} + \frac{3}{2}u\psi_x + \frac{3}{4}\left(u_x + w_1\right)\psi\right) + c_{42}(\psi_{xx} + u\psi) + \\
&\quad c_{41}\psi_x + c_{40}\psi. && \text{(19d)}
\end{aligned}$$

Here $\partial_x w_1 = \partial_y u$, $\partial_x w_2 = \partial_y w_1$ with $w_1 = w$; ψ_{4x} denotes the fourth derivative of ψ with respect to x. We use this notation from now on, if the order of differentiation is greater than three.

The k-th equation of the hierarchy (18) can be written explicitly. Let

$$A_k(u) = \sum_{j=0}^{k} a_k \frac{\partial^k}{\partial x^k}. \tag{20}$$

Expressing that (18) commutes with (17a):

$$\frac{\partial}{\partial t_k}\frac{\partial}{\partial y}\psi = \frac{\partial}{\partial y}\frac{\partial}{\partial t_k}\psi, \tag{21}$$

we obtain an explicit form for the coefficients a_j:

$$\begin{aligned}
a_k &= 1,\\
a_{k-1} &= c_{k-1},\\
a_{j-1} &= \frac{1}{2}\sum_{i=j+1}^{k}\binom{i}{j}\int a_i u_{(i-j)x}dx - \frac{1}{2}a_{j,x} + \frac{1}{2}\int a_{j,y}dx, && \text{(22)}\\
&\qquad \text{for} \quad j = k-1, \ldots, 1.
\end{aligned}$$

Here c_{k-1} is constant. Furthermore, from (22) every coefficient a_{j-1} is determined upto a constant of integration. Another consequence from (21) is the form of the k-th KP equation:

$$\frac{\partial u}{\partial t_k} = \sum_{i=1}^{k} a_i u_{ix} - a_{0,xx} + a_{0,y}. \tag{23}$$

At this point, the additional potentials $w_1, w_2, \ldots$ are introduced to remove all nonlocality, as we did in (19). These additional potentials are determined in terms of the potential $u(x, y, t)$ as follows. From the definition of $w_1(x, y, t)$,

$$w_1(x, y, t) = \int u(x, y, t) dx. \tag{24}$$

Hence, w_1 is determined upto a function of only y, say $f(y)$. We then find for w_2

$$w_2(x, y, t) = \int w_1(x, y, t) dx + x f'(y). \tag{25}$$

We require $w_2(x, y, t)$ to de quasiperiodic in x, hence $f(y)$ is necessarily a pure constant, $f(y) = c$. Similar arguments prove that all additional potentials are only determined up to additive constants. These additive constants are already taken into account in (19), where they are a consequence of the coefficients a_j in (20) being only determined up to constants of integration, as we see in (22).

The two equations (17a,17b) defining (KP) are obtained as special cases of the t_2 and the t_3 flows. In this sense we can think of x as t_1, y as t_2 and t as t_3. This 'multiple time' formalism is essential for our solution method. We call the k-th equation of (18) the k-th linear flow.

From Mulase's work [27], it follows that a KP solution of the form (1) with a finite number of phases makes stationary all but a finite number of flows in the KP hierarchy. We show below that from two of these stationary flows, one can construct both a Riemann surface and a divisor on that surface.

For this approach to be useful, two questions must be answered:

- Given only initial data, it is not obvious which flows are stationary and which are not. How does one select from (18) two flows that are stationary for the given initial data?
- Out of all possible pairs of stationary flows for the given initial data, which pair gives the simplest representation of the Riemann surface?

Here 'simplest' means the representation with the minimal number of sheets, for one of the variables. To fix notation, let (r, n) denote the indices of the two stationary flows in question, with $r < n$. We define the 'simplest' choice of (r, n) to be the one that: (a) minimizes r, regardless of its effect on n; (b) for that fixed r minimizes n. The question is: what are (r, n)?

We cannot give a simple rule that always identifies (r, n) uniquely with no further checks like those in step 5. The following constraints are known.

- The initial data has g independent phases. We show in the Appendix (Theorem 1) that this solution must be stationary with respect to at least one of the first $(g + 1)$ flows in the KP hierarchy. Therefore, r need never be greater than $g + 1$.
- As discussed in the Appendix (Theorem 1), if the g phase variables in the KP solution are all independent, then $r = g + 1$. This is the generic situation.
- Clearly, r can be less than $g + 1$. As discussed in step 2, the initial data are one-dimensional if they satisfy (16). But (16) asserts that the t_2 ($\simeq y$) flow is stationary, so $r = 2$ for one-dimensional initial data.

- The flow corresponding to $r = 1$ is trivial, so $r = 2$ is the minimal choice. Therefore, for initial data with g phases, $2 \leq r \leq g + 1$.

- Relations among the phases reduce r from its maximum of $g + 1$. For example, for a generic 3-phase solution of (KP), $r = 4$ $(= g + 1)$. As noted in [32], a 3-phase solution of KP is stationary with respect to t_3 $(\simeq t)$ if

$$\det \begin{pmatrix} k_1 & l_1 & \omega_1 \\ k_2 & l_2 & \omega_2 \\ k_3 & l_3 & \omega_3 \end{pmatrix} = 0, \tag{26}$$

  where we have used the notation of (1). In this case, $r = 3$. If the initial data also satisfy

$$\det \begin{pmatrix} k_1 & l_1 \\ k_2 & l_2 \end{pmatrix} = 0, \quad \det \begin{pmatrix} k_1 & l_1 \\ k_3 & l_3 \end{pmatrix} = 0, \tag{27}$$

  then they are one-dimensional, so $r = 2$. Thus, some 3-phase solutions of KP have $r = 4$, others have $r = 3$, and still others have $r = 2$. This is the usual situation: for initial data for (KP) with g phases, $2 \leq r \leq g + 1$, and all of these possibilities can occur (Appendix, Theorem 1).

- Now we turn our attention from r to n. Krichever [38] observed that (r, n) must be coprime (*i.e.*, no common factors) for a KP solution of rank 1. If $r = 2$ then n must be odd. In fact, if $r = 2$, $n = 2g + 1$ [23]. The case $r = 2$ is the only case for which g determines n uniquely with no further checks like those in step 5.

- For a generic KP solution of the form (1), with no relations among the phases, $r = g + 1$, $n = g + 2$. This choice has $r < n$, with (r, n) coprime.

- Unfortunately, we show in steps 4 and 5 that choosing $(r = g + 1, n = g + 2)$ generates a family of Riemann surfaces, only some of which have genus g. In the Appendix (Theorem 4), we show that for fixed (r, n), the highest genus possible is $(r - 1)(n - 1)/2$.

- We have the following confusing situation. To find a generic genus g KP solution, choose $(r = g + 1, n = g + 2)$. This choice leads to a family of Riemann surfaces that include those for generic KP solutions of genus g, but that also include others for nongeneric solutions of higher genus. These nongeneric solutions of higher genus typically have more parameters, so one must hunt to find the genus g solutions among these other solutions. A nontrivial example along these lines is given in §5.

- We show in the Appendix (Theorem 2) that the number of pairs (r, n) that can generate a Riemann surface of genus g is $g(g + 1)/2 - g + 1$. We also show that for fixed g and fixed $2 \leq r \leq g+1$, there are $(g - r + 2)$ relevant choices of n. The proof of this result also provides an exhaustive list of pairs (r, n) that can generate a KP solution of genus g and rank 1.

To summarize, for initial data with g phases, one finds all relevant pairs (r, n) consistent with g in the following way:

1. All values of r with $2 \leq r \leq g + 1$ are allowed.

2. For each r, let $n_j(r)$ be the j-th integer greater than r that is coprime with r. The lowest $(g-r+2)$ of these integers are possible values of n.

3. Exclude from this list the values of n for which $(r-1)(n-1)/2 < g$.

4. The remaining pairs (r,n) are all possible for genus g.

Note that there is always only one possibility for (r,n) when $r=2$. This is the case $(2,2g+1)$ corresponding to a one-dimensional solution. Table 1 gives an overview of the possibilities for low g.

$g=1$	
r	n
2	3

$g=2$	
r	n
3	4
2	5

$g=3$	
r	n
4	5
3	5
3	4
2	7

$g=4$	
r	n
5	6
4	7
4	5
3	7
3	5
2	9

$g=5$	
r	n
6	7
5	7
5	6
4	9
4	7
4	5
3	8
3	7
2	11

Table 1: **The possibilities for r and n for $g = 1,2,3,4,5$**

All of these possibilities for (r,n) have to be considered when solving the initial value problem for (KP) for a solution with genus g. In step 5, we give a way to decide among the different possibilities.

Identifying the proper pair (r,n) for a KP solution of genus g seems to be inherently complicated. No such complification occurs for (KdV), for which $r=2$, $n=2g+1$. Similarly, no complification arises for the nonlinear Schrödinger (NLS) equation, for which $r=2$ (in a different hierarchy) and $n=2g+2$ [34]. These complifications arise not because (KP) is in (2+1)-dimensions, but because $r \neq 2$. For example, the boussinesq equation is a (1+1)-dimensional problem with $r=3$, and we show in section 5, examples 4 and 5 that neither g nor n determine the other uniquely.

Remark: From the proof of theorem 1, it is clear that the nongeneric cases of (r,n) are contained in the generic case as limit cases. As a consequence, one could in principle always consider the generic case $(r=g+1, n=g+2)$. However, any choice of (r,n) that is not minimal results in unnecessary large expressions for the Riemann surface. Furthermore, in order for this surface to have the right genus, it needs to have a large number of singularities. In this sense, choosing (r,n) minimal can be regarded as a way to minimize the number of singularities of the Riemann surface.

Step 4. Construct the Riemann surface and the BC matrix

Write the two stationary flows as

$$\psi_{rx} + u_{r-1}\psi_{(r-1)x} + \ldots + u_1\psi_x + u_0\psi = \lambda\psi, \tag{28a}$$

$$\psi_{nx} + v_{n-1}\psi_{(n-1)x} + \ldots + v_1\psi_x + v_0\psi = \mu\psi, \tag{28b}$$

where the coefficients $u_0, u_1, \ldots, u_{r-1}$ and $v_0, v_1, \ldots, v_{n-1}$ are related to $u(x,y,t)$ as in (19a)-(19d) by compatibility with (17a) and (17b). Let $\boldsymbol{u} = (u_0, u_1, \ldots, u_{r-1})$, $\boldsymbol{v} = (v_0, v_1, \ldots, v_{n-1})$. Compatibility with (17a) implies that both u_{r-1} and v_{n-1} are constant. Conjugation with a suitable function [38] allows us to eliminate v_{n-1}, so that $\boldsymbol{v} = (v_0, v_1, \ldots, v_{n-2}, 0)$.

We want a common nontrivial solution of the two ordinary differential equations (28a) and (28b). Burchnall and Chaundy [41, 42] found the condition on the coefficients and the eigenvalues λ and μ for a common nontrivial solution to exist. They extended the concept of an elimination matrix (a.k.a. resultant or Sylvester's eliminant [46]) to linear differential operators. The procedure is as follows:

1. Take the r equations obtained from (28b) by taking $r-1$ derivatives with respect to x of it.
2. Take the n equations obtained from (28a) by taking $n-1$ derivatives with respect to x of it.
3. We now have $r+n$ homogeneous equations for the $r+n$ unknowns $\psi, \psi_x, \ldots, \psi_{(r+n-1)x}$. Write them in the form $\boldsymbol{E}(\lambda,\mu)\boldsymbol{\psi} = 0, \boldsymbol{\psi} = (\psi, \psi_x, \ldots, \psi_{(r+n-1)x})$. In this linear system, write the r equations obtained in step 1 first, in order of increasing number of derivatives. Then write the n equations obtained in step 2, again in order of increasing number of derivatives.

The $(r+n)\times(r+n)$ matrix $\boldsymbol{E}(\lambda,\mu)$ is called the elimination matrix for the equations (28a, 28b). It clearly depends on λ and μ. In order that (28a) and (28b) have a nontrivial solution, it is necessary and sufficient that

$$\Gamma : \det(\boldsymbol{E}(\lambda,\mu)) = 0. \tag{29}$$

This is an algebraic relationship between μ and λ (§2), so it defines the affine part of a Riemann surface. It is more efficient however to obtain the Riemann surface from a different, smaller matrix, called the BC matrix. We follow Previato [18] in the construction of this matrix. Break the matrix $\boldsymbol{E}(\lambda,\mu)$ up in blocks as follows:

$$\boldsymbol{E}(\lambda,\mu) = \left(\begin{array}{c|c} \boldsymbol{E}_1^{r\times r} & \boldsymbol{F}_1^{r\times n} \\ \hline \boldsymbol{E}_2^{n\times r} & \boldsymbol{F}_2^{n\times n} \end{array}\right), \tag{30}$$

where the upper indices denote the dimension of the block matrix. Define the BC matrix as

$$\boldsymbol{BC}(\lambda,\mu) = \boldsymbol{E}_1 - \boldsymbol{F}_1\boldsymbol{F}_2^{-1}\boldsymbol{E}_2. \tag{31}$$

Notice that the matrix $\boldsymbol{BC}(\lambda,\mu)$ is nothing but the Schur complement of the elimination matrix $\boldsymbol{E}(\lambda,\mu)$ [43]. Since $\boldsymbol{F}_2$ is a lower triangular matrix with diagonal entries equal to one, $\det(\boldsymbol{F}_2) = 1$, $\det(\boldsymbol{BC}(\lambda,\mu)) = \det(\boldsymbol{E}(\lambda,\mu)) = 0$ and we recover the equation for the Riemann surface Γ from the BC matrix.

Note that the BC matrix is a $r\times r$ matrix. We have significantly reduced the dimension of the matrix in question. Since $\boldsymbol{F}_2$ is lower triangular with diagonal entries equal to one, its inverse is still polynomial in the coefficients $\boldsymbol{u}$.

There is another interpretation of the BC matrix, which is perhaps more intuitive. Equation (28a) can be written in first-order $r \times r$ matrix form in terms of the vector $\boldsymbol{\Psi} = (\psi, \psi_x, \ldots, \psi_{(r-1)x})^T$. We can use this system to eliminate in (28b) any derivative of ψ of order higher than or equal to r. This reduces (28b) to an ordinary differential equation of order $r-1$. If we now rewrite this equation in $r \times r$ matrix form, we obtain

$$\boldsymbol{BC}(\lambda, \mu)\boldsymbol{\Psi} = 0. \tag{32}$$

Therefore, the BC matrix is the coefficient matrix of the $r \times r$ representation of (28b) induced by (28a). Everything we can deduce from the elimination matrix $\boldsymbol{E}(\lambda, \mu)$ can be deduced from the BC matrix $\boldsymbol{BC}(\lambda, \mu)$.

Equation (32) is central to this analysis: all of the information from the initial data is encoded in it, and every result obtained from here on follows from it. Observe from the form of (32) that the problem has been reduced to one of linear algebra. By construction, μ appears in the BC matrix only on the diagonal, and we may define $\widetilde{\boldsymbol{BC}}(\lambda)$ by:

$$\boldsymbol{BC}(\lambda, \mu) = \widetilde{\boldsymbol{BC}}(\lambda) - \mu \boldsymbol{I}_r. \tag{33}$$

Therefore, (32) is a linear algebraic eigenvalue problem for (ψ, μ). We show next that the eigenvalue of (32) defines the Riemann surface. Then in step 6, the eigenvector of (32) determines the divisor. This is precisely the information we seek.

The Riemann surface can be defined either by (29) or by

$$\Gamma : \det\left(\widetilde{\boldsymbol{BC}}(\lambda) - \mu \boldsymbol{I}_r\right) = 0. \tag{34}$$

Note that this curve is identical to the curve (29), because $\det \boldsymbol{F}_2 = 1$ (*cf.* (11)).

From Theorem 4 in the Appendix, it follows that the genus of this curve is less than or equal to $(r-1)(n-1)/2$.

We show in the Appendix (Theorem 5) that the Riemann surface Γ is independent of x, y, and t. In other words, coefficients of the BC matrix may depend on x, y, and t, but the combinations that appear in (34) do not depend on x or y or t. The proof is similar to that for the $(1+1)$-dimensional case. A consequence is that (34) provides a set of constants of the motion - namely, the coefficient in (34) of each power of (λ, μ) must be independent of (x, y, t). We use these constants in step 5, below. Note that this method to define the Riemann surface is purely local in (x, y, t). Considered as eigenvalue problems, (28a) and (28b) are each differential equations, but because we require that they have a common solution, it is not necessary to integrate these equations. Conceptually, this is a significant difference between this approach and any method that uses monodromy, like that in [17].

Step 5. Identify an appropriate pair (r, n)

The procedure outlined so far may be indefinite in two places. Depending on details of the initial data, step 1 might give the genus of the solution precisely, or it might give a lower bound on the genus. Then for a given genus, step 3 identifies a unique pair (r, n) of stationary flows only if the initial data is one-dimensional (so $r = 2$); otherwise it gives a finite list of possible pairs of stationary flows, as indicated in Table 1. The first place to test whether a pair (r, n) of stationary flows is appropriate for given initial data occurs here, at step 5.

Rewrite (28a) in first-order matrix form, $\boldsymbol{\Psi}_x = \boldsymbol{X}_r(\lambda)\boldsymbol{\Psi}$, for some $r \times r$ matrix $\boldsymbol{X}_r(\lambda)$. Equations (17a) and (17b) induce on $\boldsymbol{\Psi}$ $r-$dimensional representations for the $y-$ and $t-$evolution:

$$\begin{aligned} \boldsymbol{\Psi}_x &= \boldsymbol{X}_r(\lambda)\boldsymbol{\Psi}, && (35a)\\ \boldsymbol{\Psi}_y &= \boldsymbol{Y}_r(\lambda)\boldsymbol{\Psi}, && (35b)\\ \boldsymbol{\Psi}_t &= \boldsymbol{T}_r(\lambda)\boldsymbol{\Psi}. && (35c) \end{aligned}$$

Expressing the compatibility of (35a) with (35b) and with (35c) gives

$$\begin{aligned} \frac{\partial}{\partial y}\boldsymbol{X}_r(\lambda) - \frac{\partial}{\partial x}\boldsymbol{Y}_r(\lambda) &= [\boldsymbol{Y}_r(\lambda), \boldsymbol{X}_r(\lambda)], \\ \frac{\partial}{\partial t}\boldsymbol{X}_r(\lambda) - \frac{\partial}{\partial x}\boldsymbol{T}_r(\lambda) &= [\boldsymbol{T}_r(\lambda), \boldsymbol{X}_r(\lambda)]. \end{aligned} \qquad (36)$$

Then (35b) and (35c) are automatically compatible. Equations (35a-c) must also be compatible with (32), resulting in

$$\begin{aligned} \frac{\partial}{\partial x}\widetilde{\boldsymbol{BC}}(\lambda) &= \left[\boldsymbol{X}_r(\lambda), \widetilde{\boldsymbol{BC}}(\lambda)\right], && (37a)\\ \frac{\partial}{\partial y}\widetilde{\boldsymbol{BC}}(\lambda) &= \left[\boldsymbol{Y}_r(\lambda), \widetilde{\boldsymbol{BC}}(\lambda)\right], && (37b)\\ \frac{\partial}{\partial t}\widetilde{\boldsymbol{BC}}(\lambda) &= \left[\boldsymbol{T}_r(\lambda), \widetilde{\boldsymbol{BC}}(\lambda)\right]. && (37c) \end{aligned}$$

The compatibility conditions (36) and (37a-c) imply a set of differential equations for the solution $u(x,y,t)$ of the KP equation. One can eliminate any time-derivative from this set of equations by using (KP), if necessary with cross-differentiation. The result is a set of differential equations in x and y only that $(u(x,y,0), w_1(x,y,0), w_2(x,y,0), \ldots, w_{r-2}(x,y,0))$ must satisfy for all x and y. Since the additional potentials $(w_1, w_2, \ldots, w_{r-2})$ are determined by the potential u, this is a set of differential equations that the initial condition needs to satisfy.

These differential equations contain arbitrary constants (*e.g.*, c_j, d_j). One uses some of the differential equations at particular locations to define these constants in terms of the initial data. Then the given initial data either does or does not satisfy the entire set of differential equations, at all (x,y).

This is the test of whether a pair (r,n) is appropriate for the given initial data: $u(x,y,0)$ should satisfy the entire set of differential equations, for all (x,y) and for some choice of the free constants. If the initial data have the form (1) for some finite genus, then necessarily these data satisfy the set of differential equations associated with some pair (r,n). If they fail to satisfy the equations associated with the pair (r,n) in question, then r or n or both have been chosen too small.

If the genus g of the initial data is known, then one can build a list of possible pairs (r,n) for that value of g, ordered like those in the columns in Table 1. For the bottom pair in each column, $r = 2$. This is appropriate only for one-dimensional data; otherwise $r \geq 3$. In these lists, the number of free parameters available in the KP solution increases as one goes up the list. If a particular pair (r,n) yields a differential equation as described above which is not satisfied for the initial data then one should reject that pair, and move to a pair higher on the list. The top entry in each column corresponds to the generic KP solution of genus g, so if this pair fails, step

one failed to give the correct genus for the solution, *i.e.,* the wave vectors of the initial data are commensurable.

This test excludes pairs (r, n) of stationary flows with too few parameters for the initial data. It does not exclude a pair (r, n) with more free parameters than necessary, and it does not necessarily identify a unique pair (r, n) of stationary flows for given initial data.

If more than one pair of stationary flows induces a correct reconstruction of the initial data, then the simplest pair of stationary flows is the pair (r, n) for which $u(x, y, 0)$ satisfies the set of differential equations, and is smallest (*i.e.,* the smallest r, then for that r the smallest n).

A shorter test involves the coefficients in (27), because they are independent of (x, y, t). These coefficients depend on various combinations of u(x, y, 0) and its derivatives. Using the initial data, evaluate these coefficients at two or more arbitrarily chosen points (x_j, y_j). If any one of the coefficients takes on different values at different points, then r or n (or both) are too small for the initial data in question. This test is not as decisive as the main test described above, as it does not use all of the information provided by the compatibility conditions (36) and (37a-c).

Step 6. Calculate the divisor

Let Γ be a Riemann surface of genus g, compactified with a single point at infinity. Let $\tau = 1/\kappa$ be a local parameter in a neighborhood of this point, such that $\tau(\infty) = 0$, where we use ∞ to denote the point at infinity.

Definition 1 [22] *A Baker-Akhiezer function φ on the Riemann surface Γ corresponding to a divisor $\mathcal{D}$ on the surface and a polynomial $q(\kappa)$ is a scalar function such that: (a) φ is meromorphic everywhere on the affine part of Γ with poles only at the points P_i, $i = 1, 2, \ldots, g$ of the divisor $\mathcal{D}$; and (b) $\varphi(\kappa) exp\,(-q(\kappa))$ is analytic in a neighborhood of infinity.*

Let $\mathbf{\Psi}$ be the vector-valued wavefunction $\mathbf{\Psi} = (\psi, \psi_x, \ldots, \psi_{(r-1)x})$ solving (32), where ψ solves (17a) and (17b). In Theorem 7, we show that ψ is a Baker-Akhiezer function on the Riemann surface Γ, defined by (34), with divisor determined by the eigenvector $\boldsymbol{v}(\lambda, \mu)$ of $\widetilde{\boldsymbol{BC}}(\lambda)$, normalized by equating its first component to one, $\boldsymbol{v} = (1, v_2, v_3, \ldots, v_r)$, and evaluated at $(x, y, t) = (0, 0, 0)$. The polynomial in this case is $q(\kappa) = \kappa x + \kappa^2 y + \kappa^3 t + \sum_{j=4}^{\prime n} \kappa^j t_j$. The prime on the sum indicates all integers $r < j < n$ not coprime with r are omitted. From [22], we obtain the following representation for this Baker-Akhiezer function around the point at infinity of the Riemann surface Γ:

$$\psi = e^{\kappa x + \kappa^2 y + \kappa^3 t + \sum_{j=4}^{\prime n} \kappa^j t_j} \left(1 + \mathcal{O}(\frac{1}{\kappa})\right). \tag{38}$$

The actual calculation of the divisor is similar to that in §3. The Riemann surface Γ is obtained from (34), the eigenvalue equation for μ. The eigenvector $\mathbf{\Psi} = (\psi, \psi_x, \ldots, \psi_{(r-1)x})$ determines the divisor. Conversely, since $\mathbf{\Psi}_1 = \psi$ is a Baker-Akhiezer function, it can be reconstructed from the knowledge of the divisor. The other components of $\mathbf{\Psi}$ are obtained from differentiation in x. In the Appendix (Theorems 6 and 8) we prove that the divisor points are the solutions to the equations

$$\det\left(\boldsymbol{BC}^{(i1)}(\lambda, \mu)\right) = \det\left(\widetilde{\boldsymbol{BC}}^{(i1)}(\lambda) - \mu \boldsymbol{I}_r^{(i1)}\right) = 0. \tag{39}$$

where the BC matrix is evaluated at $(x, y, t) = (0, 0, 0)$ and $\widetilde{\boldsymbol{BC}}^{(i1)}(\lambda)$ denotes the cofactor of the i-th entry of the first column of $\widetilde{\boldsymbol{BC}}(\lambda)$. Again, notice the similarity between equations (39) and (12).

Denote $(\lambda_i, \mu_i) = P_i$, a point on the Riemann surface Γ, where (λ_i, μ_i) satisfies (39). The divisor points P_i constitute the divisor $\mathcal{D}$ for the Baker-Akhiezer function ψ: $\mathcal{D} = \sum_i P_i$. This implies that the number of points in the divisor is g

Step 7. Do the inverse scattering

The Riemann surface Γ and the divisor $\mathcal{D} = \sum_{i=1}^{g} P_i$, are the two ingredients needed for Krichever's inverse scattering procedure [8]. This justifies the label *spectral data* for them. Krichever's procedure seems quite different from the way inverse scattering is usually done, namely by way of a Riemann-Hilbert problem [44]. However, In [20], Krichever argues that the Riemann theta function construction for the finite genus solution is nothing but the effective solution of the associated Riemann-Hilbert problem. Using Krichever's construction we recover the potential $u(x, y, t)$, at an arbitrary time t in terms of Riemann theta functions, *i.e.,* the form (1). The algorithm outlined in this section hence constitutes the solution of the initial value problem for (KP) in the class of finite genus solutions of rank 1.

5 Examples

Example 1. One-dimensional solutions

From step 2 in §4 we know that a one-dimensional solution corresponds to $r = 2$. We now show that the converse is true as well.

For $r = 2$, we have from (19b)

$$\psi_{xx} + c_{21}\psi_x + u\psi + c_{20}\psi = \lambda\psi. \tag{40}$$

We can simplify this by translating $\lambda \to \lambda + c_{20}$ and denoting c_{21} by c_1. We obtain

$$\psi_{xx} + c_1\psi_x + u\psi = \lambda\psi. \tag{41}$$

Equation (41) is an ordinary differential equation in x. To express the compatibility of this equation with (17a), we write (41) in first-order matrix form as $\mathbf{\Psi}_x = \boldsymbol{X}_2(u, c_1, \lambda)\mathbf{\Psi}$, with $\mathbf{\Psi} = (\psi, \psi_x)^T$ and a 2×2 matrix $\boldsymbol{X}_2(u, c_1, \lambda)$. We use (17a) to induce the y-evolution of $\mathbf{\Psi}$. This results in a system of first-order differential equations in y for $\mathbf{\Psi}$, $\mathbf{\Psi}_y = \boldsymbol{Y}_2(u, c_1, \lambda)\mathbf{\Psi}$, with a 2×2 matrix $\boldsymbol{Y}_2(u, c_1, \lambda)$, in which:

$$\begin{aligned} \boldsymbol{X}_2(u, c_1, \lambda) &= \begin{pmatrix} 0 & 1 \\ \lambda - u & -c_1 \end{pmatrix}, \\ \boldsymbol{Y}_2(u, c_1, \lambda) &= \begin{pmatrix} \lambda & -c_1 \\ -c_1\lambda + c_1 u & \lambda + c_1^2 \end{pmatrix}. \end{aligned} \tag{42}$$

The compatiblity condition of the x- and $y-$evolution of $\mathbf{\Psi}$ is

$$\frac{\partial \boldsymbol{Y}_2}{\partial x}(u, c_1, \lambda) - \frac{\partial \boldsymbol{X}_2}{\partial y}(u, c_1, \lambda) = \left[\boldsymbol{X}_2(u, c_1, \lambda), \boldsymbol{Y}_2(u, c_1, \lambda)\right]. \tag{43}$$

Since $\boldsymbol{Y}_2(u, c_1, \lambda) = \lambda \boldsymbol{I}_2 - c_1 \boldsymbol{X}_2(u, c_1, \lambda)$, the right-hand side of this equation vanishes. This gives

$$u_y + c_1 u_x = 0. \tag{44}$$

This is (16) with $\beta = -c_1$. Hence if $r = 2$, the solution is one-dimensional.

Next, we show that for $(r = 2, n = 2g + 1)$, the Riemann surface is hyperelliptic and has genus g. Equation (41) shows that we can replace two x derivatives by one power of λ and also by one x-derivative. This allows us to write the n-th flow (28b) as $(n = 2g + 1)$

$$(P_{1;g}(\lambda) - \mu)\,\psi + P_{2;g}(\lambda)\psi_x = 0. \tag{45}$$

Here $P_{1;g}(\lambda), P_{2;g}(\lambda)$ are polynomials in λ of degree g. The BC-matrix is the 2×2 representation of the $n-$th flow. Taking a derivative of (45) and using (41) once more, we obtain

$$\begin{pmatrix} P_{1;g}(\lambda) - \mu & P_{2;g}(\lambda) \\ Q_{g+1}(\lambda) & Q_g(\lambda) - \mu \end{pmatrix} \begin{pmatrix} \psi \\ \psi_x \end{pmatrix} = 0, \tag{46}$$

for two polynomials, $Q_{g+1}(\lambda) = (\lambda - u)P_{2;g}(\lambda) + \partial_x P_{1;g}(\lambda)$, $Q_g(\lambda) = -c_1 P_{2;g}(\lambda) + P_{1;g}(\lambda) + \partial_x P_{2;g}(\lambda)$, of degree $g + 1$ and g respectively. The matrix $\widetilde{\boldsymbol{BC}}(\lambda)$ can always be made trace-free, by a translation in μ: $\mu \to \mu + (P_{1;g}(\lambda) + Q_g(\lambda))/2$. The BC matrix becomes

$$\boldsymbol{BC}(\lambda, \mu) = \begin{pmatrix} \frac{1}{2}\left(P_{1;g}(\lambda) - Q_g(\lambda)\right) - \mu & P_{2;g}(\lambda) \\ Q_{g+1}(\lambda) & \frac{1}{2}\left(Q_g(\lambda) - P_{1;g}(\lambda)\right) - \mu. \end{pmatrix} \tag{47}$$

From (34) and (47), the equation of the Riemann surface is:

$$\Gamma : \mu^2 = R_{2g+1}(\lambda), \tag{48}$$

where $R_{2g+1}(\lambda) = Q_{g+1}(\lambda)P_{2;g}(\lambda) + \left((P_{1;g}(\lambda) - Q_g(\lambda))/2\right)^2$ is a polynomial in λ of degree $2g + 1$. This Riemann surface has exactly the form of a hyperelliptic surface of genus g, given in (3). This shows that the genus of this Riemann surface is indeed g.

Finally, we check that the divisor consists of g points. From (39) and (47) we have

$$P_{2;g}(\lambda) = 0, \tag{49a}$$

$$\mu = \frac{1}{2}\left(Q_g(\lambda) - P_{1;g}(\lambda)\right), \tag{49b}$$

where $P_{1;g}(\lambda), P_{2;g}(\lambda)$ and $Q_g(\lambda)$ are evaluated at $(x, y, t) = (0, 0, 0)$. Equation (49a) gives g values of λ; for each of these values, (49b) gives one value of μ. This proves that the divisor has g points.

Example 2. Special case: $g = 1$

The simplest solution of the form (1) occurs when there is only one phase, so $g = 1$. In this case, the KP equation can be integrated directly. On the other hand, Table 1 shows that if $g = 1$ then $r = 2$ and $n = 3$. We show next that the BC matrix provides the same information that one obtains from direct integration of the KP equation.

First, if the solution contains only one phase, then it follows from (1b) that

$$ku_y = lu_x, \quad ku_t = \omega u_x. \tag{50}$$

Moreover, the KP equation can be integrated once to give

$$-4\frac{\omega}{k}u_x + 6uu_x + u_{xxx} + 3\frac{l^2}{k^2}u_x = 0. \tag{51}$$

Note that to obtain (51), we have absorbed σ into the y-derivative by the substitution $y \to \sigma y$. Equation (51) can be solved in terms of an elliptic function, which gives the complete solution for $g = 1$.

Alternatively, the two stationary flows are $r = 2$, which leads to (40), and $n = 3$, which can be written as

$$\psi_{xxx} + \frac{3}{2}u\psi_x + \frac{3}{4}\left(u_x + w_1\right)\psi + d_1\psi_x = \mu\psi, \tag{52}$$

where we have written d_1 instead of c_{31}, removed c_{30} by translating μ and removed c_{32} by a suitable conjugation of ψ [38]. As in Example 1, enforcing compatibility of (41) and (52) with (17a, b) gives

$$\begin{aligned}
u_y + c_1 u_x &= 0, && \text{(53a)}\\
u_t + d_1 u_x &= 0, && \text{(53b)}\\
w_1(x,y,t) = w(x,y,t) &= -c_1 u(x,y,t), && \text{(53c)}\\
4d_1u_x + 3c_1^2u_x + 6uu_x + u_{xxx} &= 0. && \text{(53d)}
\end{aligned}$$

To obtain (53c), we have equated the integration constant for $w(x,y,t)$ to zero. Clearly, (53a),(53b) are equivalent to (50), while (53d) is (51). Note that (53d) has the form

$$\frac{\partial H}{\partial x} = 0, \tag{54}$$

where $H(u) = 3u^2 + u_{xx} + 3c_1^2u + 4d_1u + h$, for some constant h. In step 5 of §4, we mentioned differential equations that can be used to test whether one has identified (g, r, n) correctly. For this example, these differential equations are those in (53a-d). Specifically, a KP solution of the form (1) can satisfy (53d) only if it has genus 1. This is the complete test for this example. If $g = 1$, then necessarily $r = g + 1 = 2, n = g + 2 = 3$.

The trace-free representation of $\widetilde{\boldsymbol{BC}}(\lambda)$, corresponding to (47), is

$$\widetilde{\boldsymbol{BC}}(\lambda) = \begin{pmatrix} bc_1\lambda + bc_2 & \lambda + bc_3 \\ \lambda^2 + \lambda bc_4 + bc_5 & -bc_1\lambda - bc_2 \end{pmatrix}, \tag{55}$$

with

$$\begin{aligned}
bc_1 &= \frac{c_1}{2},\\
bc_2 &= \frac{c_1^3}{2} + \frac{d_1c_1}{2} + \frac{1}{4}c_1u(x,y,t) - \frac{1}{4}u_x(x,y,t),\\
bc_3 &= c_1^2 + d_1 + \frac{1}{2}u(x,y,t),\\
bc_4 &= c_1^2 + d_1 - \frac{1}{2}u(x,y,t),\\
bc_5 &= -c_1^2u(x,y,t) + \frac{1}{4}c_1u_x(x,y,t) - \frac{1}{2}u^2(x,y,t) - d_1u(x,y,t) - \\
&\quad \frac{1}{4}u_{xx}(x,y,t).
\end{aligned} \tag{56}$$

Then the Riemann surface follows from (34):

$$\mu^2 = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0, \tag{57}$$

where the coefficients are given by

$$\begin{aligned} a_2 &= \frac{9}{4}c_1^2 + 2d_1, \\ a_1 &= -\frac{1}{4}H(u), \\ a_0 &= -\frac{1}{4}\left(c_1^2 + d_1\right)H(u) - \frac{1}{8}\int u\frac{\partial H(u)}{\partial x}dx. \end{aligned} \tag{58}$$

Note that the integral term vanishes, by (54). Therefore, $H(u)$ is the only nontrivial constant of the motion in this example; it is independent of x by (54), of y by (53a), and of t by (53b).

Finally, the explicit form of the only divisor point follows from (49a)-(49b):

$$\lambda_1 = -c_1^2 - d_1 - \frac{1}{2}u(0,0,0), \qquad \mu_1 = \frac{1}{4}u_x(0,0,0). \tag{59}$$

Example 3. The relation to monodromy theory

Our next example shows that the method presented here reproduces all of the results of monodromy theory, provided only that we drop the restriction that (39) be evaluated at only one point. To make the correspondence between the two theories as clear as possible, we restrict our attention to the KdV equation and to periodic initial data.

Here is a brief review of monodromy theory for the KdV equation. Full details can be found in several places, including [33]. By assumption, the initial data for (KdV) have the form (1), with a finite number (g) of phases, and with $l_j = 0$ in each phase (no y-dependence). In addition, $u(x,0)$ should be periodic in x with period L, so every k_j in (1) should have the form $k_j = 2\pi J/L$ for some integer J.

The scattering problem for (KdV) is

$$\psi_{xx} + u(x,0)\psi = \lambda\psi. \tag{60}$$

In monodromy theory, the forward problem consists of obtaining from (60) two spectra, which we define below. The inverse problem consists of recovering $u(x,t)$ from the two spectra. Each of the two spectra can be defined in terms of a 2×2 matrix called the monodromy matrix, defined as follows. Let $\mathbf{\Lambda}(x, x_0, \lambda)$ denote a fundamental matrix of solutions of (60), initialized at $x = x_0$. Because $u(x,0)$ is periodic with period L, these solutions must satisfy a relation of the form

$$\mathbf{\Lambda}(x + L, x_0, \lambda) = \boldsymbol{M}(\lambda, x_0)\mathbf{\Lambda}(x, x_0, \lambda) \tag{61}$$

for some matrix $\boldsymbol{M}(\lambda, x_0)$. The monodromy matrix, $\boldsymbol{M}$, is obtained by integrating (60) over one period at each λ.

The *main spectrum* consists of those values of λ at which $\boldsymbol{M}(\lambda, x_0) = \pm\boldsymbol{I}$. Denote these eigenvalues by E_j. For real-valued initial data of the form (1) with g phases, one shows that this spectrum has $(2g+1)$ eigenvalues that are real-valued, finite, and simple (*i.e.*, one eigenfunction for each eigenvalue), plus infinitely many double points. We ignore the double points, which play no role in the subsequent theory. The *auxiliary spectrum* consists of those points at which a

particular element of $\boldsymbol{M}(\lambda, x_0)$ vanishes. Denote these eigenvalues by γ_j. One shows that there are exactly g such values. One shows that each E_j is independent of both x_0 and t. Elements of the auxiliary spectrum, γ_j, depend on both x_0 and t according to complicated nonlinear ordinary differential equations. Once the two spectra $\{E_j, \gamma_j(x_0, t)\}$ are known, then the inverse problem has a marvelously simple solution:

$$u(x_0, t) = \sum_{j=0}^{2g} E_j - 2 \sum_{j=1}^{g} \gamma_j(x_0, t). \tag{62}$$

Now we relate monodromy theory to the theory presented here.

1. The initial data for (KdV) are one-dimensional, so the problem is covered by Example 1. Moreover, $u_y = 0$ for (KdV), so it follows from (16) that $c_1 = 0$. Therefore (41) reduces to (60).

2. The Riemann surface for these data is given by (48). It has $(2g + 1)$ finite branch points, where $\mu = 0$. These branch points are the simple points of the main spectrum, E_j, as was first proved in [33]. The entire Riemann surface is invariant, so necessarily its branch points are invariant as well.

3. Solving (39) at $(x = 0, t = 0)$ gives the divisor. Instead, we may solve (39) at an arbitrary point, $x = x_0$. In either case, (39) has g roots. Then retracing the proof of Theorem 7 shows that the poles of the redefined Baker-Akhiezer function correspond to the roots of (39), evaluated at $x = x_0$. But a defining property of the auxiliary spectrum, $\gamma_j(x_0, 0)$, is that the Baker-Akhiezer function has poles at these points, and nowhere else [33]. So the g points of the auxiliary spectrum are the g roots of (39), evaluated at $x = x_0$.

4. As stated above, points of the auxiliary spectrum move according to complicated ordinary differential equations as x_0 changes. An alternative to solving these differential equations is simply to evaluate (39) at other points, x_k. Because (34) is an algebraic equation in which x appears only through the (known) initial data, this is an attractive alternative. Unfortunately, we can offer no equally simple alternative to determine how $\gamma_j(x_0, t)$ evolves in t.

5. The reconstruction formula, (62), is implicit in the results of Example 1, but we need to display more detail to see it. We proceed in two steps. First, for $r = 2$, $n = 2g + 1$, one can show that the n-th flow has the form,

$$\psi_{(2g+1)x} + \left(\left(g + \frac{1}{2}u\right) + d_1\right)\psi_{(2g-1)x} + \sum_{j=0}^{2g-2} v_j \psi_{jx} = \mu\psi, \tag{63}$$

where d_1 is constant. We display only the coefficients of the first highest two derivatives in (63), because only these terms matter. With the same level of detail, the BC matrix in (46) becomes

$$\boldsymbol{BC}(\lambda, \mu) = \begin{pmatrix} P_{1;g-1}(\lambda) - \mu & \lambda^g + \lambda^{g-1}\left(d_1 + \frac{u}{2}\right) + \dots \\ \lambda^{g+1} + \lambda^g\left(d_1 - \frac{u}{2}\right) + \dots & Q_{g-1}(\lambda) - \mu \end{pmatrix}. \tag{64}$$

The polynomials on the diagonal both have reduced degree because $c_1 = 0$. After removing the trace of this matrix, one obtains an equation for the Riemann surface:

$$\mu^2 = \lambda^{2g+1} + 2d_1\lambda^{2g} + \sum_{j=0}^{2g-1} \tilde{c}_j \lambda^j. \tag{65}$$

The $(2g+1)$ finite branch points occur where $\mu = 0$ and (65) becomes a polynomial. Denote the roots of this polynomial by E_j. According to the well-known formula for the sum of the roots of a polynomial:

$$2d_1 = -\sum_{j=0}^{2g} E_j. \tag{66}$$

This gives an interpretation to the constant d_1. From (49a), the auxiliary spectrum is defined by

$$\lambda^g + \lambda^{g-1}\left(d_1 + \frac{u}{2}\right) + \sum_{j=0}^{g-2} \hat{c}_j \lambda^j = 0. \tag{67}$$

Denote the roots of this polynomial by $\gamma_j(x_0, t)$. The same formula for the sum of the roots of a polynomial yields

$$d_1 + \frac{u(x_0,t)}{2} = -\sum_{j=1}^{g} \gamma_j(x_0,t). \tag{68}$$

Combining (66) and (68) yields (62).

Example 4. $r = 3, n = 4$

From Table 1, an $(r = 3, n = 4)$ solution can have either genus 2 or genus 3. A typical $(r = 3, n = 4)$-potential has genus 3. The KP solutions of genus 2 are buried within this class as special cases. In this example we demonstrate the general method. In the next example we show how to find conditions the potential needs to satisfy to have genus 2.

The two stationary flows are

$$\begin{aligned} \psi_{xxx} + \frac{3}{2}u\psi_x + \frac{3}{4}\left(u_x + w_1\right)\psi + c_2\left(\psi_{xx} + u\psi\right) + c_1\psi_x &= \lambda\psi, & \text{(69a)} \\ \psi_{4x} + 2u\psi_{xx} + \left(w_1 + 2u_x\right)\psi_x + & \\ \left(u^2 + \frac{1}{2}w_{1,x} + \frac{1}{2}w_2 + u_{xx}\right)\psi + d_2\left(\psi_{xx} + u\psi\right) + d_1\psi_x &= \mu\psi. & \text{(69b)} \end{aligned}$$

As before, we have renamed c_{32} as c_2, c_{31} as c_1, c_{42} as d_2 and c_{41} as d_1. The parameters c_{30} and c_{40} have been removed by translation of λ and μ respectively, and c_{43} has been removed by conjugation.

The compatibility condition of these stationary flows with (17a) and (17b) gives the following relationships:

$$w_1 = w, \qquad w_2 = -u^2 - \frac{4}{3}c_1 u - \frac{4}{3}c_2 w - \frac{1}{3}u_{xx}, \tag{70a}$$

$$u_t + c_1 u_x + c_2 u_y = 0, \qquad w_t + c_1 w_x + c_2 w_y = 0, \tag{70b}$$

$$\frac{\partial H_1}{\partial x}(u,w) = 0, \qquad \frac{\partial H_2}{\partial x}(u,w) = 0, \tag{70c}$$

with

$$\begin{aligned} H_1(u,w) &= w_{xx} + 3uw + \frac{2}{3}c_2\left(u_{xx} + 3u^2\right) + 3d_2 w - 2c_1 w + \\ &\quad \frac{8}{3}c_2^2 w + 3d_1 u + \frac{8}{3}c_1 c_2 u + h_1, \end{aligned} \tag{71}$$

and

$$\begin{aligned} H_2(u,w) &= \frac{1}{12}u_{4x} + \frac{3}{4}uu_{xx} + \frac{3}{8}u_x^2 - \frac{3}{8}w^2 + \frac{1}{2}u^3 + \frac{1}{12}\left(4c_2^2 + 9d_2\right)u^2 + \\ &\quad \frac{1}{18}\left(-12c_1^2 + 8c_2^2 c_1 + 18d_2 c_1 - 9c_2 d_1\right)u + \frac{1}{36}\left(4c_2^2 + 6c_1 + 9d_2\right)u_{xx} + \\ &\quad \frac{1}{36}\left(16c_2^3 - 36c_1 c_2 + 18d_2 c_2 - 18c_2 u - 27d_1\right)w + h_2, \end{aligned} \tag{72}$$

where h_1 and h_2 are arbitrary constants. Since t_3 corresponds to t, the $(r = 3, n = 4)$ solutions are stationary solutions of the KP equation. This is confirmed by equations (70b).

If a potential $u(x,y,t)$ of the form (1) is a stationary solution of (KP) in some Galilean frame of reference, then $\omega_j = ak_j + bl_j$ in the notation of [22]. Hence $u(x,y,t) = U(x+at, y+bt)$, which combined with (70b) results in $c_1 = -a$ and $c_2 = -b$. Substituting $u(x,y,t) = U(x+at, y+bt)$ into (KP) results in a version of the Boussinesq equation [22],

$$3U_{yy} - 4aU_{xx} - 4bU_{xy} + \left(3U^2 + U_{xx}\right)_{xx} = 0, \tag{73}$$

and we recover the familiar result that the stationary reduction of the KP equation is the Boussinesq equation. If $b = 0$, (73) reduces to the standard form of the Boussinesq equation [44]. Equation (73) can of course be obtained from (KP) and the set of equations (70a).

The BC matrix is again calculated following the procedure in step 4 of §4. It is a 3×3 matrix with the following structure:

$$\begin{aligned} \boldsymbol{BC}(\lambda,\mu) &= -\mu \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \lambda^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \\ &\quad \lambda \begin{pmatrix} \alpha_1 & 1 & 0 \\ \alpha_2 & \alpha_1 & 1 \\ \alpha_3 & \alpha_4 & -2\alpha_1 \end{pmatrix} + \begin{pmatrix} \beta_1 & \beta_2 & \alpha_2 \\ \beta_3 & \beta_4 & \beta_5 \\ \beta_6 & \beta_7 & -\beta_1 - \beta_4 \end{pmatrix}. \end{aligned} \tag{74}$$

We do not give the explicit forms of $\alpha_1, \ldots, \alpha_4$ and $\beta_1, \ldots, \beta_7$.

The equation for the Riemann surface follows from (34):

$$\Gamma : \mu^3 = \mu\left(a_2\lambda^2 + a_1\lambda + a_0\right) + \lambda^4 + b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0. \tag{75}$$

In this case, the coefficients, (a_2, b_3) are explicitly independent of (x, y, t), while $(a_0, a_1, b_0, b_1, b_2)$ depend on (x, y, t) via $H_1(u, w)$ and $H_2(u, w)$. Therefore, $H_1(u, w)$ and $H_2(u, w)$ are the only nontrivial constants of the motion for $(r = 3, n = 4)$.

We now turn to the divisor. The three cofactors determined by (39) result in linearly dependent equations. We only have to consider the last two, corresponding to the cofactors of the second and last element of the first column of (74) evaluated at $(x, y, t) = (0, 0, 0)$:

$$\det\begin{pmatrix} \lambda + \beta_2 & \alpha_2 \\ \lambda\alpha_4 + \beta_7 & -\mu - 2\alpha_1\lambda - \beta_1 - \beta_4 \end{pmatrix} = 0, \tag{76a}$$

$$\det\begin{pmatrix} \lambda + \beta_2 & \alpha_2 \\ -\mu + \lambda\alpha_1 + \beta_4 & \lambda + \beta_5 \end{pmatrix} = 0, \tag{76b}$$

From the second equation, (76b), μ is quadratic in λ. Substituting this into the first equation (76a) results in a cubic equation for λ. Hence there are indeed three points in the divisor. This agrees with the observation that a solution corresponding to $(r = 3, n = 4)$ is a genus $g = 3$ solution, unless some special conditions are met. These conditions are derived below.

Example 5. Special case: $r = 3, n = 4$ but $g = 2$

From §4 we know that a generic genus 2 solution of KP corresponds to $(r = 3, n = 4)$. These solutions are embedded in a larger class of genus 3 solutions, as we confirmed in example 4. In this example, we find the conditions on the coefficients of (75) under which the surface (75) reduces to a surface of genus 2.

It is a known result in the theory of Riemann surfaces that every surface of genus 2 is hyperelliptic [24]. This implies that amongst all surfaces of the form (75), the genus 2 surfaces have a representation in which the surface is quadratic and monic in one of the two variables and polynomial in the other, as in (3). Examining when we can transform (75) to a hyperelliptic curve of genus 2 gives conditions on the coefficients of (75) such that the Riemann surface Γ has genus 2.

We write the surface Γ in homogeneous coordinates [45]. This results in a representation in which the point at infinity is no longer distinguished from the finite points. Let $\lambda = \lambda_1/\nu_1, \mu = \mu_1/\nu_1$ in the equation for Γ. Multiplying by ν_1^4 results in the following representation for Γ:

$$\nu_1\mu_1^3 = \nu_1\mu_1\left(a_2\lambda_1^2 + a_1\nu_1\lambda_1 + a_0\nu_1^2\right) + \lambda_1^4 + b_3\nu_1\lambda_1^3 + b_2\nu_1^2\lambda_1^2 + b_1\nu_1^3\lambda_1 + b_0\nu_1^4. \tag{77}$$

In this representation, the surface is homogeneous of degree 4. The genus is invariant under linear transformations on the coordinates $(\lambda_1, \mu_1, \nu_1)$ [45], so we use such a transformation to transform (77):

$$\begin{pmatrix} \lambda_1 \\ \mu_1 \\ \nu_1 \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \mu_2 \\ \nu_2 \end{pmatrix}. \tag{78}$$

This results in a new representation of Γ of degree 4. By setting $\nu_2 = 1$, we return to an affine representation of Γ. Denoting the coefficient matrix of the linear transformation (78) by $\boldsymbol{\alpha}$ and setting $(a_0, a_1, a_2, b_0, b_1, b_2, b_3)^T = \boldsymbol{b}$, this affine (λ_2, μ_2)-representation is

$$
\begin{array}{c}
c_1(\boldsymbol{\alpha},\boldsymbol{b})\mu_2^4 + \mu_2^3\left(c_2(\boldsymbol{\alpha},\boldsymbol{b})\lambda_2 + c_3(\boldsymbol{\alpha},\boldsymbol{b})\right) + \\
\mu_2^2\left(c_4(\boldsymbol{\alpha},\boldsymbol{b})\lambda_2^2 + c_5(\boldsymbol{\alpha},\boldsymbol{b})\lambda_2 + c_6(\boldsymbol{\alpha},\boldsymbol{b})\right) + \\
\mu_2\left(c_7(\boldsymbol{\alpha},\boldsymbol{b})\lambda_2^3 + c_8(\boldsymbol{\alpha},\boldsymbol{b})\lambda_2^2 + c_9(\boldsymbol{\alpha},\boldsymbol{b})\lambda_2 + c_{10}(\boldsymbol{\alpha},\boldsymbol{b})\right) + \\
c_{11}(\boldsymbol{\alpha},\boldsymbol{b})\lambda_2^4 + c_{12}(\boldsymbol{\alpha},\boldsymbol{b})\lambda_2^3 + c_{13}(\boldsymbol{\alpha},\boldsymbol{b})\lambda_2^2 + c_{14}(\boldsymbol{\alpha},\boldsymbol{b})\lambda_2 + c_{15}(\boldsymbol{\alpha},\boldsymbol{b}) = 0 \quad .
\end{array}
\tag{79}
$$

The new coefficients $c_j(\boldsymbol{\alpha},\boldsymbol{b}), j = 1, 2, \ldots, 15$ depend linearly on the entries of $\boldsymbol{b}$. They are at most quartic in the entries of $\boldsymbol{\alpha}$. We want Γ to be of the form (3) with $n = 5$ or 6, in order to have genus 2. Since Γ is a quartic curve, this is not possible in general. However, a degenerate subset of curves have the simpler form

$$
\mu_2^2 = \mu_2 Q_3(\lambda_2) + Q_4(\lambda_2), \tag{80}
$$

with $Q_3(\lambda_2)$ and $Q_4(\lambda_2)$ polynomials in λ_2 of degree 3 and 4 respectively. To achieve (80), the coefficients of $\mu_2^4, \mu_2^3\lambda_2, \mu_2^3, \mu_2^2\lambda_2^2$ and $\mu_2^2\lambda_2$ must be zero. In addition, the coefficients of μ_2^2 and $\mu_2\lambda_2^3$ must not be zero. Then a transformation $(\mu_2 \to \mu_2 + Q_3(\lambda_2)/2, \lambda \to \lambda)$ results in a surface of the form (3) with $n = 6$. This transformation does leave the genus of the curve invariant as it is birational [45].

We have imposed $c_1(\boldsymbol{\alpha},\boldsymbol{b}) = c_2(\boldsymbol{\alpha},\boldsymbol{b}) = c_3(\boldsymbol{\alpha},\boldsymbol{b}) = c_4(\boldsymbol{\alpha},\boldsymbol{b}) = c_5(\boldsymbol{\alpha},\boldsymbol{b}) = 0$. Furthermore, We need $c_6(\boldsymbol{\alpha},\boldsymbol{b}) \neq 0 \neq c_7(\boldsymbol{\alpha},\boldsymbol{b})$. Since the coefficients $c_j(\boldsymbol{\alpha},\boldsymbol{b}), j = 1, 2, \ldots, 15$ of (79) are linear in the entries of $\boldsymbol{b}$, we will consider these entries to be unknowns and solve for them. We then have a total of five linear equations for seven unknowns in $\boldsymbol{b}$:

$$
\boldsymbol{M}(\boldsymbol{\alpha})\boldsymbol{b} = \boldsymbol{N}(\boldsymbol{\alpha}). \tag{81}
$$

Let

$$
k = \frac{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}{\alpha_{11}\alpha_{32} - \alpha_{12}\alpha_{31}}, \quad l = \frac{\alpha_{32}}{\alpha_{12}}, \quad m = \frac{\alpha_{22}}{\alpha_{12}}. \tag{82}
$$

Then one finds from (81) that (b_0, b_1) are arbitrary, and that $(a_0, a_1, a_2, b_2, b_3)$ can be written in terms of (b_0, b_1, k, l, m) as follows:

$$
\begin{aligned}
a_2 &= -\frac{l^2}{m}b_1 + 3\frac{l^3}{m}b_0 + 3m^2 - 6klm - 3k^2 - \frac{1}{lm}, \\
a_1 &= 2\frac{l}{m}b_1 - 6\frac{l^2}{m}b_0 + 6k^2 l + 6km + \frac{2}{l^2 m}, \\
a_0 &= -\frac{1}{m}b_1 + 3\frac{l}{m}b_0 - 3k^2 - \frac{1}{l^3 m}, \\
b_3 &= l^2 b_1 - 2l^3 b_0 - 2m^3 + 6klm^2 - \frac{2}{l}, \\
b_2 &= -2lb_1 + 3l^2 b_0 + \frac{1}{l^2} - 6km^2.
\end{aligned}
\tag{83}
$$

This solution of (81) is independent of the transformation variables α_{13}, α_{23} and α_{33}. Since the restrictions $c_6(\boldsymbol{\alpha},\boldsymbol{b}) \neq 0$ and $c_7(\boldsymbol{\alpha},\boldsymbol{b}) \neq 0$ depend on them, α_{13}, α_{23} and α_{33} are used to assure that the coefficients of μ_2^2 and $\mu_2\lambda_2^3$ are nonzero, so that the surface is indeed hyperelliptic of genus 2.

The set of equations (83) gives the 5 coefficients $(a_2, a_1, a_0, b_3, b_2)$ in terms of only three parameters (k, l, m) and the other coefficients (b_1, b_0). This entails that once (b_1, b_0) are given, in order

for Γ to have genus 2 (and hence be hyperelliptic), the coefficients $(a_2, a_1, a_0, b_3, b_2)$ have to exist on a co-dimension 2 surface in the five-dimensional coefficient space. This can be interpreted as a total of two conditions the coefficients $(a_2, a_1, a_0, b_3, b_2)$ (and hence the potential $u(x, y, t)$) have to satisfy. These conditions can be found explicitly by eliminating (k, l, m) from (83). Since the coefficients of the Riemann surface are conserved in (x, y, t), these conditions need to be checked at only one point for (x, y, t).

Let us give a short summary of the last two examples. As pointed out in §4, step 3, a generic genus g solution of the KP equation requires $(r = g + 1, n = g + 2)$. Thus for a generic genus 2 solution, we need $(r = 3, n = 4)$. This case is worked out in example 4 where it is demonstrated that a typical solution corresponding to $(r = 3, n = 4)$ has genus 3. Only when the solution satisfies an additional two constraints (given parametrically by (83)) does it have genus 2. This illustrates that a generic genus g solution of the KP equation is not a typical solution for $(r = g+1, n = g+2)$. With this in mind, it is clear that the genus of a KP solution is not sufficient to characterize a solution in this formalism. The pair (r, n), once the initial condition $u(x, y, 0)$ is given, characterizes the genus of the initial condition, as the genus of the resulting Riemann surface. In this sense, the pair (r, n) characterizes the solution of the KP equation. Unfortunately, at this time it is not obvious how to obtain r and n straight from initial data.

Acknowledgements

We are grateful to R. MacRae, T. Shiota, and B.A. Dubrovin for helpful comments. HS acknowledges the kind hospitality of NCAR, Boulder, CO, where some of this work was carried out. This work is supported in part by the Office of Naval Research (N00014-95-10402) and by the National Science Foundation (DMS-9304390).

Appendix: Theorems and proofs

Theorem 1 *For a KP solution of the form (1) with g phases, $2 \leq r \leq g + 1$ and all of these are possible. For a generic solution, with no relations between the phases, $r = g + 1$, $n = g + 2$.*

Proof We can write a finite genus solution of KP in a form slightly different from (1). We can introduce 'higher time' variables t_k such that $u = u(t_1, t_2, t_3 \ldots,)$ solves *all* equations of the KP hierarchy [37]. In terms of theta functions, such a solution is expressed as

$$\begin{aligned} u &= 2\partial_x^2 \ln \Theta \left(\boldsymbol{K}_1 x + \boldsymbol{K}_2 y + \boldsymbol{K}_3 t + \boldsymbol{K}_4 t_4 + \boldsymbol{K}_5 t_5 + \ldots + \right. \\ &\quad \left. \boldsymbol{K}_g t_g + \boldsymbol{K}_{g+1} t_{g+1} + \boldsymbol{K}_{g+2} t_{g+2} + \boldsymbol{\varphi} \right), \end{aligned} \tag{84}$$

and $\boldsymbol{K}_j$ is the g-dimensional vector of wavenumbers corresponding to t_j (e.g. $\boldsymbol{K}_1 = (k_1, k_2, \ldots, k_g)^T$, $\boldsymbol{K}_2 = (l_1, l_2, \ldots, l_g)^T$). The quantity $\boldsymbol{\varphi}$ depends on all 'time' variables not explicitly written. These are not relevant for our considerations.

Define the square matrix $\bar{\boldsymbol{K}} = (\boldsymbol{K}_1, \boldsymbol{K}_2, \ldots, \boldsymbol{K}_g)$. If this matrix is nonsingular (*i.e.,* if the vectors $\boldsymbol{K}_j, j = 1, \ldots, g$ are linearly independent), then the two vectors $\boldsymbol{K}_{g+1}$ and $\boldsymbol{K}_{g+2}$ are linearly dependent on its columns, say,

$$\boldsymbol{K}_{g+1} = \sum_{i=1}^{g} \alpha_i \boldsymbol{K}_i, \qquad \boldsymbol{K}_{g+2} = \sum_{i=1}^{g} \beta_i \boldsymbol{K}_i. \tag{85}$$

The solution $u = u(t_1, t_2, t_3 \ldots,)$ can now easily be transformed such that it does not depend on t_{g+1} and t_{g+2}. Since g vectors in g dimensions are generically linear independent, the case of $r = g + 1$ and $r = g + 2$ can be regarded as the generic case for a genus g solution.

If the matrix $\bar{\boldsymbol{K}}$ is singular, its columns are linearly dependent and at least one of the $\boldsymbol{K}_j, j \leq g$ can be written as a linear combination of $\boldsymbol{K}_i$, with $i < j$. Analogous to the generic case above, a transformation will eliminate all t_j dependence and the j-th flow is stationary. This proves that $r \leq g + 1$, and that any value less than $g + 1$, greater than 2 is possible. □.

Lemma 1 *Consider a KP solution of the form (1), and of genus g. Denote the numbers of the stationary flows from the KP hierarchy for this solution by (r, n), with $r < n$. For fixed r, $2 \leq r \leq g + 1$, there are at most $(g - r + 2)$ possible choices of n.*

Proof This is a simple problem in linear algebra, but it takes some explaining to set it up that way. The KP solution in question has g phases, characterized by g independent vectors of wavenumbers, as in Theorem 1. These vectors define a nested sequence of subspaces of the g-dimensional space, defined by their spanning sets:

$$\{\boldsymbol{K}_1\} \subset \{\boldsymbol{K}_1, \boldsymbol{K}_2\} \subset \ldots \subset \{\boldsymbol{K}_1, \ldots, \boldsymbol{K}_g\}. \tag{86}$$

If the KP solution is also a stationary solution of the r-th flow of the KP hierarchy, then $\boldsymbol{K}_r$ necessarily lies in the subspace spanned by the first $(r - 1)$ vectors. For $r < g + 1$, this requires that the numbering of the basis vectors must be changed, because $\boldsymbol{K}_r$ is no longer independent. For each r in $2 \leq r \leq g + 1$, there is a different set of independent basis vectors, and so a different set of nested subspaces.

If the KP solution is also a stationary solution of the n-th flow, then $\boldsymbol{K}_n$ must lie in the g-dimensional space as well. For fixed g and fixed r, with $r < n$, let $\#n(g, r)$ denote the number of nested subspaces into which $\boldsymbol{K}_n$ can be placed. We now show that

$$\#n(g, r) = g - r + 2. \tag{87}$$

In the simplest possible case, $g = 1$, so $r = g + 1 = 2$. There is only one subspace, $\{\boldsymbol{K}_1\}$, so there is one choice for n, so $\#n(1, 2) = 1$, in accord with (87).

The general argument is by induction. Suppose we know $\#n(g - 1, r)$. For $(g - 1) \to g$ at fixed r, the dimension of the whole space increases by one. The nested subspaces available now consist of all of those available at $(g - 1)$, plus one more. Therefore,

$$\#n(g, r) = \#n(g - 1, r) + 1, \tag{88}$$

so that for some $f(r)$,

$$\#n(g, r) = g + f(r). \tag{89}$$

To find $f(r)$, note that when $r = g + 1$, then there is only one choice for n (*i.e.*, $n = g + 2$), so $\#n(g, g + 1) = 1$. Evaluating (89) at $r = g + 1$ shows that

$$f(r) = 2 - r. \tag{90}$$

Combining (90) and (89) establishes (87).

Equation (87) provides only an upper bound on the number of choices for n, because it ignores the requirement that (r, n) be coprime, and it ignores the upper bound on the possible genus of

a Riemann surface, which is given in Theorem 4. However, the list of choices obtained from (87), when shortened by these two additional requirements, provides the complete list of pairs (r, n) of stationary flows that can correspond to a KP solution of the form (1), and of genus g. The lists given in Table 1 were obtained in just this way. □

Theorem 2 *$\mathcal{P}_g$, the number of possible pairs (r, n) that can correspond to a solution of genus g is bounded by $g(g+1)/2 - g + 1$.*

Proof From lemma 1,

$$\begin{aligned} \mathcal{P}_g &\leq \sum_{r=2}^{g+1} (g - r + 2) \\ &= \sum_{k=1}^{g} k \\ &= \frac{g(g+1)}{2}. \end{aligned}$$

In the above, we included $g - r + 2$ possibilities for every r. Specifically, for $r = 2$, we include g possibilities. But only one these possibilities results in genus g, namely $n = 2g + 1$. The bound for $\mathcal{P}_g$ can hence be decreased by $g - 1$: $\mathcal{P}_g \leq g(g+1)/2 - (g-1)$. □

Theorem 3 *(a) The Riemann surface Γ, defined by (34), can be compactified with one point at infinity. (b) This point is a branch point with r branches.*

Proof (a) Once we make the matrix $\widetilde{\boldsymbol{BC}}(\lambda)$ trace-free, the equation for Γ can be written as

$$\Gamma : (-\mu)^r + \sum_{j=1}^{r-2} \mu^j P^{(j)}(\lambda) + Q_n(\lambda) = 0. \tag{91}$$

In this equation, $Q_n(\lambda)$ is a polynomial of degree n, the $P^{(j)}(\lambda)$, $j = 1, 2, \ldots, r-2$ are polynomials of λ of degree less than $n - j$. To find the point(s) at infinity on the surface, we rewrite the equation in homogeneous coordinates [45], by transforming $\mu \to \mu/\nu, \lambda \to \lambda/\nu$, and then multiply by ν^n, since n is the highest power in (91). This results in a polynomial relationship in (λ, μ, ν) of degree n. Points of infinity are where this relationship intersects the line at infinity, $\nu = 0$ [45]. The only term in the rewritten form of (91) without powers of ν corresponds to the n-th degree term of $Q_n(\lambda)$. We obtain

$$\lambda^n = 0. \tag{92}$$

The point at infinity expressed in homogeneous coordinates is hence $(\lambda, \mu, \nu) = (0, 1, 0)$. This proves that the surface Γ is compactified by a single point at infinity.

(b) Around this point, the asymptotic form of the curve is $(-\mu)^r \sim c(\lambda)^n$. Hence $-\mu \sim (c\lambda^n)^{1/r}$ as $\lambda \to \infty$. Since r and n are coprime, the point at infinity is a branch point of full order, *i.e.,* it joins r sheets. Note that this implies that the branching number [45] of the branch point at infinity is $r - 1$. □

Theorem 4 *The genus of the Riemann surface Γ is at most $(r-1)(n-1)/2$.*

Proof This proof uses the same notation as in Theorem 3. Consider the equation for Γ (91) as a polynomial in μ. Its discriminant [46] is the following $(2r-1)\times(2r-1)$ determinant:

$$\Delta(\lambda) =$$

$$\det\begin{pmatrix}
(-1)^r & 0 & P^{(r-2)} & P^{(r-1)} & \dots & P^{(1)} & Q_n & & & \\
 & (-1)^r & 0 & P^{(r-2)} & P^{(r-1)} & \dots & P^{(1)} & Q_n & & \\
 & & \dots & & \dots & & \dots & & \dots & \\
 & & & (-1)^r & 0 & P^{(r-2)} & P^{(r-1)} & \dots & P^{(1)} & Q_n \\
(-1)^r r & 0 & \hat{P}^{(r-2)} & \hat{P}^{(r-1)} & \dots & \hat{P}^{(1)} & & & & \\
 & (-1)^r r & 0 & \hat{P}^{(r-2)} & \hat{P}^{(r-1)} & \dots & \hat{P}^{(1)} & & & \\
 & & (-1)^r r & 0 & \hat{P}^{(r-2)} & \hat{P}^{(r-1)} & \dots & \hat{P}^{(1)} & & \\
 & & & \dots & \dots & & \dots & & \dots & \\
 & & & & (-1)^r r & 0 & \hat{P}^{(r-2)} & \hat{P}^{(r-1)} & \dots & \hat{P}^{(1)}
\end{pmatrix}, \tag{93}$$

with $\hat{P}^{(k)} = kP^{(k)}$, the coefficient of μ^{k-1} in the derivative of Γ with respect to μ. The explicit dependence of the polynomials on λ has been left out for convenience. The empty spaces in $\Delta(\lambda)$ are zeros. In the first $r-1$ rows the coefficients of (91) are written with the first coefficient on the diagonal. In the last r rows, the coefficients of the derivative of (91) with respect to μ are written, again shifted over one column as the row number is increased by one.

One can prove the degree of the term $(-1)^{r^2} r^r Q_n^{r-1}$ is the degree of $\Delta(\lambda)$, so the degree of $\Delta(\lambda)$ is $n(r-1)$. Hence there are $n(r-1)$ discriminant points. These points, together with the point at infinity (see Theorem 3), give a total branching number of at most $n(r-1)+r-1 = (n+1)(r-1)$. The total branching number is equal to $(n+1)(r-1)$ if all discriminant points are branch points. This is not always the case, so this argument gives only an upper bound on the total branching number.

The Riemann-Hurwitz formula [21, 24],

$$g = \frac{W}{2} - r + 1, \tag{94}$$

gives the genus of an r-sheeted covering of the Riemann sphere in terms of the total branching number W. In our case the Riemann sphere is the λ-sphere. We obtain

$$g \le \frac{(n+1)(r-1)}{2} - r + 1 = \frac{(r-1)(n-1)}{2}, \tag{95}$$

where equality only holds if all discriminant points are branch points. Note that since r and n are coprime, $(r-1)(n-1)$ is always even. □

Theorem 5 *The Riemann surface* Γ *is independent of* (x, y, t).

Proof The coefficients of (34), the eigenvalue equation of $\widetilde{\boldsymbol{BC}}(\lambda)$, are invariants of the matrix $\widetilde{\boldsymbol{BC}}(\lambda)$. The r invariants can be expressed in terms of $\mathrm{tr}(\widetilde{\boldsymbol{BC}}^k(\lambda)), k = 0, 1, \dots, r-1$. From (37a) we obtain

$$\widetilde{\boldsymbol{BC}}^{k-1}(\lambda)\,\frac{\partial}{\partial x}\widetilde{\boldsymbol{BC}}(\lambda) \;=\; \widetilde{\boldsymbol{BC}}^{k-1}(\lambda)\left[X_r(\lambda), \widetilde{\boldsymbol{BC}}(\lambda)\right]$$

$$
\begin{aligned}
\Rightarrow \qquad \frac{1}{k}\frac{\partial}{\partial x}\widetilde{\boldsymbol{BC}}^{k}(\lambda) &= \left[\widetilde{\boldsymbol{BC}}^{k-1}(\lambda)\, X_r(\lambda), \widetilde{\boldsymbol{BC}}(\lambda)\right] \\
\Rightarrow \qquad \frac{\partial}{\partial x}\mathrm{tr}\left(\widetilde{\boldsymbol{BC}}^{k}(\lambda)\right) &= 0,
\end{aligned}
$$

since the trace of a commutator is zero. The proof to show that the invariants are also $y-$ and $t-$independent is identical. This proves that the coefficients of the Riemann surface Γ are conserved quantities for the evolution in x, y and t of the potential. Hence the Riemann surface Γ is independent of (x, y, t). □

Remark: For convenience, we assume in Theorem 6 that calculations are done at the point $(x, y, t) = (0, 0, 0)$. For clarity, we indicate all variable dependence explicitly in this proof.

Theorem 6 *Let $\boldsymbol{v}(x, y, t, \lambda, \mu) = (v_1, v_2, \ldots, v_r)^T$ be the eigenvector of $\widetilde{\boldsymbol{BC}}(x, y, t, \lambda)$ with eigenvalue μ, normalized by $v_1(x, y, t, \lambda, \mu) \equiv 1$. Furthermore, let $\boldsymbol{\Lambda}(x, y, t, \lambda)$ be a fundamental matrix of the solutions of the equations (35a), (35b) and (35c), normalized by $\boldsymbol{\Lambda}(0, 0, 0, \lambda) = \boldsymbol{I}_r$. Then $\boldsymbol{\Psi} = \boldsymbol{\Lambda}(x, y, t, \lambda)\boldsymbol{v}(0, 0, 0, \lambda, \mu)$ satisfies the equations (35a), (35b), (35c) and (32).*

Proof Since the equations (35a), (35b) and (35c) are linear, we only need to show that $\boldsymbol{\Psi}$ satisfies (32).

By direct calculation, we verify that $\widetilde{\boldsymbol{BC}}(x, y, t, \lambda)\boldsymbol{\Lambda}(x, y, t, \lambda)$ satisfies (35a-35c), using (37a-37c). But every matrix solution of (35a-35c) can be expressed as $\boldsymbol{\Lambda}(x, y, t, \lambda)\boldsymbol{A}(\lambda)$ for some $r \times r$ matrix $\boldsymbol{A}$. Evaluating $\widetilde{\boldsymbol{BC}}(x, y, t, \lambda)\boldsymbol{\Lambda}(x, y, t, \lambda) = \boldsymbol{\Lambda}(x, y, t, \lambda)\boldsymbol{A}(\lambda)$ at $(x, y, t) = (0, 0, 0)$, we find $\boldsymbol{A}(\lambda) = \widetilde{\boldsymbol{BC}}(0, 0, 0, \lambda)$. Hence,

$$
\begin{aligned}
\widetilde{\boldsymbol{BC}}(x, y, t, \lambda)\boldsymbol{\Psi}(x, y, t, \lambda, \mu) &= \widetilde{\boldsymbol{BC}}(x, y, t, \lambda)\boldsymbol{\Lambda}(x, y, t, \lambda)\boldsymbol{v}(0, 0, 0, \lambda, \mu) \\
&= \boldsymbol{\Lambda}(x, y, t, \lambda)\widetilde{\boldsymbol{BC}}(0, 0, 0, \lambda)\boldsymbol{v}(0, 0, 0, \lambda, \mu) \\
&= \boldsymbol{\Lambda}(x, y, t, \lambda)\mu\boldsymbol{v}(0, 0, 0, \lambda, \mu) \\
&= \mu\boldsymbol{\Lambda}(x, y, t, \lambda)\boldsymbol{v}(0, 0, 0, \lambda, \mu) \\
&= \mu\boldsymbol{\Psi}(x, y, t, \lambda, \mu). \qquad \square
\end{aligned}
$$

Theorem 7 *Let $\boldsymbol{\Psi} = (\psi, \psi_x, \ldots, \psi_{(r-1)x})^T$ be the simultaneous solution of equations (35a), (35b), (35c) and (32). Let $\boldsymbol{v}(x, y, t, \lambda, \mu) = (v_1, v_2, \ldots, v_r)^T$ be the eigenvector of $\widetilde{\boldsymbol{BC}}(x, y, t, \lambda)$ with eigenvalue μ, normalized by $v_1(x, y, t, \lambda, \mu) \equiv 1$. Then (a) $\psi_x/\psi = v_2(x, y, t, \lambda, \mu)$ and (b) ψ is a Baker-Akhiezer function with divisor given by the poles of $\boldsymbol{v}(0, 0, 0, \lambda, \mu)$ and polynomial $q(\kappa) = \kappa x + \kappa^2 y + \kappa^3 t + \sum'^{n}_{j=4} \kappa^j t_j$ (the prime on the sum indicates all integers $r < j < n$ not coprime with r are excluded from the sum).*

Proof (a) The two vectors $\boldsymbol{\Psi}(x, y, t, \lambda, \mu)$ and $\boldsymbol{v}(x, y, t, \lambda, \mu)$ are both solutions of (32). Thus we have

$$
\boldsymbol{\Psi}(x, y, t, \lambda, \mu) = c(x, y, t, \lambda, \mu)\boldsymbol{v}(x, y, t, \lambda, \mu), \tag{96}
$$

for some scalar $c(x, y, t, \lambda, \mu)$. (Note: this is where the rank 1 condition is important. It guarantees the dimension of the eigenspace is 1.) Evaluating the first component of this equation gives $\psi = c$. The second component reads

$$\psi_x(x,y,t,\lambda,\mu) = c(x,y,t,\lambda,\mu)v_2(x,y,t,\lambda,\mu), \tag{97}$$

from which we obtain $\psi_x/\psi = v_2(x,y,t,\lambda,\mu)$.

(b) By construction, $\boldsymbol{v}(x,y,t,\lambda,\mu)$ is a rational function on the Riemann surface Γ, since the entries of the BC matrix are polynomial in λ and μ. From Theorem 7, we have $\boldsymbol{\Psi}(x,y,t,\lambda,\mu) = \boldsymbol{\Lambda}(x,y,t,\lambda)\boldsymbol{v}(0,0,0,\lambda,\mu)$. Since $\boldsymbol{\Lambda}(x,y,t,\lambda)$ is analytic in any bounded disk in the λ-plane, $\boldsymbol{\Psi}(x,y,t,\lambda,\mu)$ is a meromorphic function on the surface Γ away from the point at infinity. Furthermore the poles of $\psi(x,y,t,\lambda,\mu)$ are the poles of $\boldsymbol{v}(0,0,0,\lambda,\mu)$ and conversely. The analyticity of $\psi\exp(-\kappa x - \kappa^2 y - \kappa^3 t - \sum_{j=4}^{'n}\kappa^j t_j)$ at the point at infinity follows from [47]. □

Remark: The equality $\psi_x/\psi = v_2(x,y,t,\lambda,\mu)$ can be used to determine the x-dependence of ψ explicitly:

$$\psi(x,y,t,\lambda,\mu) = C(y,t,\lambda,\mu)\exp\int^x v_2(x,y,t,\lambda,\mu)dx, \tag{98}$$

where $C(y,t,\lambda,\mu)$ is a scalar, independent of x. For the periodic KdV equation, this result is well-known [23, p108].

Theorem 8 *The divisor on the Riemann surface corresponding to a finite genus KP solution satisfies(39).*

Proof From Theorem 7, the poles of $\psi(x,y,t,\lambda,\mu)$ are the poles of $v_2(0,0,0,\lambda,\mu)$.

Since we normalized $\boldsymbol{v}(x,y,t,\lambda,\mu)$ by $v_1(x,y,t,\lambda,\mu) \equiv 1$, $\boldsymbol{v}(0,0,0,\lambda,\mu)$ is uniquely determined by (32) evaluated at $(x,y,t) = (0,0,0)$. Since $\boldsymbol{v}(0,0,0,\lambda,\mu)$ is an eigenvector of $\widetilde{\boldsymbol{BC}}(\lambda)$ evaluated at $(0,0,0)$, the r equations of (32) are linearly dependent. At least one of these equations can be disregarded. Assume we can disregard the first equation. Equation (32) evaluated at $(x,y,t) = (0,0,0)$ gives

$$\left(\begin{array}{c|c} \boldsymbol{BC}(\lambda,\mu)_{11} & \\ \hline \boldsymbol{bc}^{(1)}(\lambda,\mu) & \boldsymbol{BC}^{(11)}(\lambda,\mu) \end{array}\right)\begin{pmatrix} 1 \\ v_2 \\ \vdots \\ v_r \end{pmatrix} = 0. \tag{99}$$

Here $\boldsymbol{BC}^{(11)}(\lambda,\mu)$ is the minor matrix of element (1,1) of $\boldsymbol{BC}(\lambda,\mu)$, $\boldsymbol{BC}(\lambda,\mu)_{11}$, and $\boldsymbol{bc}^{(1)}(\lambda,\mu)$ is the first column of $\boldsymbol{BC}(\lambda,\mu)$, without the first element. We get

$$\boldsymbol{BC}^{(11)}(\lambda,\mu)\hat{\boldsymbol{v}} = -\boldsymbol{bc}^{(1)}(\lambda,\mu), \tag{100}$$

with $\hat{\boldsymbol{v}}$ the vector $\boldsymbol{v}$ without the first element. The poles of $\boldsymbol{v}$ are at the poles of $\hat{\boldsymbol{v}}$. These occur when $\hat{\boldsymbol{v}}$ is undefined, *i.e.,* when $\det(\boldsymbol{BC}^{(11)}(\lambda,\mu)) = 0$. This is the first equation of (39). Disregarding any other row from $\boldsymbol{BC}(\lambda,\mu)$ results in the vanishing of another cofactor of the first column. If all cofactors of the first column vanish, the matrix $\boldsymbol{BC}(\lambda,\mu)$ is automatically singular, so it satisfies (34), *i.e,* the points in the divisor lie on the Riemann surface Γ.

References

[1] N. J. Zabusky and M. D. Kruskal, Phys. Rev. Lett., 15 (1965), 240.

[2] B. B. Kadomtsev and V. I. Petviashvili, Sov. Phys. Doklady, 15 (1970), 539.

[3] V. E. Zakharov and E. I. Shul'man, Dokl. Akad. Nauk SSSR 283 (1985), 1325.

[4] V. E. Zakharov (ed.), What Is Integrability? (Springer-Verlag, Berlin Heidelberg New York, 1991).

[5] V. Dryuma, Sov. Phys. JETP Lett. 19 (1974) 387.

[6] V. E. Zakharov and A. B. Shabat, Funct. Anal. Appl., 8 (1974), 226.

[7] I. M. Krichever, Sov. Math. Doklady, 17 (1976), 394.

[8] I. M. Krichever, Funct. Anal. Appl., 11 (1977), 11.

[9] S. V. Manakov, Physica 3D (1981), 420.
H. Segur, in 'Mathematical methods in hydrodynamics and integrability in dynamical systems', (La Jolla, Calif.,1981), 211.
A. S. Fokas and M. J. Ablowitz, Stud. Appl. Math., 69, 211.

[10] M. J. Ablowitz, D. Bar Yaacov and A. S. Fokas, Stud. Appl. Math. 69 (1983) 135.

[11] S. P. Novikov, Funct. Anal. Appl., 8 (1974), 236.

[12] P. D. Lax, Comm. Pure Appl. Math. 28 (1975), 141.

[13] E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its and V. B. Matveev, Algebro-geometric approach to nonlinear integrable equations, §1 (Springer-Verlag, Berlin Heidelberg New York, 1994).

[14] H. P. McKean and E. Trubowitz, Comm. Pure Appl. Math. 29 (1977), 143.

[15] H. P. McKean, Comm. Pure Appl. Math. 34 (1981), 599.

[16] P. G. Grinevich and I. M. Krichever, in 'Soliton Theory, a Survey of Results', ed. A. P. Fordy, (Manchester University Press, 1990), 354.

[17] I. M. Krichever, Russian. Math. Surveys 44 (1989), 145.

[18] E. Previato, Comment. Math. Helvetici 62 (1987), 292.

[19] J. Feldman, H. Knörrer and E Trubowitz, "Riemann Surfaces of Infinite Genus, IV: The Kadomcev Petviashvili Equation", ETH preprint, Zürich.

[20] I. M. Krichever, J. Soviet Math. 28(1) (1985), 51.

[21] G. Springer, Introduction to Riemann Surfaces (Addison-Wesley, Reading Massachusetts, 1957)

[22] B.A. Dubrovin, Russian Math. Surveys 36:2 (1981) 11.

[23] S. P. Novikov, S. V. Manakov, L. P. Pitaevskii and V. E. Zakharov, Theory of solitons, §2 (Consultants Bureau, New York, 1984).

[24] H. M. Farkas and I. Kra, Riemann Surfaces (Springer-Verlag, New York Berlin Heidelberg, 1992).

[25] F. Schottky, Acta Math. 27 (1903), 235.

[26] T. Shiota, Characterization of Jacobian Varieties in terms of Soliton Equations, Invent. Math. 83 (1986) 333.

[27] M. Mulase, Cohomological Structure in Soliton Equations and Jacobian Varieties, J. Differential Geometry 19, (1984) 403.

[28] H. Segur and A. Finkel, Stud. Appl. Math. 73 (1985), 183.

[29] A. I. Bobenko and L. A. Bordag, Periodic Multiphase Solutions of the Kadomtsev-Petviashvili Equation, J. Phys. A: Math. Gen. 22 (1989) 1259.

[30] E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its and V. B. Matveev, Algebro-geometric approach to nonlinear integrable equations, §3 and §5 (Springer-Verlag, Berlin Heidelberg New York, 1994).

[31] L. A. Bordag, Einige Fragen der Effektivisierung von thetafunktionalen Losungen intergrierbaren nichtlinearen Evolutionleichungen (University of Leipzig thesis, 1993).

[32] B. A. Dubrovin, R. Flickinger and H. Segur, Three-phase solutions of the Kadomtsev-Petviashvili equation (preprint, to appear in Stud. Appl. Math).

[33] B. A. Dubrovin, V. B. Matveev and S. P. Novikov, Russian. Math. Surveys 31 (1976), 59.

[34] E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its and V. B. Matveev, Algebro-geometric approach to nonlinear integrable equations, §4.6 (Springer-Verlag, Berlin Heidelberg New York, 1994).

[35] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, Stud. Appl. Math., 53 (1974) 249.

[36] V. E. Zakharov, Sov. Phys. JETP, 38 (1974) 108.

[37] M. Sato and Y. Sato, in 'Non-linear partial differential equations in applied science' (North-Holland, amsterdam-New York 1983), 259.

[38] I. M. Krichever, Funct. Anal. Appl. 12(3) (1978), 175.

[39] S. P. Novikov, S. V. Manakov, L. P. Pitaevskii and V. E. Zakharov, Theory of solitons, Appendix (Consultants Bureau, New York, 1984).

[40] Bogoyavlenskii and S.P. Novikov, Funct. Anal. Appl. 10(1) (1976), 9.

[41] J. L. Burchnall and T. W. Chaundy, Proc. Lond. Math. Soc. Ser. 2, 21 (1922) 420.

[42] J. L. Burchnall and T. W. Chaundy, Proc. R. Soc. Lond. A118 (1928) 557

[43] V. V. Prasolov, Problems and Theorems in Linear Algebra (AMS, Providence, Rhode Island, 1994)

[44] M. J. Ablowitz and P. A. Clarkson, Solitons, nonlinear evolution equations and inverse scattering (Cambridge University Press, 1991).

[45] S. S. Abhyankar, Algebraic Geometry for Scientists and Engineers (AMS, Providence, Rhode Island, 1990).

[46] B. L. van der Waerden, Algebra, volume 1 (Springer-Verlag New York, 1991)

[47] T. Miwa, in 'Soliton Theory, a Survey of Results', ed. A. P. Fordy, (Manchester University Press, 1990), 338.