NL2735 **Poisson brackets**

Let M be an n-dimensional manifold, refered to as the phase space. Let f g, and h denote analytic functions on M; A Poisson bracket of any two analytic functions on the phase space is defined as an operation which satisfies

(i) $\{\alpha f + \beta g, h\} = \alpha\{f, h\} + \beta\{g, h\}$, (linearity in the first component)

(ii) $\{f, g\} = -\{g, f\}$, (skew-symmetry)

(iii) $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$, (Jacobi identity)

(iv) $\{f, gh\} = g\{f, h\} + \{f, g\}h$, (Leibniz property)

where α, β are numbers. The first two properties ensure that a Poisson bracket is a bilinear operation on M. Properties (i)–(iii) imply that the analytic functions on M form a Lie algebra with respect to the Poisson bracket.

If local coordinates z_i, $i = 1, \ldots, n$ are chosen on M, then the Poisson bracket has the coordinate representation

$$\{f, g\} = \sum_{j,k=1}^{n} \omega_{jk}(z) \frac{\partial f}{\partial z_j} \frac{\partial g}{\partial z_k} = (\nabla f)^T \boldsymbol{\omega} \nabla g, \tag{1}$$

where $\nabla f = (\partial f/\partial z_1, \ldots, \partial f/\partial z_n)$, and the Poisson matrix $\boldsymbol{\omega}(z) = (\omega_{jk}(z))_{j,k=1}^{n}$ is a skew-symmetric square matrix, satisfying a technical condition enforced by the Jacobi identity.

Any nonconstant function C on M that Poisson commutes with all other functions on M is called a Casimir of the Poisson bracket. From (1) it follows that the existence of a Casimir requires $\boldsymbol{\omega}$ to be singular, and ∇C is in the null space of $\boldsymbol{\omega}$. Furthermore, the number of independent Casimirs is the corank of $\boldsymbol{\omega}$. For a Poisson bracket with r Casimirs $C_1, \ldots, C_r$, Darboux's theorem states that it is always possible to find coordinates $(q_1, \ldots, q_N, p_1, \ldots, p_N, C_1, \ldots, C_r)$ on M such that in these coordinates

$$\boldsymbol{\omega} = \begin{pmatrix} \mathbf{0} & \boldsymbol{I}_N & \mathbf{0} \\ -\boldsymbol{I}_N & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{2}$$

where $\boldsymbol{I}_N$ is the N dimensional identity matrix, and $\mathbf{0}$ is the zero matrix of the appropriate dimensions. In these coordinates,

$$\{f, g\} = \sum_{j=1}^{N} \left(\frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} \right). \tag{3}$$

This representation of the Poisson bracket is called the canonical Poisson bracket, and the coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ are called canonical coordinates.

The importance of Poisson brackets derives from their relationship to Hamiltonian systems: let H be a function on M. Hamiltonian dynamics with Hamiltonian function H are defined on any function f on M by

$$\dot{f} = \{f, H\}. \tag{4}$$

Using the coordinate representation (1), the Hamiltonian dynamics for the coordinates is

$$\dot{z}_j = \{z_j, H\} = \sum_{k=1}^{n} \omega_{jk} \frac{\partial H}{\partial z_k}, \tag{5}$$

which reduces to the standard definition of a Hamiltonian system if canonical coordinates are used. From (4) it is clear that any function which Poisson commutes with the Hamiltonian is conserved for the Hamiltonian system defined by the Poisson bracket and the Hamiltonian H. In particular, H is conserved. Also, any Casimir is conserved. Since the conservation of the Casimirs is independent of the choice of H, they do not contain dynamical information. Rather, as obvious from Darboux's theorem, they foliate the phase space and represent geometric restrictions on the possible motions in phase space. The Hamiltonian system can also be defined using the Hamiltonian function H and a symplectic two-form, of which $\boldsymbol{\omega}$ is the coordinate representation (Weinstein , 1984).

As an example, consider Euler's equations of a free rigid body (Weinstein , 1984). Denote the angular momentum by (M_1, M_2, M_3), and the moments of inertia by I_1, I_2, I_3. The Poisson matrix is

$$\boldsymbol{\omega} = \begin{pmatrix} 0 & M_3 & -M_2 \\ -M_3 & 0 & M_1 \\ M_2 & -M_1 & 0 \end{pmatrix}. \tag{6}$$

The Hamiltonian is $H = (M_1^2/I_1 + M_2^2/I_2 + M_3^2/I_3)/2$. The Poisson matrix has rank 2 (except at the origin), and there is one Casimir: $C_1 = M_1^2 + M_2^2 + M_3^2$.

The notion of Poisson brackets is extendable to infinite-dimensional phase spaces, so as to describe dynamics governed by evolution (partial differential) equations (Marsden and Morrison , 1984). In this case, the Poisson matrix $\boldsymbol{\omega}$ is replaced by a skew-adjoint differential operator B. If the evolution equation is first-order in the dynamical variable t, this operator is scalar. Otherwise it is a matrix operator of the same dimension as the order of the evolution equation. Instead of functions on phase space, we consider functionals

$$F[\boldsymbol{u}] = \int f[\boldsymbol{u}] dx. \tag{7}$$

Here $\boldsymbol{u}(x,t)$ is an infinite-dimensional coordinate on the phase space, indexed by the independent variable x. The square brackets denote that $f[\boldsymbol{u}]$ depends not only on $\boldsymbol{u}$, but possibly also on its derivatives with respect to x: $\boldsymbol{u}_x, \boldsymbol{u}_{xx}, \ldots$. The limits of integration depend on the boundary conditions imposed on the evolution equation. In the above, the variable x is assumed to be one dimensional. This is extended to higher dimensions in obvious fashion.

The Poisson bracket between any two functionals on phase space is the functional given by

$$\{F, G\} = \int \frac{\delta F}{\delta u} B \frac{\delta G}{\delta u} dx, \tag{8}$$

where $\delta F/\delta \boldsymbol{u}$ is the variational derivative of F with respect to $\boldsymbol{u}$:

$$\frac{\delta F}{\delta \boldsymbol{u}} = \frac{\partial f}{\partial \boldsymbol{u}} - \frac{\partial}{\partial x}\frac{\partial f}{\partial \boldsymbol{u}_x} + \frac{\partial^2}{\partial x^2}\frac{\partial f}{\partial \boldsymbol{u}_{xx}} - \dots \tag{9}$$

The Poisson bracket defined this way satisfies the properties (i), (ii), and (iv). The operator B is chosen so that the Jacobi identity (iii) is also satisfied.

A functional H on the phase space defines Hamiltonian dynamics on any functional by

$$\frac{\partial F}{\partial t} = \{F, H\} \quad \Leftrightarrow \quad \frac{\partial \boldsymbol{u}}{\partial t} = \{\boldsymbol{u}, H\} = B\frac{\delta H}{\delta \boldsymbol{u}}. \tag{10}$$

As an example, consider the Korteweg-de Vries equation $u_t = uu_x + u_{xxx}$ (Gardner , 1971; Zakharov and Faddeev , 1971). This equation with $-\infty < x < \infty$ is Hamiltonian with $B = \partial/\partial x$ and $H = \int_{-\infty}^{\infty}(u^3/6 - u_x^2/2)dx$. Thus, the Poisson bracket is

$$\{F, G\} = \int_{-\infty}^{\infty} \frac{\delta F}{\delta u}\frac{\partial}{\partial x}\frac{\delta G}{\delta u}dx, \tag{11}$$

and $\int_{-\infty}^{\infty} udx$ is its only Casimir.

The Poisson bracket formulation of a Hamiltonian system is especially significant when a quantum description of the dynamics is required. Dirac's principle of canonical quantization postulates that such a quantum description is obtained by replacing all classical quantities by their quantum mechanical operator counterparts (generalized coordinates $q \rightarrow \hat{q}$, the operation of multiplying by q; momentum $p \rightarrow -i\hbar\partial/\partial q$, *etc*, and all Poisson brackets by commutators/$(i\hbar)$). Then, in the classical limit as $\hbar \rightarrow 0$, the quantum mechanical equations reduce to classical equations, as desired by the correspondence principle.

Poisson brackets were introduced by Poisson (1809) during his investigations on perturbation theory in classical mechanics. Poisson's Traité de mécanique (two volumes, 1811 & 1833) were standard texts for many years.

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See also Conservation laws and constants of motion; Hamiltonian systems; Korteweg–de Vries equation; Lie algebras and Lie groups; Quantum theory;

Further Reading

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Poisson, S. D. 1809. Mémoire sur la variation des constantes arbitraires dans les questions de mécanique [Memoir on the variation of arbitrary constants in mechanics]. *Journal de l'École polytechnique*, cahier XV: 266-298

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