A Constructive Test for Integrability of Semi-Discrete Systems.

Bernard Deconinck
Program in Applied Mathematics,
Box 526,
University of Colorado at Boulder
Boulder, Colorado 80309-0526, USA
tel: (303) 492-3462
fax: (303) 492-4066
e-mail: deconinc@newton.colorado.edu

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Abstract

The Estabrook-Wahlquist method for establishing the integrability of partial differential equations is extended to semi-discrete (lattice) systems. If successful, the method construct the linear eigenvalue problem associated with the equation.

Keywords: integrability, semi-discrete, prolongation, lattice

1 Introduction

In [1, 2], Estabrook and Wahlquist developed a method to derive in a systematic way the linear spectral problem associated with a given partial differential equation (PDE). Subsequently, Fordy and co-workers [3, 4] made the method more algorithmic and put it in an algebraic instead of a differential geometric framework. If successful, the method finds a linear spectral problem for integrable 1+1 PDE. It also indicates nonintegrability of the given PDE. However, because the method involves some assumptions, its failure does not prove nonintegrability.

Recently there has been interest in integrable systems of infinitely many coupled ordinary differential equations, either as semi-discrete versions of an integrable PDE [5, 6] or in their own right ([7, 8, 9, 10]). In this paper, we generalize the Estabrook-Wahlquist method to apply to certain semi-discrete systems. Given such a semi-discrete system of equations, we demonstrate a systematic way to find a scattering pair (i.e. an associated linear spectral problem) for it, if it is integrable. If successful we have not only proven integrability but also constructed a starting point we can use to integrate the equations, using the inverse scattering transform technique. If the system is not integrable it may be possible to exclude the existence of scattering pairs in a certain class, hence providing strong indications for nonintegrability. As a test for integrability, the method has two big advantages over the Painlevé method: it does not depend on the coordinates and variables in which we have formulated the semi-discrete system. Also, if successful it provides a conclusive answer for integrability by constructing the scattering pair for the system.

The method is algorithmic, which we illustrate throughout this paper. The algorithm has seven steps. These steps are inherently sequential. In order to maintain this clear sequential process, we did not break some of the larger steps up into smaller steps since these smaller steps are often better carried out in parallel with the others. If the method is not succesful the algorithm breaks down during step four. This indicates nonintegrability for the given semi-discrete systems within the class of scattering pairs we considered.

In the following sections we illustrate the algorithm on two familiar integrable cases (sections 2 and 5). We also provide some theoretical insight into the connections with the corresponding continuum limit (if there is one) in section 3. Section 4 deals with the Standard Discretization of NLS, a well-known nonintegrable case. Finally, in section 6 we present scattering pairs for new systems recently proposed by Y. B. Suris [9, 10].

2 The Ablowitz-Ladik pair for the discrete NLS equation

Consider the Ablowitz-Ladik [11] discretization of the NLS equation

$$\begin{aligned} i\,\dot{q}_n &= q_{n+1} - 2q_n + q_{n-1} + q_n r_n (q_{n+1} + q_{n-1}), \\ -i\,\dot{r}_n &= r_{n+1} - 2r_n + r_{n-1} + q_n r_n (r_{n+1} + r_{n-1}). \end{aligned} \tag{1}$$

We have rescaled the potentials q_n and r_n and the continuous time variable so that the discretization parameter does not appear explicitly.

2.1 Compatibility of the scattering pair, functional dependence of the scattering matrix S_n

We want to construct a scattering pair for (1). In other words we want to find matrices S_n and T_n, which we require to be finite dimensional such that

$$\begin{aligned} v_{n+1} &= S_n v_n, \\ \dot{v_n} &= T_n v_n. \end{aligned} \quad (2)$$

Here S_n is the scattering matrix of the pair. Since (1) is the nontrivial compatibility condition of this pair, both S_n and T_n depend on the potentials q_j and r_j. We also expect the pair to depend on a scattering parameter, which should express a degree of freedom in choosing the pair. Isospectrality is then a consequence of the fact that this scattering parameter is time independent by construction.

Compatibility of (2) (or zero curvature condition) implies

$$\dot{S_n} + S_n T_n = T_{n+1} S_n. \quad (3)$$

Let's assume at this point that $S_n = S_n(q_n, r_n)$. In particular we do not include any forward or backward shifted potentials in the scattering matrix. This assumption fully determines the spatial and time dependence of S_n, since the scattering parameter is isospectral. Using the above assumption and (1) we can see that

$$\begin{aligned} &-i\tfrac{\partial S_n}{\partial q_n}(q_{n+1} - 2q_n + q_{n-1} + q_n r_n(q_{n+1} + q_{n-1}))+ \\ &i\tfrac{\partial S_n}{\partial r_n}(r_{n+1} - 2r_n + r_{n-1} + q_n r_n(r_{n+1} + r_{n-1})) + S_n T_n = T_{n+1} S_n. \end{aligned} \quad (4)$$

2.2 Explicit form of the time evolution matrix T_n

From (4), T_n does not depend on any forward shifts:

$$T_n = T_n(q_n, r_n, q_{n-1}, r_{n-1}). \quad (5)$$

Taking the partial derivative of (4) with respect to q_{n+1}, we find

$$-i\frac{\partial S_n}{\partial q_n}(1 + q_n r_n) = \frac{\partial T_{n+1}}{\partial q_{n+1}} S_n, \quad (6)$$

and three similar equations. From these

$$\frac{\partial^2 T_{n+1}}{\partial q_{n+1}^2} = 0, \quad \frac{\partial^2 T_{n+1}}{\partial r_{n+1}^2} = 0, \quad \frac{\partial^2 T_n}{\partial q_{n-1}^2} = 0, \quad \frac{\partial^2 T_n}{\partial r_{n-1}^2} = 0, \quad (7)$$

since S_n does not depend on any shifts and can be assumed to be non-singular (S_n^{-1} describes going left in space and should hence be defined, we assume the linear problem (2) is well-posed). Integrating (7) we find that T_n has the following functional dependence:

$$\begin{aligned} T_n \;=\; & q_n r_n q_{n-1} r_{n-1} \tilde{X}_1 + q_n r_n q_{n-1} \tilde{X}_2 + q_n r_n r_{n-1} \tilde{X}_3 + \\ & q_n r_n \tilde{X}_4 + q_n q_{n-1} r_{n-1} \tilde{X}_5 + r_n q_{n-1} r_{n-1} \tilde{X}_6 \\ & q_{n-1} r_{n-1} \tilde{X}_7 + q_n r_{n-1} X_1 + r_n q_{n-1} X_2 + q_n q_{n-1} X_3 + \\ & r_n r_{n-1} X_4 + q_n Y_1 + r_n Y_2 + q_{n-1} Y_3 + r_{n-1} Y_4 + Z. \end{aligned} \quad (8)$$

Here $\tilde{X}_i, X_j, Y_k$ and Z are constant matrices that show up as integration constants. Their dimension is unknown and remains to be determined.

Substituting (8) back into (4) shows that $\tilde{X}_j = 0$, for all j. The set of $\{X_i, Y_j\}$ are elements of a Lie algebra under the commutator bracket.

2.3 Constraints on the algebra

The evolution equations (1) impose constraints on the algebra determined by the $\{X_i, Y_j\}$. We now find these restrictions explicitly in order to determine the simplest, i.e. lowest-dimensional representation of the algebra. Since the algebra is determined by the commutation table of its elements we look for the constraints in this form.

We can now separate the dependence on the shifted potentials. This results in a system of five matrix equations:

$$
\begin{aligned}
-i\frac{\partial S_n}{\partial q_n}(1+q_nr_n) &= (r_nX_1+q_nX_3+Y_1)S_n, &(9)\\
i\frac{\partial S_n}{\partial r_n}(1+q_nr_n) &= (q_nX_2+r_nX_4+Y_2)S_n, &(10)\\
-i\frac{\partial S_n}{\partial q_n}(1+q_nr_n) &= -S_n(r_nX_2+q_nX_3+Y_3), &(11)\\
i\frac{\partial S_n}{\partial r_n}(1+q_nr_n) &= -S_n(q_nX_1+r_nX_4+Y_4), &(12)
\end{aligned}
$$

and

$$
2i\frac{\partial S_n}{\partial q_n} - 2i\frac{\partial S_n}{\partial r_n} = (r_nY_4+q_nY_3+Z)S_n - S_n(q_nY_1+r_nY_2+Z). \tag{13}
$$

This last equation will be used to determine Z once the other elements in the algebra and S_n are known.

The first four equations are of the form

$$
\begin{aligned}
\frac{\partial S_n}{\partial q_n} &= A_nS_n, \quad \frac{\partial S_n}{\partial r_n} = B_nS_n, &(14)\\
\frac{\partial S_n}{\partial q_n} &= S_nC_n, \quad \frac{\partial S_n}{\partial r_n} = S_nD_n. &(15)
\end{aligned}
$$

Compatibility conditions for such equations are

$$
\begin{aligned}
A_nS_n &= S_nC_n, & B_nS_n &= S_nD_n, &(16)\\
\frac{\partial A_n}{\partial r_n} - \frac{\partial B_n}{\partial q_n} &= [B_n, A_n], & \frac{\partial C_n}{\partial r_n} - \frac{\partial D_n}{\partial q_n} &= [C_n, D_n], &(17)\\
\frac{\partial B_n}{\partial q_n}S_n &= S_n\frac{\partial C_n}{\partial r_n}, & \frac{\partial A_n}{\partial r_n}S_n &= S_n\frac{\partial D_n}{\partial q_n}, &(18)\\
\frac{\partial A_n}{\partial q_n}S_n &= S_n\frac{\partial C_n}{\partial q_n}, & \frac{\partial B_n}{\partial r_n}S_n &= S_n\frac{\partial D_n}{\partial r_n}. &(19)
\end{aligned}
$$

This set of compatibility conditions is certainly not complete and for specific examples it might be necessary to use other ones, obtained from equating higher-order mixed partial derivatives. Notice that (17) stands out among the other conditions, since it does not involve S_n and depends only on the matrix coefficients of the system.

Using (17) in our specific case results in equations in which the dependence on the potentials is explicit, hence we can equate coefficients of different powers of the potential. We get two sets of commutator relationships:

$$X_4 = i\,[X_4, X_1]\,, \qquad X_3 = -i\,[X_3, X_2]\,, \tag{20}$$

$$X_1 + X_2 = i\,[Y_1, Y_2]\,, \qquad 0 = [X_2, X_1] + [X_4, X_3]\,, \tag{21}$$

$$Y_2 = i\,[X_4, Y_1] + i\,[Y_2, X_1]\,, \qquad Y_1 = i\,[X_2, Y_1] + i\,[Y_2, X_3]\,, \tag{22}$$

and

$$X_4 = i\,[X_4, X_2]\,, \qquad X_3 = -i\,[X_3, X_1]\,, \tag{23}$$

$$X_1 + X_2 = i\,[Y_3, Y_4]\,, \qquad 0 = [X_2, X_1] + [X_3, X_4]\,, \tag{24}$$

$$-Y_3 = i\,[X_3, Y_4] + i\,[Y_3, X_1]\,, \qquad -Y_4 = i\,[X_2, Y_4] + i\,[Y_3, X_4]\,, \tag{25}$$

from which

$$[X_1, X_2] = 0, \quad [X_3, X_4] = 0. \tag{26}$$

2.4 Determine the center of the algebra and locate subalgebras. Find the lowest-dimensional representation of the algebra

Combining (16) and (18) we establish

$$[X_1 - X_2, S_n] = 0. \tag{27}$$

Taking derivatives of (27) with respect to both q_n and r_n and using (27) again with the non-singularity of S_n, we see that $X_1 - X_2$ commutes with all the X_i, Y_j. We require that these elements determine the algebra, hence we have established that $X_2 - X_1$ is a central element of this algebra. We then put

$$X_1 - X_2 = \alpha I, \tag{28}$$

where I is the $N \times N$ identity matrix, N being the dimension of our representation. We show in the next section that we cannot equate all elements in the center of the algebra to the zero matrix, as is customary in the Estabrook-Wahlquist method [4]. Other conditions can be obtained from (9) and (11) by taking a q_n-derivative of both sides and equating the left hand sides. We obtain

$$\{X_3, S_n\} = 0, \qquad \{X_4, S_n\} = 0. \tag{29}$$

Here $\{A, B\}$ is the anti-commutator of A and B: $\{A, B\} = AB + BA$. The second equality is obtained the same way from (10) and (12). Taking derivatives of these two equalities we find that both X_3 and X_4 are in the center of the algebra. As a consequence they also are scalar multiples of the identity. However, from either (20) or (23) X_3 and X_4 are both traceless. Hence

$$X_3 = 0 = X_4. \tag{30}$$

There are now essentially 5 elements left in the algebra: $X_1 + X_2, Y_1, Y_2, Y_3$ and Y_4. Their commutation table is given below.

At this point we could impose a closure assumption on the algebra as one usually does when using the Estabrook-Wahlquist method to construct scattering pairs [3]. However, this would be a non-algorithmic step and we will try to avoid it if possible. Rather we attempt to find a

$[.,.]$	X_1+X_2	Y_1	Y_2	Y_3	Y_4
X_1+X_2	0	$-iY_1$	iY_2	$-iY_3$	iY_4
Y_1	iY_1	0	$-i(X_1+X_2)$		
Y_2	$-iY_2$	$i(X_1+X_2)$	0		
Y_3	iY_3			0	$-i(X_1+X_2)$
Y_4	$-iY_4$			$i(X_1+X_2)$	0

Table 1: **The commutation table for the algebra $\{X_1+X_2, Y_1, Y_2, Y_3, Y_4\}$, associated with the Ablowitz-Ladik system.**

representation for the algebra by considering representations for subalgebras. Subalgebras are in general easily found for these prolongation algebras. This procedure may not work. In this case we have to either go on to higher dimensional embeddings of the subalgebras or we can try to close the algebra at this point. We will indicate how to remedy the method if we cannot locate any subalgebras. The subalgebras we find here are $\{X_1+X_2, Y_1, Y_2\}$ and $\{X_1+X_2, Y_3, Y_4\}$. Using the well-known commutation relations for $sl(2,C)$ [12] we can identify

$$X_1+X_2 = ih, \quad Y_1 = \beta e_-, \quad Y_2 = -\frac{1}{\beta}e_+, \quad Y_3 = \gamma e_-, \quad Y_4 = -\frac{1}{\gamma}e_+, \tag{31}$$

with γ and β arbitrary constants. h, e_- and e_+ are the usual raising and lowering basis of $sl(2,C)$. Their simplest (two-dimensional) representation is

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{32}$$

2.5 Determine the scattering matrix S_n

We now explicitly know X_1, X_2 and Y_j, for all j. With this information, we can try to find a two-dimensional representation for the entire algebra. This does indeed work in this case. If this were not the case, we need to go back to step 4 of the algorithm, as pointed out above. We can integrate equations (9-12). The result is

$$S_n = Ce^{\frac{i\alpha-1}{2}\ln(1+q_n r_n)} \begin{pmatrix} 1 & -\frac{1}{\gamma}r_n \\ i\beta q_n & -\frac{\beta}{\gamma} \end{pmatrix}. \tag{33}$$

C, being an arbitrary constant, can be set equal to $i\sqrt{(\gamma/\beta)}$, since we can easily accomplish this using a trivial gauge transformation. Furthermore, if we want to avoid non-algebraic dependence of the scattering pair on the potentials, we have to choose $\alpha = -i$.

2.6 Find Z, the constant part of T_n

With S_n and the other elements in the algebra known, (13) determines Z:

$$Z = \begin{pmatrix} z_1 & 0 \\ 0 & z_4 \end{pmatrix}, \tag{34}$$

with $z_1 - z_4 = -i(\sqrt{\beta/\gamma} + \sqrt{\gamma/\beta})^2$. Only this difference is defined. This was to be expected: we can always transform time to change z_1 or z_4 as we like. Only their difference is invariant. We can put without loss of generality

$$z_1 = -\frac{i}{2}(\sqrt{\beta/\gamma} + \sqrt{\gamma/\beta})^2, \quad z_4 = \frac{i}{2}(\sqrt{\beta/\gamma} + \sqrt{\gamma/\beta})^2. \tag{35}$$

2.7 The explicit form of the scattering pair

Applying the constant gauge transformation

$$\tilde{v_n} = G v_n, \tag{36}$$

with

$$G = \begin{pmatrix} 0 & 1 \\ -\sqrt{\beta\gamma} & 0 \end{pmatrix}, \tag{37}$$

we find the Ablowitz-Ladik pair for the discrete NLS equation [11]:

$$S_n = \begin{pmatrix} z & q_n \\ -r_n & 1/z \end{pmatrix}, \quad T_n = \begin{pmatrix} -iq_n r_{n-1} - \frac{i}{2}(\frac{1}{z} - z)^2 & \frac{i}{z} q_{n-1} - izq_n \\ izr_{n-1} - \frac{i}{z} r_n & ir_n q_{n-1} + \frac{i}{2}(\frac{1}{z} - z)^2 \end{pmatrix}, \tag{38}$$

where $z = -i\sqrt{\beta/\gamma}$ plays the role of the scattering parameter. The reason we wrote the pair using (35) is that upon reintroducing the discretization parameter Δx , the above pair has the correct continuum limit as $\Delta x \to 0$. Using three assumptions (dependence of S_n, isospectrality and well-posedness of the associated linear problem), we were able to find the scattering pair for (1). The method for doing this was purely algorithmic, as we indicated during the derivation. Since we found a non-trivial scattering pair for (1), (1) is integrable, and we have successfully completed the seven steps of the algorithm.

3 Identity-Connected Lie algebras

In the previous calculation we encountered central elements of the algebra, which we did not equate to zero, unlike in the Estabrook-Wahlquist method for PDE's.

Assume that the semi-discrete system under consideration is a spatial discretization of a PDE, with gridpoints at $x_n = n\Delta x$. As a consequence, neighboring potentials are equal, up to first-order in Δx. A similar argument holds for the eigenfunctions of the linear spectral problem v_n. The transformation from v_n to v_{n+1}, which we require to be linear for integrable problems will be 'close' to the identity. We show what we mean by close:

Let's start with the scattering pair for an integrable PDE:

$$v_x = L(x,t,\lambda)v, \quad v_t = M(x,t,\lambda)v. \tag{39}$$

Let's assume that the semi-discrete system we are interested in has a scattering pair given by the discretization of (39). Upon discretizing (using a right spatial discretization for example)

$$x \to x_n = n\Delta x, \quad v(x) = v(n\Delta x) \to v_n, \quad L(x,t,\lambda) \to L_n(t,\lambda), \tag{40}$$

we obtain

$$v_{n+1} = [I + \Delta x\, L_n(t,\lambda)]\, v_n + \wr(\Delta x) = S_n(t,\lambda,\Delta x)v_n + \mathcal{O}(\Delta x). \tag{41}$$

We see indeed that the transformation from v_n to v_{n+1}, i.e. the scattering matrix, is a first-order in Δx correction to the identity. This reasoning does not hold for the PDE case, where the potential and its derivatives are the independent variables for the purpose of the Estabrook-Wahlquist method. There is no reason for these different derivatives to have approximately the same numerical value. In general, the scattering pair for a PDE can be chosen to be trace-free, since the underlying algebra is a Lie algebra. As was argued in [4], this Lie algebra can be assumed to be semi-simple. For semi-discrete systems, we pointed out above that the pair does not have to be trace-free. The procedure shows also that one can obtain integrable discretizations of a PDE by discretizing the pair of the PDE and expressing its compatibility [11] (one can obtain (1) this way). In that case, the algebra for the pair of the discretization will be the same as the algebra underlying the PDE, with an identity component added in.

We call the resulting algebra *identity-connected* [13]. It is in fact the first-order approximation to the Lie group generated by the Lie algebra. This algebraic structure is a Lie algebra, with a non-trivial center. The identity matrix, commuting with every other matrix will be in the center of the algebra and it cannot be set equal to zero as the above reasoning shows. Therefore, whenever we find an element is central to the algebra, we can only assume it is a scalar multiple of the identity matrix. Only if we know it is also trace-free can we equate it to zero.

The discretization obtained this way is not necessarily the desired one. An integrable PDE may have more than one integrable discretization and the above procedure only isolates one. Furthermore, not every interesting semi-discrete system is the discretization of an integrable PDE.

4 The Standard Discretization of the NLS equation

The Estabrook-Wahlquist method for PDE's can prove non-integrability under certain assumptions. We now illustrate that this is also possible for semi-discrete systems. To do this let us consider the Standard Discretization of the NLS equation. The following semi-discrete system is known to be non-integrable [5], but is nevertheless significant for its physical applications [8]:

$$\begin{aligned} i\,\dot{q_n} &= q_{n+1} - 2q_n + q_{n-1} + 2q_n^2 r_n, \\ -i\,\dot{r_n} &= r_{n+1} - 2r_n + r_{n-1} + 2q_n r_n^2. \end{aligned} \tag{42}$$

The algorithm is now applied to the system (42).

4.1 Compatibility of the scattering pair, functional dependence of the scattering matrix S_n

The first two steps of the algorithm are identical to those in section (2). In particular, we assume the scattering matrix at position n depends only on the potentials at position n, not on any shifted potentials. This limits the class of scattering pairs under consideration.

4.2 Explicit form of the time evolution matrix T_n

In an analysis that parallels that in section (2), we find again

$$\begin{aligned} T_n \;=\; & q_n r_{n-1} X_1 + r_n q_{n-1} X_2 + q_n q_{n-1} X_3 + r_n r_{n-1} X_4 + \\ & q_n Y_1 + r_n Y_2 + q_{n-1} Y_3 + r_{n-1} Y_4 + Z. \end{aligned} \tag{43}$$

4.3 Constraints on the algebra

Using (43) we can separate in the compatibility condition the dependence on the shifted potentials, resulting in

$$-i\frac{\partial S_n}{\partial q_n} = (r_n X_1 + q_n X_3 + Y_1)S_n, \tag{44}$$
$$i\frac{\partial S_n}{\partial r_n} = (q_n X_2 + r_n X_4 + Y_2)S_n, \tag{45}$$
$$-i\frac{\partial S_n}{\partial q_n} = -S_n(r_n X_2 + q_n X_3 + Y_3), \tag{46}$$
$$i\frac{\partial S_n}{\partial r_n} = -S_n(q_n X_1 + r_n X_4 + Y_4), \tag{47}$$

and

$$(1-q_n r_n)\left(2i\frac{\partial S_n}{\partial q_n}q_n - 2i\frac{\partial S_n}{\partial r_n}r_n\right) = (r_n Y_4 + q_n Y_3 + Z)S_n - S_n(q_n Y_1 + r_n Y_2 + Z). \tag{48}$$

To express the compatibility of these equations, we use (16-19), but with different matrices A, B, C and D. The compatibility condition (17) again provides two sets of commutator relationships,

$$0 = [X_4, X_1], \qquad 0 = [X_3, X_2], \tag{49}$$
$$X_1 + X_2 = i[Y_1, Y_2], \qquad 0 = [X_2, X_1] + [X_4, X_3], \tag{50}$$
$$0 = [X_4, Y_1] + [Y_2, X_1], \qquad 0 = [X_2, Y_1] + i[Y_2, X_3], \tag{51}$$

and

$$0 = [X_4, X_2], \qquad 0 = [X_3, X_1], \tag{52}$$
$$X_1 + X_2 = i[Y_3, Y_4], \qquad 0 = [X_2, X_1] + [X_3, X_4], \tag{53}$$
$$0 = [X_3, Y_4] + i[Y_3, X_1], \qquad 0 = [X_2, Y_4] + [Y_3, X_4]. \tag{54}$$

4.4 Determine the center of the algebra and locate subalgebras. Find the lowest-dimensional representation of the algebra

Condition (18) gives $[X_1, S_n] = 0 = [X_2, S_n]$. Taking derivatives with respect to the potentials and using (44-47) and (49, 52) we get $[X_i, X_j] = 0, [X_i, Y_j] = 0$, for $i = 1, 2$ and $j = 1, ..., 4$. Hence X_1 and X_2 are central elements. We put $X_1 = \alpha I, X_2 = \beta I$. Since from (49) $\mathrm{tr}(X_1 + X_2) = 0$,

$$X_1 = \alpha I = -X_2. \tag{55}$$

Equation (19) again leads to

$$\{X_3, S_n\} = 0 = \{X_4, S_n\}, \tag{56}$$

and subsequently (16) yields

$$Y_1 S_n + S_n Y_3 = 0, \quad Y_2 S_n + S_n Y_4 = 0. \tag{57}$$

We can use (56) in the same way we used (29). The conclusion is similar. Both X_3 and X_4 commute with all elements in the algebra. Hence each is a scalar multiple of the identity. On the other hand, from (56) we obtain that the traces of X_3 and X_4 are zero. Hence

$$X_3 = 0 = X_4. \tag{58}$$

We can now integrate (44-47) to obtain S_n. Once we have done this, it is clear we have completely exhausted equations (44-47). This integration is straightforward, essentially because all matrices in the exponent of the exponential commute. We find

$$S_n = e^{i\alpha q_n r_n} e^{iq_n Y_1 - ir_n Y_2} S_0 = S_0 e^{i\alpha q_n r_n} e^{-iq_n Y_3 + ir_n Y_4}. \tag{59}$$

If the Y_i are nilpotent matrices then the scattering matrix becomes essentially polynomial up to a scalar factor. In (59) S_0 is the value of S_n at infinity, assuming we are considering the infinite line problem with decaying boundary conditions (these boundary conditions are not essential for what follows, they are used for simplicity). Substituting the first form of S_n into (48) leads to

$$\begin{aligned} Y_1 S_n(-2q_n + 2q_n^2 r_n) \quad &+ \quad Y_2 S_n(-2r_n + 2r_n^2 q_n) = \\ (q_n Y_3 + r_n Y_4 + Z) S_n \quad &- \quad S_n(q_n Y_1 + r_n Y_2 + Z). \end{aligned} \tag{60}$$

Multiplying this equation on the left by Y_1^{n-1}, on the right by S_n^{-1} and taking the trace, we can eliminate all S_n and its inverses. Notice that this is possible, since $Y_1^n S_n = (-1)^n S_n Y_3^n$ from (57). We find a polynomial equation in the potentials. Vanishing of the coefficient of $q_n^2 r_n$ implies $\mathrm{tr} Y_1^n = 0$; the trace of Y_1 to any power is zero. As a consequence Y_1 is nilpotent [12]. Similarly, Y_2, Y_3 and Y_4 are nilpotent. Assume that Y_1 is nilpotent of degree d, then disregarding the scalar exponential factor (which cancels out in (60)) and equating the highest power in q_n, we find

$$Y_1 = 0. \tag{61}$$

This argument can be repeated to obtain that all the Y_j's are zero. This means that any pair for the Standard Discretization of NLS which depends only locally on the potentials (no shifts or nonlocal factors) is abelian and hence trivial. We can conclude that under the above assumptions the Standard Discretization is not integrable using the inverse scattering technique. The extension of the Estabrook-Whalquist method can hence determine nonintegrability under certain assumptions just as easily as does its original version for PDE's.

5 The Toda lattice

One of the first semi-discrete systems to attract a lot of attention was the Toda lattice [7]. It appeared as a physical model in its own right, not as a discretized version of a PDE (although it can be interpreted as such). The Toda lattice is for our purposes important to consider, since it is a system with nonpolynomial dependence on the potentials. The Estabrook-Wahlquist method for PDE's for such systems becomes much harder than for quasi-polynomial systems [3]. The Toda system can be written in the form

$$\dot{q_n} = p_n, \quad \dot{p_n} = e^{q_{n+1} - q_n} - e^{q_n - q_{n-1}}. \tag{62}$$

Usually when discussing this system in terms of a scattering pair, it is brought into polynomial form [14]. Integrability of a system should not depend on having the right coordinate system, therefore we will try to construct the pair using the form (62). As mentioned above this will also illustrate that the semi-discrete version of the Estabrook-Wahlquist method works in nonpolynomial cases.

5.1 Compatibility of the scattering pair, functional dependence of the scattering matrix S_n

Assuming again that $S_n = S_n(q_n, p_n)$ depends only on the potentials locally and not on any shifts, we find that $T_n = T_n(q_n, q_{n-1})$. The compatibility condition (3) gives

$$\frac{\partial S_n}{\partial q_n} p_n + \frac{\partial S_n}{\partial p_n} \left(e^{q_{n+1}-q_n} - e^{q_n - q_{n-1}}\right) + S_n T_n = T_{n+1} S_n. \tag{63}$$

5.2 Explicit form of the time evolution matrix T_n

Taking derivatives of (63) with respect to q_{n+1} and q_{n-1}, we get

$$\frac{\partial S_n}{\partial p_n} e^{q_{n+1}-q_n} = \frac{\partial T_{n+1}}{\partial q_{n+1}} S_n, \quad \frac{\partial S_n}{\partial p_n} e^{q_n - q_{n-1}} = -S_n \frac{\partial T_n}{\partial q_{n-1}}. \tag{64}$$

These equations can be used to fix the functional dependence of T_n more explicitly. For instance, from the second equation, we find $T_n = A_n(q_n)\exp(-q_{n-1}) + B_n(q_n)$, where A_n and B_n are matrix-valued functions to be determined. We get a similar equation in which the functional dependence of T_n on q_n is explicit, using the first equation in (64). Compatibility between these two expressions leads to

$$T_n = X_1 e^{q_n - q_{n-1}} + X_2 e^{q_n} + X_3 e^{-q_{n-1}} + X_4. \tag{65}$$

In this expression, the X_i are constant matrixes with dimension and representation to be determined.

5.3 Constraints on the algebra

Substituting (65) into (63), we separate the dependence on the shifted potentials. We get three equations:

$$\begin{aligned} \frac{\partial S_n}{\partial p_n} &= \left(X_1 + X_2 e^{q_n}\right) S_n, && (66) \\ \frac{\partial S_n}{\partial p_n} &= S_n \left(X_1 + X_3 e^{-q_n}\right), && (67) \end{aligned}$$

and

$$\frac{\partial S_n}{\partial q_n} p_n + S_n \left(X_2 e^{q_n} + X_4\right) = \left(X_3 e^{-q_n} + X_4\right) S_n. \tag{68}$$

We can cast these three equations in a more useful form by expressing the compatibility between them: taking the derivative with respect to p_n of (68) and with respect to q_n of (66), we can equate mixed partial derivatives. Using (66) to eliminate as many derivatives as possible, we obtain

$$\frac{\partial S_n}{\partial q_n} = \left([X_3 e^{-q_n} + X_4, X_1 + X_2 e^{q_n}] - p_n X_2 e^{q_n}\right) S_n, \tag{69}$$

Repeating this procedure using (67) instead of (66),

$$\frac{\partial S_n}{\partial q_n} = -S_n \left([X_1 + X_3 e^{-q_n}, X_2 e^{q_n} + X_4] - p_n X_3 e^{-q_n}\right). \tag{70}$$

The equations (66),(67),(69) and (70) are similar in form to for instance (9-12). We can use the compatibility conditions (16-19) as we did before.

Using (17) we obtain commutators for the generators X_i. Many of these will be nested commutators. In order to be able to deal with these, we will prolong the algebra by adding a number of elements to it. Let

$$Y_1 = [X_1, X_4], \quad Y_2 = [X_2, X_3], \quad Y_3 = [X_2, X_4], \quad Y_4 = [X_3, X_4]. \tag{71}$$

Prolonging the algebra is often necessary when using the Estabrook-Wahlquist method to find the scattering pair for a PDE, essentially to avoid dealing with double commutators. Up to now, we did not have to do this. Apart from being a nice example of nonpolynomial dependence, the Toda pair turns out to be a semi-discrete system for which it is necessary to prolong the algebra.

Now, (17) gives

$$0 = [X_1, X_3], \ 0 = [X_1, X_2], \ 0 = [X_1, Y_1 + Y_2], \tag{72}$$

$$2X_2 = [X_2, Y_1 + Y_2] + [X_1, Y_3], \ 0 = [X_2, Y_3], \tag{73}$$

$$2X_3 = [X_1, Y_1 + Y_4] + [X_3, Y_2], \ 0 = [X_3, Y_4]. \tag{74}$$

5.4 Determine the center of the algebra and locate subalgebras. Find the lowest-dimensional representation of the algebra

From (18), we find

$$S_n X_3 e^{-q_n} = e^{q_n} X_2 S_n. \tag{75}$$

Comparing this with (66-67) results in $[S_n, X_1] = 0$. Taking derivatives of (75) with respect to q_n and using (69, 70) in analogy with what we did in section (2), $[X_1, Y_j] = 0$, for all j. We can now use the Jacobi identity [12] to get a few more restrictions on the algebra. Expressing the Jacobi identity for X_1, X_2 and X_4 gives $[X_2, Y_1] = 0$. The Jacobi identity for X_1, X_3 and X_4 gives $[X_3, Y_1] = 0$. We could get more commutation relations if we so desired. This is however not strictly necessary, as we can already identity a subalgebra at this point. The commutation table of the algebra is given below, taking all known commutator relationships into account.

We see from table 2 that the subalgebra $\{X_2, X_3, Y_2\}$ is isomorphic with $sl(2, C)$. Using this identification, we find in a similar way as for (31) that

$$X_2 = \alpha e_+, \quad X_3 = -\frac{1}{\alpha} e_-, \quad Y_2 = -h. \tag{76}$$

Since X_1 is in the center of $\{X_2, X_3, Y_2\}$, it equals a scalar multiple of the identity: $X_1 = \beta I$. As a consequence $Y_1 = 0$. The lowest-dimensional representation of $sl(2, C)$ is two-dimensional.

[.,.]	X_1	X_2	X_3	X_4	Y_1	Y_2	Y_3	Y_4
X_1	0	0	0	Y_1	0	0	0	0
X_2	0	0	Y_2	Y_3	0	$2X_2$	0	
X_3	0	$-Y_2$	0	Y_4	0	$-2X_3$		0
X_4	$-Y_1$	$-Y_3$	$-Y_4$	0				
Y_1	0	0	0		0	0	0	0
Y_2	0	$-2X_2$	$2X_3$		0	0		
Y_3	0	0			0		0	
Y_4	0		0		0			0

Table 2: **The commutation table for the algebra $\{X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4\}$, associated with the Toda lattice.**

5.5 Determine the scattering matrix S_n

Using this representation and (66,67), we get for S_n

$$S_n = e^{\beta p_n} \begin{pmatrix} 1 & \alpha e^{q_n} p_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Q_{n1} & Q_{n2} \\ Q_{n3} & 0 \end{pmatrix}. \tag{77}$$

Here the second matrix only depends on q_n, not on p_n. Also, from (68), $\mathrm{tr} S_n$ is independent of q_n. This gives

$$Q_{n1} = C_1, \quad Q_{n2} = -\alpha^2 C_2 e^{q_n}, \quad Q_{n3} = C_2 e^{q_n}, \tag{78}$$

with C_1, C_2 constants. The second equation follows from (75). If we want S_n to be linear in the p_n, we can put $\beta = 0$.

5.6 Find Z, the constant part of T_n

In this case, we can use (68) to determine X_4. We find $X_{44} - X_{41} = C_1/\alpha C_2$, where we have used the same notation as before: X_{44}, X_{41} are the (2,2) and (1,1) component of X_4 respectively. Because only their difference is fixed, we may choose

$$X_{41} = 0, \qquad X_{44} = \frac{C_1}{\alpha C_2}. \tag{79}$$

5.7 The explicit form of the scattering pair

The scattering pair becomes

$$S_n = \begin{pmatrix} C_1 + \alpha C_2 p_n & -\alpha^2 C_2 e^{q_n} \\ C_2 e^{-q_n} & 0 \end{pmatrix}, \quad T_n = \begin{pmatrix} 0 & \alpha e^{q_n} \\ -\frac{1}{\alpha} e^{-q_{n-1}} & \frac{C_1}{\alpha C_2} \end{pmatrix}. \tag{80}$$

The parameter α can be eliminated by redefining C_2, and shifting the potential q_n by a constant amount. It is then obvious that the ratio C_1/C_2 is the only essential parameter in the problem and we hence recover the scattering pair given in [15].

6 Some new results

Recently, Y. B. Suris showed the integrability of two new systems [9, 10]. Integrability was demonstrated using a Hamiltonian formalism, but no scattering pair (in the form of a zero curvature representation) was presented. We now provide scattering pairs for these new systems, using the algorithm outlined in the previous sections. The first system [9] can be written in first-order form as

$$\dot{q_n} = p_n + e^{q_n - q_{n-1}}, \quad \dot{p_n} = p_{n+1} e^{q_{n+1} - q_n} - p_n e^{q_n - q_{n-1}}. \tag{81}$$

Assuming the scattering matrix S_n depending only on q_n and p_n, we found the scattering pair for (81) to be

$$S_n = \begin{pmatrix} z + p_n & -p_n e^{q_n} \\ e^{-q_n} & -1 \end{pmatrix}, \quad T_n = \begin{pmatrix} 0 & p_n e^{q_n} \\ -e^{-q_{n-1}} & z \end{pmatrix}. \tag{82}$$

To obtain this result we had to deal with determining equations for S_n that were not of the form (9-12). As a result the integrability conditions were not exactly (16-19). Nevertheless, we systematically derive a different (slightly more complicated) set of integrability conditions, expressing the compatibility of the determining equations for S_n. However, we could not identify a subalgebra and we needed to impose a closure assumption on the algebra. With the closure assumption, we can identify the algebra and find its representation. Once again the algebra is $sl(2, C)$, with an extra identity component. The resulting form for S_n and T_n is (82).

Suris' second system [10]

$$\dot{q_n} = p_n, \quad \dot{p_n} = p_n \left(e^{q_{n+1} - q_n} - e^{q_n - q_{n-1}} \right). \tag{83}$$

has the scattering pair

$$S_n = \begin{pmatrix} z + p_n & -e^{q_n} \\ -z e^{-q_n} & 1 \end{pmatrix}, \quad T_n = \begin{pmatrix} e^{q_n - q_{n-1}} & e^{q_n} \\ z e^{-q_{n-1}} & z \end{pmatrix}. \tag{84}$$

In this case the determining equations for S_n could be cast into the form (9-12), using the same method as for the Toda lattice. Also here, we needed a closure assumption to determine the representation of the algebra.

The reader can verify that the two pairs (82) and (84) result in the respective systems (81) and (83), using (3). The similarity of these scattering pairs with the pair for the Toda lattice is striking. This is not surprising, considering the similarity of the corresponding systems with the Toda lattice and the results in [9, 10]. The great benefit of the above scattering representation is that we only need to calculate with 2×2 matrices, no matter what the boundary conditions on the systems (81) and (83) are.

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References

[1] H. D. Wahlquist and F. B. Estabrook, Journal of Math. Phys. 16.1 (1975) 1.

[2] F. B. Estabrook and H. D. Wahlquist, Journal of Math. Phys. 17 (1976) 1293.

[3] R. Dodd and A. Fordy, Proc. R. Soc. Lond. A 385 (1983) 389.

[4] A. P. Fordy, Prolongation structures of nonlinear evolution equations, in: Soliton theory, a survey of results, ed. A. Fordy (Manchester University Press, 1990).

[5] M. J. Ablowitz and B. M. Herbst, SIAM J. Appl. Math 50 (1990) 339.

[6] M. J. Ablowitz and P. A. Clarkson, Solitons, nonlinear evolution equations and inverse scattering (London Mathematical Society Lecture note series 149, Cambridge University Press, 1991).

[7] M. Toda, J. Phys. Soc. Japan 22 (1967) 431.

[8] A. B. Aceves, C. De Angelis, G. G. Luther, A. M. Rubenchik and S. K. Turitsyn, Physica D 87 (1995) 262.

[9] Y. B. Suris, A new integrable system related to the relativistic Toda lattice, Preprint (1996).

[10] Y. B. Suris, A new integrable system related to the Toda lattice, Preprint (1996)

[11] M. J. Ablowitz and H. Segur, Solitons and the inverse scattering transform (Siam Studies in Applied Mathematics, 1981).

[12] N. Jacobson, Lie algebras (Dover, New York, 1979).

[13] S. Chakravarty, Private communication.

[14] H. Flaschka, Prog. Theo. Phys. 51 (1974) 703.

[15] L. D. Faddeev and L. A. Takhtajan, Hamiltonian methods in the theory of solitons (Springer-Verlag, 1987).