RECOVERING THE WATER-WAVE SURFACE FROM PRESSURE MEASUREMENTS

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Talk Abstract

A new method is proposed to recover the water-wave surface elevation from pressure data obtained at the bottom of the fluid. The new method requires the numerical solution of a nonlocal nonlinear equation relating the pressure and the surface elevation which is obtained from the Euler formulation of the water-wave problem without approximation. This new approach is compared with other approaches currently used in field observations.

Introduction

In field experiments, the surface elevation of a water-wave surface in shallow water is often determined by measuring the pressure along the bottom of the fluid, see *e.g.* [2], [5], [9], [10], [11], [12]. Using this pressure data, it is common to reconstruct the surface elevation using a well-known transfer function linear relationship between the Fourier transforms of the pressure and the elevation of the surface [8]:

$$\mathcal{F}\left\{\eta(x)\right\}(k) = \cosh(kh)\mathcal{F}\left\{p(x)\right\}(k), \qquad (1)$$

where h represents the average depth of the fluid, $\eta(x)$ is the zero-average surface elevation of the wave, and $p(x) = (P(x,-h) - \rho g h)/\rho g$ is the non-static part of the pressure $P(x,z)$ evaluated at the bottom of the fluid $z = -h$, scaled by the fluid density ρ and the acceleration of gravity g. In this relationship, we regard η and p as functions of the spatial coordinate x, with parametric dependence on time t. It is equally useful to let t vary for fixed x, as would be appropriate for a time series measurement, which results in extra factors of the wave speed $c(k)$, due to the presence of a temporal instead of spatial Fourier transform (F). Different modifications of this formula have been proposed. Most common is the use of a multiplicative correction factor to the transfer function. Bishop & Donelan argue that such correction factors are not necessary [5]. While the above linear relationship is accurate on some scales, it fails to reconstruct the surface elevation accurately in the case of large-amplitude waves, as might be expected. Errors of 15% or more are common, as is shown below.

In order to address the inaccuracies of the linear model, nonlinear methods are required. With the exception of recent work by Constantin & Strauss [4], few nonlinear results are found in the literature. Starting from a traveling wave assumption, Constantin & Strauss obtain different properties and bounds relating the pressure and surface elevation. However, they do not present a reconstruction method to accurately determine one function in terms of the other.

One way to obtain an improved pressure - to - surface elevation map is to use perturbation methods to determine nonlinear correction terms to (1). One such approach is given below, and it is included in our comparisons. Our main focus, however, is the presentation of a new nonlocal nonlinear relationship between the pressure at the bottom of the fluid, and the elevation of a traveling-wave surface that captures the full nonlinearity of Euler's Equations. The advantage of this approach is that

1. it allows for the surface to be reconstructed numerically from any given pressure data for a traveling wave,
2. it provides an environment for direct analysis of the relationship between all physically relevant parameters such as depth and wave speed,
3. and it allows for the quick derivation of perturbation expansions such as the one mentioned above.

In the case of periodic boundary conditions with period L equated to 2π, the relationship is

$$\frac{\sqrt{c^2 - 2g\eta}}{\sqrt{1+\eta_x^2}} = \sum_{k=-\infty}^{\infty} e^{ikx}\hat{P}_k \cosh\left(k\left(\eta + h\right)\right), \qquad (2)$$

where

- $\hat{P}_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx}\sqrt{c^2 - 2p(x)}\,dx$, and
- c is the speed of the traveling wave.

Similarly, the corresponding equation on the whole line (with sufficiently fast decaying boundary conditions as

$x \to \pm\infty$) is

$$\frac{\sqrt{c^2 - 2g\eta}}{\sqrt{1+\eta_x^2}} = \int_{-\infty}^{\infty} e^{ikx} \hat{P}(k) \cosh\left(k\left(\eta + h\right)\right)\, dk, \quad (3)$$

where $\hat{P}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\sqrt{c^2 - 2p(x)}\, dx$.

In what follows, we derive these nonlocal relations and demonstrate their practicality. We compare results from the linear reconstruction model (1) and those from a nonlinear model obtained using perturbation theory with results from the nonlocal formulation using numerical data for traveling waves in shallow water. We demonstrate the superiority of the nonlocal reconstruction formula for a large range of amplitudes. Comparisons with experimental data, as well as extensions including surface tension and more mathematical detail will be presented elsewhere [6].

Equations of Motion

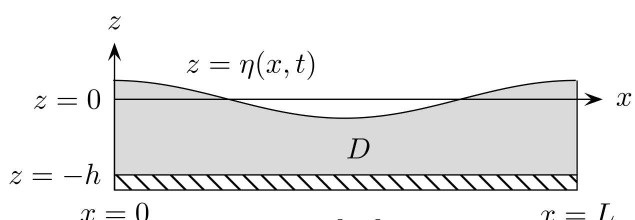

Figure 1: Fluid Domain

Consider Euler's equations describing the dynamics of the surface of an ideal fluid in two dimensions (with a one-dimensional surface). For the periodic problem, this requires us to solve Laplace's equation inside the fluid domain D with periodic boundary conditions in the horizontal x direction (see Figure 1). The equations of motion are

$$\phi_{xx} + \phi_{zz} = 0, \qquad (x,z) \in D, \quad (4)$$

$$\phi_z = 0, \qquad z = -h, \quad (5)$$

$$\eta_t + \eta_x\phi_x = \phi_z, \qquad z = \eta(x,t), \quad (6)$$

$$\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) + g\eta = 0, \qquad z = \eta(x,t), \quad (7)$$

where $\phi(x,z,t)$ represents the velocity potential of the fluid with surface elevation $\eta(x,t)$. In order to relate the pressure at the bottom of the fluid with the surface elevation, we reformulate the problem at both interfaces.

Rewriting the surface elevation

Euler's equations as written above are challenging to work with directly: they are a free-boundary problem with nonlinear boundary conditions specified at an unknown boundary. For the two-dimensional problem (*i.e.*, one-dimensional surface), there are several reformulations that reduce these complications. Ablowitz, Fokas & Musslimani [1] introduced a nonlocal reformulation of the Euler equations, valid for surface waves localized on the whole line or the whole plane. It is essentially trivial to extend this formulation to periodic boundary conditions [7].

To simplify the equations of motion defined at the surface, let $q(x,t)$ represent the velocity potential at the surface $z = \eta(x,t)$. In other words [1],

$$q(x,t) = \phi(x, \eta(x,t), t).$$

Combining the above with equation (6), we have

$$\phi_z = \eta_t + \eta_x\left(q_x - \phi_z\eta_x\right),$$

which allows us to solve directly for ϕ_x, ϕ_z, and ϕ_t in terms of η and q

$$\phi_x = \frac{q_x - \eta_x\eta_t}{1+\eta_x^2}, \quad \phi_z = \frac{\eta_t + \eta_x q_x}{1+\eta_x^2},$$
$$\phi_t = q_t - \frac{\eta_t\left(\eta_t + \eta_x q_x\right)}{1+\eta_x^2}. \quad (8)$$

By substituting the resulting expressions into the dynamic boundary condition (7), we reduce equations (6-7) into the single equation given by:

$$q_t + \frac{1}{2}q_x^2 + g\eta - \frac{1}{2}\cdot\frac{(\eta_t + q_x\eta_x)^2}{1+\eta_x^2} = 0. \quad (9)$$

Transitioning to a traveling coordinate frame moving with speed c by making the change of variables

$$\xi = x - ct, \qquad \tilde{\eta}(\xi,t) = \eta(x-ct,t),$$

$$\tilde{q}(\xi,t) = q(x-ct,t),$$

and looking for stationary solutions, (9) results in

$$-cq_\xi + \frac{1}{2}q_\xi^2 + g\eta - \frac{1}{2}\cdot\frac{\eta_\xi^2(q_\xi - c)^2}{1+\eta_\xi^2} = 0, \quad (10)$$

where we have dropped the tildes for simplicity. Noting that (10) is a quadratic equation for q_ξ, we solve for q_ξ in terms of the surface elevation to find

$$q_\xi - c = \pm\sqrt{(c^2 - 2g\eta)(1+\eta_\xi^2)}, \quad (11)$$

the $\pm$ representing waves traveling to the left or right respectively.

Upon substitution of the equation for q_ξ into the expressions for the velocity potential at the surface, we find:

$$\phi_\xi = c + \frac{\sqrt{c^2 - 2g\eta}}{\sqrt{1+\eta_\xi^2}}, \qquad \phi_z = \frac{\eta_\xi\sqrt{c^2 - 2g\eta}}{\sqrt{1+\eta_\xi^2}}, \quad (12)$$

where we have used $+$, without loss of generality.

Rewriting the Pressure at the Bottom

Returning to the original coordinate system (x, z, t), let $Q(x,t) = \phi(x, -h, t)$ (the velocity potential at the bottom). In the fluid, we know that the Bernoulli equation holds:

$$\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) + gz + \frac{P(x,z,t)}{\rho} = 0.$$

Evaluating this equation at $z = -h$, we find

$$Q_t + \frac{1}{2}Q_x^2 - gh + \frac{P(x,-h,t)}{\rho} = 0. \quad (13)$$

Moving to a traveling coordinate frame as before, we make the change of variables

$$\xi = x - ct, \qquad \tilde{Q}(\xi, t) = Q(x-ct, t),$$
$$\tilde{p}(\xi, t) = P(x - ct, -h, t)/\rho - gh,$$

and look for stationary solutions. Thus, we solve (13) for Q_ξ in terms of the pressure to find

$$\Phi_\xi = c + \sqrt{c^2 - 2p(\xi)}, \quad \Phi_z = 0 \quad (14)$$

where $p(\xi)$ represents the non-static part of the pressure at the bottom in the traveling coordinate frame ($p(\xi) = P(x-ct, -h, t)/\rho - gh$) and we have dropped the tildes for simplicity. For consistency with our previous choice, we choose the $+$ sign again.

Connecting the Elevation and Pressure

Within the domain of the fluid D,

$$\phi_{\xi\xi} + \phi_{zz} = 0,$$

where the boundary conditions given in (12) and (14) must also be satisfied.

We can write the solution of this equation as

$$\phi(\xi, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\xi}\hat{\Psi}(k)\cosh\left(k\left(z+h\right)\right)\, dk,$$

where the boundary condition for ϕ_z at $z = -h$ is satisfied. For the boundary condition at the bottom for ϕ_ξ we find,

$$c\delta(k) + \mathcal{F}\left\{\sqrt{c^2 - 2p}\right\}(k) = ik\hat{\Psi}(k), \quad (15)$$

where $\delta(k)$ is the Dirac delta function. Evaluating $\phi_\xi(\xi, z)$ at the surface ($z = \eta$), we have

$$\phi_\xi(\xi, \eta) = \int_{-\infty}^{\infty} e^{ik\xi} ik\hat{\Psi}(k)\cosh\left(k\left(\eta + h\right)\right)\, dk.$$

Using the boundary conditions given in (12), we find the nonlocal relationship (3). This provides an implicit relationship between the surface elevation of a traveling wave $\eta(x)$ and the pressure measured at the bottom of the fluid $p(x)$. The periodic analogue of this relation (2) is found in a similar way. Nondimensionalizing, we find

$$\sqrt{\frac{c^2 - 2\epsilon\eta}{1 + (\epsilon\mu\eta_\xi)^2}} =$$
$$\int_{-\infty}^{\infty} e^{ik\xi}\mathcal{F}\left(\sqrt{c^2 - 2\epsilon p(\xi)}\right)\cosh\left(\mu k\left(1 + \epsilon\eta\right)\right)\, dk,$$

where $\epsilon = \frac{a}{h}$ (a scales $\eta(\xi)$ and h represents the depth of the fluid), and $\mu = \frac{h}{l}$ (l scales x). This form of the nonlocal relation is useful as a starting point to derive various approximations. A few examples are:

- If we expand η and p as power series in ϵ, we recover the linear relationship (1), as expected: this is a small amplitude expansion, but no assumption is made about long or short waves. Since the relationship is linear, superposition is possible, allowing one to consider wave forms that are not necessarily traveling at constant speed.

- Alternatively, if we balance the relationship between μ and ϵ so that $\mu = \sqrt{\epsilon}$ (this is the KdV approximation, see [1]), we find (up to order ϵ^3)

$$\eta(x) = p - \frac{\epsilon}{2}\frac{\partial^2 p}{\partial\xi^2} +$$
$$\epsilon^2\left(\frac{1}{24}\frac{\partial^4 p}{\partial\xi^4} - p\frac{\partial^2 p}{\partial\xi^2} - \left(\frac{\partial p}{\partial\xi}\right)^2\right) + \mathcal{O}(\epsilon^3). \quad (16)$$

Numerical Comparisons of the Different Approaches

We numerically reconstruct the surface elevation $\eta(x)$ from a solution for the pressure $p(x)$ and corresponding wave speed c, using the periodic nonlocal relation (2). Traveling-wave surface elevations can be computed using methods outlined in [7]. The corresponding pressure $p(x)$ data and wave speed c are computed using similar methods. Next, using only the pressure data and wave speed, (2) is solved for the surface elevation iteratively via Newton's method. As an initial condition for the surface elevation we use the (not very good) Archimedean

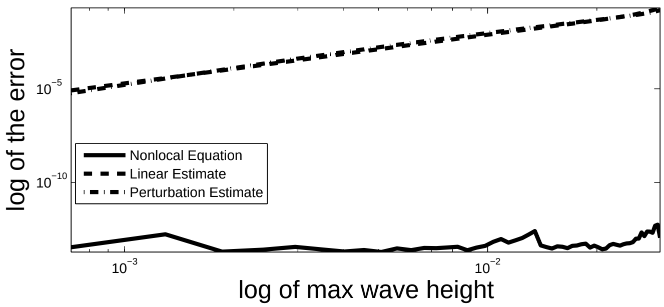

Figure 2: Log/Log plot of the error in the reconstructed surface elevation η_r as a function of the amplitude of η. The error from the linear and perturbation estimates are of the same order and overlap in the above figure.

initial guess $\eta = P/(\rho g)$. Using numerical continuation, we determine the surface elevation for increasingly larger amplitudes in the pressure data.

For $h = .1$, $g = 1$, $\rho = 1$ and $L = 2\pi$, we compute the error

$$error = \frac{||\eta - \eta_r||_\infty}{||\eta||_\infty}$$

where η represents the expected solution and η_r represents the reconstructed solution. This is done for the reconstruction using the nonlocal relationship (2), the linear approximation (1), and a modified form of the perturbation expansion (16) where the linear terms are replaced by the full relationship given in (1).

As seen in the above figure, the error in both approximations grows as the amplitude of the Stokes wave increases. In fact, both approximations show errors exceeding 15% for waves with surface elevations which are only 55% of the limiting wave height as calculated in [3]. The numerical method using the nonlocal equation does not suffer from this amplitude dependence, giving errors never larger than round-off due to machine precision. Even for large amplitude waves, we conclude that (2) provides a practical means to reconstruct the surface elevation from pressure data measured along the bottom of a fluid. Comparisons with experimental data will be reported elsewhere [6].

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