Stability and instability of nonlinear waves:

Introduction

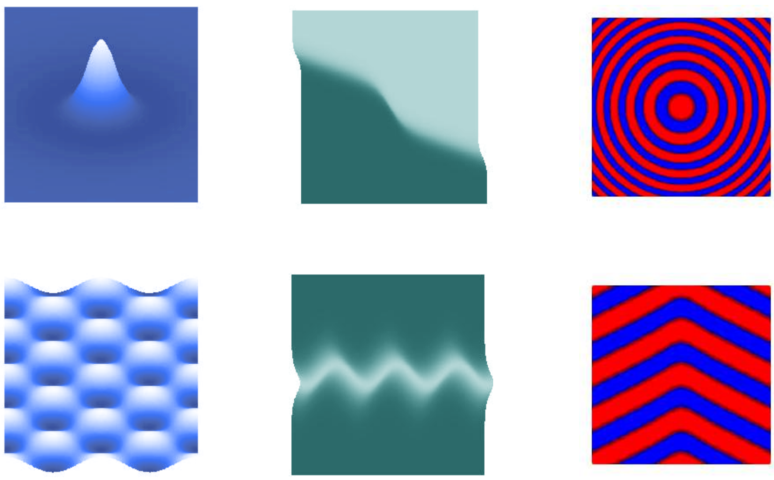
1. Nonlinear Waves
2. Stability problems
3. Stability of pulses and fronts
4. Stability of periodic waves

1 Nonlinear waves

- particular solutions of PDEs:

  – well-defined *spatial and temporal structure* –

- observed in nature, experiments, numerical simulations
- role in the dynamics of PDEs

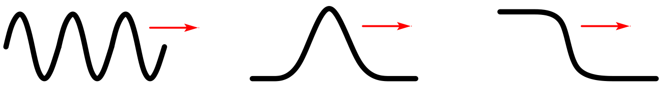


1.1 Examples

One-dimensional waves

- **Standing and travelling waves**

$$u(x - ct) \quad \text{with} \quad c = 0 \;\; \text{or} \;\; c \neq 0$$



periodic wave — pulse/solitary wave — front/kink

  - *found as solutions of an ODE with "time" $x - ct$*

- **Modulated waves**

$, \ldots$

1.2 Questions

- Existence: *no time dependence*
  - solve a steady PDE
  - 1d waves: solve an ODE
- Stability: *add time*
  - solve an initial value problem

  - initial data $u_*(x) + \varepsilon v(x)$

  ? what happens as $t \to \infty$?
- Interactions: *initial value problem*

  - initial data: superposition of two/several nonlinear waves

  ? what happens ?
- *Role in the dynamics of the PDE*

1.3 PDEs

- Dissipative problems: e.g. reaction diffusion systems

$$U_t = D\,\Delta U + F(U)$$

  $U(x,t) \in \mathbb{R}^N$; $\quad t \geq 0$ time; $\quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ space

  D diffusion matrix: $D = \mathrm{diag}\,(d_1, \ldots, d_N) > 0$

  $F(U)$ kinetics (smooth map)

- Dispersive problems: e.g. the Korteweg-de Vries equation

$$u_t = u_{xxx} + uu_x, \quad u(x,t) \in \mathbb{R},\ x \in \mathbb{R},\ t \in \mathbb{R}$$

- *Abstract initial value problem:* $\quad U_t = \mathcal{F}(U,\partial_x)$

  $U(\cdot,t) \in$ space of functions defined on $\mathbb{R}$ ($L^2(\mathbb{R})$, $C_b^0(\mathbb{R})$,...)

2 Stability problems

PDE

$$U_t = \mathcal{F}(U,\partial_x)$$

- Travelling wave $U_*(x - ct)$: set $y = x - ct$

$$-c\partial_y U_* = \mathcal{F}(U_*,\partial_y)$$

- Moving reference frame: $y = x - ct$

$$U_t = c\partial_y U + \mathcal{F}(U,\partial_x)$$

- We regard $U_*(y)$ as an *equilibrium of this infinite-dimensional dynamical system* (in a space X of functions depending upon y: $L^2(\mathbb{R})$, $C_b^0(\mathbb{R})$,...)

- Question: stability of the equilibrium U_* ?

2.1 Finite dimensions

ODE

$$U_t = F(U), \quad U(t) \in \mathbb{R}^n, \quad F : \mathbb{R}^n \to \mathbb{R}^n$$

$$\longrightarrow \quad \text{Equilibrium } U_* \text{ with } F(U_*) = 0$$

Stability

- *Lyapunov stability*

$$\text{“ } \textit{if } U(0) = U_0 \textit{ is close to } U_*, \quad \textit{then } U(t) \textit{ stays close to } U_* \textit{ for all } t \geq 0 \text{ ”}$$

Definition. An equilibrium U_* *is stable* (in the sense of Lyapunov) if

$$\forall\, \epsilon > 0 \;\; ex.\ \delta > 0 : \quad \forall\, U_0 = U(0),\ \|U_0 - U_*\| \leq \delta \implies \|U(t) - U_*\| \leq \epsilon.$$

An equilibrium U_* *is unstable* if it is not stable.

- *Asymptotic stability*

$$\text{“ } \textit{if } U(0) = U_0 \textit{ is close to } U_*, \quad \textit{then } U(t) \to U_* \textit{ as } t \to +\infty \text{ ”}$$

Definition. An equilibrium U_* *is asymptotically stable* if it is stable and

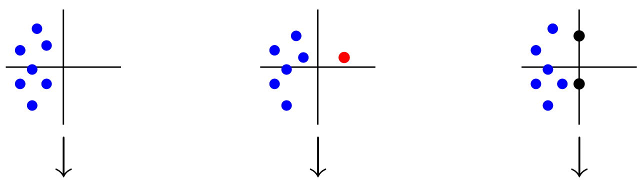
$$ex.\ \delta > 0 : \quad \forall\, U_0 = U(0),\ \|U_0 - U_*\| \leq \delta \implies \|U(t) - U_*\| \to 0,\ t \to +\infty.$$

Linear systems

$$U_t = AU, \qquad A \textit{ is an } n \times n \textit{ matrix}$$

Stability of $U_* = 0$ is determined by the eigenvalues of A:

$$\mathrm{spec}(A) = \{\lambda \in \mathbb{C} \;/\; (\lambda - A) \textit{ not } \text{invertible}\}$$



asymptotically stable — unstable — stable / unstable ($m_a(\lambda) = m_g(\lambda),\ \forall\, \lambda \in \mathrm{i}\mathbb{R}$/otherwise)

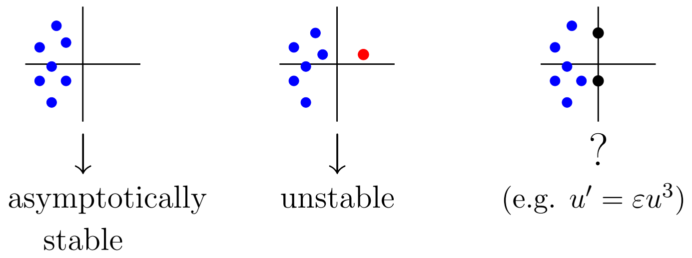
Nonlinear systems

$$U_t = F(U), \qquad F : \mathbb{R}^n \to \mathbb{R}^n \text{ smooth}$$

Equilibrium U_* with $F(U_*) = 0$.

- Linearized system: $V_t = F'(U_*)V$

Eigenvalues of $F'(U_*)$:



2.2 Notions of stability

PDE:

$$U_t = F(U), \quad U(t) \in X, \quad \text{Banach space}$$

- Equilibrium U_* with $F(U_*) = 0$
- Linearized problem: $V_t = F'(U_*)V$, $V(t) \in X$

  $F'(U_*)$ (closed) linear operator in the space X

- **Nonlinear stability** – *as before*

$$\|U(0) - U_*\| \leq \delta \implies \|U(t) - U_*\| \leq \epsilon, \ t \geq 0$$

$\longrightarrow$ asymptotic nonlinear stability

$$\|U(0) - U_*\| \leq \delta \implies \|U(t) - U_*\| \to 0$$

- **Linear stability:** *stability of the equilibrium* $V_* = 0$ *of the linearized equation* $V_t = F'(U_*)V$

$$\|V(0)\| \leq \delta \implies \|V(t)\| \leq \epsilon, \ t \geq 0$$

$\longrightarrow$ asymptotic linear stability

$$\|V(0)\| \leq \delta \implies \|V(t)\| \to 0$$

- **Spectral stability:** *spectrum of the linear operator* $F'(U_*)$

$$\operatorname{spec}(F'(U_*)) \subset \{\lambda \in \mathbb{C} \,/\, \operatorname{Re}\lambda \leq 0\}$$

$\longrightarrow$ neutral stability: $\operatorname{spec}(F'(U_*)) \subset \mathrm{i}\mathbb{R}$

Finite dimensions

- spectral stability is necessary for linear and nonlinear stability
- spectral stability $\not\Longrightarrow$ linear stability $\quad (\lambda \in \mathrm{i}\mathbb{R},\ m_{\mathrm{a}}(\lambda) \neq m_{\mathrm{g}}(\lambda))$
- linear stability $\not\Longrightarrow$ nonlinear stability $\quad (u' = \varepsilon u^3)$

Remark.

- nonlinear stability: many different methods
- linear stability: semi-group theory
- spectral stability: spectral theory for linear differential operators, dynamical systems

3 Stability of pulses and fronts

Spectral stability of pulses and fronts

- *Study of the spectrum of the linearization* $L := F'(U_*)$
- Closed linear operator $L : D(L) \subset X \to X$, X Banach space
  - resolvent set: $\rho(L) = \{\lambda \in \mathbb{C} \ / \ \lambda - L \ \text{ invertible}\}$
  - spectrum: $\operatorname{spec}(L) = \mathbb{C} \setminus \rho(L)$

Allen-Cahn equation

$$u_t = u_{xx} + u - u^3$$

- Steady waves $\longrightarrow$ ODE: $u_{xx} = -u + u^3$
  - *phase portrait* $\longrightarrow$ *2 heteroclinic orbits*
  - *translation invariance* $\longrightarrow$ *2 one-parameter families of* **fronts**

$$u_{\pm}(x+\alpha) = \tanh\left(\frac{x+\alpha}{\sqrt{2}}\right), \quad \alpha \in \mathbb{R}$$

- Linearization about the front $u_\pm$

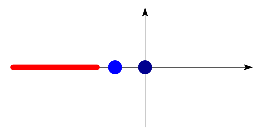
$$v_t = v_{xx} + v - 3u_\pm^2 v$$

Linear operator

$$\mathcal{L}v = v_{xx} - 2v + 3\,\mathrm{sech}^2\left(\frac{x}{\sqrt{2}}\right)v$$

Closed linear operator in $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$.

- Spectrum of $\mathcal{L}$



  - $(-\infty, -2] \longleftrightarrow$ *essential spectrum*
  - $\{-\frac{3}{2}, 0\}$ simple eigenvalues $\longleftrightarrow$ *point spectrum*
  - 0 *is an eigenvalue due to translational invariance*

$$\mathcal{L}(\partial_x u_\pm) = 0 \quad !$$

! nonlinear stability $\longrightarrow$ *orbital stability*

Korteweg-de Vries equation

$$u_t = u_{xxx} + 3uu_x$$

- Travelling waves $u(x + ct)$: set $y = x + ct$

$\Longrightarrow$ ODE: $cu_y = u_{yyy} + 3uu_y \Longrightarrow u_{yy} = cu - \frac{3}{2}u^2 + const.$

  - *phase portrait* $\longrightarrow$ *homoclinic orbit to zero, $c > 0$*
  - *translation invariance*

$\longrightarrow$ *one-parameter family of* **solitary waves**

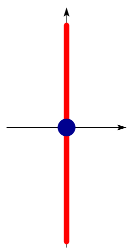
$$u_c(y + \alpha) = c\,\mathrm{sech}^2\left(\frac{\sqrt{c}(y + \alpha)}{2}\right), \quad \alpha \in \mathbb{R}$$

- Linearization about the solitary wave u_c

$$\mathcal{L}v = v_{yyy} - cv_y + 3\,(u_c v)_y$$

Closed linear operator in $L^2(\mathbb{R})$ with domain $H^3(\mathbb{R})$.

- Spectrum of $\mathcal{L}$



– 0 *is an eigenvalue due to translational invariance*

$$\mathcal{L}(\partial_y u_c) = 0 \quad !$$

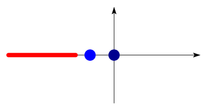
– 0 *is a* ***double*** *eigenvalue due to Galilean invariance*

$$(c \mapsto c + \alpha,\ u \mapsto u + \tfrac{\alpha}{3})$$

$$\mathcal{L}(\partial_c u_c) = \partial_y u_c \quad !$$

Remark.

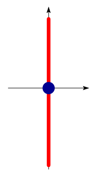
Allen-Cahn equation



$\sim$ reaction-diffusion systems
$\sim$ dissipative problems

$\rightarrow$ *sectorial operator*
$\rightarrow$ *holomorphic semi-group*

KdV equation



$\sim$ Hamiltonian systems
$\sim$ dispersive problems

$\rightarrow$ C^0 – *semi-group*

3.1 *Essential* and point spectra

Definitions. Let X be a Banach space and let $L : D(L) \subset X \to X$ be a closed linear operator.

- L **is Fredholm** if it has closed range $R(L)$ and

$$\dim \ker(L) < +\infty, \quad \operatorname{codim} R(L) < +\infty.$$

  Fredholm index: $\operatorname{ind}(L) = \dim \ker(L) - \operatorname{codim} R(L)$

- **Spectrum:** $\operatorname{spec}(L) = \{\lambda \in \mathbb{C} \ / \ \lambda - L \ \text{ is } \textit{not} \text{ invertible}\}$

- **Essential spectrum:**

$$\operatorname{spec}_{\text{ess}}(L) = \{\lambda \in \mathbb{C} \ / \ \lambda - L \ \text{ is } \textit{not} \text{ Fredholm with index zero}\}$$

- **Point spectrum:** $\operatorname{spec}_{\text{pt}}(L) = \operatorname{spec}(L) \setminus \operatorname{spec}_{\text{ess}}(L)$

$$\operatorname{spec}_{\text{pt}}(L) = \{\lambda \in \mathbb{C} \ / \ \lambda - L \ \text{ is Fredholm with index zero } \textit{and not} \text{ invertible}\}$$

Definitions. A nonlinear wave u_* is called

- **spectrally stable** if
$$\mathrm{spec}(\mathcal{L}_*) \subset \{\lambda \in \mathbb{C} \ / \ \mathrm{Re}\,\lambda \leq 0\}$$
- **essentially spectrally stable** if
$$\mathrm{spec}_{\mathrm{ess}}(\mathcal{L}_*) \subset \{\lambda \in \mathbb{C} \ / \ \mathrm{Re}\,\lambda \leq 0\}$$

Essential spectra

Theorem. [Kato (1980), Chapter IV, Theorem 5.35]

Assume that L is a relatively compact perturbation[1] *of a closed operator $L_0 : D(L_0) \subset X \to X$. Then*

- *$\lambda - L$ is Fredholm $\iff$ $\lambda - L_0$ is Fredholm and*
$$\mathrm{ind}(\lambda - L) = \mathrm{ind}(\lambda - L_0).$$
- $\mathrm{spec}_{\mathrm{ess}}(L) = \mathrm{spec}_{\mathrm{ess}}(L_0)$

Reaction-diffusion systems [Sandstede & Scheel, 20XX; Fiedler & Scheel, 2003]

$$U_t = DU_{xx} + F(U), \qquad U(x,t) \in \mathbb{R}^N$$

Hypothesis. *Existence of a (smooth)* **travelling wave**

$$U(x,t) = U_*(x - ct), \qquad c \in \mathbb{R}$$

satisfying

$$U_*(y) \to U_\pm, \quad \text{as } y \to \pm\infty$$

(U_* is either a pulse $U_- = U_+$, or a front $U_- \neq U_+$).

- Set $y = x - ct$ and find the **linearized problem**
$$V_t = DV_{yy} + cV_y + F'(U_*)V$$

Spectral problem

Find the essential spectrum of the linear operator?

$$\mathcal{L}_* V = DV_{yy} + cV_y + F'(U_*)V, \quad U_*(y) \to U_\pm, \ \text{as } y \to \pm\infty$$

- *$\mathcal{L}_*$ is a closed linear operator*

 in $X := L^2(\mathbb{R};\mathbb{R}^N)$ with domain $X^2 := H^2(\mathbb{R};\mathbb{R}^N)$ [2]

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[1] Relatively compact perturbation if

- $D(L_0) \subset D(L)$
- for any $(u_n) \subset D(L_0)$ with (u_n) and $(L_0 u_n)$ bounded $\Longrightarrow$ (Lu_n) contains a convergent subsequence.

[2] Same results when replacing $L^2(\mathbb{R};\mathbb{R}^N)$ by $L^p(\mathbb{R};\mathbb{R}^N)$, $1 < p < \infty$, or $C^0_{\mathrm{b,unif}}$.

- Use the result in the theorem and the **asymptotic operator**

$$\mathcal{L}_\infty V = DV_{yy} + cV_y + F'(U_\infty)V$$

in which

$$U_\infty(y) = \begin{cases} U_+ & \text{if } y > 0 \\ U_- & \text{if } y < 0 \end{cases}$$

Perturbation result

Theorem.

- $\mathcal{L}_*$ *is a* **relatively compact perturbation** *of* $\mathcal{L}_\infty$.
- $\lambda - \mathcal{L}_*$ *is Fredholm* $\iff$ $\lambda - \mathcal{L}_\infty$ *is Fredholm and*

$$\operatorname{ind}(\lambda - \mathcal{L}_*) = \operatorname{ind}(\lambda - \mathcal{L}_\infty).$$

- $\operatorname{spec}_{\text{ess}}(\mathcal{L}_*) = \operatorname{spec}_{\text{ess}}(\mathcal{L}_\infty)$

Examples

- Allen-Cahn equation: $\mathcal{L}_\infty v = v_{xx} - 2v$

$\longrightarrow$ operator with constant coefficients: *Fourier analysis*

$\longrightarrow$ $\operatorname{spec}_{\text{ess}}(\mathcal{L}_*) = \{-k^2 - 2 \;/\; k \in \mathbb{R}\} = (-\infty, -2]$

- Korteweg-de Vries equation: $\mathcal{L}_\infty v = v_{yyy} - cv_y$

$\longrightarrow$ operator with constant coefficients: *Fourier analysis*

$\longrightarrow$ $\operatorname{spec}_{\text{ess}}(\mathcal{L}_*) = \{-\mathrm{i}k^3 - c\mathrm{i}k \;/\; k \in \mathbb{R}\} = \mathrm{i}\mathbb{R}$

Essential spectra for pulses

Pulse $U_*(y) \to U_\infty$, as $y \to \pm\infty$ $(U_\infty = U_- = U_+)$

- asymptotic operator: constant coefficients

$$\mathcal{L}_\infty V = DV_{yy} + cV_y + F'(U_\infty)V$$

- Fourier transform

$$\widehat{\mathcal{L}}_\infty V = -k^2 DV + \mathrm{i}kcV + F'(U_\infty)V$$

- $\lambda - \widehat{\mathcal{L}}_\infty$ invertible $\iff$ $\det\left(\lambda + k^2 D - \mathrm{i}kc - F'(U_\infty)\right) \neq 0, \forall\, k \in \mathbb{R}$

$$\operatorname{spec}_{\text{ess}}(\mathcal{L}_*) = \{\lambda \in \mathbb{C} \;/\; \det\big(\lambda + k^2 D - \mathrm{i}kc - F'(U_\infty)\big) = 0, \text{ for some } k \in \mathbb{R}\}$$

Essential spectra for fronts

Front $U_*(y) \to U_\pm$, as $y \to \pm\infty$ $(U_- \neq U_+)$

- Asymptotic operator:

$$\mathcal{L}_\infty V = DV_{yy} + cV_y + F'(U_\infty)V$$

  in which

$$U_\infty(y) = \begin{cases} U_+ \text{ if } y > 0 \\ U_- \text{ if } y < 0 \end{cases}$$

- Spectral problem: $\lambda V = \mathcal{L}_\infty V$, $\lambda \in \mathbb{C}$

$$\lambda V = DV_{yy} + cV_y + F'(U_\infty)V, \quad \lambda \in \mathbb{C}$$

- *Regard the spectral problem as an ODE!* [Henry, 1981; Palmer, 1988; ...]

Dynamical system

$$\lambda V = \mathcal{L}_\infty V = DV_{yy} + cV_y + F'(U_\infty)V, \quad \lambda \in \mathbb{C}$$

- **First order ODE**

$$\begin{aligned} V_y &= W \\ W_y &= D^{-1}\left(-cW - F'(U_\infty)V + \lambda V\right) \end{aligned}$$

- **First order differential operator**

$$\mathcal{M}(\lambda)\begin{pmatrix} V \\ W \end{pmatrix} := \frac{\mathrm{d}}{\mathrm{d}y}\begin{pmatrix} V \\ W \end{pmatrix} - \mathcal{A}(y,\lambda)\begin{pmatrix} V \\ W \end{pmatrix} = 0$$

  $\mathcal{M}(\lambda)$ is a closed linear operator in $H^1(\mathbb{R};\mathbb{R}^N) \times L^2(\mathbb{R};\mathbb{R}^N)$ with domain $H^2(\mathbb{R};\mathbb{R}^N) \times H^1(\mathbb{R};\mathbb{R}^N)$

Equivalence

$$\mathcal{L}_\infty V = DV_{yy} + cV_y + F'(U_\infty)V, \quad \mathcal{L}_\infty : H^2(\mathbb{R};\mathbb{R}^N) \to L^2(\mathbb{R};\mathbb{R}^N)$$

$$\mathcal{M}(\lambda) := \frac{\mathrm{d}}{\mathrm{d}y} - \mathcal{A}(y,\lambda), \quad \mathcal{A}(y,\lambda)\begin{pmatrix} V \\ W \end{pmatrix} = \begin{pmatrix} W \\ D^{-1}\left(-cW - F'(U_\infty)V + \lambda V\right) \end{pmatrix}$$

$$\mathcal{M}(\lambda) : H^2(\mathbb{R};\mathbb{R}^N) \times H^1(\mathbb{R};\mathbb{R}^N) \to H^1(\mathbb{R};\mathbb{R}^N) \times L^2(\mathbb{R};\mathbb{R}^N)$$

Proposition.

$\lambda - \mathcal{L}_\infty$ is Fredholm (resp. invertible) $\Longleftrightarrow$ $\mathcal{M}(\lambda)$ is Fredholm (resp. invertible)

$$\mathrm{ind}(\lambda - \mathcal{L}_\infty) = \mathrm{ind}(\mathcal{M}(\lambda))$$

Fredholm index of $\mathcal{M}(\lambda)$

$$\mathcal{M}(\lambda) := \frac{\mathrm{d}}{\mathrm{d}y} - \mathcal{A}(y,\lambda), \quad \mathcal{A}(y,\lambda) = \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(-F'(U_\infty)+\lambda) & -cD^{-1} \end{pmatrix}$$

- Write

$$\mathcal{A}(y,\lambda) = \begin{cases} A_+(\lambda), & y>0 \\ A_-(\lambda), & y<0 \end{cases}, \quad A_\pm(\lambda) = \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(-F'(U_\pm)+\lambda) & -cD^{-1} \end{pmatrix}$$

  – $A_\pm(\lambda)$ are $2N \times 2N$ matrices with constant coefficients!

- Solve *two ODEs with constant coefficients* [3]

  $\longrightarrow$ $y>0$: $\frac{\mathrm{d}}{\mathrm{d}y} - A_+(\lambda)$

  $\longrightarrow$ $y<0$: $\frac{\mathrm{d}}{\mathrm{d}y} - A_-(\lambda)$

- **Eigenvalues of $A_\pm(\lambda)$: $\nu_\pm^j(\lambda)$, $j = 1, \ldots, 2N$**

- **Morse indices of $A_\pm(\lambda)$:**

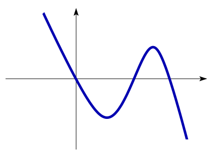
$$\mathrm{i}_\pm(\lambda) := \sum_{\mathrm{Re}\,\nu_\pm^j(\lambda)>0} m_\mathrm{a}(\nu_\pm^j(\lambda)) \;=\; \textit{dimension of the unstable subspace}$$

Theorem.

- *$\mathcal{M}(\lambda)$ is Fredholm $\Longleftrightarrow$ $\mathrm{Re}\,\nu_\pm^j(\lambda) \neq 0$, for all j.*
- $\mathrm{ind}(\mathcal{M}(\lambda)) = \mathrm{i}_-(\lambda) - \mathrm{i}_+(\lambda)$

Example: *bistable reaction-diffusion equation*

$$u_t = u_{xx} + f(u)$$



$$f(0) = f(1) = 0$$
$$f'(1) < f'(0) < 0$$
$$\int_0^1 f(u)\mathrm{d}u < 0$$

[3] dynamics determined by the eigenvalues of $A_\pm(\lambda)$

- *Existence* of a unique travelling front, speed $c_* > 0$

$$u_*(x - c_*t) \longrightarrow \begin{cases} 0, & y \to -\infty \\ 1, & y \to +\infty \end{cases}$$

- *Linear operator:* set $y = x - c_*t$

$$\mathcal{L}_* v = v_{yy} + c_* v_y + f'(u_*)v$$

- *Asymptotic operator*

$$\mathcal{L}_\infty v = v_{yy} + c_* v_y + f_\infty v, \quad f_\infty = \begin{cases} f'(0), & y < 0 \\ f'(1), & y > 0 \end{cases}$$

equivalently,

$$\mathcal{L}_\infty = \begin{cases} \mathcal{L}_-, & y < 0 \\ \mathcal{L}_+, & y > 0 \end{cases}, \quad \begin{cases} \mathcal{L}_- = \partial_{yy} + c_*\partial_y + f'(0) \\ \mathcal{L}_+ = \partial_{yy} + c_*\partial_y + f'(1) \end{cases}$$

- *Asymptotic matrices*

$$\lambda v = \mathcal{L}_\infty v \quad \Longleftrightarrow \quad \begin{cases} v_y &= w \\ w_y &= -c_* w - f_\infty v + \lambda v \end{cases}$$

$$\mathcal{M}(\lambda) = \frac{\mathrm{d}}{\mathrm{d}y} - \mathcal{A}(y, \lambda), \quad \mathcal{A}(y, \lambda) = \begin{pmatrix} 0 & 1 \\ -f_\infty + \lambda & -c_* \end{pmatrix}$$

- *Eigenvalues of* $A_-(\lambda)$

$$A_-(\lambda) = \begin{pmatrix} 0 & 1 \\ -f'(0) + \lambda & -c_* \end{pmatrix}, \quad A_+(\lambda) = \begin{pmatrix} 0 & 1 \\ -f'(1) + \lambda & -c_* \end{pmatrix}$$

The eigenvalues $\nu_-^j(\lambda),\ j = 1, 2$ satisfy

$$\det(\nu - A_-(\lambda)) = 0 \quad \Longleftrightarrow \quad \nu^2 + \nu c_* + f'(0) - \lambda = 0$$

- *Purely imaginary eigenvalues* $\nu = \mathrm{i}k$, $k \in \mathbb{R}$, *satisfy*

$$-k^2 + \mathrm{i}kc_* + f'(0) = \lambda$$

$\longrightarrow$ $A_-(\lambda)$ *has purely imaginary eigenvalues for* $\lambda \in \mathbb{C}$ *on the curve*

$$\Gamma_- \ : \ k \mapsto -k^2 + \mathrm{i}kc_* + f'(0)$$

$\longrightarrow$ $\mathcal{M}(\lambda)$ *is not Fredholm for* $\lambda \in \mathbb{C}$ *on the curve* Γ_-

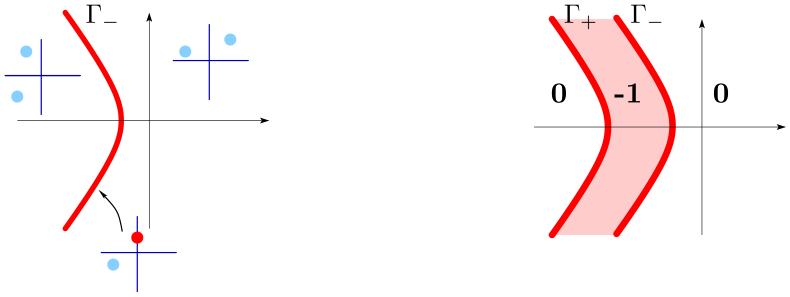


Figure 1: *Location of the eigenvalues* $\nu_-^j(\lambda)$ *of* $A_-(\lambda)$ *(left), Fredholm indices of* $\mathcal{M}(\lambda)$*, and spectrum of* $\mathcal{L}_*$ *(right)* $(\mathrm{ind}(\mathcal{M}(\lambda)) = \mathrm{i}_-(\lambda) - \mathrm{i}_+(\lambda))$

Remarks.

- *The eigenvalues* $\nu_\pm^j$ *satisfy the dispersion relation*

$$\det\left(\nu - A_\pm(\lambda)\right) = 0 \quad\Longleftrightarrow\quad \nu^2 + \nu c_* + f_\infty = \lambda$$

and can be computed directly from the asymptotic operators

$$\mathcal{L}_\infty = \begin{cases} \mathcal{L}_-, & y < 0 \\ \mathcal{L}_+, & y > 0 \end{cases}, \quad \begin{cases} \mathcal{L}_- = \partial_{yy} + c_*\partial_y + f'(0) \\ \mathcal{L}_+ = \partial_{yy} + c_*\partial_y + f'(1) \end{cases}$$

- *The curves* $\Gamma_\pm$ *are precisely the spectra of the operators with constant coefficients* $\mathcal{L}_\pm$*.*
  - *The operators* $\lambda - \mathcal{L}_*$ *are Fredholm precisely outside* $\Gamma_- \cup \Gamma_+$*.*

    We call $\Gamma_\pm$ **Fredholm borders**.
  - *The Fredholm index changes when crossing* $\Gamma_\pm$*.*
  - *The Fredholm index is zero to the right of the "first" curve* $\Gamma_\pm$*.*

3.2 Dispersion relation, Fredholm borders, and group velocity

Dispersion relation

- **Asymptotic operator**

$$\mathcal{L}_\infty = \begin{cases} \mathcal{L}_-, & y < 0 \\ \mathcal{L}_+, & y > 0 \end{cases}, \quad \begin{cases} \mathcal{L}_- = DV_{yy} + c_*V_y + F'(U_-)V \\ \mathcal{L}_+ = DV_{yy} + c_*V_y + F'(U_+)V \end{cases}$$

  – *First order operator:* $\mathcal{M}(\lambda) := \frac{\mathrm{d}}{\mathrm{d}y} - \mathcal{A}(y,\lambda)$

$$\mathcal{A}(y,\lambda) = \begin{cases} A_+(\lambda), & y > 0 \\ A_-(\lambda), & y < 0 \end{cases}, \quad A_\pm(\lambda) = \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}\left(-F'(U_\pm) + \lambda\right) & -c_*D^{-1} \end{pmatrix}$$

– *Eigenvalues* $\nu^j_\pm(\lambda)$ *of* $A_\pm(\lambda)$ *satisfy*

$$\det(\nu - A_\pm(\lambda)) = 0 \quad \Longleftrightarrow \quad \det\left(D\nu^2 + c_*\nu + F'(u_\pm) - \lambda\right) = 0$$

- **Dispersion relation**

$$d_\pm(\lambda,\nu) := \det\left(D\nu^2 + c_*\nu + F'(u_\pm) - \lambda\right)$$

  – $\lambda \;\rightarrow\;$ **temporal eigenvalue**

  – $\nu \;\rightarrow\;$ **spatial eigenvalue**

Fredholm borders

- *Spatial eigenvalues* $\nu = \mathrm{i}k$ *give the spectra of* $\mathcal{L}_\pm$

$$\Gamma_\pm = \{\lambda \;/\; d_\pm(\lambda, \mathrm{i}k) = 0,\;\; k \in \mathbb{R}\}$$

  - *The operators* $\lambda - \mathcal{L}_*$ *are Fredholm precisely outside* $\Gamma_- \cup \Gamma_+$.

    We call $\Gamma_\pm$ **Fredholm borders.**
  - *The Fredholm index changes when crossing* $\Gamma_\pm$.
  - *The Fredholm index is zero to the right of the "first" curve* $\Gamma_\pm$.

- $d_\pm$ are polynomials in λ of degree N

  $\longrightarrow$ *N complex roots* $\lambda^\ell_\pm(\mathrm{i}k)$, $\ell = 1, \ldots, N$

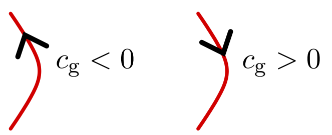
  $\longrightarrow$ *the Fredholm borders consist of* $2N$ *algebraic curves*

$$\Gamma_\pm = \bigcup_{\ell=1}^{N} \Gamma^\ell_\pm, \quad \Gamma^\ell_\pm = \{\lambda^\ell_\pm(\mathrm{i}k),\;\; k \in \mathbb{R}\}$$

Group velocity[4]

$$c^\ell_{\mathrm{g},\pm}(k) = \mathrm{Im}\left(\frac{\mathrm{d}\lambda^\ell_\pm(-\mathrm{i}k)}{\mathrm{d}k}\right)$$

- *Oriented curves* $\Gamma^\ell_\pm$

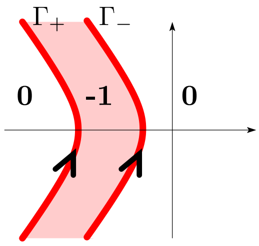


[4]Similar to the notion of group velocity in the dispersive case.

Proposition. *The Fredholm index*

- *increases by one upon crossing a curve* Γ_+^ℓ *from right to the left with respect to its orientation;*
- *decreases by one upon crossing a curve* Γ_-^ℓ *from right to the left with respect to its orientation.*

Example:



3.3 Weighted spaces

Exponential weights

- *The essential spectrum is the same in function spaces such as* L^p, C^k, H^k, ...
  – *for* **translation invariant norms**
- *The location of the Fredholm borders changes in spaces with* **exponential weights:**

$$L^2_{\underline{\eta}}(\mathbb{R}) = \{u \in L^2_{\mathbf{loc}} \;/\; \int_{\mathbb{R}_+} |\mathrm{e}^{\eta_+ y} u(y)|^2 + \int_{\mathbb{R}_-} |\mathrm{e}^{\eta_- y} u(y)|^2 < +\infty\}$$

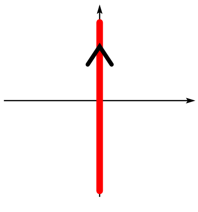
  - exponential rates $\underline{\eta} = (\eta_-, \eta_+) \in \mathbb{R}^2$
  - $\eta_- < 0 \;\longrightarrow\;$ disturbances decay exponentially at $-\infty$
    $\eta_+ > 0 \;\longrightarrow\;$ disturbances decay exponentially at $+\infty$
  - *suitable decay rates for nonlinearities:* $\eta_- < 0$ *and* $\eta_+ > 0$.

Example: *Transport equation*

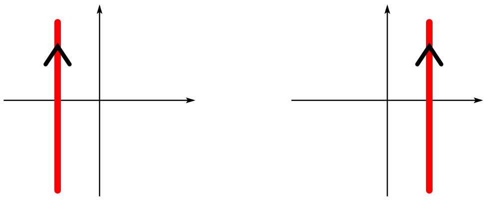
$$u_t = u_y$$

- *Spectrum in* $L^2(\mathbb{R})$

$$d(\lambda, \nu) = \lambda - \nu$$
$$\longrightarrow \quad \lambda(\mathrm{i}k) = \mathrm{i}k$$
$$\longrightarrow \quad c_{\mathrm{g}} = -1$$



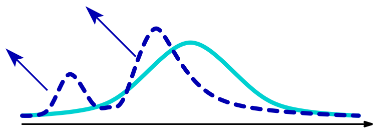
- *Weight* $e^{\eta y}$*:* $e^{\eta y}u(y) \in L^2(\mathbb{R}) \quad \longrightarrow \quad$ *operator* $\partial_y - \eta$
- *Spectrum in* $L^2_\eta(\mathbb{R})$*:* $\lambda(\mathrm{i}k) = \mathrm{i}k - \eta$



$\eta > 0$ $\qquad$ $\eta < 0$

Essential spectrum in exponentially weighted spaces

- $c_g < 0 \;\longrightarrow\;$ transport to the left $\;\longrightarrow\;$ weight $e^{\eta y}$, $\eta > 0$, *moves the spectrum to the left*



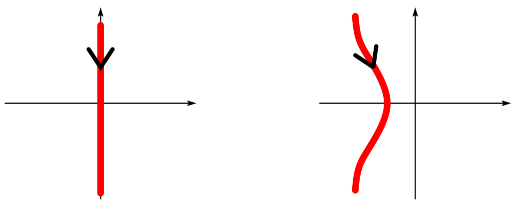
- $c_g > 0 \;\longrightarrow\;$ transport to the right $\;\longrightarrow\;$ weight $e^{\eta y}$, $\eta < 0$, *moves the spectrum to the left*

Example: *KdV equation*

$$\mathcal{L}_\infty v = v_{yyy} - cv_y \quad \Longrightarrow \quad d(\lambda,\nu) = \lambda - \nu^3 + c\nu, \quad c > 0$$

$$\lambda(-\mathrm{i}k) = \mathrm{i}k^3 + \mathrm{i}ck \quad \Longrightarrow \quad c_g = 3k^2 + c > 0$$

$\longrightarrow$ *weight* $e^{\eta y}$, $\eta < 0$*:* [Pego& Weinstein, 1994]



Theorem.

- *The essential spectrum in the weighted space* $L^2_{\underline{\eta}}(\mathbb{R})$ *is determined by the translated dispersion relations*

$$d^{\underline{\eta}}_\pm(\lambda,\nu) := d_\pm(\lambda, \nu - \eta_\pm).$$

- *Small weights* $\underline{\eta} \sim 0$ *change the Fredholm borders*

$$\Gamma^\ell_\pm(\eta_\pm) = \{\lambda^\ell_\pm(\mathrm{i}k - \eta_\pm),\ k \in \mathbb{R}\}$$

according to the formula

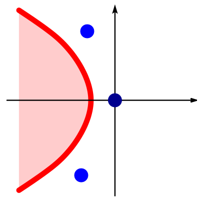
$$\frac{\partial \operatorname{Re} \lambda^\ell_\pm(-\mathrm{i}k - \eta_\pm)}{\partial \eta_\pm}\bigg|_{\eta_\pm = 0} = c^\ell_{g,\pm}(k).$$

3.4 Instability mechanisms

Essential stability

- **Spectral stability**

  $\mathrm{spec}(\mathcal{L}_*) \subset \{\lambda \in \mathbb{C} \;/\; \mathrm{Re}\,\lambda \leq 0\}$

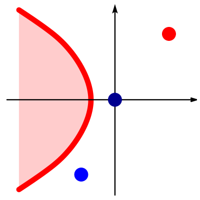


- **Essential spectral stability**

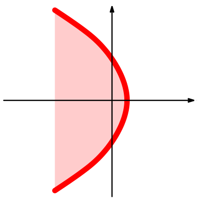
  $\mathrm{spec}_{\mathrm{ess}}(\mathcal{L}_*) \subset \{\lambda \in \mathbb{C} \;/\; \mathrm{Re}\,\lambda \leq 0\}$

  ex. $\lambda \in \mathrm{spec}_{\mathrm{pt}}, \; \mathrm{Re}\,\lambda > 0$

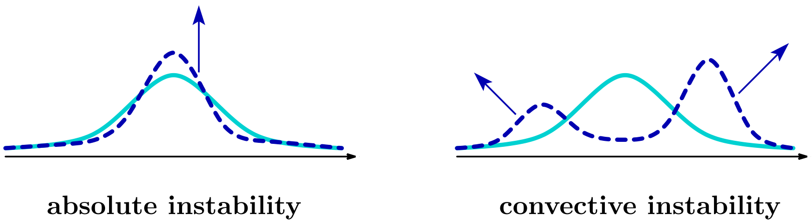
  $\longrightarrow$ finite-dimensional space of *exponentially growing disturbances*



Essential instability



What happens?



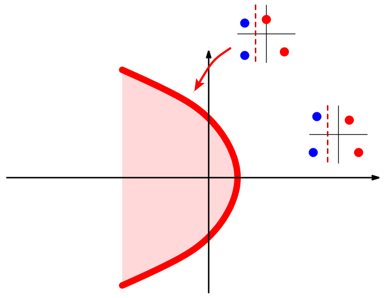
How to detect these instabilities?

[Briggs, 1964; Brevdo & Bridges, 1996; Sandstede & Scheel, 2000]

Roots of the dispersion relation $d(\lambda,\nu)=0$

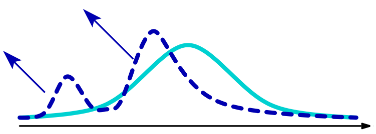
- **Essential instability:** unstable essential spectrum

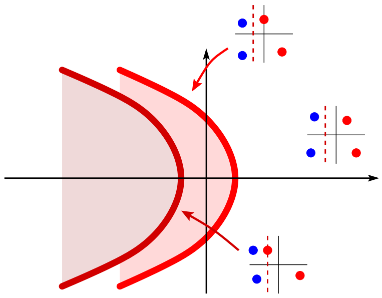
  $\longrightarrow$ *ex. root* $\nu \in \mathrm{i}\mathbb{R}$*, for* $\operatorname{Re}\lambda > 0$



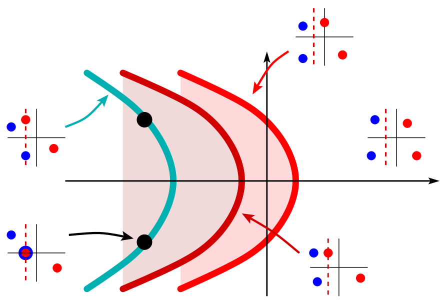
- **Transient instability:** stable essential spectrum in L^2_η

  $\longrightarrow$ *ex.* η *so that the roots* $\nu+\eta \notin \mathrm{i}\mathbb{R}$*, for* $\operatorname{Re}\lambda > 0$





Absolute spectrum



Absolute spectrum

- Eigenvalues $\nu^\ell_\pm(\lambda)$ of $A_\pm(\lambda)$

$$\operatorname{Re}\nu^1_\pm(\lambda) \geq \operatorname{Re}\nu^2_\pm(\lambda) \geq \ldots \geq \operatorname{Re}\nu^{2N}_\pm(\lambda)$$

- For $\operatorname{Re}\lambda >> 1$ (to the "right" of the essential spectrum) we have $i_-(\lambda) = i_+(\lambda) =: i_\infty(\lambda)$, so that we can order the eigenvalues

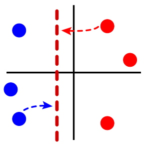
$$\operatorname{Re}\nu^1_\pm(\lambda) \geq \ldots \geq \operatorname{Re}\nu^{i_\infty}_\pm(\lambda) > 0 > \operatorname{Re}\nu^{i_\infty-1}_\pm(\lambda) \geq \ldots \geq \operatorname{Re}\nu^{2N}_\pm(\lambda)$$

Definition. $\lambda \in \mathrm{spec}_{\mathrm{abs}}(\mathcal{L}_*)$ if

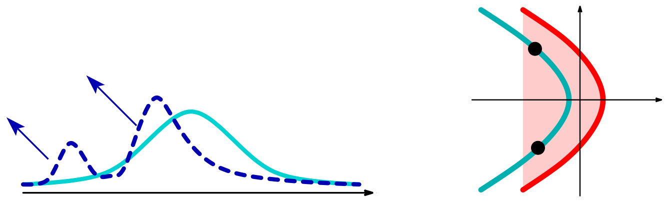
$$\mathrm{Re}\,\nu_-^{i_\infty}(\lambda) = \mathrm{Re}\,\nu_-^{i_\infty -1}(\lambda)$$

or

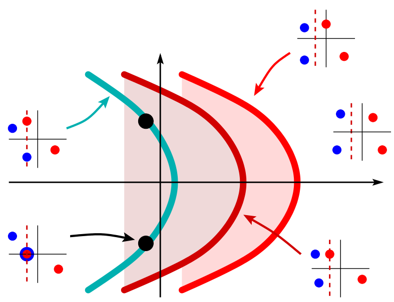
$$\mathrm{Re}\,\nu_+^{i_\infty}(\lambda) = \mathrm{Re}\,\nu_+^{i_\infty -1}(\lambda)$$



- **Transient instability:** unstable essential spectrum *and* stable absolute spectrum

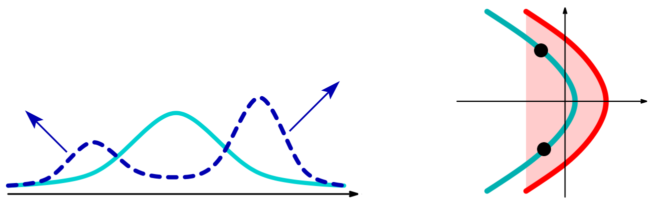


Convective (not transient) and absolute instabilities

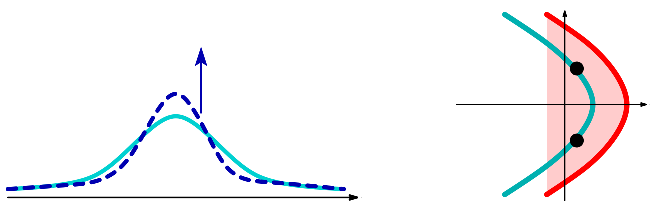


$$\mathbf{S_d} := \{\lambda \in \mathbb{C} \;/\; \nu_-^{i_\infty}(\lambda) = \nu_-^{i_\infty -1}(\lambda) \;\textit{or}\; \nu_+^{i_\infty}(\lambda) = \nu_+^{i_\infty -1}(\lambda)\} \subset \mathrm{spec}_{\mathrm{abs}}(\mathcal{L}_*)$$

- **Convective (not transient) instability:** unstable absolute spectrum *and* stable $\mathbf{S_d}$



- **Absolute instability:**

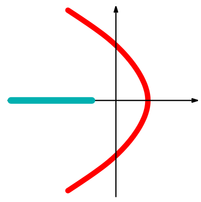


Large bounded domains

Example: *advection-diffusion equation*

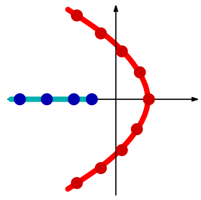
$$w_t = w_{xx} + w_x + \frac{1}{8}w =: \mathcal{L}w$$

- $L^2(\mathbb{R})$: $\mathrm{spec}_{\mathrm{ess}}\mathcal{L} = \{\lambda = -k^2 + \mathrm{i}k + \frac{1}{8},\ k \in \mathbb{R}\}$
  $\mathrm{spec}_{\mathrm{abs}}\mathcal{L} = (-\infty, -\frac{1}{8}]$



? *Bounded domain* $(-L, L)$

- Periodic boundary conditions:
  $\mathrm{spec}_{\mathrm{per}}\mathcal{L} = \{\lambda_\ell = -\left(\frac{\pi\ell}{L}\right)^2 + \mathrm{i}\frac{\pi\ell}{L} + \frac{1}{8},\ \ell \in \mathbb{Z}\} \overset{L\to\infty}{\longrightarrow} \mathrm{spec}_{\mathrm{ess}}\mathcal{L}$
- Dirichlet boundary conditions:
  $\mathrm{spec}_{\mathrm{Dir}}\mathcal{L} = \{\lambda_\ell = -\left(\frac{\pi\ell}{2L}\right)^2 - \frac{1}{8},\ \ell \in \mathbb{Z}\} \overset{L\to\infty}{\longrightarrow} \mathrm{spec}_{\mathrm{abs}}\mathcal{L}$
- Neumann boundary conditions:
  $\mathrm{spec}_{\mathrm{Neu}}\mathcal{L} \overset{L\to\infty}{\longrightarrow} \mathrm{spec}_{\mathrm{abs}}\mathcal{L} \cup \{\frac{1}{8}\}$

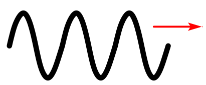


- *Periodic boundary conditions:* $\mathrm{spec}_{\mathrm{per}}\mathcal{L} \overset{L\to\infty}{\longrightarrow} \mathrm{spec}_{\mathrm{ess}}\mathcal{L}$
- *"Separated boundary conditions:"* " $\mathrm{spec}_{\mathrm{sbc}}\mathcal{L} \overset{L\to\infty}{\longrightarrow} \mathrm{spec}_{\mathrm{abs}}\mathcal{L}$ "

4 Stability of periodic waves

4.1 Spectral problems

Periodic waves



Example: *KdV equation*

$$u_t = u_{xxx} + 3uu_x$$

- Travelling waves $u(x + ct)$: set $y = x + ct$
  $\Longrightarrow$ ODE: $cu_y = u_{yyy} + 3uu_y \Longrightarrow u_{yy} = cu - \frac{3}{2}u^2 + const.$

– *phase portrait* $\longrightarrow$ *family of periodic orbits,* $p_{a,c}$*, for* $c<0$

$\longrightarrow$ *three-parameter family of* **periodic waves**

$$p_{a,c}(k_{a,c}(y+\alpha)) = \ldots,\ \ \alpha \in \mathbb{R},\ \ \textit{with } p_{a,c}(\cdot)\ 2\pi-\textit{periodic and even}$$

- Linearization about the periodic wave $p_{a,c}$

$$\mathcal{L}_{a,c}v = v_{yyy} - cv_y + 3\,(p_{a,c}v)_y$$

Spectral problems

- **Linearization** about the periodic wave $p_{a,c}$

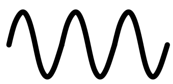
$$\mathcal{L}_{a,c}v = v_{yyy} - cv_y + 3\,(p_{a,c}v)_y$$

- **Spaces for disturbances**

  – *periodic functions*

  – *localized functions*

  – *bounded functions*



  – *Periodic perturbations:* same period as $p_{a,c}$

$$e.g.\ \mathcal{L}_{a,c} : H^3\left(0, \frac{2\pi}{k_{a,c}}\right) \longrightarrow L^2\left(0, \frac{2\pi}{k_{a,c}}\right)$$

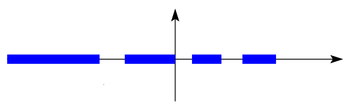
$\longrightarrow$ *linearized operator has compact resolvent*

$\longrightarrow$ *point spectrum*

  – *Localized/bounded perturbations: e.g.* $\mathcal{L}_{a,c} : H^3\left(\mathbb{R}\right) \longrightarrow L^2\left(\mathbb{R}\right)$

$\longrightarrow$ *no point spectrum*

$\longrightarrow$ *e.g. Schrödinger operators with periodic potentials*



4.2 Bloch-wave decomposition

– localized/bounded perturbations –

- *reduces the spectral problem for localized/bounded perturbations to the study of the spectra of an (infinite) family of operators with* **point spectra**

[Reed & Simon, 1978; Scarpelini, 1994, 1995; Mielke, 1997]

- *One spatial dimension:* **Floquet theory**

Example: *KdV equation*

$$\mathcal{L}_{a,c}v = v_{yyy} - cv_y + 3\,(p_{a,c}v)_y$$

$$p_{a,c}(k_{a,c}(y+\alpha)) = \dots,\ \alpha \in \mathbb{R},\ \textit{with } p_{a,c}(\cdot)\ 2\pi - \textit{periodic and even}$$

- Set $z = k_{a,c}y \longrightarrow$ operator with 2π-periodic coefficients

$$\mathcal{L}_{a,c}w = k_{a,c}^3 w_{zzz} - ck_{a,c}w_z + 3k_{a,c}\,(p_{a,c}w)_z$$

  – *Localized perturbations* $\longrightarrow$ spectrum in $L^2(\mathbb{R})$
  – *Bounded perturbations* $\longrightarrow$ spectrum in $C_b^0(\mathbb{R})$

- **Eigenvalue problem:** solve

$$\lambda w = \mathcal{L}_{a,c}w = k_{a,c}^3 w_{zzz} - ck_{a,c}w_z + 3k_{a,c}\,(p_{a,c}w)_z$$

Floquet theory

- **First order system**

$$\frac{\mathrm{d}}{\mathrm{d}z}W = A(z,\lambda)W, \qquad W = \begin{pmatrix} w \\ w_1 = w_z \\ w_2 = w_{1z} \end{pmatrix}$$

  $A(z,\lambda)$ matrix with 2π-periodic coefficients

- **Floquet theory:** *any solution is of the form*

$$W(z) = Q_\lambda(z)\mathrm{e}^{C(\lambda)z}W(0)$$

  – $Q_\lambda(\cdot)$ *is a 2π-periodic matrix function*
  – $C(\lambda)$ *matrix with constant coefficients* [5]

Spectral problem

$$\frac{\mathrm{d}}{\mathrm{d}z}W = A(z,\lambda)W, \qquad W(z) = Q_\lambda(z)\mathrm{e}^{C(\lambda)z}W(0)$$

[5] eigenvalues of $C(\lambda)$: *Floquet exponents*

The ODE has a nontrivial bounded solution for $\lambda \in \mathbb{C}$

$\iff \quad \lambda \in \ker_{C_b^0(\mathbb{R})}(\mathcal{L}_{a,c})$

$\iff$ *ex. solution of the form* $W(z) = Q(z)\mathrm{e}^{\mathrm{i}\gamma z}$*,* $Q(\cdot)$ $2\pi -$ *periodic,* $\gamma \in \left[-\frac{1}{2}, \frac{1}{2}\right)$

$\iff$ *the eigenvalue problem has a nontrivial solution*

$$w(z) = q(z)\mathrm{e}^{\mathrm{i}\gamma z}, \quad \gamma \in \left[-\frac{1}{2}, \frac{1}{2}\right), \quad q(\cdot) \;\; 2\pi - \textit{periodic}$$

$\iff$ *ex. nontrivial* 2π**-periodic solution** *to*

$$\lambda q = k_{a,c}^3(\partial_z + \mathrm{i}\gamma)^3 q - ck_{a,c}(\partial_z + \mathrm{i}\gamma)q + 3k_{a,c}(\partial_z + \mathrm{i}\gamma)\,(p_{a,c}q)$$

for $\gamma \in \left[-\frac{1}{2}, \frac{1}{2}\right)$

$\iff$ *the* **linear operator** $\lambda - \mathcal{L}_{a,c,\gamma}$**,**

$$\mathcal{L}_{a,c,\gamma} q = k_{a,c}^3(\partial_z + \mathrm{i}\gamma)^3 q - ck_{a,c}(\partial_z + \mathrm{i}\gamma)q + 3k_{a,c}(\partial_z + \mathrm{i}\gamma)\,(p_{a,c}q)$$

has a nontrivial kernel in $L^2(0, 2\pi)$

$\iff \quad \lambda \in \mathrm{spec}_{L^2(0,2\pi)}(\mathcal{L}_{a,c,\gamma}), \quad \gamma \in \left[-\frac{1}{2}, \frac{1}{2}\right)$

Lemma.

- $\ker_{C_b^0(\mathbb{R})}(\mathcal{L}_{a,c}) = \bigcup_{\gamma \in \left[-\frac{1}{2}, \frac{1}{2}\right)} \mathrm{spec}_{L^2(0,2\pi)}(\mathcal{L}_{a,c,\gamma})$
- $\ker_{C_b^0(\mathbb{R})}(\mathcal{L}_{a,c}) = \mathrm{spec}_{C_b^0(\mathbb{R})}(\mathcal{L}_{a,c}) = \mathrm{spec}_{L^2(\mathbb{R})}(\mathcal{L}_{a,c})$

Proof. Solve $\lambda w - \mathcal{L}_{a,c} w = f$ using *Floquet theory* and the *variation of constant formula* ....

Theorem.

$$\mathrm{spec}_{C_b^0(\mathbb{R})}(\mathcal{L}_{a,c}) = \mathrm{spec}_{L^2(\mathbb{R})}(\mathcal{L}_{a,c}) = \bigcup_{\gamma \in \left[-\frac{1}{2}, \frac{1}{2}\right)} \mathrm{spec}_{L^2(0,2\pi)}(\mathcal{L}_{a,c,\gamma})$$

Remarks.

- $\mathcal{L}_{a,c,\gamma}$ *have only point spectra*
- *At* $\gamma = 0$ *we find* $\mathcal{L}_{a,c}$ *in* $L^2(0, 2\pi)$*, i.e. the operator for periodic perturbations.*

  Determine the **point spectra** *of* $\mathcal{L}_{a,c,\gamma}$ **?**

4.3 KdV equation: stability of small periodic waves

The **KdV equation**

$$u_t = u_{xxx} + 3uu_x$$

possesses a *three-parameter family of travelling periodic waves*

$$p_{a,c}(k_{a,c}(x + ct + \alpha)) = \ldots, \ \alpha \in \mathbb{R}, \ c < 0,$$
$$\textit{with } p_{a,c}(\cdot) \ 2\pi - \textit{periodic and even}$$

- *Translation invariance* $\longrightarrow$ $\alpha = 0$
- *Scaling invariance* $\longrightarrow$ $c = -1$
  - $y = x + ct \longrightarrow u_t + cu_y = u_{yyy} + 3uu_y$
  - $u(y,t) = |c|v(|c|^{1/2}y, |c|^{3/2}t) \longrightarrow v_t = v_{yyy} + v_y + 3vv_y$
- **One-parameter family of *even* periodic waves**

$$p_a(k_a y) = \ldots, \ \textit{with } p_a(\cdot) \ 2\pi - \textit{periodic and even}$$

Small periodic waves

- *Parameterization of small waves (a small)*

$$\begin{aligned} p_a(z) &= a\,cos(z) + \frac{a^2}{4}(\cos(2z) - 3) + \mathrm{O}(a^3) \\ k_a &= 1 - \frac{15}{16}a^2 + \mathrm{O}(a^4) \end{aligned}$$

- *Linearized operator:* $\mathcal{L}_a = k_a^3 \widetilde{\mathcal{L}}_a$

$$\widetilde{\mathcal{L}}_a w = w_{zzz} + \frac{1}{k_a^2} w_z + \frac{3}{k_a^2}(p_a w)_z$$

- *Spectrum*

$$\mathrm{spec}(\widetilde{\mathcal{L}}_a) = \bigcup_{\gamma \in [-\frac{1}{2}, \frac{1}{2})} \mathrm{spec}_{L^2(0,2\pi)}(\widetilde{\mathcal{L}}_{a,\gamma})$$

Spectral stability

$$\widetilde{\mathcal{L}}_a w = w_{zzz} + \frac{1}{k_a^2} w_z + \frac{3}{k_a^2}(p_a w)_z$$

Theorem. *For a sufficiently small,*

- $\mathrm{spec}(\widetilde{\mathcal{L}}_a) \subset \mathrm{i}\mathbb{R}$
- *the periodic travelling waves are spectrally stable.*

Proof. Show that

$$\mathrm{spec}_{L^2(0,2\pi)}(\widetilde{\mathcal{L}}_{a,\gamma}) \subset \mathrm{i}\mathbb{R}, \quad \forall\, \gamma \in \left[-\frac{1}{2}, \frac{1}{2}\right)$$

in which

$$\widetilde{\mathcal{L}}_{a,\gamma} = (\partial_z + \mathrm{i}\gamma)^3 + \frac{1}{k_a^2}(\partial_z + \mathrm{i}\gamma) + \frac{3}{k_a^2}(\partial_z + \mathrm{i}\gamma)\,(p_a \cdot)$$

Step 0: perturbation argument

- *a small* $\longrightarrow$ $\widetilde{\mathcal{L}}_{a,\gamma}$ *is a "small perturbation" of* $\widetilde{\mathcal{L}}_{0,\gamma}$

$$\widetilde{\mathcal{L}}_{a,\gamma} = \widetilde{\mathcal{L}}_{0,\gamma} + \widetilde{\mathcal{L}}_{1,\gamma}, \qquad \widetilde{\mathcal{L}}_{0,\gamma} = (\partial_z + \mathrm{i}\gamma)^3 + (\partial_z + \mathrm{i}\gamma)$$

  - $\widetilde{\mathcal{L}}_{0,\gamma}$ *operator with constant coefficients*
  - $\widetilde{\mathcal{L}}_{1,\gamma}$ *operator with* 2π*-periodic coefficients and*

$$\|\widetilde{\mathcal{L}}_{1,\gamma}\|_{H^1 \to L^2} = \mathrm{O}(a)$$

– $\widetilde{\mathcal{L}}_{a,\gamma}$ is a *small relatively bounded* perturbation of $\widetilde{\mathcal{L}}_{0,\gamma}$ –

Step 1: spectrum of $\widetilde{\mathcal{L}}_{0,\gamma}$

- *Fourier analysis*

$$\mathrm{spec}(\widetilde{\mathcal{L}}_{0,\gamma}) = \{\mathrm{i}\left((n+\gamma) - (n+\gamma)^3\right),\ n \in \mathbb{Z}\}$$

  (see Figure 3)

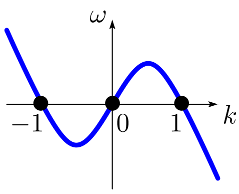


Figure 2: *Dispersion relation* $\omega = (k - k^3)$ *(*$k = n + \gamma$*,* $n \in \mathbb{Z}$*,* $\gamma \in \left[-\frac{1}{2}, \frac{1}{2}\right)$ $\longrightarrow$ *eigenvalue* $\mathrm{i}\omega_{n,\gamma}$*)*

Step 2: $\gamma_* \le |\gamma| \le \frac{1}{2}$

- all eigenvalues are simple *(see Figure 3)*
- *picture persists for small a* **?**

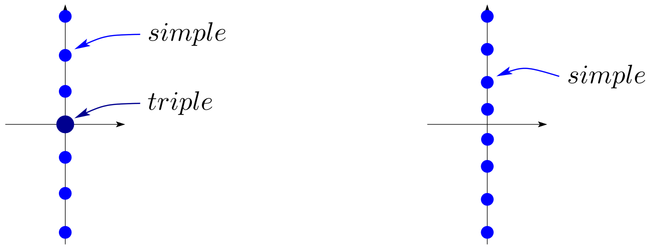
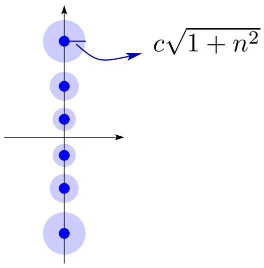


Figure 3: Spectrum of $\widetilde{\mathcal{L}}_{0,\gamma}$ for $\gamma = 0$ (left) and $\gamma \neq 0$ (right).

Difficulties

- infinitely many simple eigenvalues
- *relatively* bounded perturbation

Proposition 2.1.



For all $\gamma_* > 0$, $c > 0$, ex. $a_1 > 0$ such that

$$\operatorname{spec}(\widetilde{\mathcal{L}}_{a,\gamma}) \subset \bigcup_{n\in\mathbb{Z}} B(\mathrm{i}\omega_{n,\gamma}, c\sqrt{1+n^2}),$$

for $|a| \leq a_1$ and $\gamma_* \leq |\gamma| \leq \frac{1}{2}$.

Proof. Use the resolvent formula

$$\left(\lambda - \widetilde{\mathcal{L}}_{a,\gamma}\right)^{-1} = \left(\lambda - \widetilde{\mathcal{L}}_{0,\gamma}\right)^{-1}\left(\mathrm{id} - \widetilde{\mathcal{L}}_{1,\gamma}\left(\lambda - \widetilde{\mathcal{L}}_{0,\gamma}\right)^{-1}\right)^{-1}$$

and the estimates

$$\|\left(\lambda - \widetilde{\mathcal{L}}_{0,\gamma}\right)^{-1}\|_{L^2\to H^1} \leq \frac{1}{c}, \qquad \|\widetilde{\mathcal{L}}_{1,\gamma}\|_{H^1\to L^2} = \mathrm{O}(a)$$

which show that $\lambda - \widetilde{\mathcal{L}}_{a,\gamma}$ is invertible for λ outside these balls.

Proposition 2.2. *Fix $\gamma_* > 0$ and choose $c > 0$ small. Then*

- *the balls $B(\mathrm{i}\omega_{n,\gamma}, c\sqrt{1+n^2})$ are mutually disjoints;*
- *$\widetilde{\mathcal{L}}_{a,\gamma}$ has precisely* **one simple eigenvalue inside each ball** *$B(\mathrm{i}\omega_{n,\gamma}, c\sqrt{1+n^2})$, for a sufficiently small, and $\gamma_* \leq |\gamma| \leq \frac{1}{2}$. This eigenvalue is* **purely imaginary**.

Proof. Choose a ball $B(\mathrm{i}\omega_{n,\gamma}, c\sqrt{1+n^2})$.

- *$\widetilde{\mathcal{L}}_{a,\gamma}$ has precisely one simple eigenvalue inside this ball.*

- Construct spectral projectors[6] $\Pi^n_{0,\gamma}$ for $\widetilde{\mathcal{L}}_{0,\gamma}$, and $\Pi^n_{a,\gamma}$ for $\widetilde{\mathcal{L}}_{a,\gamma}$
- Show that
$$\|\Pi^n_{a,\gamma} - \Pi^n_{0,\gamma}\| < \min\left(\frac{1}{\|\Pi^n_{0,\gamma}\|}, \frac{1}{\|\Pi^n_{a,\gamma}\|}\right)$$
- Conclude that $\Pi^n_{a,\gamma}$ **and** $\Pi^n_{0,\gamma}$ **have the same finite rank**, equal to 1.

• *This eigenvalue is purely imaginary.*

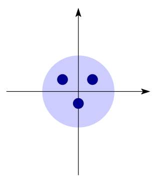
- $\lambda \in \mathrm{spec}(\widetilde{\mathcal{L}}_{a,\gamma}) \iff -\overline{\lambda} \in \mathrm{spec}(\widetilde{\mathcal{L}}_{a,\gamma})$ since p_a is even:
$$\begin{aligned} \widetilde{\mathcal{L}}_{a,\gamma} w(z) = \lambda w(z) &\iff \overline{\widetilde{\mathcal{L}}_{a,\gamma}}\, \overline{w(z)} = \overline{\lambda}\, \overline{w(z)} \\ &\iff \widetilde{\mathcal{L}}_{a,-\gamma}\, \overline{w(z)} = \overline{\lambda}\, \overline{w(z)} \\ &\overset{z \to -z}{\iff} -\widetilde{\mathcal{L}}_{a,\gamma}\, \overline{w(-z)} = \overline{\lambda}\, \overline{w(-z)} \\ &\iff \widetilde{\mathcal{L}}_{a,\gamma}\, \overline{w(-z)} = -\overline{\lambda}\, \overline{w(-z)} \end{aligned}$$
- This shows that the spectrum is symmetric with respect to the imaginary axis, so that the simple eigenvalue lies on the imaginary axis.

Step 3: $|\gamma| \le \gamma_*$ *(γ small)*

• **Step 3.1:** *treat the (infinitely many) simple eigenvalues (corresponding to Fourier modes $|n| \ge 2$) as in Step 2.*

• **Step 3.2:** *consider the triple eigenvalue in the origin with Fourier modes $n = 0, \pm 1$.* A standard perturbation argument shows that $\widetilde{\mathcal{L}}_{a,\gamma}$ *has three eigenvalues inside the ball* $B(0,1)$, provided a and γ are sufficiently small.

Step 3.2: $n = 0, \pm 1$, $|\gamma| \le \gamma_*$
Locate the three eigenvalues inside $B(0,1)$?



Consider the associated **spectral subspace (three-dimensional) and**

• compute a **basis** $\{\xi_0(a,\gamma),\ \xi_1(a,\gamma),\ \xi_2(a,\gamma)\}$;

• compute the 3×3-matrix $\mathcal{M}_{a,\gamma}$ representing the **action of** $\widetilde{\mathcal{L}}_{a,\gamma}$ on this subspace;

• show that this matrix has three purely imaginary **eigenvalues** for small a and γ.

$\boxed{a = 0}$ *operator with constant coefficients*

[6] Dunford integral formula $\Pi^n_{a,\gamma} = \frac{1}{2\pi \mathrm{i}} \int_{C_n} (\lambda - \widetilde{\mathcal{L}}_{a,\gamma})^{-1} \mathrm{d}\lambda$

- Basis

$$\xi_0(0,\gamma)=\cos(z),\quad \xi_1(0,\gamma)=-\sin(z),\quad \xi_2(0,\gamma)=1$$

- Matrix

$$\mathcal{M}_{0,\gamma}=\begin{pmatrix} -2\mathrm{i}\gamma-\mathrm{i}\gamma^3 & 3\gamma^2 & 0\\ -3\gamma^2 & -2\mathrm{i}\gamma-\mathrm{i}\gamma^3 & 0\\ 0 & 0 & \mathrm{i}\gamma-\mathrm{i}\gamma^3 \end{pmatrix}$$

$\boxed{\gamma=0}$ *linearization about* p_a

- Translation invariance + Galilean invariance

$$\widetilde{\mathcal{L}}_{a,0}\left(\partial_z p_a\right)=0,\quad \widetilde{\mathcal{L}}_{a,0}(1)=\frac{3}{k_a^2}\partial_z p_a,\quad \widetilde{\mathcal{L}}_{a,0}\left(\partial_a p_a-\frac{2k_a'}{k_a}\left(\frac{1}{3}+p_a\right)\right)=0$$

- Basis

$$\xi_0(a,0)=\partial_a p_a-\frac{2k_a'}{k_a}\left(\frac{1}{3}+p_a\right),\quad \xi_1(a,0)=\frac{1}{a}\partial_z p_a,\quad \xi_2(a,0)=1$$

- *Matrix*

$$\mathcal{M}_{a,0}=\frac{1}{k_a^2}\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 3a\\ 0 & 0 & 0 \end{pmatrix}$$

$\boxed{a\gamma\neq 0}$ *compute expansions*

- Basis at order 2

$$\begin{aligned}
\xi_0(a,\gamma) &= \cos(z)+\frac{a}{2}\cos(2z)-\frac{a}{4}+\frac{15a^2}{4}\cos(z)+\frac{9a^2}{64}\cos(3z)-\frac{\mathrm{i}a\gamma}{2}\sin(2z)+\dots\\
\xi_1(a,\gamma) &= -\sin(z)-\frac{a}{2}\sin(2z)-\frac{9a^2}{64}\sin(3z)-\frac{\mathrm{i}a\gamma}{2}\cos(2z)+\dots\\
\xi_2(a,\gamma) &= 1+\dots
\end{aligned}$$

- Matrix at order 3

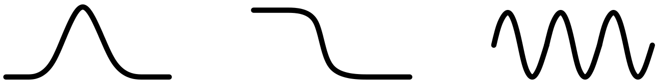
$$\mathcal{M}_{a,\gamma}=\frac{1}{k_a^2}\begin{pmatrix} -2\mathrm{i}\gamma+3\mathrm{i}a^2\gamma-\mathrm{i}\gamma^3 & 3\gamma^2 & 3\mathrm{i}a\gamma\\ -3\gamma^2 & -2\mathrm{i}\gamma+3\mathrm{i}a^2\gamma-\mathrm{i}\gamma^3 & 3a\\ 3\mathrm{i}a\gamma/4 & 3a\gamma^2/4 & \mathrm{i}\gamma-3\mathrm{i}a^2\gamma/2-\mathrm{i}\gamma^3 \end{pmatrix}+\dots$$

...... *and conclude.*

Spectral stability of nonlinear waves

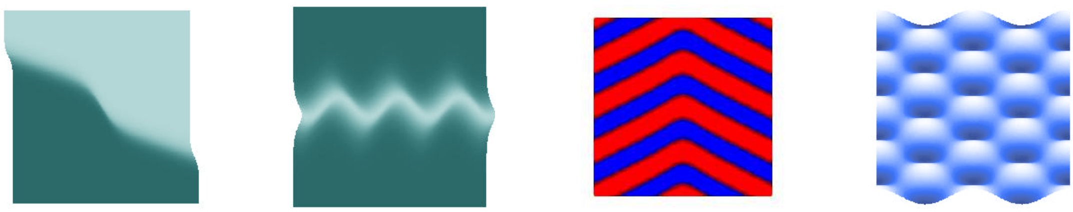
– methods from the theory of dynamical systems –

- Stability of **one-dimensional waves**



- These methods can be adapted to study the stability of **modulated waves**

and of certain types of **two-dimensional waves** such as *waves with a distinguished spatial direction* and *doubly periodic waves*



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