Numerical Evans Function Computation

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INTRODUCTION

Traveling Wave

Consider the evolution equation

$$u_t = \mathcal{F}(u, u_x, u_{xx}, \ldots).$$

A traveling wave with speed s is a solution

$$u(x,t) = \hat{u}(x - st).$$

Or equivalently a stationary solution of

$$u_t - su_x = \mathcal{F}(u, u_x, u_{xx}, \ldots).$$

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Existence of Profile

- A *front* is a heteroclinic point
- A *pulse* is a homoclinic point
- A *wavetrain* is a periodic solution

Nonlinear Stability

Examine flow “near” the stationary solution

$$u(x,0) = \hat{u}(x) + v(x,0),$$

where $\|v(x,0)\|$ is small.

Asymptotically orbital stability when

$$\lim_{t\to\infty} u(x,t) = \hat{u}(x+\delta).$$

Example

- Isentropic Navier Stokes

$$v_t - u_x = 0$$
$$u_t + p(v) = \left(\frac{u_x}{v}\right)_x$$

(play movie)

Spectral Stability

Linearization about the profile

$$v_t = \underbrace{(d\mathcal{F}(\hat{u}) - s\partial_x)v}_{Lv} + \underbrace{Q(v, v_x, v_{xx}, \ldots)}_{\text{Higher order}}.$$

Spectral Stability when $\sigma(L) \cap P = \emptyset$, where

$$P = \{\lambda \mid \Re e(\lambda) \geq 0\} \backslash \{0\}.$$

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EVANS FUNCTION

Eigenvalue Problem

Write the eigenvalue problem

$$\lambda v = Lv, \qquad -\infty < x < \infty,$$

as a first-order system

$$\begin{cases} W' = A(x,\lambda)W, & W \in \mathbb{C}^n \\ W(\pm\infty) = 0. \end{cases}$$

Assume: $A(x,\lambda)$ is consistently split and asymptotically constant in x.

Evans Function

Define

$$D(\lambda) = \underbrace{W_1^+ \wedge \ldots \wedge W_k^+}_{S^+(\lambda)} \wedge \underbrace{W_{k+1}^- \wedge \ldots \wedge W_n^-}_{U^-(\lambda)}$$

where $\{W_i^+\}_{i=1}^k$ and $\{W_j^-\}_{j=k+1}^n$ are analytic bases of the stable/unstable manifolds of $x = \pm\infty$, respectively.

Problems with Stiffness

Maintaining linear independence of $\{W_i^+\}_{i=1}^k$ and $\{W_j^-\}_{j=k+1}^n$ is hard.

Common Options:

- Orthogonalization – bad for winding numbers due to loss of analyticity.
- Compound-Matrix Method – numerically prohibitive for moderate values of n.

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COMPOUND-MATRIX METHOD

Compound-Matrix Method I

- Developed by Gilbert & Bakkus (1966) and Ng & Reid (1979).
- First used to numerically compute Evans functions by Pego (1995).
- Use in Evans functions rediscovered and further developed by Brin & Zumbrun (1998) and Bridges & collaborators (2002).

Compound Matrix Method II

Lift to wedge-product space

$$W' = A^{(k)}(x,\lambda)W, \quad W \in \mathbb{C}^{\binom{n}{k}}.$$

Flow of smallest simple mode at $x = \infty$ corresponds to the k-form spanning the stable manifold. Similar approach holds for $x = -\infty$.

Lifting to wedge-product space I

- Let $\{w_i\}_{i=1}^{n}$ be a basis for $\mathbb{C}^n$. Then

$$\{w_{i_1} \wedge \ldots \wedge w_{i_k} \mid i_1 < \ldots < i_k\}$$

is a basis for $\Lambda^k(\mathbb{C}^n)$.

Hence $A : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ induces a linear map

$$A^{(k)} : \Lambda^k(\mathbb{C}^n) \longrightarrow \Lambda^k(\mathbb{C}^n)$$

satisfying…

Lifting to wedge-product space II

$$A^{(k)} w_{i_1} \wedge \ldots \wedge w_{i_k}$$

$$= \sum_{j=1}^{k} w_{i_1} \wedge \ldots \wedge A w_{i_j} \wedge \ldots \wedge w_{i_k}.$$

Lifting to wedge-product space III

If in an eigenbasis $Aw_i = \mu_i w_i$,

then $A^{(k)} w_{i_1} \wedge \ldots \wedge w_{i_k}$

$$= \sum_{j=1}^{k} w_{i_1} \wedge \ldots \wedge A w_{i_j} \wedge \ldots \wedge w_{i_k}$$

$$= \left(\sum_{j=1}^{k} \mu_{i_j} \right) w_{i_1} \wedge \ldots \wedge w_{i_k}.$$

Example

"Good" Boussinesq equation

$$u_{tt} = u_{xx} - u_{xxxx} - (u^2)_{xx},$$

admits the profile

$$\bar{u}(\xi) = \tfrac{3}{2}(1 - s^2)\mathrm{sech}^2\left(\tfrac{\sqrt{1-s^2}}{2}\xi\right), \quad \xi = x - st,$$

which is known to be stable when $\frac{1}{2} \leq |s| < 1$ and unstable when $|s| < \frac{1}{2}$.

Example (cont)

Linearization about the profile yields

$$\lambda^2 u - 2s\lambda u' = (1 - s^2)u'' - u'''' - (2\bar{u}u)'',$$

hence

$$A(x,\lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\lambda^2 - 2\bar{u}_{xx} & 2\lambda s - 4\bar{u}_x & (1-s^2) - 2\bar{u} & 0 \end{pmatrix}, \quad W = \begin{pmatrix} u \\ u' \\ u'' \\ u''' \end{pmatrix}.$$

Example (cont)

Lifting to wedge-product space yields

$$A^{(2)}(x,\lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 2\lambda s - 4\bar{u}_x & (1-s^2) - 2\bar{u} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \lambda^2 + 2\bar{u}_{xx} & 0 & 0 & (1-s^2) - 2\bar{u} & 0 & 1 \\ 0 & \lambda^2 + 2\bar{u}_{xx} & 0 & -2\lambda s + 4\bar{u}_x & 0 & 0 \end{pmatrix}.$$

Compound-Matrix Blow-Up

Note:

- Matrix-vector multiplication scales as $\binom{n}{k}^2$.
- Note:

$$\binom{8}{4} = 70, \binom{14}{7} = 3432, \binom{20}{10} = 184,756.$$

- Sparse matrix-vector multiplication scales as

$$(k(n-k)+1)\binom{n}{k}.$$

POLAR-COORDINATE METHOD

Evans Function Reformulation I

Let $W_+ = (W_1^+ \dots W_k^+)$ and $W_- = (W_{k+1}^- \dots W_n^-)$ then for a $k \times k$ matrix α_+ and a $(n-k) \times (n-k)$ matrix α_-, we have $W_- = \Omega_- \alpha_-$ and $W_+ = \Omega_+ \alpha_+$ where $\Omega_+^H \Omega_+ = I_k$ and $\Omega_-^H \Omega_- = I_{n-k}$.

Evans Function Reformulation II

Then

$$\mathrm{W}_1^+ \wedge \cdots \wedge W_k^+ = \underbrace{(\det \alpha_+)}_{\gamma_+}(\Omega_1^+ \wedge \cdots \wedge \Omega_k^+)$$

and

$$W_{k+1}^- \wedge \cdots \wedge W_n^- = \underbrace{(\det \alpha_-)}_{\gamma_-}(\Omega_{k+1}^- \wedge \cdots \wedge \Omega_n^-).$$

Hence, we define the Evans function to be
$D(\lambda) = \det\,(W_+, W_-)\,|_{x=0} = \gamma_+ \gamma_- \det\,(\Omega_+, \Omega_-)\,|_{x=0}$.

Continuous Orthogonalization

Choose $k \times k$ matrix-valued function $\alpha(x)$ such that

$$W(x) = \Omega(x)\alpha(x), \quad W, \Omega \in \mathbb{C}^{n \times k}, \Omega^H \Omega = I_k.$$

Then $W' = A(x,\lambda)W$ becomes

$$\Omega'(x) = A(x,\lambda)\Omega(x) + \Omega(x)\underbrace{\alpha'(x)\alpha(x)^{-1}}_{g(x)}.$$

Drury Method I

Let

$$g = -\Omega^H A \Omega$$

then

$$\begin{aligned} \Omega' &= A\Omega - \Omega\Omega^H A\Omega \\ &= (I - \Omega\Omega^H)A\Omega. \end{aligned}$$

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Drury Method II

- Invariant on Stiefel Manifold

$$\mathcal{V}_k(\mathbb{C}) = \{\Omega \in \mathbb{C}^{n\times k} \mid \Omega^H \Omega = I_k\}$$

or

$$\mathcal{E}^{-1}(0) \quad \text{where} \quad \mathcal{E}(\Omega) := \Omega^H\Omega - I_k.$$

- Numerically unstable

$$\mathcal{E}' = g^H\mathcal{E} + \mathcal{E}g.$$

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Davey Method I

Let

$$g = -\underbrace{(\Omega^H\Omega)^{-1}\Omega^H}_{\Omega^\dagger} A\Omega$$

then

$$\begin{aligned}\Omega' &= A\Omega - \Omega\Omega^\dagger A\Omega \\ &= (I - \Omega\Omega^\dagger)A\Omega.\end{aligned}$$

Davey Method II

- Numerically stable $\mathcal{E}' = 0$.
- But still not attracting on the Stiefel manifold.
- One variation is to add a damping term

$$\Omega' = (I - \Omega\Omega^\dagger)A\Omega + \gamma\Omega(I - \Omega^H\Omega),$$

then $\mathcal{E}' = -2\gamma(I + \mathcal{E})\mathcal{E}$.

Polar Coordinate Method I

Note

$$A\Omega\alpha = \Omega' + \Omega\alpha'.$$

We can left multiply by $\Omega^\dagger$ to get

$$\alpha' = \Omega^\dagger A\Omega\alpha$$

since

$$\Omega^\dagger\Omega' = \Omega^\dagger(I - \Omega\Omega^\dagger)A\Omega = 0.$$

Polar Coordinate Method II

By Abel's theorem, we get

$$\gamma' = \text{tr}\ (\Omega^\dagger A \Omega)\gamma, \quad \gamma = \det\ \alpha.$$

All combined, we have

$$\begin{cases} \Omega' = (I - \Omega\Omega^\dagger)A\Omega \\ \gamma' = \text{tr}\ (\Omega^\dagger A \Omega)\gamma \end{cases}$$

Shooting on both ends yields

$$D(\lambda) = \gamma_+\gamma_- \det\ (\Omega_+, \Omega_-)\ |_{x=0}.$$

Operational Count

- With damping the polar-coordinate method grows as
$$kn^2 + 4k^2 n.$$
- With sparse solver, the compound-matrix method grows as
$$(k(n-k)+1)\binom{n}{k}.$$
- Break even between n=7 and n=8.

OTHER NUMERICAL ISSUES

Other Issues

- Kato’s Method
- Integrated Coordinates
- Geometric Integrators
- Contours…how big is big enough?

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EXAMPLES

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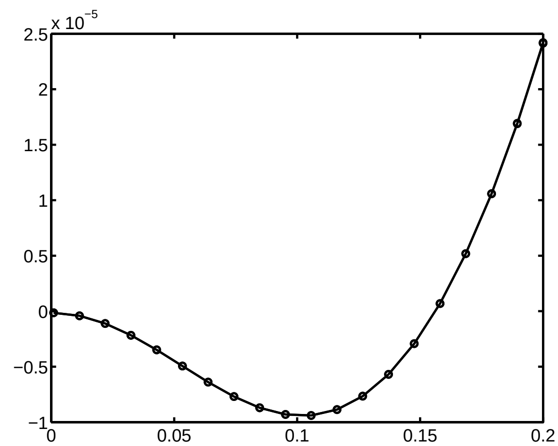
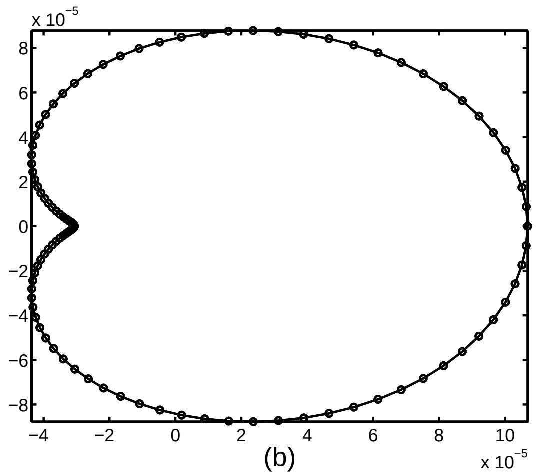
“Good” Boussinesq (revisited)



(a)

We compute the Evans function along real axis from $\lambda = 0$ to $\lambda = 0.2$

“Good” Boussinesq (re-revisited)



The image of the closed contour $\Gamma(t) = 0.16 + 0.05e^{2\pi i t}$.

Isentropic Navier Stokes

(show Matlab figures)

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Future Directions

- Push the limits of the method.
- Test different continuous orthogonalization methods (e.g., geometric integrators).
- Semi-spectral methods instead of shooting?
- Further development of STABLAB.