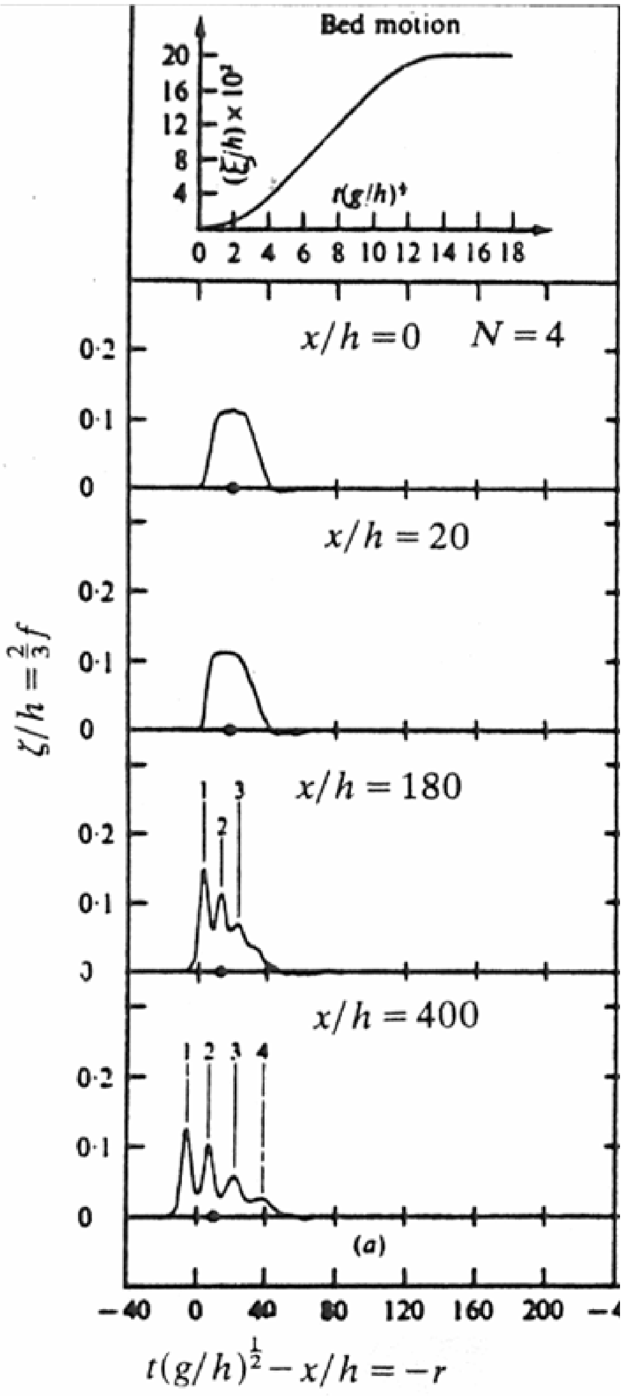

Workshop: stability and instability of nonlinear waves

Sept. 6-8, 2006

4 lectures on

Nonlinear stability

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A. Formulate the problem

1. Some definitions and questions

x - a spatial coordinate in n-dimensions, t - time

$g(x,t)$ - an M-component function of (x,t)

$$\partial_t g = N(g) \qquad (1)$$

We assume that if $g(x,0)$ is given in a suitable space, then (1) determines $g(x,t)$ for all $t > 0$.

$G(x)$ - an equilibrium (or stationary) solution, so

$$N(G) = 0. \qquad (2)$$

Is $G(x)$ stable to small but arbitrary perturbations in initial data?

Other Questions

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Why is there a “stability theory” ?

<u>Opinion</u>: Because we can’t solve the initial-value problem for (1) in any general sense.

a) Comprehend the solutions of (1) by analyzing the stability of isolated solutions, like $G(x)$.

b) Stability is a qualitative question, so answering it should (might?) be easier than solving (1).

Other Questions

Q2. Stability of $G(x)$ means that any other solution of (1) that starts close to $G(x)$ stays close forever (where "close" is defined properly). But if it stays close forever, we should be able to linearize (1) about $G(x)$.

Why is nonlinear stability necessary?

Other Questions

Q2. Stability of G(x) means that any other solution of (1) that starts close to G(x) stays close forever (where "close" is defined properly). But if it stays close forever, we should be able to linearize (1) about G(x).

Why is nonlinear stability necessary?

Answer: This reasoning is correct for most problems, and linear theory is usually qualitatively correct. But there are counter-examples, where linear theory is completely misleading.

Example: linear theory fails [A7, p.2]

Variables: $x_1(t)$, $x_2(t)$, $r^2 = x_1^2 + x_2^2$, $r \geq 0$

Real-valued parameters: α, β

$$\frac{dx_1}{dt} = \alpha \cdot x_2 + \beta \cdot x_1 r, \qquad \frac{dx_2}{dt} = -\alpha \cdot x_1 + \beta \cdot x_2 r. \qquad (3)$$

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Linearized:

$$\frac{dx_1}{dt} = \alpha \cdot x_2, \qquad \frac{dx_2}{dt} = -\alpha \cdot x_1.$$

=>

$$\frac{d}{dt}(x_1{}^2 + x_2{}^2) = \frac{d}{dt}(r^2) = 0.$$

Linearly stable

Nonlinear analysis

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=> $$\frac{d}{dt}\left(\frac{x_1{}^2 + x_2{}^2}{2}\right) = 0 + \beta(x_1{}^2 + x_2{}^2) \cdot r$$

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=> $$\frac{dr}{dt} = \beta \cdot r^2$$

Nonlinear analysis

$$\frac{dr}{dt} = \beta \cdot r^2$$

$\beta > 0$

$$r(t) = \frac{1}{\beta(t^* - t)} > 0$$

r(t) blows up in finite time for *any* initial condition with r > 0.

$\beta < 0$

$$r(t) = \frac{1}{|\beta|(t + t_0)} > 0$$

r(t) $\to$ 0 as t $\to \infty$

asymptotic stability

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do not commute.

- What happens to volume in phase space?

2. General concept:
volume in phase space

a) Let (x_1, x_2,..x_N) be coordinates in *N*-dim. phase space

Consider a hypothetical fluid there.

A fluid particle at (x_1,…x_N) has a velocity

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b) Divergence of velocity field $\nabla \cdot \vec{v} = \frac{\partial v_1}{\partial x_1} + \ldots + \frac{\partial v_N}{\partial x_N}$

- If $\nabla \cdot \vec{v} > 0$ everywhere, volume is expanding
- If $\nabla \cdot \vec{v} < 0$ everywhere, volume is contracting
- If $\nabla \cdot \vec{v} = 0$ everywhere, volume is conserved
- Other possibilities exist

Volume in phase space

c) Back to example problem

$$v_1 = \frac{dx_1}{dt} = \alpha \cdot x_2 + \beta \cdot x_1 \sqrt{x_1{}^2 + x_2{}^2},$$
$$v_2 = \frac{dx_2}{dt} = -\alpha \cdot x_1 + \beta \cdot x_2 \sqrt{x_1{}^2 + x_2{}^2}.$$

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$$v_1 = \frac{dx_1}{dt} = \alpha \cdot x_2 + \beta \cdot x_1 \sqrt{x_1^{\;2} + x_2^{\;2}},$$
$$v_2 = \frac{dx_2}{dt} = -\alpha \cdot x_1 + \beta \cdot x_2 \sqrt{x_1^{\;2} + x_2^{\;2}}.$$

Show

$$\nabla \cdot \vec{v} = 3\beta \cdot r$$

$\beta > 0 \quad \Rightarrow$ expanding

$\beta < 0 \quad \Rightarrow$ contracting

$\beta = 0 \quad \Rightarrow$ conservative

Volume in phase space

d) In these lectures, develop one theory for volume-preserving flows (Hamiltonian), and another theory for volume-contracting flows (dissipative).

Volume in phase space

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Q3: Can linear theory be misleading in other ways?

Answer: Yes

- See HW problem 3 for a Hamiltonian system in which linear theory gives a completely wrong answer about stabilty
- KAM theory

3. Hamiltonian systems

(See Ch. 1 of [A8] for a nice introduction to Hamiltonian mechanics

a) Finite dimensional, canonical systems

Phase space has 2N dimensions, $N < \infty$

Coordinates: $(p_1, p_2,..,p_N; q_1, q_2,\ldots,q_N)$

A dynamical system moves a point on this phase space according to specific rules:

$$\frac{dp_j}{dt} = P_j(\vec{p},\vec{q};t), \quad \frac{dq_j}{dt} = Q_j(\vec{p},\vec{q};t), \quad j = 1,\ldots,N$$

Hamiltonian systems

b) This system of 2N ODEs is Hamiltonian if there is a 2-continuously differentiable function of $\vec{p}$, $\vec{q}$, $H(\vec{p},\vec{q};t)$, such that all the ODEs can be written in the form

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j}, \quad j = 1,\ldots,N \qquad (4)$$

c) Some consequences of (4)

- If $H = H(\vec{p}(t),\vec{q}(t))$ with no explicit time-dependence, then H is a constant of the motion. (Oddly, this is the usual situation.)

Proof: Compute $\frac{dH}{dt}$, using (4) and the chain rule.

Hamiltonian systems

c) Consequences of (4), continued

- Any flow of the form (4) conserves volume in phase space.

<u>Proof</u>: Define velocity components in the usual way, and calculate $\nabla \cdot \vec{v}$, making use of (4) and the differentiability of H. Show

$$\nabla \cdot \vec{v} = \sum_{j=1}^{N} \frac{\partial}{\partial p_j} \left(\frac{dp_j}{dt}\right) + \frac{\partial}{\partial p_j} \left(\frac{dp_j}{dt}\right) = 0.$$

Hamiltonian systems

c) Consequences of (4), continued

- For fixed j, (p_j,q_j) are coordinates on a 2-D plane within the 2N-D phase space. On that plane,

$$\frac{\partial}{\partial p_j}\left(\frac{dp_j}{dt}\right)+\frac{\partial}{\partial q_j}\left(\frac{dq_j}{dt}\right)=0, \quad j=1,\ldots,N.$$

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These are some of Poincare’s integral invariants

(see ▪38,44,45 of [A1]).

Unfortunately, these integral invariants tell us *nothing* about a particular solution of the equations. Volume in phase space tells us about the evolution of a collection of solutions.

Hamiltonian systems

c) Consequences of (4), continued

- Volume conservation in phase space means that if some solutions approach a given solution as $t \to \infty$, other solutions must move away from it.

Asymptotic stability never occurs in a Hamiltonian system.

Hamiltonian systems

c) A big consequence of (4)

- Let $G(\tau)$ denote a specific solution of a Hamiltonian system, (4). Let $g(t)$ denote any other solution of (4). Denote the distance between $g(t)$ and $G(t)$ at a specific time by

$$\|G-g\|_*$$

Def'ns: We say that $G(t)$ is stable (in the sense of Lyapunov) if for every $\varepsilon > 0$, there is a $\delta > 0$ such that if

$$\|G-g\|_a < \delta \quad \text{at } t = 0,$$

then necessarily

$$\|G-g\|_b < \varepsilon \quad \text{for all } t > 0.$$

A solution that is not stable is unstable.

Hamiltonian systems

d) Infinite-dimensional Hamiltonian systems

- x - represents physical space (in 1,2,3,…,10 dim)
  t - time

$$\vec{p} = p_1(x,t), p_2(x,t), \ldots, p_N(x,t)$$
$$\vec{q} = q_1(x,t), q_2(x,t), \ldots, q_N(x,t)$$

evolve according to

$$\partial_t p_j = P_j(\vec{p}, \vec{q}; x, t)$$
$$\partial_t q_j = Q_j(\vec{p}, \vec{q}; x, t)$$

for j = 1,2,..,N (5)

$$x \in D, \quad t > 0.$$

P, Q may contain spatial derivatives and/or integrals. In addition, there are usually boundary conditions, needed for well-posedness.

Hamiltonian systems

d) Infinite-dimensional Hamiltonian systems

- Let $H(\vec{p},\vec{q}) = \int [h(\vec{p},\vec{q};x,t]dx$ be a real-valued functional, mapping the phase space to the reals. h might contain spatial derivatives or integrals, and we require that $h(\vec{p},\vec{q};x,t)$ be 2-differentiable

  in p,q. We <u>define</u> $\frac{\delta H}{\delta p}, \frac{\delta H}{\delta q}$ so that

$$H(\vec{p}+\delta\vec{p},q+\delta\vec{q}) = H(\vec{p},\vec{q}) + \int [(\frac{\delta H}{\delta p})\cdot\delta p + [(\frac{\delta H}{\delta q})\cdot\delta\vec{q}]dx + O(|\delta\vec{p}|^2,|\delta\vec{q}|^2) \qquad (6)$$

(See HW problems 4, 5 for examples.)

Hamiltonian systems

d) Infinite-dimensional Hamiltonian systems

- We say that the evolution equations in (5) are <u>Hamiltonian</u> if they can be written in the form

$$\begin{aligned} \partial_t q_j &= \frac{\delta H}{\delta p_j}, \\ \partial_t p_j &= -\frac{\delta H}{\delta q_j}, \end{aligned} \qquad j = 1, \ldots, N \tag{7}$$

Hamiltonian systems

- $$\partial_t q_j = \frac{\delta H}{\delta p_j}, \quad j = 1,\dots,N \qquad (7)$$
  $$\partial_t p_j = -\frac{\delta H}{\delta q_j},$$

- More jargon: The system (7) is called <u>canonical</u>, and (p_j,q_j) are called <u>conjugate variables</u>. Here is another way to write (7). Define

$$g = \begin{pmatrix} \vec{p} \\ \vec{q} \end{pmatrix}, \quad \frac{\delta H}{\delta g} = \begin{pmatrix} \dfrac{\delta H}{\delta p} \\ \dfrac{\delta H}{\delta q} \end{pmatrix}, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

  Then (7) can be written as

$$\partial_t g = J \frac{\delta H}{\delta g} \qquad (8)$$

Hamiltonian systems

e) Noncanonical Hamiltonian systems

- Let $g(x,t)$ represent the variables in the evolution equations in question. We say that the evolution equations are Hamiltonian if they can be written in the form

$$\partial_t g = J \frac{\delta H}{\delta g} \quad , \qquad (8)$$

in terms of a Poisson tensor, J. The canonical formulation is a special case of this.

Hamiltonian systems

e) Noncanonical Hamiltonian systems

- The *correct* way to define Hamiltonian mechanics is in terms of a Poisson bracket, [•,•]. The Poisson bracket is related to the Poisson tensor and the inner product, <•,•> through

$$[M,N] = \left< \frac{\delta M}{\delta g}, J\frac{\delta N}{\delta g} \right> \tag{9}$$

- A prominent example of a noncanonical Hamiltonian system is the Korteweg-deVries equation

$$\partial_t u + 6u\partial_x u + \partial_x{}^3 u = 0 \tag{10}$$

with

$$J = \partial_x, \quad H = \int [\tfrac{1}{2}(\partial_x u)^2 - u^3] dx$$

Gardner [A4] first proved the validity of this formulation, and used it.