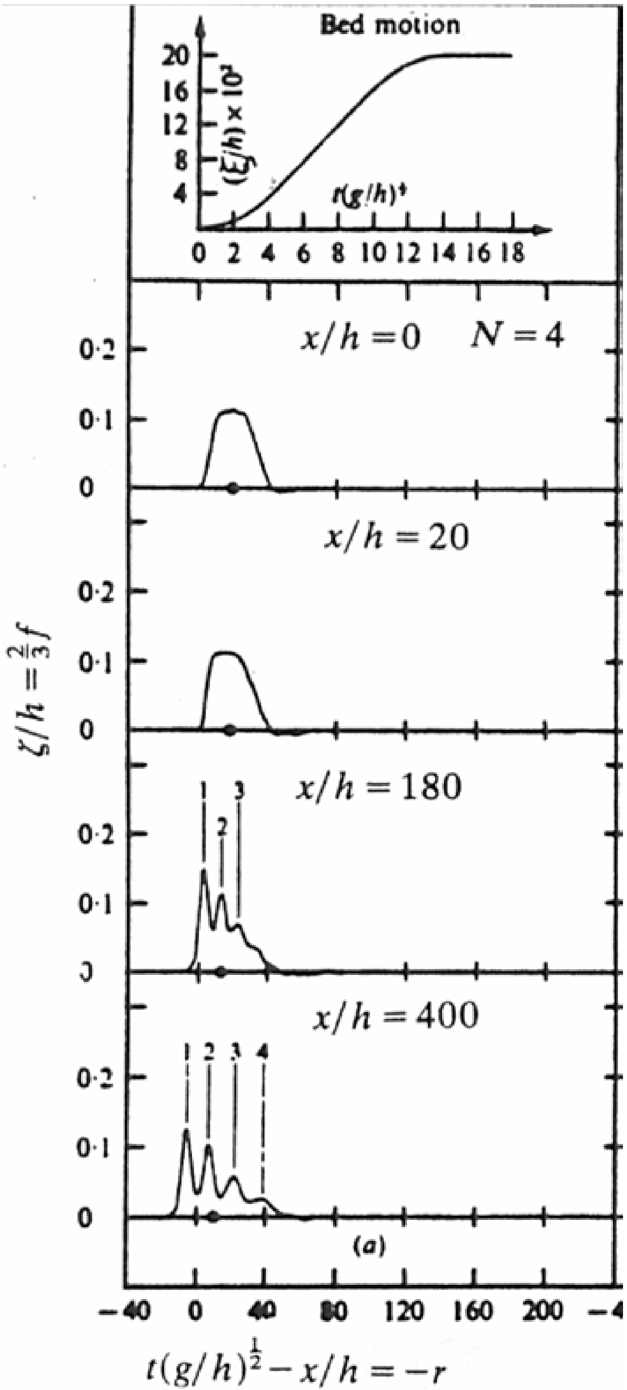

Lecture 2:

Nonlinear stability in Hamiltonian systems

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1. Statement of the problem

a) Consider a Hamiltonian system of nonlinear partial differential equations for $g(x,t)$:

$$\partial_t g = J \frac{\delta H}{\delta g} \qquad (1)$$

Let $G(x)$ be an equilibrium (or stationary) solution of (1):

$$J \frac{\delta H}{\delta g} \Big|_{g=G} = 0$$

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Q: Is $G(x)$ stable to small perturbations in the initial conditions? Or is there a family of solutions of (1), $g(x,t;\alpha)$, such that $\|G(x) - g(x,0;\alpha)\|$ can be made arbitrarily small at $t = 0$, but

$$\|G(x) - g(x,t;\alpha)\| > 2 \quad \text{(say) at some } t > 0?$$

Statement of the problem

b) The choice of norm (or norms) used to prove stability for a PDE is an important part of the problem. There are known examples of solutions of PDEs that are stable in one norm, but not in another. (See [B6].)

Statement of the problem

c) Two, very different, strategies

(i) Linearize (1) about $G(x)$. If done correctly, the linearized equations inherit the Hamiltonian structure of (1).

- Show that $G(x)$ is linearly stable.
- Boost this result into a result for nonlinear stability, by showing that the nonlinear terms can be controlled by the terms in the linear equations, for small enough perturbations.

(But it ain’t necessarily so - see Cherry [A3].)

Statement of the problem

(ii) Strategy 2

Find "enough" constants of the motion, possibly including the Hamiltonian, $H(g)$, and others. If you're lucky, some combination of constants,

$$K(g) = H + 3C_1 - \frac{17}{\pi} C_2 + 5$$

defines a norm on (G-g):

$$\|G - g\|^2 = K(g)$$

$K(g)$ is a constant of the motion, so it can be made arbitrarily small for all $t > 0$ by making it small enough at $t = 0$. **DONE!**

Statement of the problem

(iia) variation on strategy 2 (Arnold [B1, B2] and others)

$K(g) = H + 3C_1 - \frac{17}{\pi} C_2 + 5$ is not a norm, but it bounds a norm in the following way:

- $K(G) = 0$.
- $K(g) > 0$ for any $g \neq G$ in the function space.
- There exist constants $\{m, c, C\}$ and a norm ||•||, such that

$$m > 0, \quad 0 < c \leq C < \infty,$$

$$c\|G - g\|^m \leq K(g) \leq C\|G - g\|^m$$

Show from this that $G(x)$ is stable in the sense of Lyapunov. (See problem 1 of set B.)

This method known to Lagrange of ODEs (see B8, p. 208).

Statement of the problem

d) Critique of each method

(i) Establish linearized stability. Then boost this to nonlinear stability.

- Advantage: linear stability is very well developed. (See this conference.)
- Disadvantage: Cherry [A2] showed that linear stability $\neq>$ nonlinear stability.
- Disadvantage: Proving nonlinear stability can be quite hard.

Statement of the problem

d) Critique of each method

(ii) Arnold’s method

- Advantage: Conceptually simple
- Advantage: Skip linearized dynamics altogether. (“Stability theory ought ot be simpler than solving full PDE.”)
- Advantage: Method is algorithmic (as we’ll see)
- Disadvantage: In it’s current form, it only works for some problems. If you don’t find the “right” conservation laws, you’re dead.

Statement of the problem

d) Critique of each method

(iii) For either method

- Both methods assume solutions exist for all time. Proving existence is separate. Example [B9]:

$$i\partial_t \psi + \nabla^2 \psi + 2|\psi|^2 \psi = 0 \qquad \text{in 2-D or 3-D}$$

Constants:

$$M = \int |\psi|^2 \, dx,$$

$$H = \int [|\nabla \psi|^2 - |\psi|^4] dx$$

But if $H < 0$, solution blows up in finite time!

Statement of the problem

d) Critique of each method

(iii) For either method

- Either method provides sufficient conditions for nonlinear stability, when it works. Failure of method does not imply instability.

Statement of the problem

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• Either method provides sufficient conditions for nonlinear stability, when it works. Failure of method does not imply instability.

e) Open question: The two methods have almost nothing in common. Is there a way to combine the two approaches, to obtain a method stronger than either by itself?

2. Arnold's method
"Energy-Casimir method" in [B5]

a) Q: What is a Casimir?

A In usual formulation of Hamiltonian mechanics:

- Eq'ns of motion: $$\partial_t F = [F,H] = < \frac{\delta F}{\delta g}, J\frac{\delta H}{\delta g} >$$

- Constants of motion: $[F,H]=0$

- If J is canonical, there are no Casimirs.

  If J is not canonical, J may have a null space.

  Then $$J\frac{\delta C}{\delta g}=0$$

independent of H. C is a Casimir.

2. Arnold's method
"Energy-Casimir method" in [B5]

a) Q: What is a Casimir?

Example: Korteweg-deVries equation: $J = \partial_x$

Let $C = \int g(x,t)dx$

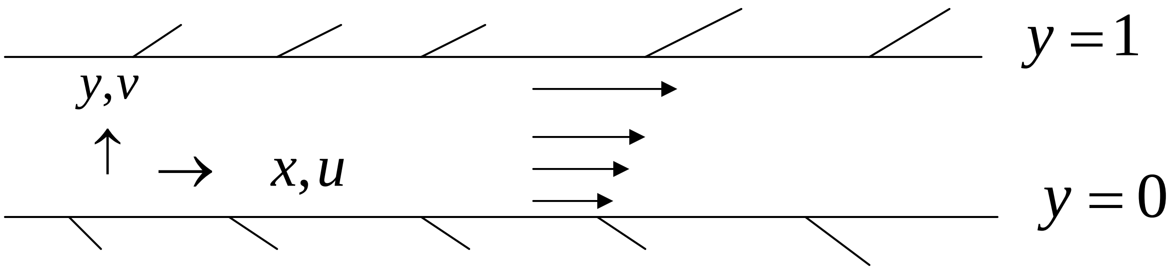
Then $\frac{\delta C}{\delta g} = 1 \quad \Rightarrow \quad J\frac{\delta C}{\delta g} = 0$

C is a Casimir - a quantity automatically conserved, independent of H. (See HW #2)

Arnold's method

b) model problem [B1,B2]

2-D flow of an ideal fluid between parallel walls



• eq'ns

$$\partial_t u + u\partial_x u + v\partial_y u + \partial_x p = 0,$$

$$\partial_t v + u\partial_x v + v\partial_y v + \partial_y p = 0,$$

$$\partial_x u + \partial_y v = 0,$$

$$v = 0 \qquad y = 0, y = 1$$

periodic b.c. in x

Arnold's method

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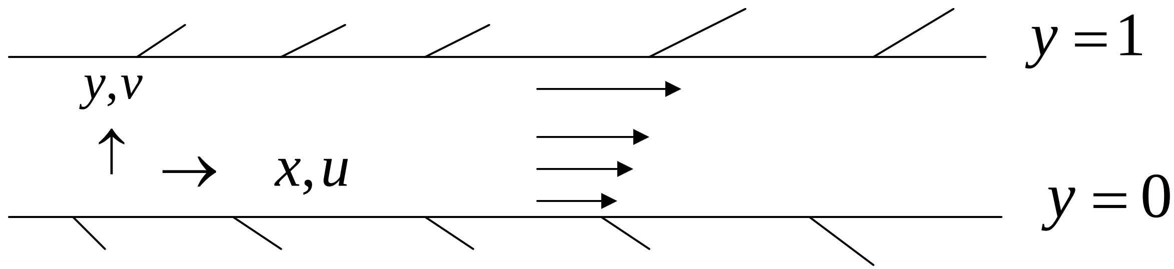

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$$\partial_x u + \partial_y v = 0,$$

$$v = 0 \quad y = 0, y = 1$$

- stationary sol'n

$$u = U(y),$$

$$v = 0,$$

$$p = 0.$$

periodic b.c. in x

Is this flow stable?

Arnold's method

c) Rewrite the problem

- Introduce stream function, $\psi(x,y,t)$

$$u = \partial_y \psi, \quad v = -\partial_x \psi.$$

$$\Rightarrow \quad \psi = \psi_o + \int_{(0,0)}^{(x,y)} (u \cdot dy - v \cdot dx)$$

on $y = 0$, ($dy = 0$, $v = 0$), ψ = const

on $y = 1$, ψ = const

Arnold's method

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on $y = 0$, ($dy = 0$, $v = 0$), ψ = const

on $y = 1$, ψ = const

- Eliminate pressure - define vorticity

$$\omega = -\nabla \times \vec{u} = -\partial_x v + \partial_y u = \nabla^2 \psi$$

Arnold's method

c) Rewrite the problem

- Introduce vorticity

$$\omega = -\nabla \times \vec{u} = -\partial_x v + \partial_y u = \nabla^2 \psi$$

- (1) eq'ns of motion

$$\partial_t \omega + u \partial_x \omega + v \partial_y \omega = 0$$

Arnold's method

c) Rewrite the problem

• Introduce vorticity

$$\omega = -\nabla \times \vec{u} = -\partial_x v + \partial_y u = \nabla^2 \psi$$

• (1) eq'ns of motion

$$\partial_t \omega + u \partial_x \omega + v \partial_y \omega = 0$$

(2) following a fluid particle $\Rightarrow \quad \dfrac{D\omega}{Dt} = 0$

$\Rightarrow$ Each fluid particle labelled by its vorticity

Arnold's method

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(2) following a fluid particle $\Rightarrow$ $\dfrac{D\omega}{Dt} = 0$

$\Rightarrow$ Each fluid particle labelled by its vorticity

(3) $$\partial_t \omega + [\partial_y \psi \cdot \partial_x \omega - \partial_y \psi \cdot \partial_x \omega] = 0$$

$$Jac(\psi, \omega)$$

Arnold's method

c) Rewrite the problem

$$\partial_t \omega + \underbrace{[\partial_y \psi \cdot \partial_x \omega - \partial_y \psi \cdot \partial_x \omega]}_{Jac(\psi,\omega)} = 0$$

For the stationary flow, $G(x):\quad \begin{aligned} u &= U(y), \\ v &= 0, \\ p &= 0, \end{aligned}$ $\quad \begin{aligned} \overline{\psi} &= \int_0^y U(\eta)\, d\eta \\ \overline{\omega} &= U'(y) \end{aligned}$

Arnold's method

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$$\partial_t \overline{\omega} = 0 \quad \Rightarrow \quad Jac(\overline{\psi}, \overline{\omega}) = 0$$

$\Rightarrow \nabla\overline{\psi}, \nabla\overline{\omega}$ are parallel

Arnold's method

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$\Rightarrow \nabla \overline{\psi}, \nabla \overline{\omega}$ are parallel

Arnold assumes that $\dfrac{d\overline{\omega}}{dy} = U''(y) \neq 0$

Rayleigh [B7] had shown that U"(y)=0 => instability

3. The algorithm

Holm, Marsden, Ratiu, Weinstein [B5]

Step 1: Start with a Hamiltonian problem

3. The algorithm

Holm, Marsden, Ratiu, Weinstein [B5]

Step 1: Start with a Hamiltonian problem

Arnold skips step 1.

(See [B5] - Hamiltonian structure is complicated, not needed.)

The algorithm

Holm, Marsden, Ratiu, Weinstein [B5]

Step 2: Find "enough" constants of the motion. Combine them with Lagrange multipliers.

- kinetic energy: $$H = \int [\frac{u^2 + v^2}{2}] dxdy$$
- horizontal momentum:

$$\partial_t u + \partial_x (u^2 + p) + \partial_y (uv) = 0 \quad \Rightarrow \quad \int [u] dxdy$$

- on $y = 0$, $y = 1$:

$$\partial_t u + \partial_x (u^2 + p) = 0 \quad \Rightarrow \quad \begin{array}{l} \int [u]_{y=0} dx \\ \int [u]_{y=1} dx \end{array}$$

The algorithm

Holm, Marsden, Ratiu, Weinstein [B5]

Step 2: Find "enough" constants of the motion. Combine them with Lagrange multipliers.

- ω is carried by each fluid particle, and particles are conserved, so $\int [\omega] dxdy$

- More generally, any smooth function of w is conserved. $\Rightarrow \int \Phi(\omega) dxdy$

The algorithm

Holm, Marsden, Ratiu, Weinstein [B5]

Step 2: Combine them with Lagrange multipliers.

$$K = \int [\frac{u^2 + v^2}{2} + \mu u + \Phi(\omega)] dxdy$$
$$+ \lambda_0 \int [u]_{y=0} dx + \lambda_1 \int [u]_{y=1} dx$$

Step 3: If possible, choose Lagrange multipliers so that $\delta K = 0$ on the stationary solution.

(If not possible, stop. You lose.)

The algorithm

Holm, Marsden, Ratiu, Weinstein [B5]

Step 3: If possible, choose Lagrange multipliers so that $\delta K = 0$ on the stationary solution.

Q: What variables can be varied?

A: ω, or ψ, or (u,v) but such that $\nabla \cdot u = 0$

- Do it!

The algorithm

Holm, Marsden, Ratiu, Weinstein [B5]

Step 3: If possible, choose Lagrange multipliers so that $\delta K = 0$ on the stationary solution.

Result:

$$\delta K = \int [U(y) + \mu - \partial_y(\Phi'(\overline{\omega}))][\delta u]dxdy$$

We need

$$\partial_y(\Phi'(\overline{\omega})) = U(y) + \mu$$

The algorithm

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$$\partial_y(\Phi'(\overline{\omega})) = U(y) + \mu$$

Define

$$\Psi(y) = \int^y U(\eta)d\eta$$

Then

$$\frac{d\Psi}{dy} = U(y) + \mu = \partial_y(\Phi'(\overline{\omega})) \quad \Rightarrow \quad \Psi(y) = \Phi'(\overline{\omega})$$

The algorithm

Holm, Marsden, Ratiu, Weinstein [B5]

Step 4: No flexibility is left. Take the second variation, $\delta^2 K$, and hope that $\delta^2 K$ is positive (or negative) definite. (If not, you lose.)

Find:

$$\delta^2 K = \frac{1}{2}\int [(\delta u)^2 + (\delta v)^2 + \Phi''(\overline{\omega})(\partial_y \delta u - \partial_x \delta v)^2]\,dxdy$$

The algorithm

Holm, Marsden, Ratiu, Weinstein [B5]

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Find:

$$\delta^2 K = \frac{1}{2}\int [(\delta u)^2 + (\delta v)^2 + \Phi''(\overline{\varpi})(\partial_y \delta u - \partial_x \delta v)^2] dxdy$$

If $\Phi''(\overline{\varpi}) \geq 0$, then $\delta^2 K$> 0 unless $\delta u \equiv 0, \delta v \equiv 0$.

"Formal stability" in [B5].

The algorithm

Holm, Marsden, Ratiu, Weinstein [B5]

Step 4: Second variation.

Result: If $\Phi''(\overline{\omega}) > 0$, then $\delta^2 K$ is positive definite! Hooray.

The algorithm

Holm, Marsden, Ratiu, Weinstein [B5]

Step 4: Second variation.

Result: If $\Phi''(\overline{\omega}) \geq 0$, then $\delta^2 K$ is positive definite! Hooray.

Facts:

- $\delta^2 K$ is the Hamiltonian of the linearized flow.
- $\delta^2 K$ is also a quadratic form on (δu, δv).
- In this problem, there are no degeneracies, so $\delta^2 K$ being positive definite guarantees linear stability!!
- $\Phi''(\overline{\omega}) \geq 0$ => {U''(y) $\neq$0 => linear stability}.

The algorithm

Holm, Marsden, Ratiu, Weinstein [B5]

Steps 5 & 6: Convexity & nonlinear stability

- Let (δu, δv) be small but finite. ($U(y)+\delta$u,δv) satisfies equations of motion (so $\nabla \cdot (\delta\vec{u}) = 0$).
- $K(U+\delta$u, δv) is a constant of the motion.
- $\Delta K = K(U+\delta$u, δv)- $K(U, 0)$ =

$$= \frac{1}{2}\int [(\delta u)^2 + (\delta v)^2 + \Phi''(\overline{\omega})(\delta\tilde{\omega})^2]dxdy$$

The algorithm

Steps 5 & 6: Convexity & nonlinear stability

$$\Delta K = \frac{1}{2}\int [(\delta u)^2 + (\delta v)^2 + \Phi''(\overline{\omega})(\delta\tilde{\omega})^2]\,dxdy$$

Last step. Further restrict $\Phi''(\overline{\omega}) = \dfrac{U(y)+\mu}{U''(y)}$

So that $0 < c \leq \Phi''(\overline{\omega}) \leq C < \infty$

Then ΔK bounds a (Sobolev-type) norm, and proves nonlinear stability.

Done.