C. Nonlinear stability of KdV soliton

1. Last time -

   Arnold's method on Arnold's problem

   - Everything worked, easily
   - Appropriate norm dictated by problem
   - $\delta^2 K \geq 0$ at step 4 $\Rightarrow$ Linear stability (weird?)

2. Today - nonlinear stability of KdV soliton

   - See Appendix of [A8] for very detailed analysis (& readable)
   - Talk to R. Kollar for geometry of problem

1. Statement of problem

KdV $\quad \partial_\tau u + 6u\,\partial_\xi u + \partial_\xi^3 u = 0 \qquad -\infty < \xi < \infty$

$\tau > 0$

u & all derivatives $\to 0$ as $|\xi| \to \infty$

Hamiltonian structure

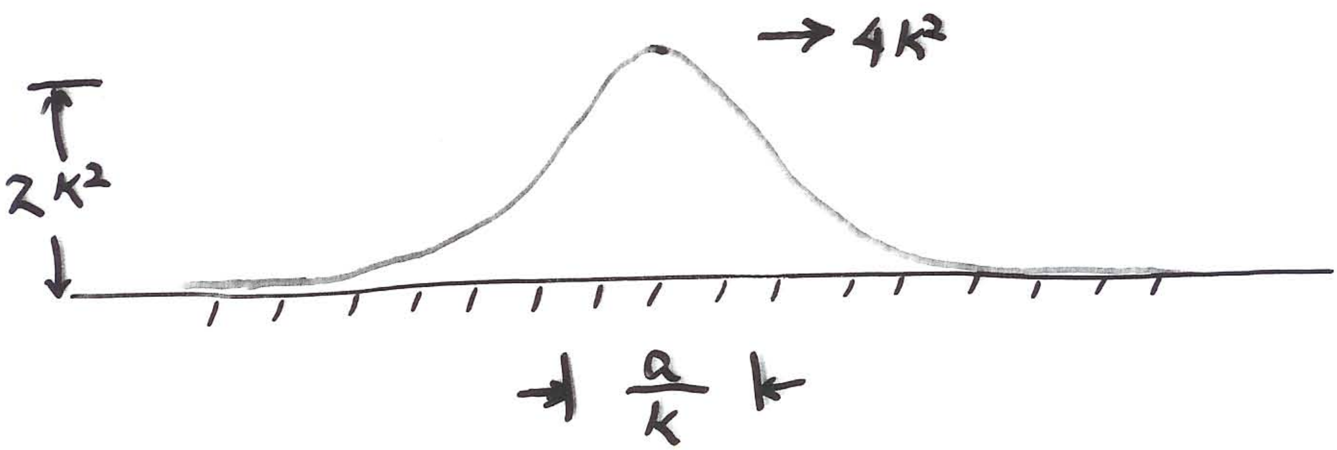
$$H = \int \left[\tfrac{1}{2}(\partial_\xi u)^2 - u^3 \right] d\xi$$

$$J = \partial_\xi$$

$$\text{KdV} \quad \leftrightarrow \quad \partial_\tau u = J \frac{\delta H}{\delta u}$$

Solitary wave solution (soliton)

$$u(\xi,\tau) = 2K^2 \operatorname{sech}^2(K(\xi - 4K^2\tau + \theta))$$



<u>Q</u>: Is this solution of KdV stable

to small perturbations of initial data?

b) Comments & complications (many)

- 2 free constants : K, θ

  K - "strength" of soliton

  θ - phase constant

- orbital stability

perturbation = $K+\delta K$, $\delta K \ll K$

$\Rightarrow$ 2 waves diverge slowly

$\Rightarrow$ $\| G(\xi - 4K^2\tau) - u(\xi,\tau) \| \rightarrow 2\|G\|$

as $\tau \rightarrow \infty$

~~unstable~~ ?

- "orbital stability" means:

$\underset{y}{\text{infimum}} \| G(\xi - 4K^2\tau) - u(\xi + y, \tau) \|$

(value judgement)

(causes problems later)

• constants of motion

$I_1 = \int [u] \, d\xi$ Casimir

$I_2 = \int [u^2] \, d\xi$ L_2 norm

$I_3 = -H = \int [u^3 - \frac{1}{2}(\partial_\xi u)^2]$ Hamiltonian

⋮

more

⋮

• important constraint

For a KdV soliton, $I_2 = \frac{16}{3} K^3$

$\Rightarrow$ For every positive value of I_2, there is a soliton with that value.

Require $\underbrace{I_2(u)}_{\text{perturbed solution}} = \underbrace{I_2(G)}_{\text{soliton}}$

Use constants of motion in 2 ways

- build Lyapunov functional
- constrain initial data

- A nonlinear constraint (Benjamin)

Let G represent solitary wave

$u = G + \delta u$ = perturbed solution

$$\int (G+\delta u)^2 \, d\xi = \int G^2 \, d\xi$$

$$\Rightarrow \boxed{2\int [G \cdot \delta u] \, d\xi = -\int (\delta u)^2 \, d\xi}$$

- For linear theory

$$\delta u << G$$

$$\Rightarrow \int [G \cdot \delta u] \, d\xi \sim 0$$

perturbation orthogonal to solitary wave

- For nonlinear theory

δu small, but finite

<u>not</u> orthogonal

$$\int [G \cdot \delta u] \, d\xi < 0$$

• Last consideration –

revise KdV equation (optional)

Define $x = \xi - 4K^2\tau$, $t = \tau$

$$\Rightarrow \quad \partial_\xi = \partial_x , \qquad \partial_\tau = \partial_t - 4K^2 \partial_x$$

$$u(\xi,\tau) = v(x,t)$$

Then

KdV: $\partial_t v + 6v\partial_x v - 4K^2 \partial_x v + \partial_x^3 v = 0$

solitary wave: $G(x) = 2K^2 \operatorname{sech}^2(Kx + \theta)$

G(x) satisfies:

$$\partial_x [\partial_x^2 G + 3G^2 - 4K^2 G] = 0$$

Hamiltonian:

$$\hat{H} = \int [\tfrac{1}{2}(\partial_x v)^2 - v^3 + 2K^2 v^2]\, dx$$

$$J = \partial_x$$

$$I_1 = \int v\, dx , \quad I_2 = \int v^2 dx$$

Ready!

3. The Arnold machine, modified

Step 1: Hamiltonian eq'n?

Yes $\quad \partial_t v = J \frac{\delta \hat{H}}{\delta v} \leftrightarrow$ KdV

Step 2: Constants of motion

$$K(v) = \hat{H} + c_1 I_1 + c_2 I_2$$

$$= \int_{-\infty}^{\infty} [\tfrac{1}{2}(\partial_x v)^2 - v^3 + 2K^2 v^2 + c_1 v + c_2 v^2] dx$$

Step 3: 1^{st} variation

Require $\delta K|_{v=G} = 0$ (if possible)

$$K(G+\delta v) = K(G) + \delta K + O(\delta v^2)$$

$$\delta K = \int [\delta v][-\partial_x^2 G - 3G^2 + 4K^2 G + c_1 + 2c_2 G] dx$$

$\Rightarrow \quad c_1 = 0, \quad c_2 = 0,$

$$-\partial_x^2 G - 3G^2 + 4K^2 G = 0$$

so $K(v) = \int_{-\infty}^{\infty} [\tfrac{1}{2}(\partial_x v)^2 - v^3 + 2K^2 v^2] dx$

No free parameters in $K(v)$

Step 4: 2^{nd} variation

$$K(G+\delta v) = K(G) + 0 + \delta^2 K$$
$$- \int (\delta v)^3 \, dx,$$

$$\delta^2 K = \int_{-\infty}^{\infty} \left[\tfrac{1}{2}(\partial_x(\delta v))^2 - 3G(\delta v)^2 + 2K^2(\delta v)^2 \right] dx$$

≥ 0 ✓ $\qquad \leq 0 \qquad \geq 0$ ✓

???

Concerns: $\delta^2 K \geq 0$?

$\int (\delta v)^3 \, dx$?

Response: Control $\int (\delta v)^3 dx$ by

choosing $\| \cdot \|$

Define 1^{st} Sobolev norm for $v(x,\cdot)$ on $\mathbb{R}$

$$\| v \|_{(1)} := \left\{ \int_{-\infty}^{\infty} \left[v^2 + (\partial_x v)^2 \right] dx \right\}^{\frac{1}{2}}$$

Why $\|\cdot\|_{(1)}$?

consider $v(x)$, cont. diff. on $\mathbb{R}$,
with $v(x) \to 0$ as $x \to -\infty$
& $v(x) \in H^1$

$$\int_{-\infty}^{x} 2v \frac{dv}{dy}\, dy = \int_{-\infty}^{x} 2v \frac{dv}{dy}\, dy$$

$$v^2(x) \leq 2\int_{-\infty}^{x} |v| \cdot \left|\frac{dv}{dy}\right| dy$$

$$\leq 2 \int_{-\infty}^{\infty} |v| \cdot \left|\frac{dx}{dy}\right| dy$$

$$\sup_x \left[v^2(x)\right] \leq \int_{-\infty}^{\infty} |v|^2 + \left|\frac{dv}{dy}\right|^2 dy$$

$$\Rightarrow \quad \boxed{\|v\|_\infty \leq \|v\|_{(1)}}$$

$$\text{so} \left| -\int_{-\infty}^{\infty} (\delta v)^3 dx \right| \leq \int_{-\infty}^{\infty} |\delta v|^3 dx$$

$$\leq \sup_x |\delta v| \int_{-\infty}^{\infty} |\delta v|^2 dx$$

$$\leq \|v\|_{(1)} \cdot \|v\|_{(1)}^2 = \|v\|_{(1)}^3$$

H^1 norm can control higher order term.

What about $\delta^2 K$?

$$\delta^2 K = \frac{1}{2}\int_{-\infty}^{\infty}\left[\,\partial_x(\delta v)^2 + (4\kappa^2 - 6G)(\delta v)^2\right]dx$$

$\delta^2 K$ is a quadratic form on (δv)

Define L so $\quad \delta^2 K = \frac{1}{2}\langle \delta v, L\,\delta v\rangle$

$$L = -\partial_x^2 + 4\kappa^2 - 12\kappa^2\,\text{sech}^2(\kappa x + \theta)$$

L is self-adjoint. Find its spectrum

$$Lf = \lambda f$$

$\lambda_1 = -5\kappa^2 \qquad f_1(x) = \text{sech}^3(\kappa x + \theta)$

$\lambda_2 = 0 \qquad f_2(x) = \text{sech}^2(\kappa x + \theta)\cdot\tanh(\kappa x + \theta)$

$\lambda_3 = 3\kappa^2 \qquad f_3(x) = 4\,\text{sech}(\kappa x + \theta) - 5\,\text{sech}^3(\kappa x + \theta)$

$\lambda_4 = 4\kappa^2 \qquad f_4(x) = 2\tanh(\kappa x + \theta) - 5\,\text{sech}^2(\kappa x + \theta)\tanh(\kappa x + \theta)$

λ_4 begins continuous spectrum

2 problems in spectrum

$\lambda_1 = -5K^2 < 0 \qquad f_1 = \operatorname{sech}^3(Kx + \theta)$

$\lambda_2 = 0 \qquad f_2 = c\, \partial_x G(x)$

<u>Result:</u> Let $\delta v(x) \in H^1$, & $y \in \mathbb{R}$

If $\int \partial_x G(x) \cdot \delta v(x)\, dx = 0$

& $\int G(x) \cdot \delta v(x)\, dx = -\frac{1}{2} \int (\delta v)^2\, dx$

Then there exist constants (c_1, c_2) depending only on $G(x)$ so that

$$\delta^2 K = \frac{1}{2} \langle \delta v, L\, \delta v \rangle \geq c_1 \|\delta v\|_{(1)}^2 - c_2 \left(\|\delta v\|_{(1)}^3 + \|\delta v\|_{(1)}^4 \right)$$

(see [AB], p 245 for calculation)

Good news:

By making δv small enough in $\|\cdot\|_{(1)}$,

we can make $\delta^2 K \geq c_3 \|\delta v\|_{(1)}$; $c_3 > 0$

This controls the cubic term in

$$K(G+\delta v) - K(G)$$

Bad news The "sliding norm" is not yet a norm. More work is required (see Appendix of [A 8].)

More general picture ([C5], [C6], [A2])

Several problems of stability of solitary wave G, with speed c, can be stated as:

$H(u)$ = Hamiltonian of flow

$$I_2(u) = \tfrac{1}{2}\int u^2\,dx$$

set $d(c) = H|_G + c\, I_2|_G$

$d''(c) > 0 \Leftrightarrow$ nonlinear stability