D. Lyapunov stability in dissipative eq'ns

1. statement of problem

a) $\partial_t u = N(u)$ $\quad u(x,t) \in S$ (f'n space)

(maybe) $u \to 0$ as $t \to \infty$ in some norm

- $G(x,t)$ is a known sol'n

  maybe $G = G(x)$, stationary

  maybe $G \to 0$ as $t \to \infty$

b) $G(x,t)$ is <u>asymptotically stable</u>

(in a norm $\|\cdot\|$) if for every $\epsilon > 0$

there is $\delta > 0$ such that

$\|G - u\| < \delta$ at $t = 0$

$\Rightarrow \|G - u\| < \epsilon$ for all $t > 0$

<u>and</u> $\|G - u\| \to 0$ as $t \to \infty$

for any $u \in S$

c) If $G \to 0$ as $t \to \infty$, only asymptotic stability is relevant. Probably we want $(u - G) \to 0$ <u>faster</u> than $G \to 0$

2. What is a Lyapunov functional?

- $F(u; G)$ is a functional on u

$$F : \mathcal{S} \to \mathbb{R}$$

- $F(u; G)\big|_{u = G} = 0$
- $F(u; G) > 0$ if $u \neq G$, $t < \infty$
- $\frac{dF}{dt} \leq 0$ & $F \to 0$ as $t \to \infty$

(perhaps can be weakened

- There is a norm $(\|\cdot\|)$ & constants (m, c, C): $m \geq 1$, $0 < c \leq C < \infty$

such that for all $t \geq 0$

$$0 < c\|u - G\|^m \leq F(u; G) \leq C\|u - G\|^m$$

How to find $F(u; G)$?

a) Often called "energy method"

(recall "energy-Casimir method")

b) Finding a good Lyapunov f'nal is an art form (see D. D. Joseph)

c) Proposal: For a mathematical model of a physical problem, consider thermodynamics.

- 1st Law of Thermodynamics

  In an isolated system, energy is conserved.

- 2nd Law of Thermodynamics

  S = entropy

  $$\frac{dS}{dt} \geq 0$$

<u>Q:</u> $F(u; G) = -S$?

Example #1 - Heat eq'n

a) $\partial_t u = K \nabla^2 u$, $K > 0$, const

+ b.c.

[same result in any dimension.
Consider $n=1$]

<u>1-D:</u> $\partial_t u = K \partial_x^2 u$, $u \to 0$ as $|x| \to$ or $\partial_x u = 0$ on boundary

- $\frac{d}{dt} \int u \, dx = K \partial_x u \Big|_{\text{boundaries}}$

$= 0$

$\Rightarrow \int u \, dx = \text{const}$

- $u \cdot \partial_t u = u \cdot K \partial_x^2 u$

$\partial_t (\frac{1}{2} u^2) = \partial_x (K u \partial_x u) - K (\partial_x u)^2$

$\Rightarrow \frac{d}{dt} \int u^2 \, dx = -2K \int (\partial_x u)^2 \, dx \leq$

$\Rightarrow \frac{d}{dt} \int u^2 \, dx < 0$ unless $u = \text{cons}$

- $\Rightarrow$ uniqueness of solutions

& $\Rightarrow$ $u = 0$ is asymptotically stable

3. Heat eq'n

b) Diffusion of heat in an isolated container of non-reacting gas with no mean motion.

T_0 = reference temperature

E_0 = Internal energy at temp. T_0

$u(x,t) = T - T_0$ (T = absolute temp)

Internal energy $c_p u + E_0$

c_p ← specific heat at const pressure

- Interpret $\int u\,dx = \text{const}$

<u>Ans:</u> 1st Law of Thermo

$$\frac{dE}{dt} = \frac{d}{dt}\int c_p u\,dx = c_p \frac{d}{dt}\int u\,dx = 0$$

$c_p u(x,t)$ is energy density

- Interpret $\frac{d}{dt}\int u^2\,dx$?

Consider <u>entropy</u> of ideal gas [D2]

$$s = s_0 + c_p \ln\left(\frac{T}{T_0}\right) - R \ln\left(\frac{p}{p_0}\right)$$

For system at const pressure, & small deviations in temperature from T_0

$$s = s_0 + c_p \ln\left(\frac{T_0 + u}{T_0}\right) + s_1$$

$$s = s_2 + c_p \ln\left(1 + \frac{u}{T_0}\right)$$

$$s \sim s_2 + c_p \left[\frac{u}{T_0} - \frac{1}{2}\left(\frac{u}{T_0}\right)^2 + O\left(\left(\frac{u}{T_0}\right)^3\right)\right]$$

$$\int s\,dx = \int (s_2)\,dx + \frac{c_p}{T_0}\int u\,dx$$

$$- \frac{c_p}{2T_0^2}\int u^2\,dx + O(u^3)$$

$$\frac{d}{dt}\int s\,dx = 0 + 0$$

$$- \frac{c_p}{2T_0^2}\frac{d}{dt}\int u^2\,dx$$

$\Rightarrow \frac{d}{dt}\int u^2\,dx \leq 0$ <u>is</u> 2nd Law of Thermodynamics

4. 2nd example: Boltzmann's H-theorem [D1, D3]

a) R. C. Tolman, <u>Principles of statistical Physics</u>, 1938, p 134

The derivation of this theorem and the appreciation of its significance may be regarded as among the greatest achievements of physical science.

4. Boltzmann's H-theorem

In a spatially uniform state of a dilute gas with spherical molecules

- $f(\vec{c}, t)$ = probability density of finding a molecule with speed $\vec{c}$ at time t.

$$\Rightarrow \quad f \geq 0, \qquad \int f \, d\hat{c} = 1$$

- Kinetic energy of a molecule is $\frac{m|c|^2}{2}$

Total energy of gas in domain D

$$E = \int_D d\hat{x} \int d\hat{c} \; f(\vec{c}, t) \cdot \frac{m|c|^2}{2}$$

Energy is conserved

- Boltzmann's eq'n

$$\partial_t f(\vec{c}, t) = \int (f' f_1' - f f_1) \underbrace{\kappa(\vec{c}_1)}_{\geq 0} d\hat{c},$$

4. Boltzmann's H-theorem

- Define $H = \int f \ln f \, d\hat{c}$

  (H = - entropy)

- $$\frac{dH}{dt} = \int (\ln f + 1) \frac{\partial f}{\partial t} \, d\hat{c}$$
  $$= \int \ln f \cdot \frac{\partial f}{\partial t} \, dc$$

  $$= \iint \ln f \, (f' f_1' - f f_1) \, \mathcal{K}(\hat{c}_1) \, d\hat{c}_1 \, d\hat{c}$$

- argument about "detailed balance"

$$\Rightarrow \frac{dH}{dt} = \iint \underbrace{\ln\left(\frac{f f_1}{f_1' f'}\right)(f' f_1' - f f_1)}_{} \underbrace{\mathcal{K}(\hat{c}_1)}_{\geq 0} \, d\hat{c}_1 \, d\hat{c}$$

$$\ln\left(\frac{a}{b}\right)(b-a) < 0 \quad \text{if } a \neq b$$
$$= 0 \quad \text{if } a = b$$

- H-theorem

$$\frac{dH}{dt} \leq 0 \quad \& \quad \frac{dH}{dt} = 0 \text{ only if}$$