Finite Volume Methods for Hyperbolic Problems

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Preface

Hyperbolic partial differential equations arise in a broad spectrum of disciplines where wave motion or advective transport is important: gas dynamics, acoustics, elastodynamics, optics, geophysics, and biomechanics, to name but a few. This book is intended to serve as an introduction to both the theory and the practical use of high-resolution finite-volume methods for hyperbolic problems. These methods have proved to be extremely useful in modeling a broad set of phenomena, and I believe that there is need for a book introducing them in a general framework that is accessible to students and researchers in many different disciplines.

Historically, many of the fundamental ideas were first developed for the special case of compressible gas dynamics (the Euler equations), for applications in aerodynamics, astrophysics, detonation waves, and related fields where shock waves arise. The study of simpler equations such as the advection equation, Burgers' equation, and the shallow water equations has played an important role in the development of these methods, but often only as model problems, with the ultimate goal being application to the Euler equations. This orientation is still reflected in many of the texts on these methods. Of course the Euler equations remain an extremely important application, and are presented and studied in this book, but there are also many other applications where challenging problems can be successfully tackled by understanding the basic ideas of high-resolution finite-volume methods. Often it is *not* necessary to understand the Euler equations in order to do so, and the complexity and peculiarities of this particular system may obscure the more basic ideas.

In particular, the Euler equations are *nonlinear*. This nonlinearity, and the consequent shock formation seen in solutions, leads to many of the computational challenges that motivated the development of these methods. The mathematical theory of nonlinear hyperbolic problems is also quite beautiful, and the development and analysis of finite-volume methods requires a rich interplay between this mathematical theory, physical intuition, and numerical analysis. As a result it is a challenging and satisfying field of study, and much of this book focuses on nonlinear problems.

However, all of Part I and much of Part III (on multidimensional problems) deals entirely with linear hyperbolic systems. This is partly because many of the concepts can be introduced and understood most easily in the linear case. A thorough understanding of linear hyperbolic theory, and the development of high-resolution methods in the linear case, is extremely useful in fully understanding the nonlinear case. In addition, I believe there are many linear wave-propagation problems (*e.g.*, in acoustics, elastodynamics, or electromagnetics) where these methods have a great deal of potential that has not been fully exploited, particularly for problems in heterogeneous media. I hope to encourage students to explore some of these areas, and researchers in these areas to learn about finite-volume methods. I have tried to make it possible to do so without delving into the additional complications of the nonlinear theory.

Studying these methods in the context of a broader set of applications has other pedagogical advantages as well. Identifying the common features of various problems (as unified by the hyperbolic theory) often leads to a better understanding of this theory and greater ability to later apply these techniques to new problems. The finite-volume approach can itself lead to greater insight into the physical phenomena and mathematical techniques. The derivation of most conservation laws gives first an integral formulation that is then converted to a differential equation. A finite-volume method is based on the integral formulation, and hence is often closer to the physics than the partial differential equation.

Mastering a set of numerical methods in conjunction with learning the related mathematics and physics has a further advantage: it is possible to apply the methods immediately in order to observe the behavior of solutions to the equations, and thereby gain intuition for how these solutions behave. To facilitate this hands-on approach to learning, virtually every example in the book (and many examples not in the book) can be solved by the reader using programs and data that are easy to download from the web. The basis for most of these programs is the CLAWPACK software package, which stands for Conservation LAW PACKage. This package was originally developed for my own use in teaching and so is intimately linked with the methods studied in this book. By having access to the source code used to generate each figure, it is possible for the interested reader to delve more deeply into implementation details that aren't presented in the text. Animations of many of the figures are also available on the webpages, making it easier to visualize the time-dependent nature of these solutions. By downloading and modifying the code, it is also possible to experiment with different initial or boundary conditions, with different mesh sizes or other parameters, or with different methods on the same problem.

CLAWPACK has been freely available for several years and is now extensively used for research as well as teaching purposes. Another function of this book is to serve as a reference to users of the software who desire a better understanding of the methods employed and the ways in which these methods can be adapted to new applications. It is *not*, however, designed to be a User's Manual for the package, and it is not necessary to do any computing in order to follow the presentation in the book.

There are many different approaches to developing and implementing high-resolution finite-volume methods for hyperbolic equations. In this book I concentrate primarily on one particular approach, the "wave-propagation algorithm" that is implemented in CLAWPACK, but numerous other methods and the relation between them are discussed at least briefly. It would be impossible to survey all such methods in any detail, and instead my aim is to provide enough understanding of the underlying ideas that the reader will have a good basis for learning about other methods from the literature. With minor modifications of the CLAWPACK code it is possible to implement many different methods and easily compare them on the same set of problems.

This book is the result of an evolving set of lecture notes that I have used in teaching this material over the past 15 years. An early version was published in 1989 after giving the course at ETH in Zürich [276]. That version has proved popular among instructors and students, perhaps primarily because it is short and concise. Unfortunately the same claim cannot be made for the present book. I have tried, however, to write the book in such a way that self-contained subsets can be extracted for teaching (and learning) this material. The latter part of many chapters gets into more esoteric material that may be useful to have available for reference but is not required reading. In addition, many whole chapters can be omitted without loss of continuity in a course that stresses certain aspects of the material. In particular, to focus on linear hyperbolic problems and heterogeneous media, a suggested set of chapters might be 1 – 9 and 18 – 21, omitting the sections in the multidimensional chapters that deal with nonlinearity. Other chapters may also be of interest, but can be omitted without loss of continuity. To focus on nonlinear conservation laws, the basic theory can be found in Chapters 1 – 8, 11 – 15, and 18 – 21. Again other topics can also be covered if time permits, or this can be shortened further by concentrating on scalar equations or one-dimensional problems, for example.

This book may also be useful in a course on hyperbolic problems where the focus is not on numerical methods at all. The mathematical theory in the context of physical applications is developed primarily in Chapters 1 – 3, 9, 11, 13, 14, 16, 18, and 22, chapters that contain little discussion of numerical issues. It may still be advantageous to use CLAWPACK to further explore these problems and develop physical intuition, but this can be done without a detailed study of the numerical methods employed.

To be continued...