Solutions

Let $c = \sqrt{gh_0}$ and $Z = c/g = \sqrt{h_0/g}$.
Z is analogous to the impedance in linear acoustics.

The eigenvalues are $\lambda^1 = -c$ and $\lambda^2 = +c$.
The eigenvectors are

$$r^1 = \begin{bmatrix} -Z \\ 1 \end{bmatrix}, \qquad r^2 = \begin{bmatrix} Z \\ 1 \end{bmatrix}.$$

You could choose any other normalization, multiplying each by a nonzero constant. The formulas for the waves and the solution to the Riemann problem should come out the same no matter how you normalize.

The matrices R and R^{-1} are:

$$R = \begin{bmatrix} -Z & Z \\ 1 & 1 \end{bmatrix}, \qquad R^{-1} = \frac{1}{2Z} \begin{bmatrix} -1 & Z \\ 1 & Z \end{bmatrix}.$$

Solutions

Let

$$\delta = \begin{bmatrix} h_r - h_\ell \\ u_r - u_\ell \end{bmatrix}.$$

To find the waves we want to find α^1 and α^2 so that

$$\alpha^1 r^1 + \alpha^2 r^2 = \delta.$$

This is a linear system $R\alpha = \delta$ with solution $\alpha = R^{-1}\delta$.

$$\alpha = \frac{1}{2Z} \begin{bmatrix} -\delta_1 + Z\delta_2 \\ \delta_1 + Z\delta_2 \end{bmatrix}.$$

So, for example, the left-going wave is

$$\mathcal{W}^1 = \alpha^1 r^1 = \frac{1}{2Z}(-\delta_1 + Z\delta_2) \begin{bmatrix} -Z \\ 1 \end{bmatrix}.$$

Solutions

The intermediate state in the Riemann solution is $q_m = q_\ell + \mathcal{W}^1$.

Working this out gives the general solution:

$$\begin{aligned} h_m &= \frac{1}{2}(h_\ell + h_r) - \frac{Z}{2}(u_r - u_\ell) \\ u_m &= \frac{1}{2}(u_\ell + u_r) - \frac{1}{2Z}(h_r - h_\ell) \end{aligned}$$

Dam break: In particular, if $u_\ell = u_r = 0$ then

$$h_m = \frac{1}{2}(h_\ell + h_r), \qquad u_m = -\frac{1}{2Z}(h_r - h_\ell).$$

Note that the intermediate depth is the average of the two sides, the intermediate velocity is positive if $h_\ell > h_r$.

Solutions

General solution:

$$h_m = \frac{1}{2}(h_\ell + h_r) - \frac{Z}{2}(u_r - u_\ell)$$
$$u_m = \frac{1}{2}(u_\ell + u_r) - \frac{1}{2Z}(h_r - h_\ell)$$

Reflecting wall: if $h_\ell = h_r = \bar{h}$ and $u_\ell = -u_r = \bar{u}$ then

$$h_m = \bar{h} + Z\bar{u}, \qquad u_m = 0.$$

Note that the intermediate depth is greater than $\bar{h}$ if $\bar{u} > 0$, but less than $\bar{h}$ if $\bar{u} < 0$ (flow away from the wall).