Adjoint Methods for Guiding Adaptive Mesh Refinement in Wave Propagation Problems

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Objectives

The AMRClaw and GeoClaw software use block-structured adaptive mesh refinement to selectively refine around propagating waves [2]. For problems where a small region of the solution is of primary interest, solving the time−dependent adjoint equation and using a suitable inner product with the forward solution allows more precise refinement of the relevant waves.

Clawpack and Adaptive Mesh Refinement

Clawpack is a collection of finite volume methods to solve hyperbolic systems of conservation laws. Adaptive mesh refinement (AMR), implemented for Clawpack in AMRClaw, clusters grid points in areas of interest, such as discontinuities or regions where the solution has a complicated structure.

The mesh refinement used in AMRClaw utilizes various strategies for determining which areas of the solution need refinement. These strategies include:

- A Richardson error estimation procedure that compares the solution on the existing grid with the solution on a coarser grid, and refines cells where this error is greater than a specified tolerance [2].
- For geophysical flows, refining cells where the surface elevation of the water is perturbed from sea level above some set tolerance [1].

The Adjoint Method

Suppose we are interested in calculating the value of a functional

$$J = \int_a^b \varphi(x)^T q(x, t_f) dx,$$

where the q is the solution to the time dependent equation

$$q_t + A(x) q_x = 0$$

subject to some initial and boundary conditions. Note that

$$\int_a^b \int_{t_0}^{t_f} \varphi(x)^T (q_t + A(x) q_x)\, dx dt = 0.$$

If we

- integrate by parts,
- set $\varphi(x) = \hat{q}(x, t_f)$,
- require that $\hat{q}_t + (A(x)^T \hat{q})_x = 0$,
- and select the appropriate boundary terms for $\hat{q}$

this equation simplifies to

$$\int_a^b \hat{q}^T(x, t_f) q(x, t_f)\, dx = \int_a^b \hat{q}^T(x, t_0) q(x, t_0)\, dx.$$

Note that this requires solving the adjoint equation backward in time since data is given at the final time t_f.

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Implications

- The integral over the inner product between $\hat{q}$ and q at the final time is equal to this integral at any earlier time $t = t_0$.
- The locations where the inner product $\hat{q}(x, t_0)^T q(x, t_0)$ is large at time t_0 are the areas that will have a significant effect on the inner product at time $t = t_f$.
- Hence, the adjoint approach identifies at any given time step the areas that will most influence the final inner product.
- We can apply AMR to only these areas.

Acoustics Example

Consider the linear acoustics equations in one dimension in a piecewise constant medium,

$$q_t(x, t) + A q_x(x, t) = 0,$$

with wall boundary conditions and some initial condition. Here,

$$A(x) = \begin{bmatrix} 0 & K(x) \\ \frac{1}{\rho(x)} & 0 \end{bmatrix}, \quad q(x, t) = \begin{bmatrix} p(x, t) \\ u(x, t) \end{bmatrix},$$

p is the pressure, u is the velocity, K is the bulk modulus, and ρ is the density. The adjoint problem for this system is

$$\hat{q}_t + \left(A^T \hat{q}\right)_x = 0 \quad x \in [a, b],\ t > 0$$
$$\hat{u}(a, t) = 0,\ \hat{u}(b, t) = 0 \quad t \geq 0,$$

where we can pick the function $\varphi(x) = \hat{q}(x, t_f)$ to highlight some region of interest. Consider an example where $a = -5$, $b = 3$, $K = 1$ and

$$\rho = \begin{cases} 1 & \text{if } x < 0, \\ 4 & \text{if } x > 0. \end{cases}$$

As initial data for $q(x, t)$ we take a Gaussian hump in pressure about $x = -2$ and a zero velocity. For $\hat{q}(x, t_f)$ we take a square pulse in $\hat{p}$ centered about $x = 2$, and $\hat{u} = 0$ at $t_f = 20$. The results below show that:

- The inner product clearly identifies which regions will influence our region of interest at the final time.
- Using AMR in all the regions where the 1-norm of q is large would result in refinement of areas that will have no effect on our area of interest at the final time.

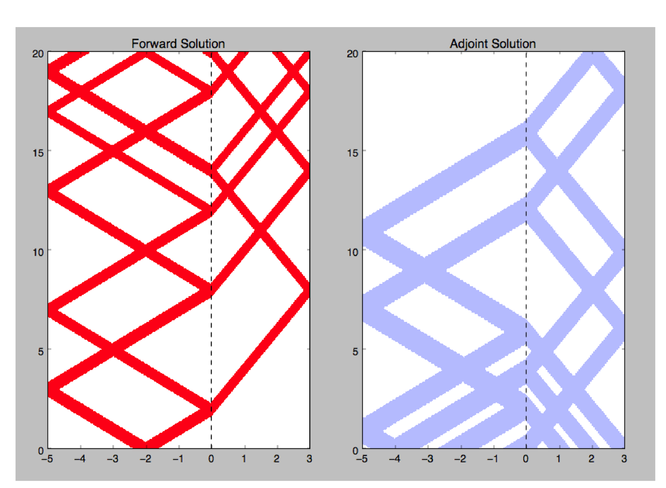


Figure 1: One-dimensional acoustics. The 1-norm of q (forward) and $\hat{q}$ (adjoint) in the (x, t) plane are shown in the left and right figures respectively.

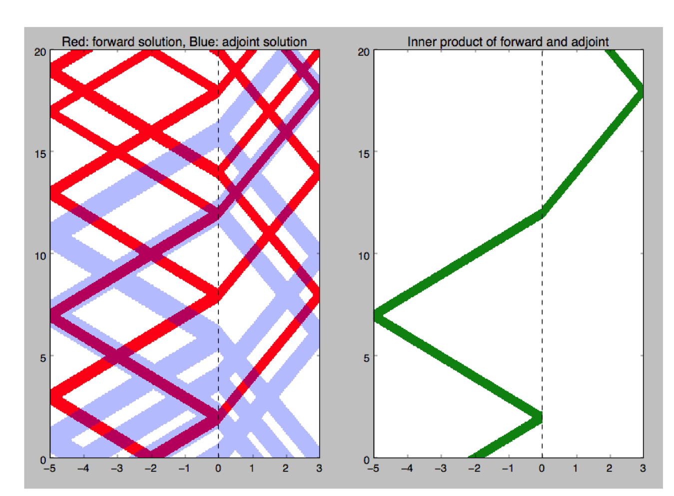


Figure 2: The 1-norm of q and $\hat{q}$ in the (x, t) plane are overlayed in the left figure, and their inner product is shown in the right, showing the region needing refinement.

Incorporating a Time Range

Suppose we want good accuracy in a region of interest over a range of times $t_s \leq t \leq t_f$. Then at time t we should refine where

$$\max_{T(t) \leq \tau \leq t} \int_a^b \hat{q}(x, \tau) q(x, \tau)\, dx$$

exceeds some tolerance, where $T(t) = \max(t + t_s - t_f,\ 0)$. (Using the fact that the adjoint for this problem is autonomous in time.)

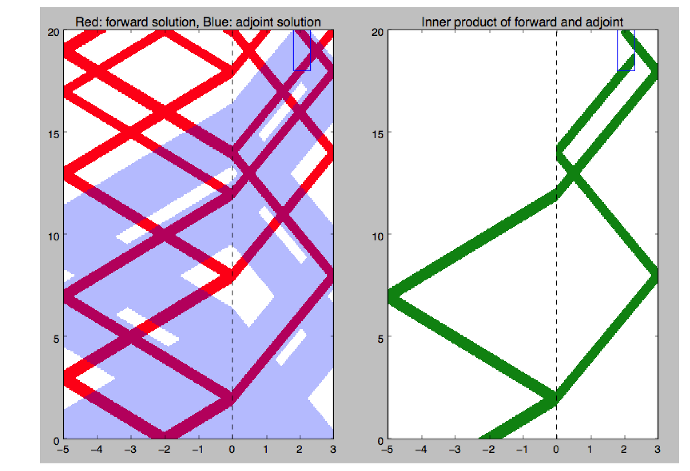
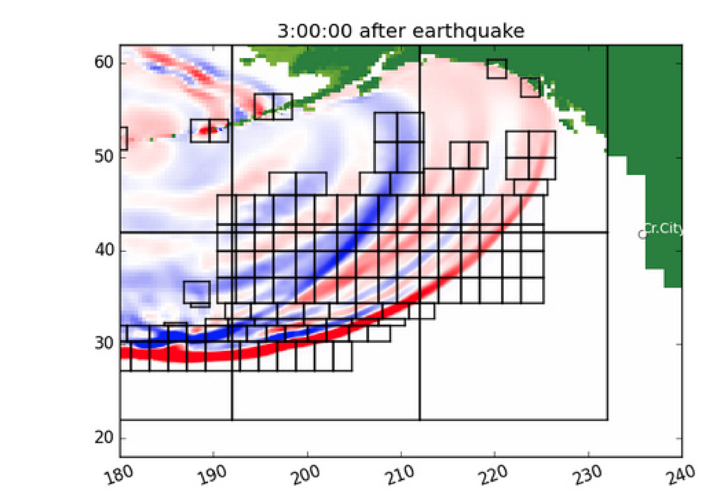


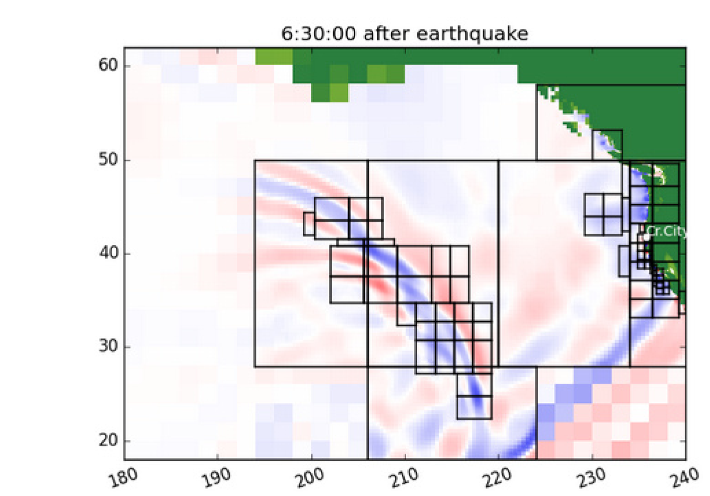
Figure 3: Left: 1-norm of q and $\hat{q}$ for $t_s = 18$. Right: Max inner product over time range, showing additional wave that must be refined.

Tsunami Propagation

This approach has been applied in GeoClaw, shown below for a tsunami originating on the Aleutian-Alaska Subduction Zone. The function φ is a square pulse around the tide gauge in Crescent City, California.

Note that using the adjoint method to guide AMR focuses the refinement on the parts of the wave that will actually reach the gauge. For this example, the linearlized shallow water equations were used for the adjoint problem.





Conclusion

Using the adjoint method to guide adaptive mesh refinement can reduce the computational expense of solving a system of equations by enabling targeted refinement of the regions of the domain that will influence a specific area of interest. A paper is in preparation [3] to present this work.

References

[1] M. J. Berger, D. L. George, R. J. LeVeque, and K. T. Mandli. The geoclaw software for depth-averaged flows with adaptive refinement. *Advances in Water Resources*, 24:1195–1206, 2011.

[2] M. J. Berger and R. J. LeVeque. Adaptive mesh refinement using wave-propagation algorithms for hyperbolic systems. *SIAM J. Numer. Anal.*, 35(6):2298–2316, October 1998.

[3] B. N. Davis and R. J. LeVeque. Adjoint methods for guiding adaptive mesh refinement in wave propagation problems. In preparation, 2015.

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