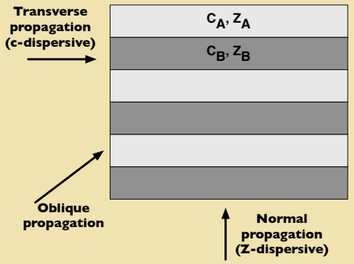
Dispersion of acoustic waves in a layered medium

$$u_{tt} = \nabla \cdot \left(\frac{1}{\rho(\mathbf{x})} \nabla (K(\mathbf{x}) u) \right)$$

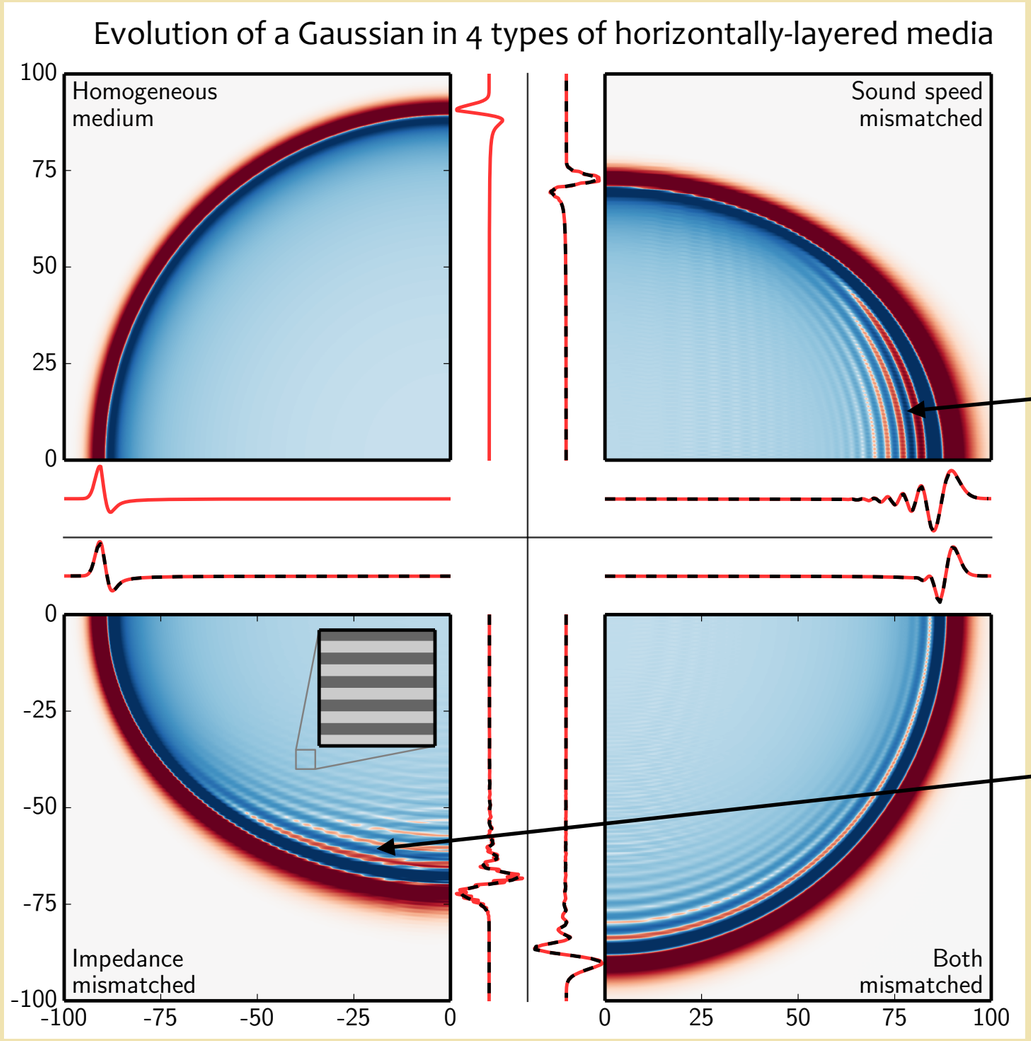
Sound speed: $c = \sqrt{\frac{K}{\rho}}$

Impedance: $Z = \sqrt{K\rho}$

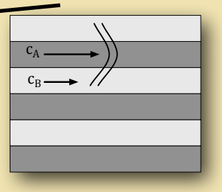


Dispersion of **normally-propagating** waves is caused by **reflection** and depends on **impedance** variation (*Z-dispersion*)

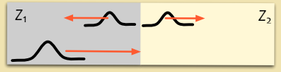
Dispersion of **transverse-propagating** waves is caused by **diffraction** and depends on **sound speed** variation (*c-dispersion*)



Dispersed by diffraction



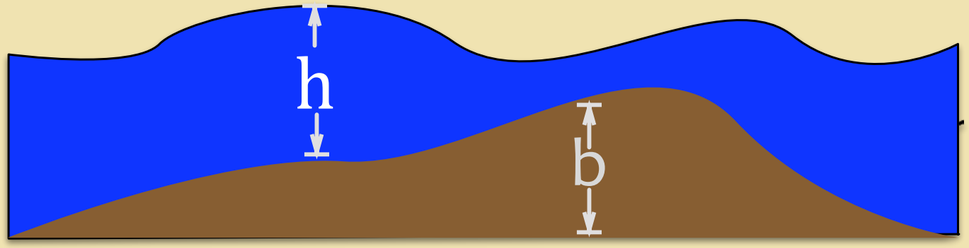
Dispersed by reflection



Shallow water waves

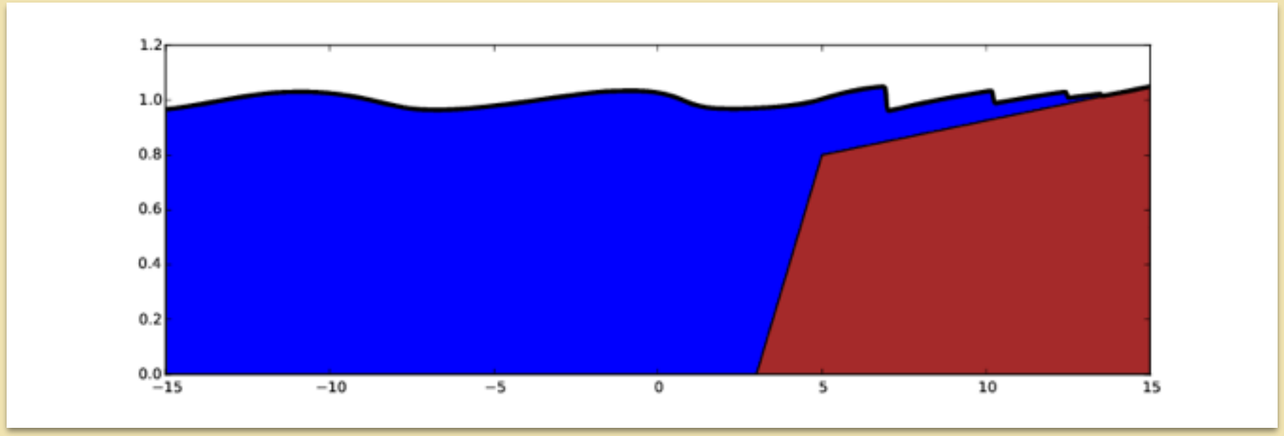
Shallow water equations:

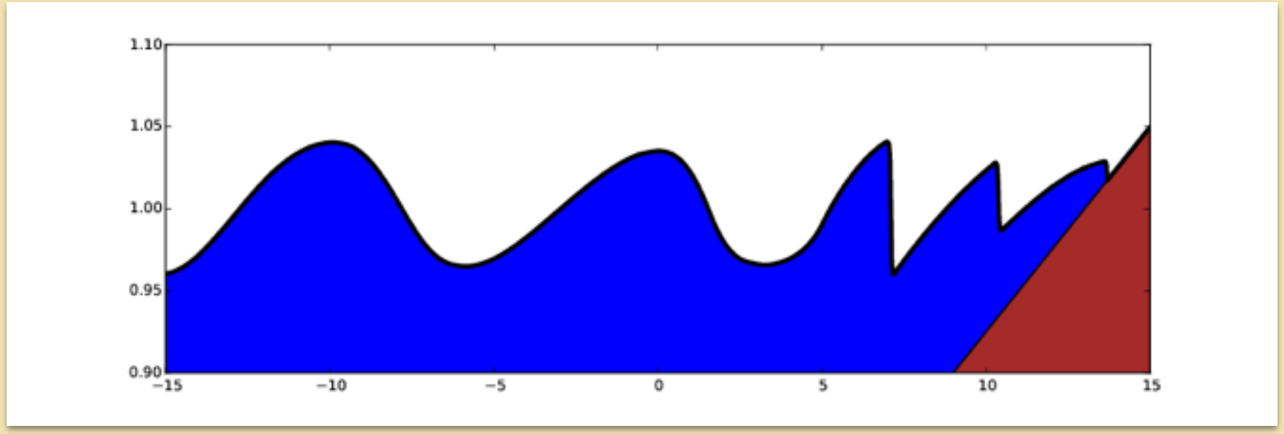
$$h_t + (hu)_x + (hv)_y = 0$$
$$(hu)_t + (hu^2 + gh^2/2)_x + (huv)_y = -ghb_x$$
$$(hv)_t + (huv)_x + (hv^2 + gh^2/2)_y = -ghb_y$$



h: depth
u, v: velocity
g: gravitational constant
b: bottom elevation

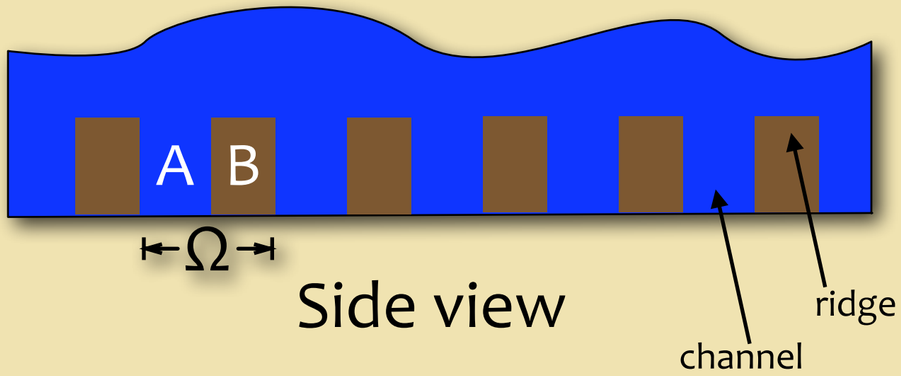
characteristic velocity: $c = u \pm \sqrt{gh}$

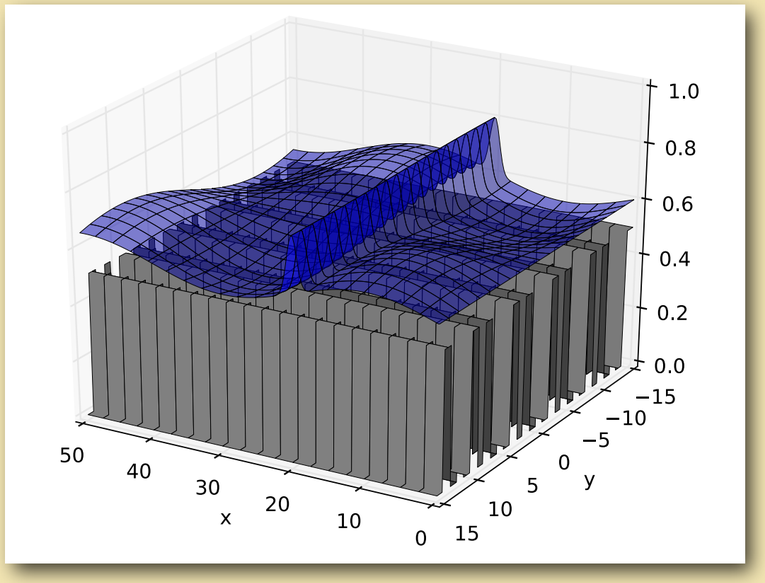


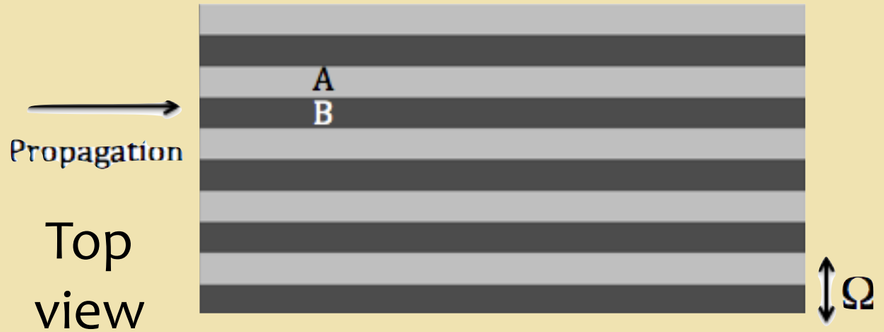


Shallow water waves propagate in a nonlinear way, leading to wave breaking

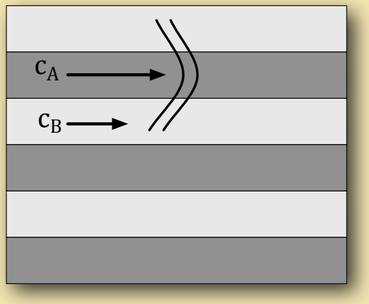
Shallow water waves over periodic bathymetry







$$c = u \pm \sqrt{gh}$$



Faster propagation over channels
Slower propagation over ridges

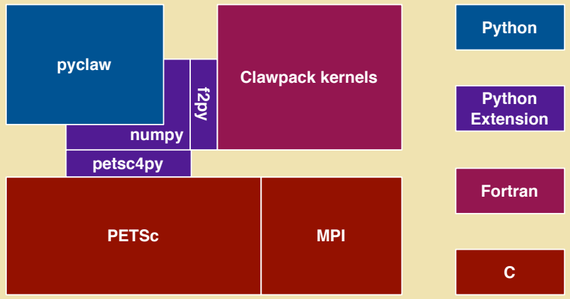
-> leads to diffractive dispersion

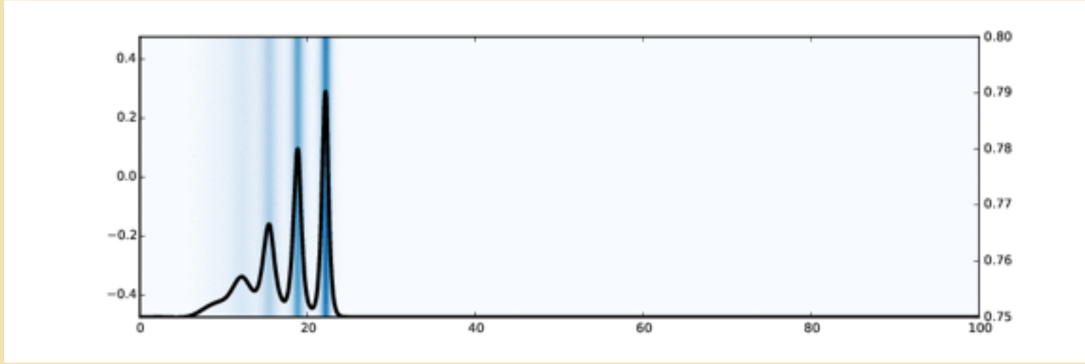
Shallow water “diffractons"

Numerical methods

Clawpack

- Godunov-type finite volume methods
- Parallelism via PyClaw
- f-wave shallow water Riemann solver

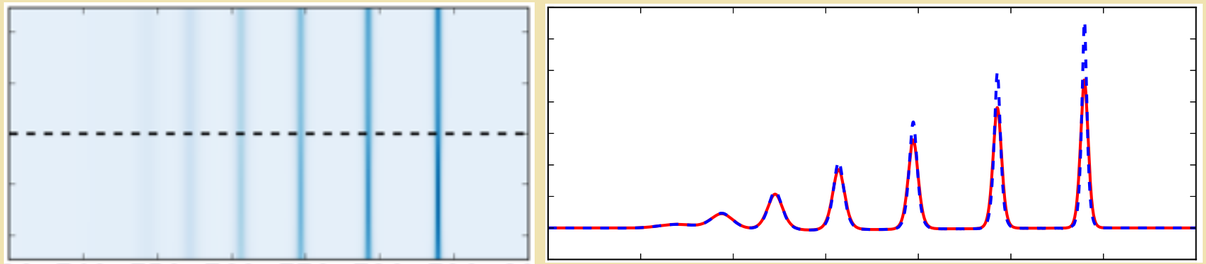




- *Shading: wave height*
- *Black curve: trace of solution at fixed y-value*

Nonlinearity + effective dispersion => solitary waves
No wave breaking

Any initial pulse evolves into a train of solitary waves:



Try it yourself:
http://nbviewer.jupyter.org/gist/ketch/9250942

Main findings

- Both **reflection** and **diffraction** can create a **dispersive** effect in periodic media
- Strong reflection or diffraction can prevent shock formation and lead to **solitary wave** formation
- This is predicted to occur in a broad range of physical models, including nonlinear **elasticity**, shallow **water waves**, and compressible **fluid dynamics**

References

- Santosa & Symes, *SIAP* 1991
- LeVeque & Yong, *SIAP* 2003
- Ketcheson & LeVeque, *Comm. Math. Sci.* 2012
- Quezada de Luna & Ketcheson, *SIAP* 2014
- Ketcheson & Quezada de Luna, *MM&S* 2015

See
http://davidketcheson.info/publications.html
(click “nonlinear waves”)