Spin and
Quantum Measurement

David H. McIntyre
Oregon State University

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Chapter 1  STERN-GERLACH EXPERIMENTS

It was not a dark and stormy night when Otto Stern and Walther Gerlach performed their now famous experiment in 1922. The Stern-Gerlach experiment demonstrated that measurements on microscopic or quantum particles are not always as certain as we might expect. Quantum particles behave as mysteriously as Erwin's socks—sometimes forgetting what we have already measured. Erwin's adventure with the mystery socks is farfetched because you know that everyday objects do not behave like his socks. If you observe a sock to be black, it will remain black no matter what other properties of the sock you observe. However, the Stern-Gerlach experiment goes against these ideas. You will see that microscopic or quantum particles do not behave like the classical objects of your everyday experience. The act of observing a quantum particle affects its measurable properties in a way that is foreign to our classical experience.

In these first three chapters, we focus on the Stern-Gerlach experiment because it is a conceptually simple experiment that demonstrates many basic principles of quantum mechanics. We will discuss a variety of experimental results and the quantum theory that has been developed to predict those results. The mathematical formalism of quantum mechanics is based upon six postulates that we will introduce as we develop the theoretical framework. We use the Stern-Gerlach experiment to learn about quantum mechanics theory for two primary reasons: (1) It demonstrates how quantum mechanics works in principle by illustrating the postulates of quantum mechanics, and (2) It demonstrates how quantum mechanics works in practice through the use of Dirac notation and matrix mechanics to solve problems. By using a simple example, we can focus on the principles and the new mathematics, rather than having the complexity of the physics obscure these new aspects.

1.1  Stern-Gerlach experiment

In 1922 Otto Stern and Walther Gerlach performed a seminal experiment in the history of quantum mechanics. In its simplest form, the experiment consists of an oven that produces a beam of neutral atoms, a region of inhomogeneous magnetic field, and a detector for the atoms, as depicted in Fig. 1.1. Stern and Gerlach used a beam of silver atoms and found that the beam was split into two in its passage through the magnetic field. One beam was deflected upwards and one downwards in relation to the direction of the magnetic field gradient.

To understand why this result is so at odds with our classical expectations, we must first analyze the experiment classically. The results of the experiment suggest an interaction between a neutral particle and a magnetic field. We expect such an interaction if the particle possesses a magnetic moment $\mu$. The energy of this interaction is given by $E = -\mu \cdot B$, which results in a force $F = \nabla(\mu \cdot B)$. In the Stern-Gerlach
Figure 1.1  Stern-Gerlach experiment to measure spin projection of neutral particles along the z-axis. The magnetic cross-section at right shows the inhomogeneous field used in the experiment.

For the experiment, the magnetic field gradient is primarily in the z-direction, and the resulting z-component of the force is

\[ F_z = \frac{\partial}{\partial z} (\mu \cdot B) \]

\[ = \mu_z \frac{\partial B_z}{\partial z} . \]

This force is perpendicular to the direction of motion and deflects the beam in proportion to the magnitude of the magnetic moment in the direction of the magnetic field gradient.

Now consider how to understand the origin of the atom's magnetic moment from a classical viewpoint. The atom consists of charged particles, which, if in motion, can produce loops of current that give rise to magnetic moments. A loop of area \( A \) and current \( I \) produces a magnetic moment

\[ \mu = IA \]  \hspace{1cm} (1.2)

in MKS units. If this loop of current arises from a charge \( q \) traveling at speed \( v \) in a circle of radius \( r \), then

\[ \mu = \frac{q}{2\pi r} \pi r^2 \]

\[ = \frac{qr v}{2} , \]

\[ = \frac{q}{2m} L \]  \hspace{1cm} (1.3)

where \( L = mrv \) is the orbital angular momentum of the particle. In the same way that the earth revolves around the sun and rotates around its own axis, we can also imagine a charged particle in an atom having orbital angular momentum \( L \) and intrinsic rotational angular momentum, which we call \( S \). The intrinsic angular momentum also creates current loops, so we expect a similar relation between the magnetic moment \( \mu \) and \( S \).
The exact calculation involves an integral over the charge distribution, which we will not do. We simply assume that we can relate the magnetic moment to the intrinsic angular momentum in the same fashion as Eq. (1.3), giving

\[ \mu = g \frac{q}{2m} S, \]

where the dimensionless gyroscopic ratio \( g \) contains the details of that integral.

A silver atom has 47 electrons, 47 protons, and 60 or 62 neutrons (for the most common isotopes). The magnetic moments depend on the inverse of the particle mass, so we expect the heavy protons and neutrons (\( \approx 2000 m_e \)) to have little effect on the magnetic moment of the atom and so we neglect them. From your study of the periodic table in chemistry, you recall that silver has an electronic configuration \( 1s^22s^22p^63s^23p^64s^23d^{10}4p^64d^{10}5s \), which means that there is only the lone \( 5s \) electron outside of the closed shells. The electrons in the closed shells can be represented by a spherically symmetric cloud with no orbital or intrinsic angular momentum (unfortunately we are injecting some quantum mechanical knowledge of atomic physics into this classical discussion). That leaves the lone \( 5s \) electron as a contributor to the magnetic moment of the atom as a whole. An electron in an \( s \) state has no orbital angular momentum, but it does have intrinsic angular momentum, which we call spin. Hence the magnetic moment of this electron, and therefore of the entire neutral silver atom, is

\[ \mu = -\frac{e g}{2m_e} S, \]

where \( e \) is the magnitude of the electron charge. The classical force on the atom can now be written as

\[ F_z = -\frac{e g}{2m_e} S_z \frac{\partial B_z}{\partial z}. \]

The deflection of the beam in the Stern-Gerlach experiment is thus a measure of the component or projection \( S_z \) of the spin along the \( z \)-axis, which is the orientation of the magnetic field gradient.

If we assume that each electron has the same magnitude \( |S| \) of the intrinsic angular momentum or spin, then classically we would write the projection as \( S_z = |S| \cos \theta \), where \( \theta \) is the angle between the \( z \)-axis and the direction of the spin \( S \). In the thermal environment of the oven, we expect a random distribution of spin directions and hence all possible angles \( \theta \). Thus we expect some continuous distribution (the details are not important) of spin projections from \( S_z = -|S| \) to \( S_z = +|S| \), which would yield a continuous spread in deflections of the silver atomic beam. Rather, the experimental result is that there are only two deflections, indicating that there are only two possible values of the spin projection of the electron. The magnitudes of these deflections are consistent with values of the spin projection of
\[ S_z = \pm \frac{\hbar}{2}, \]  

where \( \hbar \) is Planck's constant \( h \) divided by \( 2\pi \) and has the numerical value

\[ \hbar = 1.0546 \times 10^{-34} \, J \cdot s \]
\[ = 6.5821 \times 10^{-16} \, eV \cdot s. \]

This result of the Stern-Gerlach experiment is evidence of the quantization of the electron's spin angular momentum projection along an axis. This quantization is at odds with our classical expectations for this measurement. The factor of \( 1/2 \) in Eq. (1.7) leads us to refer to this as a spin 1/2 system. In this example, we have chosen the \( z \)-axis along which to measure the spin projection, but we could have chosen any other axis and would have obtained the same results.

Now that we know the fine details of the Stern-Gerlach experiment, we simplify the experiment for the rest of our discussions by focusing on the essential features. A simplified schematic representation of the experiment is shown in Fig. 1.2, which depicts an oven that produces the beam of atoms, a Stern-Gerlach device with two output ports for the two possible values of the spin projection, and two counters to detect the atoms leaving the output ports of the Stern-Gerlach device. The Stern-Gerlach device is labeled with the axis along which the magnetic field is oriented. The up and down arrows indicate the two possible measurement results for the device; they correspond respectively to the results \( S_z = \pm \hbar/2 \) in the case where the field is oriented along the \( z \)-axis. There are only two possible results in this case, so they are generally referred to as spin up and spin down. The physical quantity that is measured, \( S_z \) in this case, is called an observable. In our detailed discussion of the experiment above, we chose the field gradient in such a manner that the spin up states were deflected upwards. In this new simplification, the deflection itself is not an important issue. We simply label the output port with the desired state and count the particles leaving that port. The Stern-Gerlach device sorts (or filters or selects) the incoming particles into the two possible outputs \( S_z = \pm \hbar/2 \) in the same way that Erwin sorted his socks according to color or length.

In Fig. 1.2, the output beams have been labeled with a new symbol called a ket. We use the ket \( |+\rangle \) as a mathematical representation of the quantum state of the atoms that exit the upper port corresponding to \( S_z = +\hbar/2 \). The lower output beam is labeled with the ket \( |-\rangle \), which corresponds to \( S_z = -\hbar/2 \). The kets describe the quantum state

Figure 1.2  Simplified schematic of Stern-Gerlach experiment, depicting source of atoms, Stern-Gerlach analyzer, and counters.
mathematically and they contain all the information that we can know about the state. This ket notation was developed by P. A. M. Dirac and is central to the approach to quantum mechanics that we will take in this text. We will discuss the mathematics of these kets in full detail later. For now, it is sufficient for us to consider the ket as simply labeling the quantum state. With regard to notation, you will find many different ways of writing the $|\pm\rangle$ kets ($|\pm\rangle$ refers to both the $|+\rangle$ and $|-\rangle$ kets). The information contained within the ket symbol is used to label the ket and to distinguish the ket from other different kets. For example, the kets $|+\rangle$, $|+\hbar/2\rangle$, $|S_z = +\hbar/2\rangle$, $|+\hat{z}\rangle$, and $|\uparrow\rangle$ are all equivalent ways of writing the same thing, which in this case signifies that we have measured the $z$-component of the spin and found it to be $+\hbar/2$ or spin up. Though we may label these kets in different ways, they all refer to the same physical state and so they all behave the same mathematically. The first postulate of quantum mechanics tells us that kets in general describe the quantum state mathematically and that they contain all the information that we can know about the state. We denote a general ket as $|\psi\rangle$.

**Postulate 1**
The state of a quantum mechanical system is described mathematically by a normalized ket $|\psi\rangle$ that contains all the information we can know about the state.

We have chosen the particular simplified schematic representation of Stern-Gerlach experiments shown in Fig. 1.2 because it is the same representation used in the SPINS software program that you may use to simulate these experiments. The SPINS program allows you to perform all the experiments described in this text. In the program, the components are simply connected together to represent the paths the atoms take. The directions and deflections of the beams in the program are not relevant, and so we follow that lead in our depiction of the experiment hereafter. That is, whether the spin up output beam is drawn as deflected upwards, or downwards, or not at all is not relevant. The labeling on the output port is enough to tell us what that state is. Thus the extra ket label $|+\rangle$ on the spin up output beam in Fig. 1.2 is redundant and will be dropped soon.

The SPINS program permits alignment of Stern-Gerlach analyzing devices along all three axes and also at any angle $\phi$ measured from the $x$-axis in the $x$-$y$ plane. This would appear to be difficult, if not impossible, given that the atomic beam in Fig. 1.1 is directed along the $y$-axis, making it unclear how to align the magnet in the $y$-direction and measure a deflection. In our depiction and discussion of Stern-Gerlach experiments, we ignore this technical complication.

In the SPINS program, as in real Stern-Gerlach experiments, the numbers of atoms detected in particular states can be predicted by probability rules that we will discuss later. To simplify our schematic depictions of Stern-Gerlach experiments, the numbers shown for detected atoms are those obtained by using the calculated probabilities without any regard to possible statistical uncertainties. That is, if the theoretically predicted probabilities of two measurement possibilities are each 50%, then our schematics will display equal numbers for those two possibilities, whereas in a real
experiment, statistical uncertainties might yield a 55%/45% split in one experiment and a 47%/53% split in another, etc. In your SPINS program simulations, you will note these statistical uncertainties and so you will need to perform enough experiments to convince yourself that you have a sufficiently good estimate of the probability (see Appendix A for more information on statistics).

Now let's consider a series of simple Stern-Gerlach experiments with slight variations that help to illustrate the main features of quantum mechanics. We first describe the experiments and their results and draw some qualitative conclusions about the nature of quantum mechanics. Then we introduce the formal mathematics of the ket notation and show how it can be used to predict the results of each of the experiments.

1.1.1 Experiment 1

The first experiment is shown in Fig. 1.3 and consists of a source of atoms, two Stern-Gerlach devices both aligned along the \( z \)-axis, and counters for the output ports of the analyzers. The atomic beam coming into the 1st Stern-Gerlach device is split into two beams at the output, just like the original experiment. Now instead of counting the atoms in the upper output beam, the spin projection is measured again by directing those atoms into the 2nd Stern-Gerlach device. The result of this experiment is that no atoms are ever detected coming out of the lower output port of the 2nd Stern-Gerlach device. All atoms that are output from the upper port of the 1st device also pass through the upper port of the 2nd device. Thus we say that when the 1st Stern-Gerlach device measures an atom to have \( \hat{S}_z = +\hbar/2 \), then the 2nd device also measures \( \hat{S}_z = +\hbar/2 \) for that atom. This result is not surprising, but sets the stages for results of experiments to follow.

Though both Stern-Gerlach devices in Experiment 1 are identical, the 1st device is often referred to as the **polarizer** and the 2nd one as the **analyzer**, since the 1st one "polarizes" the beam along the \( z \)-axis and the second one "analyzes" the resultant beam. This is analogous to what can happen with optical polarizers. Some also refer to the 1st analyzer as a **state preparation device**, since it prepares the quantum state that is then measured with the analyzer. By preparing the state in this manner, the details of the source of atoms can be ignored. Thus our main focus in Experiment 1 is what happens at the analyzer, since we know that any atom entering the analyzer is described by the \( |+\rangle \) ket prepared by the polarizer. All the experiments we will describe employ a polarizer to

![Figure 1.3](image) Experiment 1 measures the spin projection along the \( z \)-axis twice in succession.
prepare the state, though the SPINS program has a feature where the state of the atoms coming from the oven is determined but unknown and the user can perform experiments to figure out the unknown state.

1.1.2 Experiment 2

The second experiment is shown in Fig. 1.4 and is identical to Experiment 1 except that the analyzer—the 2nd Stern-Gerlach device—has been rotated by 90° to be aligned with the x-axis. Now the analyzer measures the spin projection along the x-axis rather than the z-axis. Atoms input to the analyzer are still described by the ket $|+\rangle$ since the polarizer is unchanged. The result of this experiment is that atoms appear at both possible output ports of the analyzer. Atoms leaving the upper port of the analyzer have been measured to have $S_x = +\hbar/2$ and atoms leaving the lower port have $S_x = -\hbar/2$. On average, each of these ports has 50% of the atoms that left the upper port of the analyzer. As shown in Fig. 1.4, the output states of the 2nd analyzer have new labels $|+\rangle_x$ and $|-\rangle_x$, where the x subscript denotes that the spin projection has been measured along the x-axis. We assume that if no subscript is present on the quantum ket, then the spin projection is along the z-axis. This use of the z-axis as the default is common throughout our work and also in much of physics.

A few items are noteworthy about this experiment. First, we notice that there are still only two possible outputs of the Stern-Gerlach analyzer. The fact that it is aligned along a different axis doesn’t affect the fact that we only get two possible results for the case of a spin 1/2 particle. Second, note that the results of this experiment would be unchanged if we used the lower port of the polarizer. That is, atoms entering the analyzer in state $|-\rangle$ would also result in half the atoms in each of the $|\pm\rangle_x$ output ports. Finally, note that we cannot predict which of the analyzer output ports any particular atom will come out. This can be demonstrated experimentally by recording the individual counts out of each port. The arrival sequences at any counter are completely random. We can only say that there is a 50% probability that an atom from the polarizer will exit the upper analyzer port and a 50% probability that it will exit the lower port. The random arrival of atoms at the detectors can be seen clearly in the SPINS program simulations.

This probabilistic nature is at the heart of quantum mechanics. One might be tempted to say that we just don’t know enough about the system to predict which port the atom will be registered in. That is to say, there may be some other variables, of which we are ignorant, that would allow us to predict the results. Such a viewpoint is known as a

![Figure 1.4](image-url) Experiment 2 measures the spin projection along the z-axis and then along the x-axis.
hidden variable theory, and such theories have been proven to be incompatible with quantum mechanics. John Bell proved that such a quantum mechanical system cannot be described by a hidden variable theory, which amounts to saying that the system cannot have things we don't know about. It is a pretty powerful statement to be able to say that there are not things that we cannot know about a system. The conclusion to draw from this is that even though quantum mechanics is a probabilistic theory, it is a complete description of reality. We will have more to say about this later.

Note that the 50% probability referred to above is the probability that an atom input to the analyzer exits one particular output port. It is not the probability for an atom to pass through the whole system of Stern-Gerlach devices. Later we will have occasion to ask about such a probability and then we will say so. Note also that the results of this experiment (the 50/50 split at the analyzer) would be the same for any combination of two orthogonal axes of the polarizer and analyzer.

1.1.3 Experiment 3

Now consider Experiment 3, shown in Fig. 1.5, which extends Experiment 2 by adding a third Stern-Gerlach device aligned along the $z$-axis. (In this case, we refer to each device as an analyzer and label them first, second, or third.) Atoms entering the new third analyzer have been measured by the first Stern-Gerlach analyzer to have spin projection up along the $z$-axis, and by the second analyzer to have spin projection up along the $x$-axis. The third analyzer then measures how many atoms have spin projection up or down along the $z$-axis. Classically, one would expect that the final measurement would yield the result spin up along the $z$-axis, since that was measured at the first analyzer. That is to say: classically the first 2 analyzers tell us that the atoms have $S_z = +h/2$ and $S_x = +h/2$, so the third measurement must yield $S_z = +h/2$. But that doesn't happen, as Erwin learned with his quantum socks in the Prologue. The quantum mechanical result is that the atoms are split with 50% probability into each output port at the third analyzer. Thus the last two analyzers behave like the two analyzers of Experiment 2 (except with the order reversed), and the fact that there was an initial measurement that yielded $S_z = +h/2$ is somehow forgotten or erased.

This result demonstrates another key feature of quantum mechanics: the measurement perturbs the system. The third analyzer has perturbed the system such that the spin projection along the $z$-axis does not have a unique value, even though we

![Figure 1.5](image.png) Experiment 3 measures the spin projection three times in succession.
measured it with the first analyzer. Erwin saw this when he sorted, or measured, his socks by color and then by length. When he looked, or measured, a third time, he found that the color he had measured originally was now random—the socks had forgotten about the first measurement. One might ask: Can I be more clever in designing the experiment such that I don't perturb the system? The short answer is no. There is a fundamental incompatibility in trying to measure the spin projection of the atom along two different directions. So we say that $S_x$ and $S_z$ are incompatible observables. We cannot know the values of both simultaneously. The state of the system can be described by the ket $|+\rangle = |S_z = +h/2\rangle$ or by the ket $|+\rangle_x = |S_x = +h/2\rangle$, but it cannot be described by a ket $|S_z = +h/2, S_x = +h/2\rangle$ that specifies values of both projections. Having said this, it should be noted that not all pairs of quantum mechanical observables are incompatible. It is possible to do some experiments without perturbing some other aspects of the system. And we will see later that whether two observables are compatible or not is very important in how we analyze a quantum mechanical system.

Not being able to measure both the $S_z$ and $S_x$ spin projections is clearly distinct from the classical case whereby we can measure all three components of the spin vector, which tells us which direction the spin is pointing. In quantum mechanics, we cannot know which direction the spin is pointing. So when we say the spin is up, we really mean only that the spin projection along that one axis is up (vs. down). The spin is not really pointing in any given direction. This is an example of where you must check your classical intuition at the door.

1.1.4 Experiment 4

Experiment 4 is depicted in Fig. 1.6 and is a slight variation on Experiment 3. Before we get into the details, note a few changes in the schematic drawings. As promised, we have dropped the ket labels on the beams since they are redundant. We have deleted the counters on all but the last analyzer and instead simply block the unwanted beams and give the average number of atoms passing from one analyzer to the next. Note also that in Experiment 4c two output beams are combined as input to the following analyzer. This is simple in principle and in the SPINS program, but can be difficult in practice. The recombination of the beams must be done properly so as to avoid "disturbing" the beams. If you care to read more about this problem, see Feynman's Lectures on Physics, volume 3. We will have more to say about the "disturbance" later. For now we simply assume that the beams can be recombined in the proper manner.

Experiment 4a is identical to Experiment 3. In Experiment 4b the upper beam of the middle analyzer is blocked and the lower beam is sent to the third analyzer. In Experiment 4c, both beams are combined with our new method and sent to the third analyzer. It should be clear from our previous experiments that Experiment 4b has the same results as Experiment 4a. We now ask what the results of Experiment 4c are. If we were to use classical probability analysis, then Experiment 4a would indicate that the probability for an atom leaving the first analyzer to take the upper path through the second analyzer and then exit through the upper port of the third analyzer is 25%, where
Figure 1.6  Experiment 4 measures the spin projection three times in succession and uses one (a and b) or two beams (c) from the middle analyzer.

we are now referring to the total probability for those two steps. Likewise, Experiment 4b would indicate that the probability to take the lower path through the second analyzer and exit through the upper port of the third analyzer is also 25%. Hence the total probability to exit from the upper port of the third analyzer when both paths are available, which is Experiment 4c, would be 50%, and likewise for the exit from the lower port.

However, the quantum mechanical result in Experiment 4c is that all the atoms exit the upper port of the third analyzer and none exits the lower port. The atoms now appear to "remember" that they were initially measured to have spin up along the z-axis. By combining the two beams from the middle analyzer, we have avoided the quantum mechanical perturbation that was evident in Experiments 3, 4a, and 4b. The result is now the same as Experiment 1, which means it is as if the middle analyzer is not there.

To see how odd this is, look carefully at what happens at the lower port of the third analyzer. In this discussion, we refer to percentages of atoms leaving the first analyzer, since that analyzer is the same in all three experiments. In Experiments 4a and 4b, 50% of the atoms are blocked after the middle analyzer and 25% of the atoms exit the lower port of the third analyzer. In Experiment 4c, 100% of the atoms pass from the second analyzer to the third analyzer, yet fewer atoms come out of the lower port. In fact, no atoms make it through the lower port! So we have a situation where allowing more ways or paths to reach a counter results in fewer counts. Classical probability
theory cannot explain this aspect of quantum mechanics. It is as if you opened the window further to get more sunlight in and the room went dark.

However, you may already know of a way to explain this effect. Imagine a procedure whereby combining two effects leads to cancellation rather than enhancement. The concept of wave interference, especially in optics, comes to mind. In the Young's double slit experiment, light waves pass through two narrow slits and create an interference pattern on a distant screen, as shown in Fig. 1.7. Either slit by itself produces a nearly uniform illumination of the screen, but the two slits combined produced bright and dark fringes. We explain this by adding together the electric field vectors of the light from the two slits, then squaring the resultant vector to find the light intensity. We say that we add the amplitudes and then square the total amplitude to find the resultant intensity.

We follow a similar prescription in quantum mechanics. We add together amplitudes and then take the square to find the resultant probability. Before we do this, we need to explain what we mean by an amplitude in quantum mechanics and how we calculate it.

1.2 Quantum State Vectors

Postulate 1 stipulates that kets are to be used for a mathematical description of a quantum mechanical system. These kets are abstract vectors that obey many of the rules you know about ordinary spatial vectors. Hence they are often called quantum state vectors. As we will show later, these vectors must employ complex numbers in order to properly describe quantum mechanical systems. Quantum state vectors are part of a vector space whose dimensionality is determined by the physics of the system at hand. In the Stern-Gerlach example, the two possible results for a spin projection measurement dictate that the vector space has only two dimensions. That makes this problem mathematically as simple as it can be, which is why we have chosen to study it. Because the quantum state vectors are abstract, it is hard to say much about what they are, other than how they behave mathematically and how they lead to physical predictions.

Figure 1.7 Young's double slit interference experiment.
In the two-dimensional vector space of a spin 1/2 system, the two kets $|\pm\rangle$ form a basis, just like the unit vectors $\hat{i}, \hat{j}$, and $\hat{k}$ form a basis for describing vectors in three dimensional space. However, the analogy we want to make with these spatial vectors is only mathematical, not physical. The spatial unit vectors have three important mathematical properties that are characteristic of a basis: the basis vectors are normalized, orthogonal, and complete. Spatial vectors are normalized if their magnitudes are unity and they are orthogonal if they are geometrically perpendicular to each other. The basis is complete if any general vector in the space can be written as a linear superposition of the basis vectors. These properties of spatial basis vectors can be summarized as follows:

\[
\begin{align*}
\hat{i} \cdot \hat{i} &= \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 & \text{normalization} \\
\hat{i} \cdot \hat{j} &= \hat{i} \cdot \hat{k} = \hat{j} \cdot \hat{k} = 0 & \text{orthogonality }, \\
A &= a_\hat{i} \hat{i} + a_\hat{j} \hat{j} + a_\hat{k} \hat{k} & \text{completeness}
\end{align*}
\]

where $A$ is a general vector. Not that the dot product, also called the scalar product, is central to the description of these properties.

Continuing the mathematical analogy between spatial vectors and abstract vectors, we require that these same properties (at least conceptually) apply to quantum mechanical basis vectors. For the $S_z$ measurement, there are only two possible results, corresponding to the states $|+\rangle$ and $|\rangle$, so these two states comprise a complete set of basis vectors. This basis is known as the $S_z$ basis. We will focus on this basis for now and refer to other possible basis sets later. The completeness of the basis kets $|\pm\rangle$ implies that a general quantum state vector $|\psi\rangle$ is a linear combination of the two basis kets:

\[
|\psi\rangle = a |+\rangle + b |\rangle,
\]

where $a$ and $b$ are complex scalar numbers multiplying each ket. This addition of two kets yields another ket in the same abstract space. The complex scalar can appear either before or after the ket without affecting the mathematical properties of the ket (i.e., $a |+\rangle = |+\rangle a$). Note that is customary to use the symbol $\psi$ for a general quantum state. You may have seen $\psi(x)$ used before as a quantum mechanical wave function. However, the state vector or ket $|\psi\rangle$ is not a wave function. Kets do not have any spatial dependence as wave functions do. We will study wave functions after we are done with spins.

To discuss orthogonality and normalization (known together as orthonormality) we must first define scalar products as they apply to these new kets. As we said above, the machinery of quantum mechanics requires the use of complex numbers. You may have seen other fields of physics use complex numbers. For example, sinusoidal oscillations can be described using the complex exponential $e^{i\omega t}$ rather than $\cos(\omega t)$. However, in such cases, the complex numbers are not required, but are rather a convenience to make the mathematics easier. When using complex notation to describe classical vectors like electric and magnetic fields, dot products are changed slightly such
that one of the vectors is complex conjugated. A similar approach is taken in quantum mechanics. The analog to the complex conjugated vector of classical physics is called a bra in the Dirac notation of quantum mechanics. Thus corresponding to a general ket \( |\psi\rangle \) there is a bra, or bra vector, which is written as \( \langle \psi | \). If a general ket \( |\psi\rangle = a|+\rangle + b|\rangle \), then the corresponding bra \( \langle \psi | \) is defined as

\[
\langle \psi | = a^* \langle + | + b^* \langle | , \tag{1.11}
\]

where the basis bras \( \langle + | \) and \( \langle | \) correspond to the basis kets \( |+\rangle \) and \( |\rangle \), respectively, and the coefficients \( a \) and \( b \) have been complex conjugated.

The scalar product in quantum mechanics is defined as the product of a bra and a ket taken in the proper order—bra first, then ket second:

\[
\langle \langle bra | | ket \rangle \rangle \tag{1.12}
\]

When the bra and ket are combined together in this manner we get a bracket (bra c ket)—a little physics humor—that is written in shorthand as

\[
\langle bra | ket \rangle \tag{1.13}
\]

Thus, given the basis kets \( |+\rangle \) and \( |\rangle \), one inner product, for example, is written as

\[
\langle \langle + | | + \rangle \rangle = \langle + | + \rangle \tag{1.14}
\]

and so on. Note that we have eliminated the extra vertical bar in the middle. The scalar product in quantum mechanics is generally referred to as an inner product or a projection.

So, how do we calculate the inner product \( \langle + | + \rangle \)? We do it the same way we calculate the dot product \( \hat{i} \cdot \hat{i} \). We define it to be unity because we like basis vectors to be unit vectors. There is a little more to it than that because in quantum mechanics (as we will see shortly) using normalized basis vectors is more rooted in physics than in our personal preferences for mathematical cleanliness. But for all practical purposes, if someone presents a set of basis vectors to you, you may assume that they are normalized. So the normalization of the spin \( \frac{1}{2} \) basis vectors is expressed in this new notation as \( \langle + | + \rangle = 1 \) and \( \langle | - | \rangle = 1 \).

Now, what about orthogonality? The spatial unit vectors \( \hat{i}, \hat{j}, \text{ and } \hat{k} \) used in geometry are orthogonal to each other because they are at 90° with respect to each other. That orthogonality is expressed mathematically in the dot products \( \hat{i} \cdot \hat{j} = \hat{i} \cdot \hat{k} = \hat{j} \cdot \hat{k} = 0 \). For the spin basis kets \( |+\rangle \) and \( |\rangle \), there is no geometry involved. The spin basis kets \( |+\rangle \) and \( |\rangle \) are orthogonal in the mathematical sense, which we express with the inner product as \( \langle + | - \rangle = 0 \). Again, we cannot prove to you that these basis vectors are orthogonal, but we assume that a well-behaved basis set obeys orthogonality. Though there is no geometry in this property for quantum mechanical basis vectors, the fundamental idea of orthogonality is the same—if a general vector "points" in the direction of a basis vector, then there is no component in the "direction" of the other unit vectors.
In summary, the properties of normalization, orthogonality, and completeness can be expressed in the case of a two-state quantum system as:

\[
\begin{align*}
\langle +|+ \rangle &= 1 \\
\langle -|- \rangle &= 1 \\
\langle +| - \rangle &= 0 \\
\langle -|+ \rangle &= 0
\end{align*}
\]

normalization

\[
\langle +|- \rangle = 0 \quad \text{orthogonality .} \tag{1.15}
\]

\[
|\psi\rangle = a|+\rangle + b|-\rangle \quad \text{completeness}
\]

Note that a product of kets (e.g., $|+\rangle|+\rangle$) or a product of bras (e.g., $\langle+|\langle+|$) is meaningless in this new notation, while a product of a ket and a bra in the "wrong" order (e.g., $|+\rangle\langle+|$) has a meaning that we will define later. Equations (1.15) are sufficient to define how the basis kets behave mathematically. Note that the inner product is defined using a bra and a ket, though it is common to refer to the inner product of two kets, where it is understood that one is converted to a bra first. The order does matter, as we will see shortly.

Using this new notation, we can learn a little more about general quantum states and derive some expressions that will be useful later. Consider the general state vector $|\psi\rangle = a|+\rangle + b|-\rangle$. Take the inner product of this ket with the bra $\langle +|$ and obtain

\[
\langle +|\psi \rangle = \langle +| (a|+\rangle + b|-\rangle)
\]

\[
= \langle +|a|+\rangle + \langle +|b|-\rangle,
\]

\[
= a\langle +|+\rangle + b\langle +|-\rangle
\]

\[
= a
\]

using the property that scalars can be moved freely through bras or kets. Likewise, it can be shown that $\langle -|\psi \rangle = b$. Hence the coefficients multiplying the basis kets are simply the inner products or projections of the general state $|\psi\rangle$ along each basis ket, albeit in an abstract complex vector space, rather than the concrete three dimensional space of normal vectors. Using these results, we can rewrite the general state as

\[
|\psi\rangle = a|+\rangle + b|-\rangle
\]

\[
= \{\langle +|\psi \rangle\}|+\rangle + \{\langle -|\psi \rangle\}|-\rangle
\]

\[
= \langle +|\psi \rangle|+\rangle + \langle -|\psi \rangle|-\rangle \tag{1.17}
\]

where the rearrangement of the last equation again uses the property that scalars (e.g., $\langle +|\psi \rangle$) can be moved through bras or kets.

For a general state vector $|\psi\rangle = a|+\rangle + b|-\rangle$ we defined the corresponding bra to be $\langle \psi| = a^\ast \langle +| + b^\ast \langle -|$. Thus, the inner product of the state $|\psi\rangle$ with the basis ket $|+\rangle$ taken in the reverse order compared to Eq. (1.16) yields
Thus we see that an inner product with the states reversed results in a complex conjugation of the inner product:

$$\langle + | \psi \rangle = \langle \psi | + \rangle^* .$$

This important property holds for any inner product. For example, the inner product of two general states is

$$\langle \phi | \psi \rangle = \langle \psi | \phi \rangle^*. \quad (1.20)$$

Now we come to a new mathematical aspect of quantum vectors that differs from the use of vectors in classical mechanics. The rules of quantum mechanics (postulate 1) require that all state vectors describing a quantum system be normalized, not just the basis kets. This is clearly different from ordinary spatial vectors, where the length or magnitude of a vector means something and only the unit vectors $\hat{i}, \hat{j},$ and $\hat{k}$ are normalized to unity. This new rule means that in the quantum mechanical state space only the direction—in an abstract sense—is important. If we apply this normalization requirement to a general state $|\psi\rangle$, then we obtain

$$\langle \psi | \psi \rangle = \{ a^* \langle + | + \rangle + b^* \langle - | - \rangle \} \{ a | + \rangle + b | - \rangle \} = 1$$

$$\Rightarrow a^* a \langle + | + \rangle + a^* b \langle + | - \rangle + b^* a \langle - | + \rangle + b^* b \langle - | - \rangle = 1,$$

$$\Rightarrow a^* a + b^* b = 1$$

$$\Rightarrow |a|^2 + |b|^2 = 1 \quad (1.21)$$

or using the expressions for the coefficients obtained above,

$$|\langle + | \psi \rangle|^2 + |\langle - | \psi \rangle|^2 = 1. \quad (1.22)$$

**Example 1.1**

Normalize the vector $|\psi\rangle = A(|+\rangle + 2i|\rangle)$. The complex constant $A$ is often referred to as the **normalization constant**. To normalize $|\psi\rangle$, we set the inner product of the vector with itself equal to unity and then solve for $A$—note the requisite complex conjugations.
The overall phase of the normalization constant is not physically meaningful (HW), so we follow the standard convention and choose it to be real and positive. This yields \( A = \frac{1}{\sqrt{5}} \). The normalized quantum state vector is then

\[
|\psi\rangle = \frac{1}{\sqrt{5}} (|+\rangle + 2i|\rangle),
\]

\[(1.24)\]

Now comes the crucial element of quantum mechanics. We postulate that each term in the sum of Eq. (1.22) is equal to the probability that the quantum state described by the ket \( |\psi\rangle \) is measured to be in the corresponding basis state. Thus

\[
P_{S_z=+h/2} = |\langle + | \psi \rangle|^2
\]

\[(1.25)\]

is the probability that the state \( |\psi\rangle \) is found to be in the state \( |+\rangle \) when a measurement of \( S_z \) is made, meaning that the result \( S_z = +h/2 \) is obtained. Likewise,

\[
P_{S_z=-h/2} = |\langle - | \psi \rangle|^2
\]

\[(1.26)\]

is the probability that the measurement yields the result \( S_z = -h/2 \). The subscript on the probability indicates the measured value. For the spin projection measurements, we will usually abbreviate this to, for example, \( P_z \) for a \( S_z = +h/2 \) result or \( P_y \) for a \( S_y = -h/2 \) measurement.

We now have a prescription for predicting the outcomes of the experiments we have been discussing. For example, the experiment shown in Fig. 1.8 has the state \( |\psi\rangle = |+\rangle \) prepared by the first Stern-Gerlach device and then input to the second Stern-Gerlach device aligned along the z-axis. Therefore the probabilities of measuring the input state \( |\psi\rangle = |+\rangle \) to have the two output values are as shown. Because the spin-1/2 system has only two possible measurement results, these two probabilities must sum to

![Figure 1.8 Probabilities of spin projection measurements.](image)
unity—there is a 100% probability of recording some value in the experiment. This basic rule of probabilities is why the rules of quantum mechanics require that all state vectors be properly normalized before they are used in any calculation of probabilities. The experimental predictions shown in Fig. 1.8 illustrate the 4th postulate of quantum mechanics, which is presented below.

**Postulate 4 (Spin 1/2 system)**

The probability of obtaining the value \( \pm \hbar/2 \) in a measurement of the observable \( S_z \) on a system in the state \( |\psi\rangle \) is

\[
P_{\pm} = |\langle \pm |\psi\rangle|^2,
\]

where \( |\pm\rangle \) is the basis ket of \( S_z \) corresponding to the result \( \pm \hbar/2 \).

This is labeled as the 4th postulate because we have written this postulate using the language of the spin-1/2 system and will generalize it later using the 2nd and 3rd postulates. A general spin projection measurement is shown in Fig. 1.9, along with a histogram that compactly summarizes the measurement results.

An inner product, \( \langle + |\psi\rangle \) for example, is called a **probability amplitude** or sometimes just an **amplitude**. Note that the convention is to put the input or initial state on the right and the output or final state on the left: \( \langle out | in \rangle \), so one would read from right to left in describing a problem. Since the probability involves the complex square of the amplitude, and \( \langle out | in \rangle = \langle in | out \rangle^\ast \), this convention is not critical for calculating probabilities. Nonetheless, it is the accepted practice and is important in situations where several amplitudes are combined.

Armed with these new quantum mechanical rules and tools, let's analyze the experiments discussed earlier. Using the experimental results and the new rules we have introduced we can learn more about the mathematical behavior of the kets and the relationships among them. We will focus on the first two experiments for now and return to the others in the next chapter.

**Figure 1.9** (a) Spin projection measurement for general input state and (b) histogram of measurement results.
1.2.1 Analysis of Experiment 1

In Experiment 1, the initial Stern-Gerlach analyzer prepared the system in the $|+\rangle$ state and the second analyzer measured this state to always be in the $|+\rangle$ state and never in the $|-\rangle$ state. The results of the experiment are summarized in the histogram in Fig. 1.10. We can use the 4th postulate to predict the results of this experiment. We take the inner product of the input state $|+\rangle$ with each of the possible output states $|+\rangle$ and $|-\rangle$. Because we know that these states are normalized and orthogonal, we can calculate the probabilities:

$$\mathcal{P}_+ = |\langle +|+\rangle|^2 = 1$$
$$\mathcal{P}_- = |\langle -|+\rangle|^2 = 0.$$  \hspace{1cm} (1.27)

These predictions agree exactly with the experimental results shown in Fig. 1.10. A $|+\rangle$ state will always be measured to have $S_z = +\hbar/2$.

![Figure 1.10](image)

Figure 1.10  Histogram of $S_z$ spin projection measurements for experiment 1.

1.2.2 Analysis of Experiment 2

In Experiment 2, the initial Stern-Gerlach analyzer prepared the system in the $|+\rangle$ state and the second analyzer performed a measurement of the spin projection along the $x$-axis, finding 50% probabilities for each of the two possible states $|+\rangle_x$ and $|-\rangle_x$, as shown in the histogram in Fig. 1.11(a). For the experiment, we cannot predict the results of the measurements, because we do not yet have enough information about how the states $|+\rangle_x$ and $|-\rangle_x$ behave mathematically. Rather, we will use the results of the experiment to determine these states. Recalling that the experimental results would be the same if the first analyzer prepared the system to be in the $|-\rangle$ state (see Fig. 1.11(b)), we have four results for the two experiments:

$$\mathcal{P}_{1,xx} = |\langle +|+\rangle_x|^2 = \frac{1}{2}$$
$$\mathcal{P}_{1,-x} = |\langle -|+\rangle_x|^2 = \frac{1}{2}$$
$$\mathcal{P}_{2,xx} = |\langle +|-\rangle_x|^2 = \frac{1}{2}$$
$$\mathcal{P}_{2,-x} = |\langle -|-\rangle_x|^2 = \frac{1}{2}$$  \hspace{1cm} (1.28)
Figure 1.11 Histograms of $S_x$ spin projection measurements for experiment 2 for different input states (a) $|\psi_{in}\rangle = |+\rangle$ and (b) $|\psi_{in}\rangle = |-\rangle$.

Because the kets $|+\rangle$ and $|-\rangle$ form a complete basis, the kets describing the $S_x$ measurement, $|+\rangle_x$ and $|-\rangle_x$, can be written in terms of them. We do not yet know the specifics of the $|\pm\rangle_x$ states, so we use general expressions

$$
|+\rangle_x = a|+\rangle + b|-\rangle,
$$

$$
|-\rangle_x = c|+\rangle + d|-\rangle,
$$

and now our task is to use the results of Experiment 2 to determine the coefficients $a$, $b$, $c$, and $d$. We start by using the general expression for $|+\rangle_x$ in Eqn. (1.29) to calculate the probability that the $|+\rangle$ input state is measured to be in the $|+\rangle_x$ output state, i.e., to have $S_x = +h/2$, and equating it to the measured probability in Eqn.(1.28)

$$
P_{1,+,s} = |\langle +|+\rangle|^2 = \frac{1}{2}
$$

$$
= |\langle a^*|+\rangle + b^\ast \langle |-\rangle|+\rangle|^2 = \frac{1}{2}
$$

$$
= |a|^2 = \frac{1}{2}
$$

$$
= |d|^2 = \frac{1}{2}
$$

where we convert the $|+\rangle_x$ ket to a bra $\langle +|$ in order to calculate the inner product. Similarly, one can calculate the other three probabilities to arrive at $|b|^2 = |c|^2 = |d|^2 = \frac{1}{2}$. Because each coefficient is complex, each has an amplitude and phase. However, the overall phase of a quantum state vector is not physically meaningful (problem 1.3). Only the relative phase between different components of the state vector is physically measurable. Hence, we can choose one coefficient of each vector to be real and positive without any loss of generality. This allows us to write the desired states as

$$
|+\rangle_x = \frac{1}{\sqrt{2}} \left[ |+\rangle + e^{i\alpha} |-\rangle \right]
$$

$$
|-\rangle_x = \frac{1}{\sqrt{2}} \left[ |+\rangle + e^{i\beta} |-\rangle \right].
$$

Note that these are already normalized since we used all of the experimental results, which reflects the fact that the probability for all possible results of an experiment must sum to unity.
We have used all the experimental results from Experiment 2, but the $|\pm\rangle_x$ kets are still not determined. We need some more information. If we perform Experiment 1 with both analyzers aligned along the $x$-axis, the results will be as you expect—all $|+\rangle_x$ states from the first analyzer will be measured to have $S_x = +\hbar/2$ at the second analyzer, i.e., all atoms exit in the $|+\rangle_x$ state and none in the $|-\rangle_x$. The probability calculations for this experiment are

\[
P_{+x} = |\langle x | + \rangle|^2 = 1
\]
\[
P_{-x} = |\langle x | - \rangle|^2 = 0
\]

which tell us mathematically that the $|\pm\rangle_x$ states are orthonormal to each other, just like the $|\pm\rangle$ states. This also implies that the $|\pm\rangle_x$ kets form a basis, the $S_x$ basis, which you might expect because they correspond to the distinct results of a different spin projection measurement. The general expressions we used for the $|\pm\rangle_x$ kets are already normalized, but are not yet orthogonal. That is the new piece of information we need. The orthogonality condition leads to

\[
\langle x | - + \rangle = 0
\]
\[
\frac{1}{2} \left[ \langle x | + e^{-i\beta} \rangle \right] \frac{1}{2} \left[ \langle x | + e^{-i\alpha} \rangle \right] = 0
\]
\[
\frac{1}{2} \left[ 1 + e^{i(\alpha-\beta)} \right] = 0
\]
\[
e^{i(\alpha-\beta)} = -1
\]
\[
e^{i\alpha} = -e^{i\beta}
\]

where the complex conjugation of the second coefficient of the $\langle x | -$ bra should be noted. We now have an equation relating the remaining coefficients $\alpha$ and $\beta$, but need some more information to determine their values. Unfortunately, there is no more information to be had, so we are free to choose the value of the phase $\alpha$. This freedom comes from the fact that we have required only that the $x$-axis be perpendicular to the $z$-axis, which limits the $x$-axis only to a plane rather than to a unique direction. We follow convention here and choose the phase $\alpha = 0$. Thus we can express the $S_x$ basis kets in terms of the $S_z$ basis kets as

\[
|+\rangle_x = \frac{1}{\sqrt{2}} \left[ |+\rangle + |-\rangle \right]
\]
\[
|-\rangle_x = \frac{1}{\sqrt{2}} \left[ |+\rangle - |-\rangle \right]
\]

We generally use the $S_z$ basis as the preferred basis for writing general states, but could use any basis we choose. If we were to use the $S_x$ basis, then we could write the $|\pm\rangle_x$ kets as general states in terms of the $|\pm\rangle_x$ kets. This can be done by solving Eqs. (1.34) for the $|\pm\rangle_x$ kets, yielding

\[
|+\rangle = \frac{1}{\sqrt{2}} \left[ |+\rangle_x + |-\rangle_x \right]
\]
\[
|-\rangle = \frac{1}{\sqrt{2}} \left[ |+\rangle_x - |-\rangle_x \right]
\]
With respect to the measurements performed in Experiment 2, Eqn. (1.35) tells us that the $|+\rangle$ state is a combination of the states $|+\rangle_x$ and $|\rangle_x$. The coefficients tell us that there is a 50% probability for measuring the spin projection to be up along the $x$-axis, and likewise for the down possibility, which is in agreement with the histogram of measurements shown in Fig. 1.11(a). A combination of states is usually referred to as a **superposition** state.

### 1.2.3 Superposition states

To understand the importance of a quantum mechanical superposition of states, consider the $|+\rangle_x$ state found above. This state is written in terms of the $S_z$ basis states as

$$|+\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle + |\rangle).$$

Consider measurements on this state, as shown in Fig. 1.12(a). If we measure the spin projection along the $x$-axis for this state, then we record the result $S_x = +\hbar/2$ with 100% probability (Experiment 1 with both analyzers along the $x$-axis). If we measure the spin projection along the orthogonal $z$-axis, then we record the two results $S_z = \pm \hbar/2$ with 50% probability each (Experiment 2 with the first and second analyzers along the $x$- and $z$-axes, respectively). Based upon this second set of results, one might be tempted to consider the $|+\rangle_x$ state as describing a beam that contains a mixture of atoms with 50% of the atoms in the $|+\rangle$ state and 50% in the $|\rangle$ state.

Let's now carefully examine the results of experiments on this proposed mixture beam, as shown in Fig. 1.12(b). If we measure the spin projection along the $z$-axis, then each atom in the $|+\rangle$ state yields the result $S_z = +\hbar/2$ with 100% certainty and each atom in the $|\rangle$ state yields the result $S_z = -\hbar/2$ with 100% certainty. The net result is that 50% of the atoms yield $S_z = +\hbar/2$ and 50% yield $S_z = -\hbar/2$. This is exactly the same result as that obtained with all atoms in the $|+\rangle_x$ state. If we instead measure the spin projection along the $x$-axis, then each atom in the $|+\rangle$ state yields the two results $S_x = \pm \hbar/2$ with 50% probability each (Experiment 2 with the first and second analyzers along the $z$- and $x$-axes, respectively). The atoms in the $|\rangle$ state yield the same results. The net result is that 50% of the atoms yield $S_x = +\hbar/2$ and 50% yield $S_x = -\hbar/2$. This is in stark contrast to the results of Experiment 1, which tells us that once we have prepared the state to be $|+\rangle_x$, then subsequent measurements yield $S_x = +\hbar/2$ with certainty.

Hence we must conclude that the system described by the $|+\rangle_x$ state is not the same as a mixture of atoms with some in the $|+\rangle$ state and some in the $|\rangle$ state. Rather, each atom in the $|+\rangle_x$ beam is in a state that itself is a combination of the $|+\rangle$ and $|\rangle$ states. A superposition state is often called a **coherent superposition** since the relative phase of the two terms is important. For example, if the beam were in the $|\rangle_x$ state, then there would be a relative minus sign between the two coefficients, which would result in an $S_z = -\hbar/2$ measurement but would not affect the $S_x$ measurement.
Figure 1.12 (a) Superposition measurement and (b) mixture measurements.

We will not have any further need to speak of mixtures, so any combination of states we use is a superposition. Note that we cannot even write down a ket describing the mixture case. So, if someone gives you a quantum state written as a ket, then it must be a superposition and not a mixture. The random option in the SPINS program produces a mixture, while the unknown states are all superpositions.

Example 1.2

Consider the input state

\[ |\psi_in\rangle = 3|+\rangle + 4|-\rangle. \]

Normalize this state vector and find the probabilities of measuring the spin projection along the \( z \)-axis to be \( S_z = \pm h/2 \).
To normalize this state, introduce an overall multiplicative factor and solve for this factor by imposing the normalization condition:

\[ |\psi_{in}\rangle = a\left[3|+\rangle + 4|-\rangle\right] \]
\[ \langle \psi_{in}|\psi_{in}\rangle = 1 \]
\[ \{a^*\left[3\langle +|+\rangle + 4\langle -|\rangle\right]\{a\left[3|+\rangle + 4|-\rangle\}\} = 1 \]
\[ a^*a\left[9|+|+\rangle + 12\langle +|-\rangle + 12\langle -|+\rangle + 16\langle -|-\rangle\right] = 1 \]
\[ a^*a[25] = 1 \]
\[ |a|^2 = \frac{1}{25} \]

Because an overall phase is physically meaningless, we can choose \( a \) to be real and positive. Hence the normalized input state is

\[ |\psi_{in}\rangle = \frac{3}{5}|+\rangle + \frac{4}{5}|-\rangle. \] (1.39)

The probability of measuring \( S_z = +\hbar/2 \) is

\[ P_+ = \langle +|\psi_{in}\rangle|^2 \]
\[ = \left[\langle \frac{3}{5}|+\rangle + \frac{4}{5}|-\rangle\right]^2 \]
\[ = \left|\frac{3}{5}\langle +|+\rangle + \frac{4}{5}\langle +|-\rangle\right|^2 \]
\[ = \left|\frac{3}{5}\right|^2 = \frac{9}{25} \] (1.40)

The probability of measuring \( S_z = -\hbar/2 \) is

\[ P_- = \langle -|\psi_{in}\rangle|^2 \]
\[ = \left[\langle -\frac{3}{5}|+\rangle + \frac{4}{5}|-\rangle\right]^2 \]
\[ = \left|\frac{3}{5}\langle -|+\rangle + \frac{4}{5}\langle -|-\rangle\right|^2 \]
\[ = \left|\frac{3}{5}\right|^2 = \frac{9}{25} \] (1.41)

Note that the two probabilities add to unity, which indicates that we normalized the input state properly. A histogram of the predicted measurement results is shown in Fig. 1.13.