1 Polarization of Light

1.1 Classical Description

Light polarized in the $x$ direction has an electric field vector $\vec{E} = E_0 \hat{x} \cos(kz - \omega t)$
Light polarized in the $y$ direction has an electric field vector $\vec{E} = E_0 \hat{y} \cos(kz - \omega t)$
Light polarized at 45 degrees (call this direction $x'$) has an electric field vector

$$\vec{E}_0 \hat{x}' \cos(kz - \omega t) = E_0 \left( \frac{1}{\sqrt{2}} \hat{x} \cos(kz - \omega t) + \frac{1}{\sqrt{2}} \hat{y} \cos(kz - \omega t) \right)$$

1.1.1 Polarization Analyzers

The component of the electric field along the transmission direction of a Polaroid filter is transmitted. After the filter the light is polarized along the filter transmission direction. Polaroid filters “prepare” light in a state of linear polarization.

The intensity of light is proportional to the square of the electric field; therefore after the filter the intensity is proportional to the square of the cosine of the angle between the incoming electric field and the Polaroid transmission direction. $|E_x|^2 = |E|^2 \cos^2 \theta$ and intensity is proportional to $\cos^2 \theta$.

See French and Taylor[?] p. 234

An ideal Polaroid filter has these properties:

a) The outgoing light leaving a Polaroid sheet is polarized along the transmission direction of the Polaroid.

b) The Polaroid sheet transmits or selects the component of the incoming electric field which is along its transmission direction. For transmission along $x$ $E_x = \hat{x} \cdot \vec{E}$

c) An ideal Polaroid transmits 100% of the light polarized in its transmission direction and no light polarized in the perpendicular direction.

A Polaroid sheet prepares light in a state of linear polarization. A second Polaroid sheet analyzes the polarization state.

1.1.2 Polarization States

We can represent the polarization states by “state vectors” in a two dimensional vector space. Light polarized in the $x$ direction can be represented by $|x\rangle$ and light polarized in the $y$ direction by $|y\rangle$.

Polarization at 45 degrees can be represented by the state

$$|\theta = 45^\circ\rangle = \frac{1}{\sqrt{2}} (|x\rangle + |y\rangle)$$
Polarization at an angle $\theta$ in the $x, y$ plane, the direction $\hat{n}_{\theta}$, can be represented by

$$|\theta\rangle = \cos \theta |x\rangle + \sin \theta |y\rangle$$

Right and Left circular polarization can be described by

$$|R\rangle = \frac{1}{\sqrt{2}} (|x\rangle + i |y\rangle) \quad \text{and} \quad |L\rangle = \frac{1}{\sqrt{2}} (|x\rangle - i |y\rangle)$$

Notice that the state vectors have complex coefficients in this case which correspond to a phase shift.

In the state notation the $x$, $y$, $\theta$, $R$, and $L$ in the brackets are labels describing the states.

Basis states can be chosen to be $|x\rangle$ and $|y\rangle$ and all states can be written as linear combinations of two orthogonal basis states.

The most general polarization state is $|\Phi\rangle = \lambda |x\rangle + \mu |y\rangle$ where $\lambda$ and $\mu$ are in general complex with $|\lambda|^2 + |\mu|^2 = 1$. Then $|\Phi\rangle$ is normalized.

In the case of linear polarization $\lambda$ and $\mu$ are real and can be written as $\cos \theta$ and $\sin \theta$.

$$\sin^2 \theta + \cos^2 \theta = 1$$

The basis states, $|x\rangle$ and $|y\rangle$, form an orthonormal basis.

The basis is said to be “complete” if any state can be written as a linear combination of basis states.

Another orthonormal basis can be written as

$$|\theta\rangle = \cos \theta |x\rangle + \sin \theta |y\rangle$$
$$|\theta\rangle_\perp = -\sin \theta |x\rangle + \cos \theta |y\rangle.$$ 

For circular polarization we can use the basis states $|R\rangle$ and $|L\rangle$.

### 1.1.3 Preparing and Analyzing Polarized Light

When light reaches a Polaroid sheet the outgoing state is the state vector of the incoming light projected along the state representing the Polaroid transmission direction.

The amplitude of the outgoing state is the inner product of the state representing the incoming light and the state representing the analyzer selection.

Inner products are written as $\langle y | \theta \rangle$ where $\langle y \rangle$ is the dual of $|y\rangle$.

Compare with ordinary vectors, $E_x = \hat{x} \cdot \vec{E}$

If a Polaroid filter has its transmission axis in the $y$ direction; we represent the Polaroid by $\langle y \rangle$.

If the incoming light is polarized at angle $\theta$ and the analyzer transmission direction is in the $y$ direction then the outgoing amplitude is the inner product of the state representing the Polaroid and the state of the incoming light

$$\langle y | \theta \rangle = \langle y | (\cos \theta |x\rangle + \sin \theta |y\rangle) = \cos \theta \langle y | x \rangle + \sin \theta \langle y | y \rangle = \sin \theta$$

We use an orthonormal basis: $\langle x | x \rangle = 1 \quad \langle y | y \rangle = 1 \quad \langle x | y \rangle = 0$

After the analyzer the polarization direction is $y$ and the amplitude is $\sin \theta$.

The state of the outgoing light is described by $|y\rangle$.

Classically the intensity of light is proportional to the square of the electric field so the outgoing intensity is proportional to the magnitude squared of the amplitude $|\langle y | \theta \rangle|^2$.

In this example the intensity after the Polaroid is $I = I_0 \sin^2 \theta$. 

2
1.1.4 Examples

A polaroid sheet has its transmission axis at an angle $\theta$ with respect to the $x$ direction. The state representing the Polaroid is

$$\langle \theta | = \cos \theta \langle x | + \sin \theta \langle y |$$

To find the amplitude for light transmission through a Polaroid we must project the state of the incident light beam on the state representing the Polaroid.

If the incident light is polarized in the $x$ direction it is represented by $|x\rangle$. If we send this light through a Polaroid filter whose transmission direction is at angle $\theta$ then the amplitude passing the Polaroid is

$$\langle \theta | x \rangle = \cos \theta \quad \text{using } \langle x | x \rangle = 1 \quad \langle y | y \rangle = 1 \quad \langle x | y \rangle = 0$$

**Incident Light Polarized at 45 degrees**

For 45 degree Linear Polarization the amplitude is

$$\langle \theta | 45^\circ \rangle = (\cos \theta \langle x | + \sin \theta \langle y |) \left( \frac{1}{\sqrt{2}} (|x\rangle + |y\rangle) \right) \quad \langle \theta | 45^\circ \rangle = \frac{1}{\sqrt{2}} (\cos \theta + \sin \theta)$$

The magnitude squared of the amplitude (the intensity) is

$$|\langle \theta | 45^\circ \rangle|^2 = \frac{1}{2} (1 + 2 \sin \theta \cos \theta) = \frac{1}{2} (1 + \sin 2\theta)$$

If $\theta = 45^\circ$ we get intensity $= 1$ and if $\theta = 0$ or $\theta = 90^\circ$ we get intensity $= \frac{1}{2}$.

If $\theta = -45^\circ$ we get intensity $= 0$ since $\sin (-\pi/2) = -1$. The analyzer Polaroid is perpendicular to the Polaroid which prepared the light (the states are orthogonal).

1.2 Photons

We expect to find the same Energy (intensity) relationships with the photon picture as with the classical wave picture. Each photon carries a quantum of energy ($E = h\nu$) and the total energy will be proportional to the number of photons. The light intensity is proportional to the number of photons.

In the photon description of light we associate a polarization state with a beam of photons which has passed a Polaroid filter or has been prepared in some way.

We can assign a polarization state $|x\rangle, |y\rangle, |\theta\rangle, |R\rangle$, etc. to a beam of photons.

If a beam of photons is in the state $|x\rangle$ and it reaches a Polaroid filter whose transmission direction is $y$ (represented by $\langle y |$) then no photons will get through the filter.

If the filter were $|x\rangle$ then (for an ideal filter) all the photons should get through.

For a filter at 45 degrees half the photons are transmitted.

A photon either goes through the Polaroid filter or it does not. There are no fractional photons. The probability that a photon is transmitted through a Polaroid sheet depends on the polarization state in which it was prepared.

We prepare a beam of photons in a state $|\theta\rangle$ by sending the light through a Polaroid filter whose transmission axis is at angle $\theta$ measured with respect to the $x$ axis.

Any photon observed after the Polaroid filter will be in the state $|\theta\rangle$. 

3
1.2.1 Amplitude and Probability

Suppose we have a beam of photons prepared in the state $|\theta\rangle$. $|\theta\rangle = \cos \theta |x\rangle + \sin \theta |y\rangle$

The Amplitude for finding a photon after a Polaroid analyzer is given by the above inner product rules for finding amplitudes. For example for a Polaroid whose transmission axis is $x$ the amplitude is $\langle x|\theta\rangle$. The amplitude is $\cos \theta$ in this case.

The “amplitude” is sometimes called the “probability amplitude”.

The Probability for finding a photon after the analyzer is the magnitude squared of the amplitude; the probability is $|\langle x|\theta\rangle|^2$. The probability would be $\cos^2 \theta$ in this case.

Recall that probabilities are numbers in the range 0 to 1.

The probability that a photon in the state $|\theta\rangle = \cos \theta |x\rangle + \sin \theta |y\rangle$ will be transmitted by a Polaroid sheet whose transmission axis is in the $x$ direction is $\cos^2 \theta$. This is expected since the Intensity observed after the Polaroid must agree with the classical picture. (The energy is proportional to the square of the Electric field and is proportional to the number of photons.)

In general only a fraction of the original number of photons is observed after the filter.

After the analyzer any observed photon will be in state $|x\rangle$ but we know (almost) nothing about the state of the photon before the analyzer.

We must not conclude that the photon was in state $|x\rangle$ before the analyzer.

“quantum theory is incompatible with the proposition that measurements are processes by which we discover some unknown and pre-existing property.”

F. Lalaoë[?] quoting A. Peres.[?]

1.2.2 Summary

1. The polarization state of a photon is represented by a “state vector”.
2. A Polaroid sheet is represented by a state vector which corresponds to the state of a photon which has been transmitted through the Polaroid filter.
3. The Amplitude of the outgoing light is the inner product of these two state vectors.
4. The Probability for photon transmission is the squared-magnitude of the amplitude.
5. If a photon emerges from a Polaroid sheet its state corresponds to the transmission state of the Polaroid. The state after the Polaroid tells us nothing about the incoming state.
6. The Intensity of the outgoing light is proportional to the number of transmitted photons.

1.3 Quantum Cryptography

A better name for this application of Quantum Mechanics is “Quantum Key Distribution” since what is transmitted is not a message but a key with which to encode and decode messages. The key consists of a string of bits: “0” and “1” and the key has to be known by the sender and the receiver.

The key can be used to encode messages (also strings of bits) by, for example, making an “exclusive OR” between the key and the message to be encrypted. The recipient can decode the transmitted message with another exclusive OR using the key.

If the message is to be secure we want to know if it was intercepted. It is in this security aspect that Quantum Mechanics plays the fundamental important role.
1.3.1 An example of secure Key Distribution

Two people, Alice and Bob, at different locations want to share a secure cryptographic key which they will use to encode messages. We will show that if an evesdropper, Eve, intercepts the key then the evesdropping will be detected.

a) A train of individual photons is produced by a light source and photons sent one at a time.

b) These photons are sent through a Polaroid sheet whose orientation is controlled by the “sender” called Alice.

c) The photons sent by Alice reach a Polaroid sheet whose orientation is controlled by the “receiver” called Bob.

We will assume that the Polaroids are “ideal” and that the photon detectors are perfect; that is they are 100% efficient and have no noise.

As a first example to demonstrate secure key distribution (BB84) assume that Alice chooses, at random, horizontal (H) or vertical (V) or +45° or −45° Polaroid orientations. Thus each photon is in either an H state or a V state or a 45° or −45° state.

Alice records the polarization orientation she used; she assigns bit values to each orientation.

Alice assigns “0” to V and to −45° and she assigns “1” to H and to +45°.

The receiver, Bob, chooses his Polaroid orientations to be either H or V, or +45° or −45°, again at random. He does not know what orientation Alice has chosen. He keeps a record of whether or not he observes a photon and of the orientation he used.

After a train of photons has been sent Alice and Bob communicate by phone (a non-secure channel). They compare the Polaroid basis (H,V) or (+45°, −45°) that each has used. For those cases in which they have used the same basis (both used H or V or both used +45° or −45°) they retain the bits to be used as the key. For non-compatible bases they discard the bits. Notice in the table that for those cases where they have used the same basis they agree on bit values for detected photons so the random key has been transmitted from Alice to Bob.

1.3.2 Detecting an Evesdropper

Eve has a Polaroid sheet which she orients at random (since she does not know the orientation used by Alice) and intercepts the photons sent by Alice. She re-sends the photons emerging from her Polaroid to Bob. However the outgoing photon which Bob receives is now (usually) in a different state; the state of Eve’s Polaroid not the state sent by Alice.

To detect evesdropping Alice and Bob use a subset of the photons and compare the states seen by Bob with what is expected (sent by Alice). The fraction of detected photons and their states will be different if an evesdropper has intercepted and re-sent photons.

This is where Quantum Mechanics plays the important role: sending a photon through a Polaroid (that is: detecting it) changes its state.

Notice that this works only if individual photons are sent. If multiple photons or a beam of light were sent then a part of the beam could be analyzed without affecting the remainder which could be sent on to Bob.
When individual photons are detected by the evesdropper one might wonder whether she can copy (clone) the photons and send a duplicate to Bob and this avoid being detected. Eve could then also make a large number of copies and use those to determine the photon state. There is a fundamental theorem in Quantum Mechanics: the “no cloning” theorem, that shows that it is impossible to make a perfect duplicate (or clone) of an arbitrary quantum state.

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### Key Distribution: The BB84 Protocol

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</table>

Eve's present

**Alice, Bob Agree**

Eve's present