1 Polarization of Light

1.1 Classical Description

Light polarized in the $x$ direction has an electric field vector $\vec{E} = E_0 \hat{x} \cos(kz - \omega t)$

Light polarized in the $y$ direction has an electric field vector $\vec{E} = E_0 \hat{y} \cos(kz - \omega t)$

The wave number $k = 2\pi/\lambda$ where $\lambda$ is the wavelength.

$\omega = 2\pi f$ where $f$ is the frequency. $f\lambda = c$. $\hat{x} = \hat{i}$ and $\hat{y} = \hat{j}$ are unit vectors.

Light polarized at 45 degrees (call this direction $x'$) has an electric field vector

$$E_0 \hat{x}' \cos(kz - \omega t) = E_0 \left( \frac{1}{\sqrt{2}} \hat{x} \cos(kz - \omega t) + \frac{1}{\sqrt{2}} \hat{y} \cos(kz - \omega t) \right)$$

1.1.1 Polarization Analyzers

The component of the electric field along the transmission direction of a Polaroid filter is transmitted. After the filter the light is polarized along the filter transmission direction. Polaroid filters "prepare" light in a state of linear polarization.

The intensity of light is proportional to the square of the electric field; therefore after the filter the intensity is proportional to the square of the cosine of the angle between the incoming electric field and the Polaroid transmission direction.

$$|E_x| = |E| \cos \theta \text{ and intensity is proportional to } \cos^2 \theta.$$  

See French and Taylor[1] p. 234

An ideal Polaroid filter has these properties:

a) The outgoing light leaving a Polaroid sheet is polarized along the transmission direction of the Polaroid.

b) The Polaroid sheet transmits or selects the component of the incoming electric field which is along its transmission direction. For transmission along $x$ $E_x = \hat{x} \cdot \vec{E}$

c) An ideal Polaroid transmits 100% of the light polarized in its transmission direction and no light polarized in the perpendicular direction.

A Polaroid sheet prepares light in a state of linear polarization. A second Polaroid sheet analyzes the polarization state.

1.1.2 Polarization States

We can represent the polarization states by "state vectors" in a two dimensional vector space. Light polarized in the $x$ direction can be represented by $|x\rangle$ and light polarized in the $y$ direction by $|y\rangle$. 

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Polarization at 45 degrees can be represented by the state

$$|\theta = 45^\circ\rangle = \frac{1}{\sqrt{2}} (|x\rangle + |y\rangle)$$

Polarization at an angle $\theta$ in the $x,y$ plane, the direction $\hat{n}_\theta$, can be represented by

$$|\theta\rangle = \cos \theta |x\rangle + \sin \theta |y\rangle$$

Right and Left circular polarization can be described by

$$|R\rangle = \frac{1}{\sqrt{2}} (|x\rangle + i|y\rangle) \quad \text{and} \quad |L\rangle = \frac{1}{\sqrt{2}} (|x\rangle - i|y\rangle)$$

Notice that the state vectors have complex coefficients in this case which correspond to a 90 degree phase shift.

In the state notation the $x, y, \theta, R, L$ in the brackets are labels describing the states.

The most general polarization state is $|\Phi\rangle = \lambda |x\rangle + \mu |y\rangle$ where $\lambda$ and $\mu$ are in general complex with $|\lambda|^2 + |\mu|^2 = 1$. Then $|\Phi\rangle$ is normalized.

The basis states, $|x\rangle$ and $|y\rangle$, form an orthonormal basis. $\langle x|x\rangle = 1$ and $\langle x|y\rangle = 0$

The basis is said to be “complete” if any state can be written as a linear combination of basis states.

Another orthonormal basis can be written as

$$|\theta\rangle = \cos \theta |x\rangle + \sin \theta |y\rangle$$
$$|\theta_\perp\rangle = -\sin \theta |x\rangle + \cos \theta |y\rangle.$$ 

For circular polarization we can use the basis states $|R\rangle$ and $|L\rangle$.

1.1.3 Preparing and Analyzing Polarized Light

When light reaches a Polaroid sheet the outgoing state is the state vector of the incoming light projected along the state representing the Polaroid transmission direction.

The amplitude of the outgoing state is the inner product of the state representing the incoming light and the state representing the analyzer selection.

Inner products are written as $\langle y|\theta\rangle$ where $\langle y|$ is the dual of $|y\rangle$. The inner product is a scalar.

Compare with ordinary vectors, $E_x = \hat{x} \cdot \vec{E}$

If a Polaroid filter has its transmission axis in the $y$ direction; we represent it by $\langle y|$. If the incoming light is polarized at angle $\theta$ and the analyzer transmission direction is in the $y$ direction then the outgoing amplitude is the inner product of the state representing the Polaroid and the state of the incoming light

$$\langle y|\theta\rangle = \langle y| (\cos \theta |x\rangle + \sin \theta |y\rangle) = \cos \theta \langle y|x\rangle + \sin \theta \langle y|y\rangle = \sin \theta$$

We use an orthonormal basis: $\langle x|x\rangle = 1 \quad \langle y|y\rangle = 1 \quad \langle x|y\rangle = 0$
After the analyzer the polarization direction is $y$ and the amplitude is $\sin \theta$. The state of the outgoing light is described by $|y\rangle$.

Classically the intensity of light is proportional to the square of the electric field so the outgoing intensity is proportional to the magnitude squared of the amplitude $|\langle y|\theta \rangle|^2$. In this example the intensity after the Polaroid is $I = I_0 \sin^2 \theta$.

1.1.4 Examples

A polaroid sheet has its transmission axis at an angle $\theta$ with respect to the $x$ direction. The state representing the Polaroid is

$$\langle \theta \rangle = \cos \theta \langle x \rangle + \sin \theta \langle y \rangle$$

To find the amplitude for light transmission through a Polaroid we must project the state of the incident light beam on the state representing the Polaroid.

If the incident light is polarized in the $x$ direction it is represented by $|x\rangle$. If we send this light through a Polaroid filter whose transmission direction is at angle $\theta$ then the amplitude passing the Polaroid is

$$\langle \theta |x\rangle = \cos \theta \quad \text{using } \langle x|x \rangle = 1 \quad \langle y|y \rangle = 1 \quad \langle x|y \rangle = 0$$

Incident Light Polarized at 45 degrees

For 45 degree Linear Polarization the amplitude is

$$\langle \theta|45^\circ \rangle = (\cos \theta \langle x \rangle + \sin \theta \langle y \rangle) \left( \frac{1}{\sqrt{2}} (|x\rangle + |y\rangle) \right) \quad \langle \theta|45^\circ \rangle = \frac{1}{\sqrt{2}} (\cos \theta + \sin \theta)$$

The magnitude squared of the amplitude (the intensity) is

$$|\langle \theta|45^\circ \rangle|^2 = \frac{1}{2} (1 + 2 \sin \theta \cos \theta) = \frac{1}{2} (1 + \sin 2\theta)$$

If $\theta = 45^\circ$ we get intensity = 1 and if $\theta = 0$ or $\theta = 90^\circ$ we get intensity = $\frac{1}{2}$.

If $\theta = -45^\circ$ we get intensity = 0 since $\sin (-\pi/2) = -1$. The analyzer Polaroid is perpendicular to the Polaroid which prepared the light (the states are orthogonal).

1.2 Photons

We expect to find the same Energy (intensity) relationships with the photon picture as with the classical wave picture. From the analysis of the photoelectric effect each photon carries a quantum of energy $E = hf = hc/\lambda$. The total energy will be proportional to the number of photons. The light intensity is proportional to the number of photons.

Approximate numerical values are $1 \text{ eV} \approx 1.6 \times 10^{-19} \text{ J}$ $hc \approx 1240 \text{ eV - nm}$ $1 \text{ nm} = 10^{-9} \text{ m}$.
\[ E \approx \frac{1240}{\lambda} \] where \( E \) is in eV and \( \lambda \) is in nanometers.

For visible light \( \lambda \) is of order 500 nm so \( E \approx 2.5 \) eV

In the photon description of light we associate a polarization state with a beam of photons which has passed a Polaroid filter or has been prepared by atomic transitions. We can assign a polarization state \(|x\rangle, |y\rangle, |\theta\rangle, |R\rangle\), etc. to a beam of photons.

An **unpolarized** beam can be viewed as a mixture of polarizations states, for example a mixture of 50% of the photons in the state \(|x\rangle\) and 50% in the state \(|y\rangle\).

Be careful - this is completely different from a polarized beam described as a linear combination of states \( \frac{1}{2}(|x\rangle + |y\rangle) \).

If a beam of photons is in the state \(|x\rangle\) and it reaches a Polaroid filter whose transmission direction is \(y\) (represented by \(\langle y|\)) then no photons will get through the filter. If the filter were \(\langle x|\) then (for an ideal filter) all the photons should get through. For a filter at 45 degrees half the photons are transmitted.

A photon either goes through the Polaroid filter or it does not. There are no fractional photons. The **probability** that a photon is transmitted through a Polaroid sheet depends on the polarization state in which it was prepared.

We prepare a beam of photons in a state \(|\theta\rangle\) by sending the light through a Polaroid filter whose transmission axis is at angle \(\theta\) measured with respect to the \(x\) axis. Any photon observed after the Polaroid filter will be in the state \(|\theta\rangle\).

### 1.2.1 Amplitude and Probability

Suppose we have a beam of photons prepared in the state \(|\theta\rangle\). \(|\theta\rangle = \cos \theta |x\rangle + \sin \theta |y\rangle\)

The **Amplitude** for finding a photon after a Polaroid analyzer is given by the above inner product rules for finding amplitudes. For example for a Polaroid whose transmission axis is \(x\) the amplitude is \(\langle x|\theta\rangle\). The amplitude is \(\cos \theta\) in this case.

The “amplitude” is sometimes called the “probability amplitude”.

The **Probability** for finding a photon after the analyzer is the magnitude squared of the amplitude; the probability is \(|\langle x|\theta\rangle|^2\). The probability would be \(\cos^2 \theta\) in this case.

Recall that probabilities are numbers in the range 0 to 1.

The probability that a photon in the state \(|\theta\rangle = \cos \theta |x\rangle + \sin \theta |y\rangle\) will be transmitted by a Polaroid sheet whose transmission axis is in the \(x\) direction is \(\cos^2 \theta\). This is expected since the Intensity observed after the Polaroid must agree with the classical picture. (The energy is proportional to the square of the Electric field and is proportional to the number of photons.)

In general only a fraction of the original number of photons is observed after the filter.

**After** the analyzer any observed photon will be in state \(|x\rangle\) but we know (almost) nothing about the state of the photon **before** the analyzer.

We must not conclude that the photon was in state \(|x\rangle\) **before** the analyzer.

“quantum theory is incompatible with the proposition that measurements are processes by which we discover some unknown and pre-existing property.”

F. Lalöe[?] quoting A. Peres.[?]
1.2.2 Summary

1. The polarization state of a photon is represented by a “state vector”.
2. A Polaroid sheet is represented by a state vector which corresponds to the state of a photon which has been transmitted through the Polaroid filter.
3. The Amplitude of the outgoing light is the inner product of these two state vectors.
4. The Probability for photon transmission is the squared-magnitude of the amplitude.
5. If a photon emerges from a Polaroid sheet its state corresponds to the transmission state of the Polaroid. The state after the Polaroid tells us nothing about the incoming state.
6. The Intensity of the outgoing light is proportional to the number of transmitted photons.
2 Algebra of State Vectors

What does the State Vector mean?

”The state of a quantum mechanical system, including all the information you can know about it, is represented mathematically by a normalized ket $|\psi\rangle$.”

D. McIntyre: Quantum Mechanics

The state vector may also represent the experimental procedure for preparing the system. There are other interpretations of the state vector: it may refer to an ensemble of identically prepared systems or even to a single system. Le Bellac (Quantum Physics, page 97 footnote) says

“... the state vector describes the physical reality of an individual quantum system. This point of view is far from universally shared... This diversity of viewpoints has no effect on the practical application of quantum mechanics.”

These state vectors like $|x\rangle$ are called “kets”, a name proposed by Dirac; it is part of the word “bracket”. This way of writing state vectors in Quantum Mechanics is called Dirac Notation.

2.1 Inner Product

With ordinary vectors the inner product can be written as $A \cdot B$ or $B \cdot A$ but with state vectors we define for each vector a “dual” vector. There is a one-to-one correspondence between a vector and its dual vector. From a more mathematical perspective the dual vector provides a mapping from the ket vector onto complex numbers (the inner product). see Cohen-Tannoudji[?] page 110. The dual vector corresponding to $c|\theta\rangle$ is $c^*\langle\theta|$. Notice the complex conjugate of the constant $c$.

The inner products of the orthonormal basis states are $\langle x|x \rangle = 1$ and $\langle y|y \rangle = 1$ and $\langle y|x \rangle = 0$.

Suppose

$|\psi\rangle = \lambda|x\rangle + \mu|y\rangle$

$\langle \psi| = \lambda^*\langle x| + \mu^*\langle y|$

$|\phi\rangle = \alpha|x\rangle + \beta|y\rangle$

$\langle \phi| = \alpha^*\langle x| + \beta^*\langle y|$

where $|x\rangle$ and $|y\rangle$ are orthogonal and normalized.

Show that

$\langle \psi|\phi \rangle = \langle \phi|\psi \rangle^*$

we see that $\langle \psi|\psi \rangle$ is real.
3 Principles of Quantum Mechanics

3.1 Postulates - Preliminary Version

1. The state of a system is represented by a vector $|\Phi\rangle$. This is chosen to be normalized and $|\langle\Phi|\Phi\rangle|^2 = 1$. This is called the state vector of the system. $\langle\Phi|$ is the dual vector.

2. If $|\Phi\rangle$ and $|\Psi\rangle$ represent two physical states the probability amplitude for finding the system prepared in state $\Phi$ to be observed in the state $\Psi$ is given by the inner product $\langle\Psi|\Phi\rangle$. The probability is the magnitude squared of the amplitude, $|\langle\Psi|\Phi\rangle|^2$.

After testing for state $|\Psi\rangle$ the system is in the state $|\Psi\rangle$ with the probability given above.

3.2 Measurement

A system has been prepared in the state $|\Phi\rangle$. The state of the system after the testing or “measurement” is $|\Psi\rangle$ and we can take the point of view that the original state has been projected along this “direction” in the vector space.

This projection of the state after a measurement is sometimes called “state-vector collapse” or wave function collapse or reduction.

This state-collapse is sometimes considered a postulate of Quantum Mechanics. In this perspective we do not discuss the details of the measurement process itself.

Suppose we prepare a state $|\Phi\rangle$ and expand it in a series of basis states $|x\rangle$.

$$|\Phi\rangle = \sum_x a_x |x\rangle$$

The $a_x$ are the amplitudes for each basis state. The amplitude is $a_x = \langle x|\Phi\rangle$.

Mermin\cite{Mermin} says

“...the link between $|\Phi\rangle$ and the value of $x$ revealed by the measurement is this: the probability of getting the output $x$ is just $p_x = |a_x|^2$, where $a_x$ is the amplitude of $|x\rangle$ in the expansion of $|\Phi\rangle$. This connection between amplitudes and the probabilities of measurement outcomes is known as the Born rule, after the physicist Max Born.”

Paraphrasing Mermin

“...The postmeasurement state $|x\rangle$ contains no trace of the information present in the premeasurement state $|\Phi\rangle$. (beyond revealing that the amplitude $a_x \neq 0$) .... the state $|\Phi\rangle$ collapses or is reduced to the state $|x\rangle$ by the measurement.

The amplitude is $a_x = \langle x|\Phi\rangle$.