6 Square wells

6.1 Some notes from the previous day

remember that in the previous day we arrived at two forms of $\psi(x)$ one for $E > V$ and one for $E < V$ where E is the energy of the particle and V is the constant potential. Here they are without their derivation:

- For $E > V$
$$\psi(x) = A_1 e^{ikx} + A_2 E^{-ikx}$$
$$\text{such that } k = \sqrt{\tfrac{2m(E-V)}{\hbar^2}}$$
- For $E < V$
$$\psi(x) = B_1 e^{\rho x} + B_2 e^{-\rho x}$$
$$\text{such that } \rho = \sqrt{\tfrac{2m(V-E)}{\hbar^2}}$$

another note from the previous lecture is the form of the Schrödinger Equation:

$$i\hbar\frac{\partial\Psi}{\partial t} = \left(\frac{p^2}{2m} + V(x)\right)\Psi(x,t) = H\Psi(x,t)$$

The time independent Schrödinger Equation is:

$$E\Psi = H\Psi$$

7 Infinite well

7.1 what is an Infinite well

A simple case of a potential function is a potential well. A potential well function is such that $V(x)$ is zero from $x = 0$ to $x = a$ and it shoots to $+\infty$ at both $x = 0$ and $x = a$ it basically represents a one dimensional situation where a particle with energy E is confined withing $x = 0$ and $x = a$ and has no probability of being found outside of $0 < x < a$.

7.2 starting on the wave function

Since we are only interested in the stationary states we write Ψ as $\Psi = \psi(x)\phi(t)$ and we find $\phi(t) = e^{-iEt/\hbar}$ and we go to solve only the space dependent function:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

which can be rewritten as:

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = -k^2\psi$$

$$\text{with } k = \sqrt{\tfrac{2mE}{\hbar^2}}$$

7.3 boundary condition $\Rightarrow$ energy states

the next step is to realize the boundary conditions as $\psi(0) = \psi(a) = 0$ meaning that there is no chance that the particle could be found outside of the well. with this taken care of we remember from the previous day that in a stationary bound state $\psi(x) = A\sin(kx)$. A is a normalization factor. To use our boundary conditions we must have $\psi(0) = 0$ but that is already done since $\sin(0) = 0$ and to set $\sin(ka)$ we must set $ka = n\pi$ such that $n = 1, 2, \ldots$ then this restricts the values that k can take: $k = \frac{n\pi}{a}$ which is very intresting and a "side-effect" of this is that since Energy is related to k then energy can only have certain values!!!

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 n^2 \pi^2}{8ma^2} \tag{1}$$

as you can see in the above equation energy can only take certain values, the lowest Energy (E_1) corresponds to $n = 1$ and the higher energies can be written as: $4E_1, 9E_1, 16E_1 \ldots$

7.4 normalization

Because $|\psi(x)|^2$ represents the probability density for finding a particle at x, and we are setting our system up so that the particle must be within $x = 0$ and $x = a$ then we must define

$$\int_0^a |\psi(x)|^2\, dx = 1$$

which leads to $|A|^2 \int_0^a \sin^2(x)dx = n$ and to find $\int_0^a \sin^2(x)dx$ we realize that $\int_0^{2\pi} \sin^2(x)dx = \int_0^{2\pi} \cos^2(x)dx$ which in turn leads to

$$\int_0^{2\pi} \sin^2(x)dx = \frac{1}{2}\left(\int_0^{2\pi} \sin^2(x) + \cos^2(x)dx\right) = \frac{1}{2}\int_0^{2\pi} 1dx = \pi$$

note: although the physics are very different our system has the same mathematical form as the vibrating spring normal modes.

8 an analytical look at Finite wells

finite wells are wells just like our infinite wells except that V has a finite value outside the well. the interesting fact to note now about them is that $\psi(0)$ or $\psi(a)$ do not have to equal zero and hence there will be a possibility that we find particle in the "classically forbidden" region. Just as with the infinite wells ψ is sinusoidal inside the well but it falls off exponentially outside of the well. Another interesting mathematical fact to note about this "classically forbidden" region is that a possible solution is for our exponential to be growing and we only picked it as falling due to our Normalization constraint. More about this latter.

9 few notes about the formal aspects of QM

instead of writing the notes on this subject I will refer the reader to *Introduction to Quantum Mechanics 2nd Ed.* David J. Griffiths (AKA "the griffiths book") section 1.3 where the issue of average value and its meaning are discussed.

10 question

it was asked weather we could somehow "watch" the particle and just see where it is. the answer to which is that they behave differently when we do. namely once we see that a particle is at a position x it is left in an eigenstate of x which evolves in time.

The Wave Function applies not to a single particle but to an Ensemble of particles.

in the vocabulary of our wells this means that we can use each well only once, and in order to see the effect of the wave function we must have a (large) number of identical wells.

quick note on the average of energy:

$$< H >= \int \psi^* H \psi dx = E \int \psi^* \psi dx = E$$

if ψ is a stationary state with energy E

11 postulates of quantum mechanics

Dr. Rothberg briefly talked about the postulates of quantum mechanics but, not wanting to reinvent the wheel, I refer the reader to see QM texts such as the ones authored by Liboff or Cohen-Tannoudji

12 An atomic example

Imagine an infinite well situation with a width of 0.1nm $= 10^{-10}$m $= 10^{-8}$cm then let us calculate the ground state energy via: (using units of nm and eV)

$$E = \frac{\pi^2 \hbar^2}{2ma^2}$$

multiplying both the top and the bottom by c^2

$$E = \frac{\pi^2 (\hbar c)^2}{2(mc^2)a^2} = \frac{\pi^2 (200)^2}{2(\frac{1}{2}10^6)a^2} = \frac{\pi^2 4 \times 10^4}{10^6 \times .01} \approx \frac{10 \times 4}{1} \approx 40\text{eV}$$