Micromechanics of Magnetoelectroelastic Composite Materials: Average Fields and Effective Behavior

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ABSTRACT: A micromechanics approach is developed to analyze the average fields and effective moduli of heterogeneous media that exhibit full coupling between stationary elastic, electric, and magnetic fields. Exact relations regarding the internal field distribution inside a heterogeneous magnetoelectroelastic solid are first established, followed by exact connections between the effective magnetoelectroelastic and thermal moduli of two-phase composites. The Mori-Tanaka mean field approach is then applied to obtain closed form expressions for the effective moduli of fibrous and laminated composites. Finally, numerical results for $BaTiO_3$-$CoFe_2O_4$ composites are presented and discussed.

1. INTRODUCTION

THE desire to develop materials exhibiting properties superior to those currently existing has motivated the advancement of composite materials technology. There are many advantages to using composite materials over more traditional materials, such as the possibility of weight or volume reduction in a structure while maintaining a comparable or improved performance level. In recent years, an increasing interest has been directed toward *smart* or *intelligent* composite materials that are capable of responding in a desired way to internal or environmental changes. An example is the development of piezoelectric composites where significant progress has been made since the 1970s and now applications such as underwater hydrophones and medical ultrasonic imaging devices are common. These materials exhibit a remarkable product property, created through the interaction between the phases, which is a field coupling in the composite that is absent in the constituent materials. One example of such a product property is pyroelectricity. It can be achieved by combining a material with a large thermal expansion coefficient with a piezoelectric material. The composite can exhibit pyroelectricity even though neither of the constituents does. Such composites are currently used in numerous thermal-imaging devices and sensors. Also, the magnetoelectric effect in composite materials consisting of a piezoelectric phase and a piezomagnetic phase has drawn attention. Van Run et al. (1974) reported the fabrication of a $BaTiO_3$-$CoFe_2O_4$ composite with a magnetoelectric coefficient two orders larger than that of Cr_2O_3, which had the highest magnetoelectric coefficient among single-phase materials known at that time. Bracke and Van Vliet (1981) reported a broad band magnetoelectric transducer with a flat frequency response using composite materials. Since then, numerous researchers have investigated the magnetoelectric coupling in a piezoelectric-piezomagnetic composite both theoretically and experimentally (Harshe et al., 1993a,b; Avellaneda and Harshe, 1994; Nan, 1994; Benveniste, 1995). Harshe et al. and Avellaneda and Harshe studied the 2-2, 3-0, and 0-3 magnetoelectric composite theoretically on a case by case basis. They obtained expressions for the effective magnetoelectric coefficient and a figure of merit for magnetoelectric coupling. Nan proposed two models to estimate the effective properties of piezoelectric-piezo- magnetic composite materials. His models, however, fail to satisfy the exact connections between different components of the effective moduli obtained by Benveniste for piezoelectric composites (Benveniste, 1995), thus casting doubt on their theoretical rigor. To our knowledge, none of these modeling efforts consider thermal effects. Along different, but related lines, Li and Dunn (1998a) solved inclusion and inhomogeneity problems in an infinite magnetoelectroelastic medium. Their key results are explicit expressions for the generalized Eshelby tensors, which are readily used for micromechanics modeling of heterogeneous solids. Indeed, the objective of this work is to use these results to study the average fields and effective behavior of magnetoelectroelastic composites with full coupling between the elastic, electric, and magnetic fields.

As will be apparent, the present work is a generalization of recent work directed toward heterogeneous piezoelectric media, and so it is worthwhile to briefly review them. Work in this area falls into two broad categories: 1) direct estimates of average fields and effective properties, and 2) the development of internal consistency relations between the effective moduli and internal fields. The work of Cao et al. (1992a,b), Sottos et al. (1993), Zhang et al. (1993), Wang (1992), Dunn and Taya (1993a,b), and Chen (1994, 1996) fall into the for-

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mer category while the work of Benveniste and Dvorak (1992), Schulgasser (1992), Benveniste (1993a,b,c, 1994), Dunn (1993a,b), and Chen (1993) fall into the latter.

In this work, we present the basic equations and notation used for magnetoelectroelastic media in Section 2. Some exact relations regarding the effective behavior and the average fields are then derived in Section 3. These include an exact connection between the effective magnetoelectroelastic and thermal moduli of two-phase composites. In Section 4 we use the reasonably well-known Mori-Tanaka (1973) mean field approach coupled with the magnetoelectroelastic Eshelby tensor obtained by Li and Dunn (1998a) to obtain explicit expressions for the effective magnetoelectroelastic and thermal moduli of two technologically important composite microgeometries: continuous cylindrical fibers and laminates. These results are then discussed in Section 6 where extensive numerical results are also presented.

2. BASIC EQUATIONS

We consider magnetoelectroelastic media that exhibit linear, static, anisotropic coupling between the magnetic, electric, and elastic fields, but temperature enters the problem only as a parameter through the constitutive equations. In this case, the constitutive equations can be expressed as

$$\begin{aligned} \sigma_{ij} &= C_{ijkl}\varepsilon_{kl} + e_{ijl}(-E_l) + q_{ijl}(-H_l) - \lambda_{ij}\theta \\ D_i &= e_{ikl}\varepsilon_{kl} - \kappa_{il}(-E_l) - a_{il}(-H_l) - p_i\theta \qquad (1) \\ B_i &= q_{ikl}\varepsilon_{kl} - a_{il}(-E_l) - \mu_{il}(-H_l) - m_i\theta \end{aligned}$$

Here σ_{ij} and ε_{ij} are the stress and strain; D_i and E_i are the electric displacement and field; B_i and H_i are the magnetic flux and field. C_{ijkl}, κ_{il}, and μ_{il} are the elastic stiffness, the dielectric, and magnetic permeability tensors. They directly connect like fields, e.g., stresses to strains. Elastic fields are coupled to the electric and magnetic fields through the piezoelectric, e_{ijl}, and piezomagnetic, q_{ijl}, coefficients, respectively. Electric and magnetic fields are coupled through the magnetoelectric coefficients, a_{il}. Finally, elastic, electric, and magnetic fields are coupled to a temperature change θ through the thermal stress tensor λ_{ij}, the pyroelectric coefficient p_i, and the pyromagnetic coefficient m_i. The symmetry conditions satisfied by the moduli are given by Nye (1957).

In the analysis that follows, it is convenient to treat the elastic, electric, and magnetic fields on equal footing. To this end, the notation introduced by Barnett and Lothe (1975) for piezoelectric analysis and generalized to incorporate magnetic coupling by Alshits et al. (1992) is utilized. This notation is identical to conventional indicial notation with the exception that lowercase subscripts take on the range $1 \rightarrow 3$, while uppercase subscripts take on the range $1 \rightarrow 5$ and repeated uppercase subscripts are summed over $1 \rightarrow 5$. With this notation, the field variables take the following forms:

$$\Sigma_{iJ} = \begin{cases} \sigma_{ij} & J = 1,2,3 \\ D_i & J = 4 \\ B_i & J = 5 \end{cases} \qquad Z_{Mn} = \begin{cases} \varepsilon_{mn} & M = 1,2,3 \\ -E_n & M = 4 \\ -H_n & M = 5 \end{cases} \qquad (2)$$

The magnetoelectroelastic and thermal moduli are expressed as

$$E_{iJMn} = \begin{cases} C_{ijmn} & J,M = 1,2,3 \\ e_{ijn} & M = 4, J = 1,2,3 \\ q_{ijn} & M = 5, J = 1,2,3 \\ e_{imn} & J = 4, M = 1,2,3 \\ -\kappa_{in} & J = 4, M = 4 \\ -a_{in} & J = 4, M = 5 \\ q_{imn} & J = 5, M = 1,2,3 \\ -a_{in} & J = 5, M = 4 \\ -\mu_{in} & J = 5, M = 5 \end{cases}$$

$$\Pi_{iJ} = \begin{cases} \lambda_{iJ} & J = 1,2,3 \\ p_i & J = 4 \\ m_i & J = 5 \end{cases} \qquad (3)$$

With this shorthand notation, the constitutive equations can be written as

$$\Sigma_{iJ} = E_{iJKl}Z_{Kl} - \Pi_{iJ}\theta \qquad (4)$$

Of course, one can easily make alternative choices for the independent and dependent variables and formulate the basic equations using the same formalism.

3. EXACT RELATIONS REGARDING AVERAGE FIELDS AND EFFECTIVE BEHAVIOR

3.1 General Heterogeneous Medium

Consider a heterogeneous magnetoelectroelastic medium subjected to homogeneous potential boundary conditions, Z^0_{Ji}, and a uniform temperature change θ. By homogeneous boundary conditions it is meant that when they are applied to a homogeneous solid they result in homogeneous fields. The volume averaged fields in the composite are connected by the effective moduli E^*_{iJMn} and Π^*_{iJ}:

$$\langle \Sigma_{iJ} \rangle = E^*_{iJMn}\langle Z_{Mn} \rangle - \Pi^*_{iJ}\theta \qquad (5)$$

where

$$\langle (\bullet) \rangle = \frac{1}{V}\int_V (\bullet)dV$$

denotes a volume average over the heterogeneous medium. Under the action of such boundary conditions, the average strain theorem of elasticity (see, for example, Aboudi, 1991) can be generalized to show

$$\langle Z_{Mn} \rangle = Z^0_{Mn} \tag{6}$$

Furthermore, as a generalization of the Hill condition for elastic heterogeneous solids (Hill, 1963; Kreher, 1988) and heterogeneous piezoelectric solids (Li and Dunn 1998b), it is easy to verify that

$$\langle \Sigma_{iJ} Z_{Ji} \rangle = \langle \Sigma_{iJ} \rangle \langle Z_{Ji} \rangle \tag{7}$$

Equations (6) and (7) are obtained through the use of the divergence theorem on the surface of the medium and are based on the assumptions that (1) ε_{ij}, E_i, and H_i are derivable from the continuous elastic displacement, electric, and magnetic potential, respectively; (2) σ_{ij}, D_i, and B_i satisfy equilibrium and Gauss's law, respectively; and (3) the composite is subjected to homogeneous boundary conditions that would produce homogeneous magnetoelectroelastic fields in a homogeneous medium.

It is advantageous to split the elastic, electric, and magnetic fields in the heterogeneous medium into two parts: one due to external loading Z^0_{Ji} (denoted by a superscript I) and the other due to the temperature change θ (denoted by a superscript II):

$$Z_{Ji} = Z^I_{Ji} + Z^{II}_{Ji} \qquad \Sigma_{iJ} = \Sigma^I_{iJ} + \Sigma^{II}_{iJ} \tag{8}$$

In view of these definitions, the average field theorem of Equation (6) implies

$$\langle Z^I_{Ji} \rangle = Z^0_{Ji}, \quad \langle Z^{II}_{Ji} \rangle = 0 \tag{9}$$

Consequently, Equation (4) can be decomposed into two equations:

$$\Sigma^I_{iJ} = E_{iJMn} Z^I_{Mn} \tag{10}$$

$$\Sigma^{II}_{iJ} = E_{iJMn} Z^{II}_{Mn} - \Pi_{iJ}\theta \tag{11}$$

Generalizing Kreher's (1988) terminology, we call field I the *loading field* and field II the *residual field.* Because both fields satisfy the equilibrium and gradient equations, the generalized Hill condition applies to both the loading and residual fields.

Taking into account the boundary conditions and the effective constitutive equations, we can show

$$\langle Z^I_{Mn} \rangle = Z^0_{Mn} \tag{12a}$$

$$\langle \Sigma^I_{iJ} \rangle = E^*_{iJMn} Z^0_{Mn} \tag{12b}$$

$$\langle Z^{II}_{Mn} \rangle = 0 \tag{13a}$$

$$\langle \Sigma^{II}_{iJ} \rangle = -\Pi^*_{iJ}\theta \tag{13b}$$

Substituting Equations (12) and (13) into the Hill condition (7), we obtain four scalar equations:

$$\langle \Sigma^I_{iJ} Z^I_{Ji} \rangle = Z^0_{Ji} E^*_{iJMn} Z^0_{Mn} \tag{14}$$

$$\langle \Sigma^{II}_{iJ} Z^I_{Ji} \rangle = -Z^0_{Ji} \Pi^*_{iJ}\theta \tag{15}$$

$$\langle \Sigma^I_{iJ} Z^{II}_{Ji} \rangle = 0 \tag{16}$$

$$\langle \Sigma^{II}_{iJ} Z^{II}_{Ji} \rangle = 0 \tag{17}$$

From Equations (15) and (16), we obtain

$$Z^0_{Ji}\Pi^*_{iJ} = \langle Z^I_{Ji} \Pi_{iJ} \rangle \tag{18}$$

Equations (14)–(18) establish the rigorous connection between the effective properties and a statistical description of the microstructure for heterogeneous magnetoelectroelastic media. One can continue this line of analysis and obtain exact relations for the second-order moments of the internal fields, however, we do not pursue this line of inquiry here. In the next section, we will use Equation (18) to establish an exact connection between the effective magnetoelectroelastic moduli and the effective thermal moduli for a two-phase composite.

3.2 Two-Phase Composites

Here we specialize the results of the previous section to a composite consisting of a matrix with a single dispersed phase. We require that the material properties of the dispersed phase are constant with respect to a fixed sample coordinate system. Thus orientational variations of an anisotropic dispersed phase are prohibited. The two phases are characterized by their volume fractions c_1 and c_2, magnetoelectroelastic moduli $E_{iJMn}|_1$ and $E_{iJMn}|_2$, and thermal moduli $\Pi_{iJ}|_1$ and $\Pi_{iJ}|_2$, where the subscript 1 is used for the matrix and 2 is used for the dispersed phase. As a result, in the fixed sample coordinate system, $E_{iJMn}(\mathbf{x})$ and $\Pi_{iJ}(\mathbf{x})$ assume only two values.

We have already noted that $\langle Z^I_{Mn} \rangle = Z^0_{Mn}$. Furthermore, Z^0_{Mn} is also equal to the volume-weighted average of Z^I_{Mn} over each phase. As a result we have

$$c_1 \langle Z^I_{Mn}|_1 \rangle + c_2 \langle Z^I_{Mn}|_2 \rangle = Z^0_{Mn} \tag{19a}$$

Applying an analogous result for $\langle \Sigma^I_{iJ} \rangle$, and using the constitutive equations for each phase and the composite yields

$$c_1 E_{iJMn}\big|_1\left\langle Z^I_{Mn}\big|_1\right\rangle + c_2 E_{iJMn}\big|_2\left\langle Z^I_{Mn}\big|_2\right\rangle = E^*_{iJMn} Z^0_{Mn} \tag{19b}$$

Provided that ΔF_{MncD} [defined by $(E_{iJMn}|_1 - E_{iJMn}|_2)\cdot \Delta F_{MncD} = I_{iJcD}$ where I_{iJcD} is a grouping of the 2nd-rank and 4th-rank identity tensors] exists, Equations (19) can be arranged to yield

$$\begin{aligned}\left\langle Z^I_{Mn}\big|_1\right\rangle &= \frac{1}{c_1}\Delta F_{MniJ}\left(E^*_{iJAb} - E_{iJAb}|_2\right)Z^0_{Ab}\\ \left\langle Z^I_{Mn}\big|_2\right\rangle &= -\frac{1}{c_2}\Delta F_{MniJ}\left(E^*_{iJAb} - E_{iJAb}|_1\right)Z^0_{Ab}\end{aligned} \tag{20}$$

In an analogous manner we obtain

$$\begin{aligned}\left\langle Z^{II}_{Mn}\big|_1\right\rangle &= \frac{1}{c_1}\Delta F_{MniJ}\left(\Pi^*_{iJ} - \overline{\Pi}_{iJ}\right)\theta\\ \left\langle Z^{II}_{Mn}\big|_2\right\rangle &= -\frac{1}{c_2}\Delta F_{MniJ}\left(\Pi^*_{iJ} - \overline{\Pi}_{iJ}\right)\theta\end{aligned} \tag{21}$$

where $\overline{\Pi}_{iJ} = c_1\Pi_{iJ}|_1 + c_2\Pi_{iJ}|_2$. Substituting Equation (20) into Equation (18), followed by some manipulation, yields

$$\begin{aligned}\Pi^*_{iJ} = \Pi_{iJ}|_2 &+ \left(E^*_{iJMn} - E_{iJMn}|_2\right)\Delta F_{MncD}\\ &\times\left(\Pi_{cD}|_1 - \Pi_{cD}|_2\right)\end{aligned} \tag{22}$$

Equation (22) is an exact result that rigorously connects the effective thermal moduli to the effective magnetoelectroelastic moduli. Similar results for heterogeneous elastic solids were first obtained by Levin (1967) and Rosen and Hashin (1970). For heterogeneous piezoelectric solids, Dunn (1993b) and Benveniste (1993c) derived analogous results using two different approaches (Benveniste has actually obtained results for multi-phase composites). Note that in the derivation here, no specific microstructure was assumed. These exact relations are thus applicable to two-phase composites with a wide range of microstructural geometry. Also note that we have not specified how one obtains the effective magnetoelectroelastic moduli of the composite, but we have simply assumed that they can be obtained. This may be done either experimentally or through detailed micromechanics modeling, the latter being the subject of the following section.

Finally, a general expression for the effective magnetoelectroelastic moduli of perfectly bonded two-phase composites can be obtained using Equation (19b) and the constitutive equations in each phase to yield

$$E^*_{iJMn} = E_{iJMn}|_1 + c_2\left(E_{iJAb}|_2 - E_{iJAb}|_1\right)A_{AbMn} \tag{23}$$

In Equation (23) A_{AbMn} is the concentration factor that relates the average strain and potential gradients in phase 2 to that in the composite, i.e.,

$$\left\langle Z^I_{Mn}\big|_2\right\rangle = A_{MnAb} Z^0_{Ab} \tag{24}$$

The estimation of A_{MnAb} is thus the key to predicting the effective magnetoelectroelastic moduli E^*_{iJMn}. In the next section we will generalize the Mori-Tanaka (1973) theory for elastic composites to predict the effective magnetoelectroelastic moduli of a two-phase composite.

4. MICROMECHANICS MODELS

To make progress, we consider a two-phase composite consisting of a matrix containing dispersed ellipsoidal particles that are perfectly aligned. By modeling the shape of the dispersed phase as ellipsoidal, we can model a wide range of microstructural geometries. Extreme cases are lamina and continuous fiber reinforced composites. The simplest approximation of A_{MnAb} is $A_{MnAb} = I_{MnAb}$ which represents a generalization of the well-known Voigt (1889) approximation. The dilute approximation is then the next simplest micromechanics approximation. The key assumption made in the dilute approximation is that the interaction among the dispersed phases in a matrix-based composite can be ignored, and the concentration factor A_{MnAb} is obtained from the solution of the auxiliary problem of a single particle embedded in an infinite matrix. Mathematically, this solution can be expressed as

$$A^{dil}_{MnAb} = \left[I_{MnAb} + S_{MnLk} E^{-1}_{LkiJ}\big|_1\left(E_{iJAb}|_2 - E_{iJAb}|_1\right)\right]^{-1} \tag{25}$$

Here the superscript -1 denotes an inversion operation, and S_{MnAb} are magnetoelectroelastic Eshelby tensors. For ellipsoidal inclusions, they are functions of the shape of the inclusion and the magnetoelectroelastic moduli of the matrix. They are composed of one 4th rank tensor, four 3rd rank tensors, and four 2nd rank tensors. Explicit expressions of S_{MnAb} for a transversely isotropic solid containing an aligned cylindrical inclusion or a penny-shape inclusion are given by Li and Dunn (1998a). For more general cases, the Eshelby tensors can be evaluated numerically.

Although quite simple, it is widely recognized that the dilute approximation is in general only accurate for small reinforcement volume fractions. To model more accurately the effects of modest volume fractions, say up to about 40 percent, we appeal to the effective field theory originally developed by Mori and Tanaka (1973). Their original work was concerned with estimating the average internal stress in a matrix containing precipitates with eigenstrains. Since then, the method has been substantially generalized and successfully applied to many problems in the mechanics and physics of composite materials. The key assumption in Mori-Tanaka theory is that A_{MnAb} is given by the solution for a single particle embedded in an infinite matrix subjected to an applied magnetoelectroelastic field equal to the as yet unknown average field in the matrix. This assumption is easily expressed as

$$\langle Z^l_{Mn}|_2\rangle = A^{dil}_{MnAb}\langle Z^l_{Ab}|_1\rangle \tag{26}$$

With Equations (19a), (24), and (26), the concentration factor, A^{MT}_{MnAb}, can be written in the form:

$$A^{MT}_{MnAb} = A^{dil}_{MnKl}(c_1 I_{KlAb} + c_2 A^{dil}_{KlAb})^{-1} \tag{27}$$

Equations (23), (25), and (27) allow us to model the effective moduli of magnetoelectroelastic composites with various microstructural geometries.

5. EXPLICIT EXPRESSIONS FOR THE EFFECTIVE MODULI

We have obtained explicit expressions for the effective magnetoelectroelastic moduli using the Mori-Tanaka theory for two composite microgeometries: continuous fibers aligned in the x_3-direction and lamina oriented in the x_1–x_2-plane (layered in the x_3-direction). In both cases, full coupling exists between the elastic, electric, and magnetic fields, and the material properties of both phases are at most transversely isotropic with the unique axis along the x_3-direction. In both cases, the resulting independent material constants are the elastic constants C_{11}, C_{12}, C_{13}, C_{33}, and C_{44}, piezoelectric constants e_{15}, e_{31}, and e_{33}, piezomagnetic constants q_{15}, q_{31}, and q_{33}, dielectric constants κ_{11} and κ_{33}, magnetoelectric constants a_{11} and a_{33}, magnetic constants μ_{11} and μ_{33}, thermal stress constants λ_{11} and λ_{33}, pyroelectric constant p_3, and pyromagnetic constant m_3. For the elastic constants, we use the well-known Hill moduli k and m instead of C_{11} and C_{12}, where $k = (C_{11} + C_{12})/2$ and $m = (C_{11} - C_{12})/2$. The effective moduli are obtained by substituting in the closed-form expressions of Li and Dunn (1998a) for the Eshelby tensors into Equation (25), and then simplifying the combination of Equations (23), (25), and (27). After the effective magnetoelectroelastic moduli are obtained in this way, the effective thermal moduli are obtained through the exact connections of Equation (22). The explicit results are presented here for the two composite microgeometries. Constants defined for the purpose of simplifying these expressions are tabulated in the Appendix.

5.1 Fibrous (Circular Cylinder) Composite

Elastic Moduli

$$k^* = \frac{k_2 k_1 + c_2 k_2 m_1 + c_1 k_1 m_1}{c_1 k_2 + c_2 k_1 + m_1}$$

$$m^* = \frac{m_1(k_1 m_2 + c_2 k_1 m_2 + c_1 k_1 m_1 + 2m_2 m_1)}{c_1 k_1 m_2 + k_1 m_1 + c_2 k_1 m_1 + 2c_1 m_2 m_1 + 2c_2 m_1^2}$$

$$C^*_{13} = C_{13}|_1 + \frac{c_2(C_{13}|_2 - C_{13}|_1)(k_1 + m_1)}{c_1 k_2 + c_2 k_1 + m_1}$$

$$C^*_{33} = C_{33}|_1 + c_2\left[C_{33}|_2 - C_{33}|_1 - \frac{c_1(C_{13}|_1 - C_{13}|_2)^2}{c_1 k_2 + c_2 k_1 + m_1}\right]$$

$$C^*_{44} = C_{44}|_1 + j[(e_{15}|_2 - e_{15}|_1)(fe - gh)$$
$$+ (C_{44}|_2 - C_{44}|_1)(ih - fd) + (q_{15}|_2 - q_{15}|_1)(gd - ie)]$$

Piezoelectric Moduli

$$e^*_{31} = e_{31}|_1 + \frac{c_2(e_{31}|_2 - e_{31}|_1)(k_1 + m_1)}{c_1 k_2 + c_2 k_1 + m_1}$$

$$e^*_{33} = e_{33}|_1 + c_2\left[e_{33}|_2 - e_{33}|_1 + \frac{c_1(C_{13}|_1 - C_{13}|_2)(e_{31}|_2 - e_{31}|_1)}{c_1 k_2 + c_2 k_1 + m_1}\right]$$

$$e^*_{15} = e_{15}|_1 + j[(e_{15}|_2 - e_{15}|_1)(ga - fc)$$
$$+ (C_{44}|_2 - C_{44}|_1)(fb - ia) + (q_{15}|_2 - q_{15}|_1)(ic - gb)]$$

Piezomagnetic Moduli

$$q^*_{31} = q_{31}|_1 + \frac{c_2(q_{31}|_2 - q_{31}|_1)(k_1 + m_1)}{c_1 k_2 + c_2 k_1 + m_1}$$

$$q^*_{33} = q_{33}|_1 + c_2\left[q_{33}|_2 - q_{33}|_1 + \frac{c_1(C_{13}|_1 - C_{13}|_2)(q_{31}|_2 - q_{31}|_1)}{c_1 k_2 + c_2 k_1 + m_1}\right]$$

$$q^*_{15} = q_{15}|_1 + j[(e_{15}|_2 - e_{15}|_1)(ch - ae)$$
$$+ (C_{44}|_2 - C_{44}|_1)(ad - bh) + (q_{15}|_2 - q_{15}|_1)(be - cd)]$$

Dielectric Moduli

$$\kappa^*_{11} = \kappa_{11}|_1 + j[(\kappa_{11}|_2 - \kappa_{11}|_1)(ga - fc)$$
$$+ (e_{15}|_2 - e_{15}|_1)(ia - fb) + (a_{11}|_2 - a_{11}|_1)(ic - gb)]$$

$$\kappa^*_{33} = \kappa_{33}|_1 + c_2\left[\kappa_{33}|_2 - \kappa_{33}|_1 + \frac{c_1(e_{31}|_2 - e_{31}|_1)^2}{c_1 k_2 + c_2 k_1 + m_1}\right]$$

Magnetoelectric Moduli

$$a^*_{11} = a_{11}|_1 + j[(\kappa_{11}|_2 - \kappa_{11}|_1)(ae - ch) + (e_{15}|_2 - e_{15}|_1)(bh - ad) + (a_{11}|_2 - a_{11}|_1)(be - cd)]$$

$$a^*_{33} = a_{33}|_1 + c_2\left[a_{33}|_2 - a_{33}|_1 - \frac{c_1(e_{31}|_2 - e_{31}|_1)(q_{31}|_1 - q_{31}|_2)}{c_1k_2 + c_2k_1 + m_1}\right]$$

Magnetic Permeability Moduli

$$\mu^*_{11} = \mu_{11}|_1 + j[(a_{11}|_2 - a_{11}|_1)(ch - ae) + (q_{15}|_2 - q_{15}|_1)(bh - ad) + (\mu_{11}|_2 - \mu_{11}|_1)(be - cd)]$$

$$\mu^*_{33} = \mu_{33}|_1 + c_2\left[\mu_{33}|_2 - \mu_{33}|_1 + \frac{c_1(\mu_{31}|_2 - \mu_{31}|_1)^2}{c_1k_2 + c_2k_1 + m_1}\right]$$

Thermal Moduli

$$\lambda^*_{11} = \lambda_{11}|_1 + \frac{c_2C_{11}|_1\ (\lambda_{11}|_2 - \lambda_{11}|_1)}{C_{11}|_1 + c_1k_2 - c_1k_1}$$

$$\lambda^*_{33} = c_1\lambda_{33}|_1 + c_2\lambda_{33}|_2 - c_1c_2\frac{(C_{13}|_2 - C_{13}|_1)(\lambda_{11}|_2 - \lambda_{11}|_1)}{C_{11}|_1 + c_1k_2 - c_1k_1}$$

$$p^*_3 = c_1p_3|_1 + c_2p_3|_2 - c_1c_2\frac{(e_{31}|_2 - e_{31}|_1)(\lambda_{11}|_2 - \lambda_{11}|_1)}{C_{11}|_1 + c_1k_2 - c_1k_1}$$

$$m^*_3 = c_1m_3|_1 + c_2m_3|_2 - c_1c_2\frac{(q_{31}|_2 - q_{31}|_1)(\lambda_{11}|_2 - \lambda_{11}|_1)}{C_{11}|_1 + c_1k_2 - c_1k_1}$$

5.2 Laminated Composite

Elastic Moduli

$$\begin{aligned}C^*_{11} = k_1 + m_1 - r\{&(ah - de)[(C_{13}|_2 - C_{13}|_1)l - (e_{31}|_2 - e_{31}|_1)j] + (ne - ih)[(C_{13}|_2 - C_{13}|_1)c \\ &- (e_{31}|_2 - e_{31}|_1)b] + (id - na)[(C_{13}|_2 - C_{13}|_1)g - (e_{31}|_2 - e_{31}|_1)f] + (cj - lb)[(C_{11}|_2 - C_{11}|_1)h \\ &- (q_{31}|_2 - q_{31}|_1)e] + (gb - cf)[(C_{11}|_2 - C_{11}|_1)n - (q_{31}|_2 - q_{31}|_1)i] + (lf - gi)[(C_{11}|_2 - C_{11}|_1)d \\ &- (q_{31}|_2 - q_{31}|_1)a]\}\end{aligned}$$

$$C^*_{13} = C_{13}|_1 + \frac{r}{t}[(e_{31}|_2 - e_{31}|_1)(nf - jh) + (C_{13}|_2 - C_{13}|_1)(lh - ng) + (q_{31}|_2 - q_{31}|_1)(jg - lf)]$$

$$C^*_{33} = C_{33}|_1 + \frac{r}{t}[(e_{33}|_2 - e_{33}|_1)(nf - jh) + (C_{33}|_2 - C_{33}|_1)(lh - ng) + (q_{33}|_2 - q_{33}|_1)(jg - lf)]$$

$$C^*_{44} = \frac{C_{44}|_1\ C_{44}|_2}{c_2C_{44}|_1 + c_1C_{44}|_2}$$

$$C^*_{66} = c_1m_1 + c_2m|_2$$

Piezoelectric Moduli

$$e^*_{31} = e_{31}|_1 - \frac{r}{t}[(e_{31}|_2 - e_{31}|_1)(nb - jd) + (C_{13}|_2 - C_{13}|_1)(ld - nc) + (q_{31}|_2 - q_{31}|_1)(jc - lb)]$$

$$e^*_{33} = e_{33}|_1 - \frac{r}{t}[(e_{33}|_2 - e_{33}|_1)(nb - jd) + (C_{33}|_2 - C_{33}|_1)(ld - nc) + (q_{33}|_2 - q_{33}|_1)(jc - lb)]$$

$$e^*_{15} = e_{15}|_1 + \frac{c_2C_{44}|_1\ (e_{15}|_2 - e_{15}|_1)}{c_2C_{44}|_1 + c_1C_{44}|_2}$$

Piezomagnetic Moduli

$$q^*_{31} = q_{31}|_1 + \frac{r}{t}[(e_{31}|_2 - e_{31}|_1)(bh - df) + (C_{13}|_2 - C_{13}|_1)(dg - ch) + (q_{31}|_2 - q_{31}|_1)(cf - bg)]$$

$$q^*_{33} = q_{33}|_1 + \frac{r}{t}[(e_{33}|_2 - e_{33}|_1)(bh - df) + (C_{33}|_2 - C_{33}|_1)(dg - ch) + (q_{33}|_2 - q_{33}|_1)(cf - bg)]$$

$$q^*_{15} = q_{15}|_1 + \frac{c_2C_{44}|_1\ (q_{15}|_2 - q_{15}|_1)}{c_2C_{44}|_1 + c_1C_{44}|_2}$$

Dielectric Moduli

$$\kappa^*_{11} = c_1\kappa_{11}|_1 + c_2\kappa_{11}|_2 + c_1c_2\frac{(e_{15}|_2 - e_{15}|_1)^2}{c_2C_{44}|_1 + c_1C_{44}|_2}$$

$$\kappa_{33}^{*} = \kappa_{33}|_1 - \frac{r}{t}[(\kappa_{33}|_2 - \kappa_{33}|_1)(nb - jd)$$

$$- (e_{33}|_2 - e_{33}|_1)(ld - nc) + (a_{33}|_2 - a_{33}|_1)(jc - lb)]$$

Magnetoelectric Moduli

$$a_{11}^{*} = c_1 a_{11}|_1 + c_2 a_{11}|_2 + c_1 c_2 \frac{(e_{15}|_2 - e_{15}|_1)(q_{15}|_2 - q_{15}|_1)}{c_2 C_{44}|_1 + c_1 C_{44}|_2}$$

$$a_{33}^{*} = a_{33}|_1 + \frac{r}{t}[(\kappa_{33}|_2 - \kappa_{33}|_1)(bh - df)$$

$$- (e_{33}|_2 - e_{33}|_1)(dg - ch) + (a_{33}|_2 - a_{33}|_1)(cf - bg)]$$

Magnetic Permeability Moduli

$$\mu_{11}^{*} = c_1 \mu_{11}|_1 + c_2 \mu_{11}|_2 + c_1 c_2 \frac{(q_{15}|_2 - q_{15}|_1)^2}{c_2 C_{44}|_1 + c_1 C_{44}|_2}$$

$$\mu_{33}^{*} = \mu_{33}|_1 + \frac{r}{t}[(a_{33}|_2 - a_{33}|_1)(bh - df)$$

$$- (q_{33}|_2 - q_{33}|_1)(dg - ch)$$

$$+ (\mu_{33}|_2 - \mu_{33}|_1)(cf - bg)]$$

Thermal Moduli

$$\lambda_{11}^{*} = c_1 \lambda_{11}|_1 + c_2 \lambda_{11}|_2 + r\{(p_3|_1 - p_3|_2)$$

$$\times [h(ja - ib) + e(nb - jd) + f(id - na)]$$

$$+ (\lambda_{33}|_1 - \lambda_{33}|_2)[h(ic - la)$$

$$+ e(ld - nc) + g(na - id)]$$

$$+ (m_3|_1 - m_3|_2)[e(jc - lb) + f(la - ic) + g(ib - ja)]\}$$

$$\lambda_{33}^{*} = \lambda_{33}|_1 + \frac{r}{t}[(p_3|_1 - p_3|_2)(jh - nf) + (\lambda_{33}|_1 - \lambda_{33}|_2)$$

$$\times (ng - lh) + (m_3|_1 - m_3|_2)(lf - jg)]$$

$$p_3^{*} = p_3|_1 - \frac{r}{t}[(p_3|_1 - p_3|_2)(jd - nb) + (\lambda_{33}|_1 - \lambda_{33}|_2)$$

$$\times (nc - ld) + (m_3|_1 - m_3|_2)(lb - jc)]$$

$$m_3^{*} = m_3|_1 - \frac{r}{t}[(p_3|_1 - p_3|_2)(bh - df) + (\lambda_{33}|_1 - \lambda_{33}|_2)$$

$$\times (dg - ch) + (m_3|_1 - m_3|_2)(cf - bg)]$$

6. NUMERICAL RESULTS AND DISCUSSION

We have obtained closed form expressions of effective magnetoelectroelastic moduli and thermal moduli for both fibrous and laminated composites. These results have been checked by several means. First, all the effective moduli recover the moduli of the constituent phases at the two volume fraction limits. Second, by ignoring piezomagnetic and magnetoelectric effects in the constituent phases, the effective properties recover Chen's (1994, 1996) results for piezoelectric composites. Finally, for the fibrous composite, our results satisfy all of the exact connections between the effective magnetoelectroelastic moduli of a two-phase composite obtained by Benveniste (1995). This is in contrast to the results of Nan (1994) which violate some of Benveniste's exact connections.

From the closed form expressions, we see that the effective magnetoelectric coefficients, a_{11} and a_{33}, depend on $(e_{15}|_2 - e_{15}|_1)(q_{15}|_2 - q_{15}|_1)$ and $(e_{31}|_2 - e_{31}|_1)(q_{31}|_2 - q_{31}|_1)$, respectively. This is perhaps the most direct demonstration of the existence of a product property for these composites. The magnetoelectric effect exists in the composite even if neither phase exhibits the magnetoelectric effect. In such a case, the magnetoelectric effect in the composite arises from the interaction between the piezoelectric and piezomagnetic phases. Similar phenomena exist for the pyroelectric and pyromagnetic effects, which directly depend on $(e_{31}|_2 - e_{31}|_1)(\lambda_{11}|_2 - \lambda_{11}|_1)$ and $(q_{31}|_2 - q_{31}|_1)(\lambda_{11}|_2 - \lambda_{11}|_1)$, respectively. This illustrates that the pyroelectric effect can be produced by the interaction between the piezoelectric effect and thermal expansion, while the pyromagnetic effect can be produced by interaction between the piezomagnetic effect and thermal expansion. These observations also provide guidelines on how to enhance the magnetoelectric, pyroelectric, and pyromagnetic effects in a composite medium.

To further demonstrate the working of our theory, we have performed calculations to obtain the effective magnetoelectroelastic moduli of a composite consisting of a $CoFe_2O_4$ matrix reinforced by $BaTiO_3$. As noted in the introduction, this composite has been studied by other researchers, both experimentally and theoretically. The magnetoelectroelastic moduli of the two phases are presented in Table 1, where the x_1-x_2 plane is isotropic and the unique axis is along the x_3-direction.

Figures 1–6 show the effective elastic, dielectric, magnetic, piezoelectric, piezomagnetic, and magnetoelectric moduli of the $BaTiO_3$-$CoFe_2O_4$ fibrous and laminated composites as a function of the $BaTiO_3$ volume fraction. For the

Table 1. Material properties of $BaTiO_3$ and $CoFe_2O_4$.

	C_{11} (GPa)	C_{12} (GPa)	C_{13} (GPa)	C_{33} (GPa)	C_{44} (GPa)
$BaTiO_3$	166	77	78	162	43
$CoFe_2O_4$	286	173	170	269.5	45.3
	e_{15} (C/m^2)	e_{31} (C/m^2)	e_{33} (C/m^2)	κ_{11} (C^2/Nm^2)	κ_{33} (C^2/Nm^2)
$BaTiO_3$	11.6	−4.4	18.6	11.2×10^{-9}	12.6×10^{-9}
$CoFe_2O_4$	0	0	0	0.08×10^{-9}	0.093×10^{-9}
	q_{15} (m/A)	q_{31} (m/A)	q_{33} (m/A)	μ_{11} (Ns^2/C^2)	μ_{33} (Ns^2/C^2)
$BaTiO_3$	0	0	0	5×10^{-6}	10×10^{-6}
$CoFe_2O_4$	550	580.3	699.7	-590×10^{-6}	157×10^{-6}

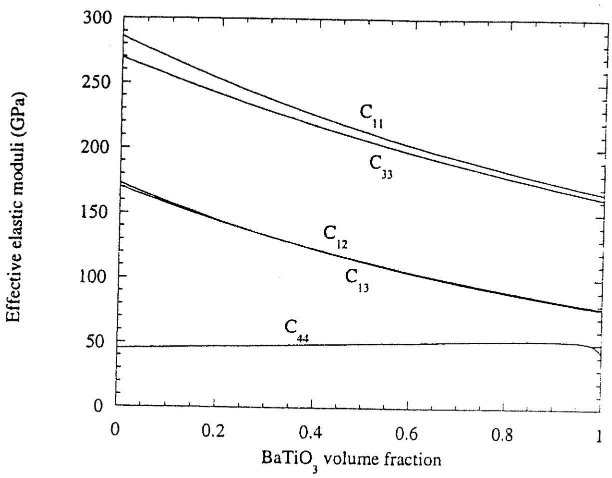

Figure 1a. *Effective elastic moduli of the fibrous composite versus the volume fraction of $BaTiO_3$.*

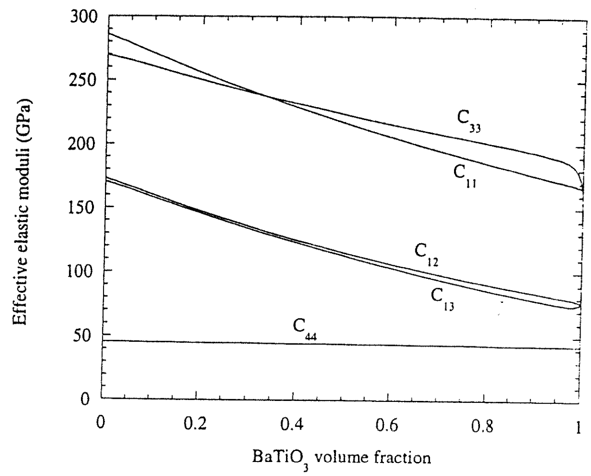

Figure 1b. *Effective elastic moduli of the fibrous composite versus the volume fraction of $BaTiO_3$.*

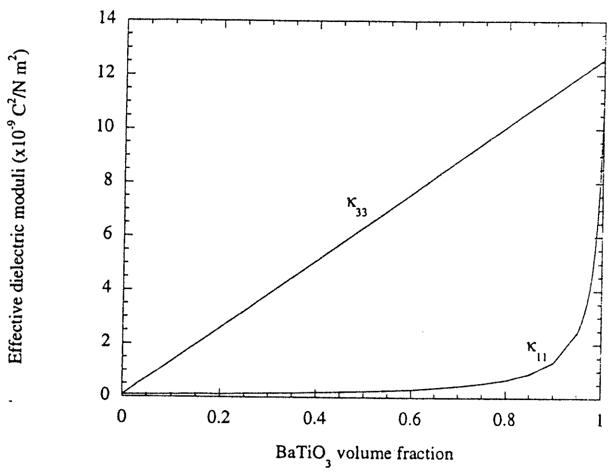

Figure 2a. *Effective dielectric moduli of the laminate versus the volume fraction of $BaTiO_3$.*

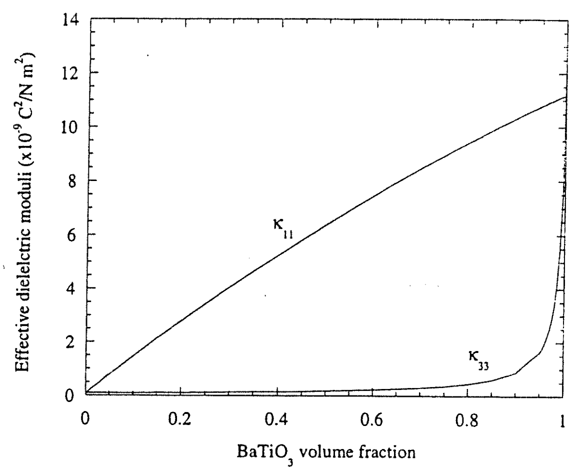

Figure 2b. *Effective dielectric moduli of the laminate versus the volume fraction of $BaTiO_3$.*

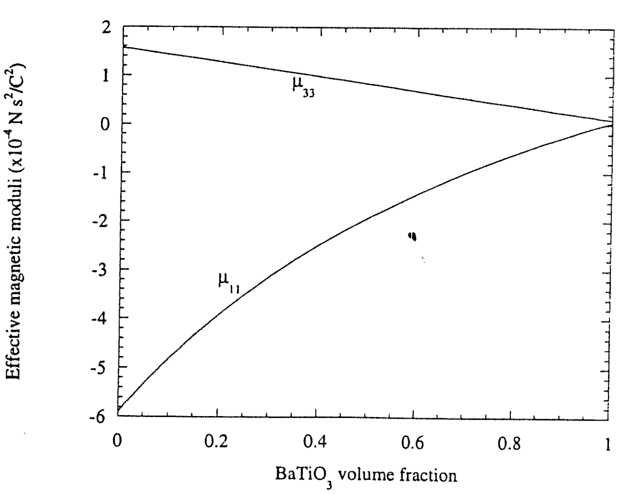

Figure 3a. *Effective magnetic moduli of the fibrous composite versus the volume fraction of $BaTiO_3$.*

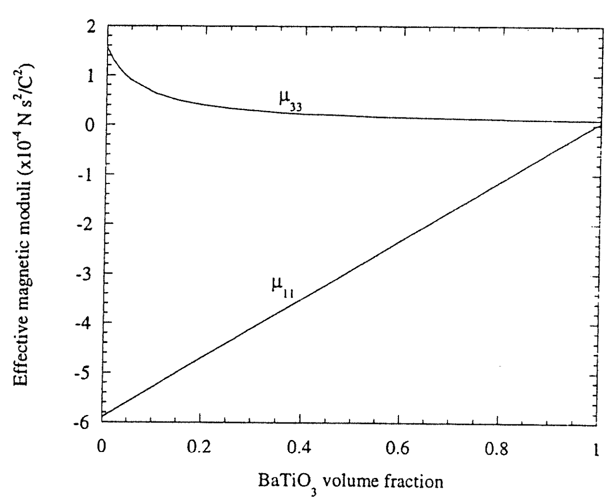

Figure 3b. *Effective magnetic moduli of the laminate versus the volume fraction of $BaTiO_3$.*

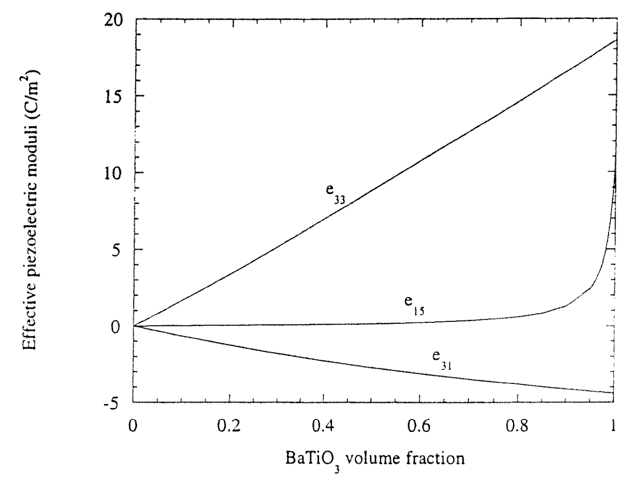

***Figure 4a.** Effective piezoelectric moduli of the fibrous composite versus the volume fraction of $BaTiO_3$.*

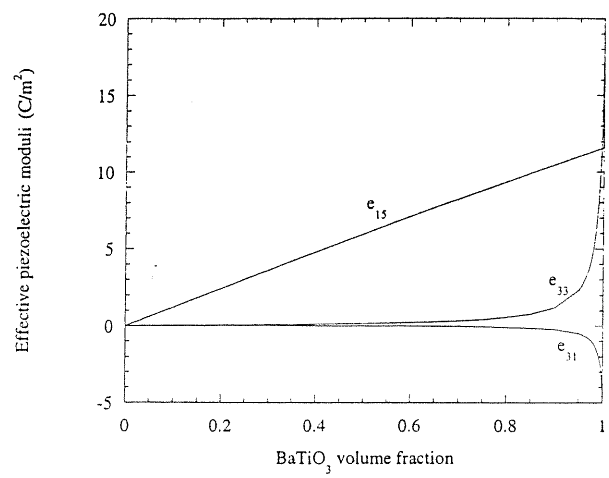

***Figure 4b.** Effective piezoelectric moduli of the fibrous composite versus the volume fraction of $BaTiO_3$.*

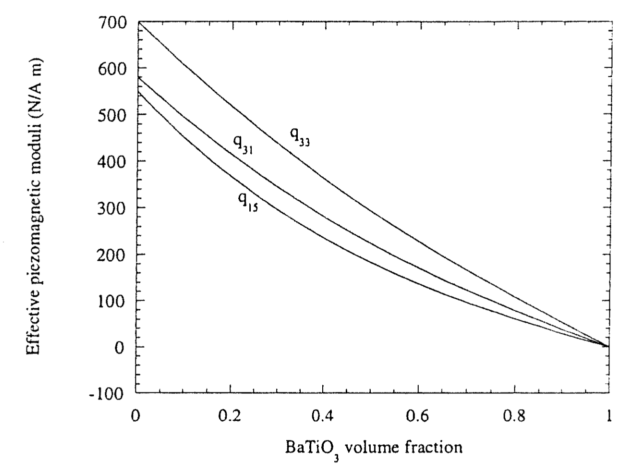

***Figure 5a.** Effective piezomagnetic moduli of the laminate versus the volume fraction of $BaTiO_3$.*

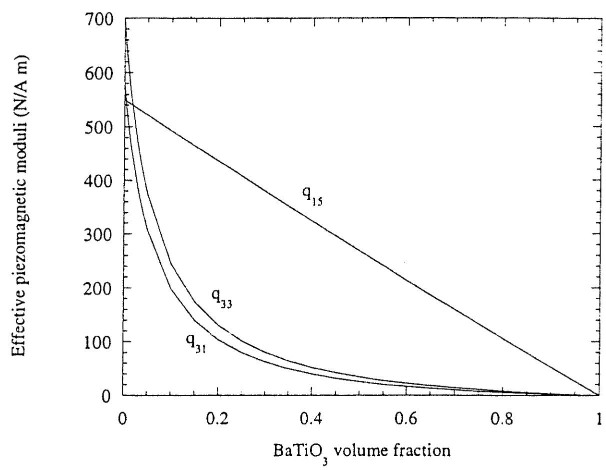

***Figure 5b.** Effective piezomagnetic moduli of the laminate versus the volume fraction of $BaTiO_3$.*

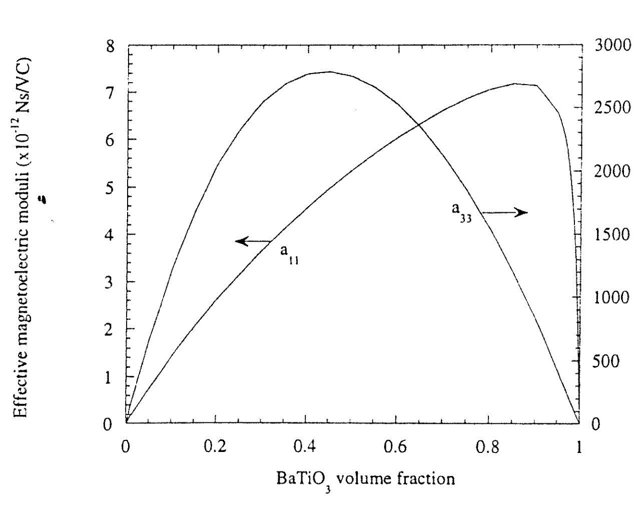

***Figure 6a.** Effective magnetoelectric moduli of the fibrous composite versus the volume fraction of $BaTiO_3$.*

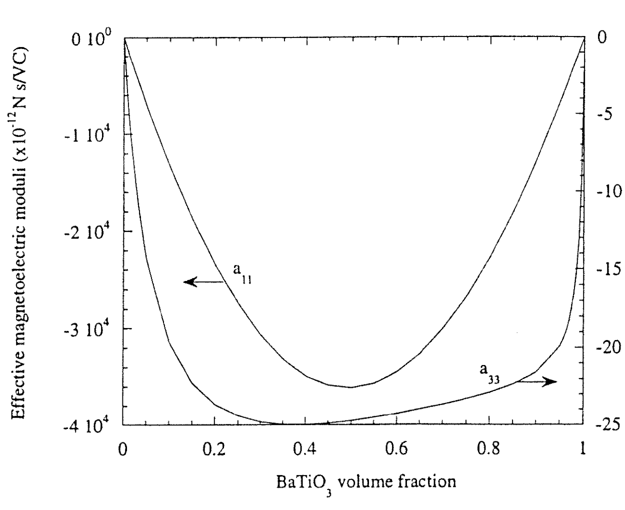

***Figure 6b.** Effective magnetoelectric moduli of the laminate versus the volume fraction of $BaTiO_3$.*

fibrous composite, the effective moduli $C_{44}, \kappa_{11}, \mu_{11}, e_{15}$, and q_{15} show nonlinear behavior with respect to the volume fraction, while for the laminate, the effective moduli C_{33}, C_{13}, $\kappa_{33}, \mu_{33}, e_{31}, e_{33}, q_{31}$, and q_{33} show nonlinear behavior with respect to the volume fraction. The magnetoelectric moduli a_{11} and a_{33} change nonlinearly with volume fraction in both composites. All other effective moduli vary essentially linearly with volume fraction. The fundamental reason for this is that the microgeometry of the fibrous composite requires that D_1, B_1, E_3, and H_3 are continuous across the phase boundaries (generators parallel the x_3-direction), while the laminate requires that E_1, H_1, D_3, and B_3 are continuous across the phase boundaries (generators normal to the x_3-direction).

To understand the behavior of the effective C_{44}, e_{15}, and q_{15} of the fibrous composite, assume that the composite is subjected to a uniform far-field ε_{23}. This will directly induce a shear stress σ_{23} in both phases, D_2 in the piezoelectric phase and B_2 in the piezomagnetic phase. Due to the continuity of D_2 and B_2 at the phase boundaries, D_2 and B_2 will be induced in the piezomagnetic and piezoelectric phases, respectively. This will in turn induce E_2 in the piezomagnetic phase and H_2 in the piezoelectric phase. Due to the average field theorem, E_2 and H_2 will then be induced in the piezoelectric and piezomagnetic phases, respectively, to enforce the requirement that the overall electric and magnetic fields inside the composite vanish. This will then cause an additional σ_{23} in both phases, D_2 in the piezoelectric phase and B_2 in piezomagnetic phase, due to the piezoelectric and piezomagnetic effect. It is this second-order effect that causes the nonlinear behavior of C_{44}, e_{15}, and q_{15} with respect to the volume fraction. For the laminated composite, D_2 in the piezoelectric phase and B_2 in the piezomagnetic phase will not induce D_2 and B_2 in the piezomagnetic and piezoelectric phases, respectively, and so there will be no second order effect fields generated. This is why C_{44}, e_{15}, and q_{15} vary roughly linearly with volume fraction in the laminate. A similar argument explains why κ_{11} and μ_{11} change nonlinearly in the fibrous composite, but not in the laminate.

To understand the behavior of the effective C_{13}, C_{33}, e_{33}, and q_{33} of the laminated composite, assume that the composite is subjected to a uniform far-field ε_{33}. This will directly induce stresses σ_{11} and σ_{33} in both phases, D_3 in the piezoelectric phase and B_3 in the piezomagnetic phase. Due to the continuity of D_3 and B_3 at the phase boundaries, D_3 and B_3 will be induced in the piezomagnetic and piezoelectric phases, respectively. This will cause E_3 in the piezomagnetic phase and H_3 in the piezoelectric phase, which will then induce E_3 and H_3 in the piezoelectric and piezomagnetic phases, respectively, to maintain the overall zero electric and magnetic field in the composite. This will then cause additional stresses σ_{11} and σ_{33} in both phases, D_3 in the piezoelectric phase and B_3 in the piezomagnetic phase. This second-order effect results in the nonlinear behavior of C_{13}, C_{33}, e_{33}, and q_{33} of the laminate, but, for the same reason discussed in the previous paragraph, not for the fibrous composite. Also, a similar argument explains why κ_{33} and μ_{33} vary nonlinearly with volume fraction of the laminate, but not in the fibrous composite. However, strong nonlinear dependence on the volume fraction is not observed in the effective C_{11} and C_{12} of laminate because e_{31} and q_{31} are of opposite sign and so the second-order stresses caused by piezoelectric and piezomagnetic phases tend to cancel each other.

The most interesting behavior, however, is the overall magnetoelectric effect that is present in the composite, but not in either of the individual phases. In the fibrous composite, a_{33} is three orders of magnitude larger than a_{11}, while in laminate, a_{11} is three orders of magnitude larger than a_{33}. If in a fibrous composite, boundary conditions consistent with a uniform far-field E_3 are applied, σ_{11} and σ_{33} will be induced in the piezoelectric phases. As required by traction continuity at the phase boundaries, σ_{11} is also induced in the piezomagnetic phase. This second order stress causes H_3 in the piezomagnetic phase, and then in the piezoelectric phase because H_3 is uniform inside the composite. However, from the uniform far-field E_3 boundary condition, we know that the overall H_3 is zero in the composite. Thus, B_3 must be introduced in the piezomagnetic phase to cancel the H_3 caused by the second order stress. However, if boundary conditions consistent with a uniform far-field E_1 are applied, σ_{13} will be induced in the piezoelectric phase, and then in the piezomagnetic phase due to traction continuity at the phase boundary. This σ_{13} will induce H_1 in the piezomagnetic phase and then H_1 will be induced in the piezoelectric phase to maintain the requirement of overall zero magnetic field in the composite. This H_1 will induce B_1 in the piezoelectric phase. The large difference between a_{33} and a_{11} of the composite is then due to the difference in magnetic constants between the piezoelectric and piezomagnetic phases. For a_{33}, B_3 is caused by the piezomagnetic phase, while for a_{11}, B_1 is caused by the piezoelectric phase, as discussed above. The magnetic constant of the piezomagnetic phase is two orders of magnitude larger than that of the piezoelectric phase. A similar argument explains why a_{11} is two orders of magnitude larger than a_{33} for the laminate.

Finally, in Figures 7 and 8 we compare our results to those of Avellaneda and Harshe (1994) for the magnetoelectric coefficient and figure of merit for a laminated (2-2) composite. Our model cannot be compared directly with theirs since they assume the piezomagnetic phases are short-circuited and thus no electric field is realized in the piezomagnetic phases. However, by setting the dielectric constants of the piezomagnetic phase to be infinite, we can simulate this situation. We carried this out and simplified our expression for the magnetoelectric coefficient in this case as

$$a_{33}^{*} = \left\{ \frac{c_1^2[q_{33}|_2^2 + (C_{33}|_2 - C_{33}|_1)(\mu_{33}|_2 - \mu_{33}|_1)]}{-c_1c_2e_{33}|_1\, q_{33}|_2\mu_{33}|_1} + \frac{c_1[C_{33}|_1(\mu_{33}|_2 - 2\mu_{33}|_1) + C_{33}|_2\mu_{33}|_2] + C_{33}|_1\mu_{33}|_1}{-c_1c_2e_{33}|_1\, q_{33}|_2\, \mu_{33}|_1} \right\}^{-1} \quad (28)$$

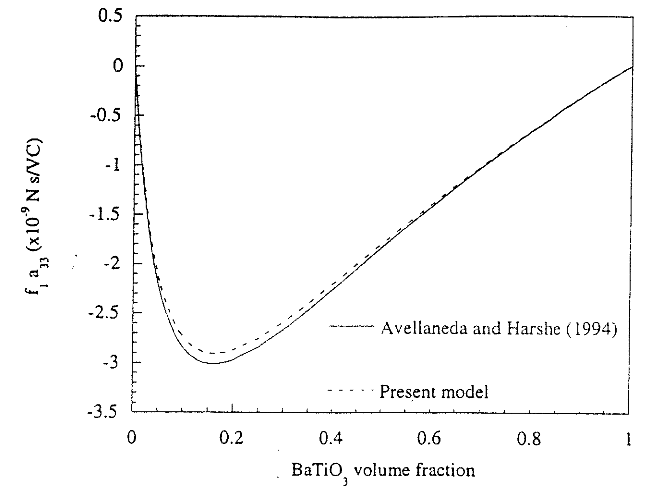

Figure 7. *Comparison of the effective magnetoelectric constant a_{33} of the laminated composite versus the volume fraction of $BaTiO_3$.*

Figures 7 and 8 show the comparison between our results and those of Avellaneda and Harshe (1994) for the magnetoelectric coefficient and the figure of merit $k_{me}^2 = a_{33}^{*2}/\kappa_{33}^*\mu_{33}^*$, respectively. Our results agree favorably with theirs, and the results suggest that short-circuiting the piezomagnetic phases in a composite laminate is an effective way to enhance the magnetoelectric coefficient. A similar argument explains why the higher magnetic permeability of the piezoelectric phase compared to that of the piezomagnetic phase results in a higher magnetoelectric coefficient. This was also observed by Avellaneda and Harshe.

7. CONCLUSIONS

We presented a micromechanics approach to analyze the average fields and effective moduli of heterogeneous media that exhibit full coupling between stationary elastic, electric, and magnetic fields. Exact relations regarding the internal field distribution inside a heterogeneous magnetoelectroelastic solid were established, along with exact connections between the effective magnetoelectroelastic and thermal moduli of two-phase composites. The Mori-Tanaka mean field approach was applied to obtain closed-form estimates for the effective moduli of fibrous and laminated composites. Finally, numerical results for $BaTiO_3$-$CoFe_2O_4$ composites were presented and discussed.

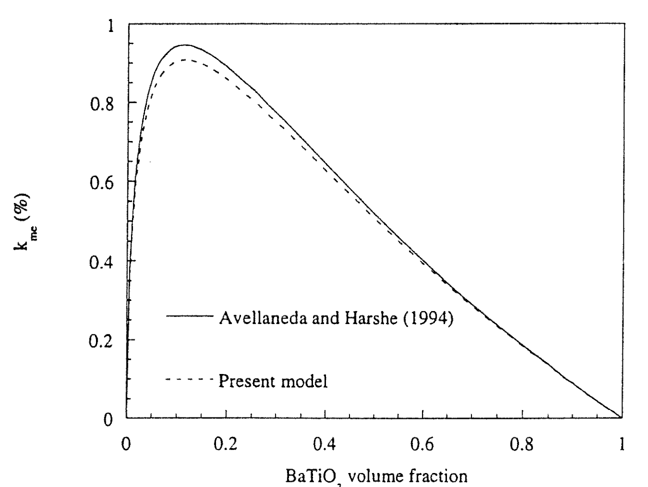

Figure 8. *Comparison of figure of merit of the laminated composite versus the volume fraction of $BaTiO_3$.*

APPENDIX

Constants for the Fibrous Composite

$$j = 2c_2 \frac{[C_{44}|_1 \kappa_{11}|_1 \mu_{11}|_1 + e_{15}^2|_1 \mu_{11}|_1 + \kappa_{11}|_1 q_{15}^2|_1 - 2a_{11}|_1 e_{15}|_1 q_{15}|_1 - a_{11}^2|_1 C_{44}|_1)]}{(ic - gb)h + (fb - ia)e + (ga - fc)d}$$

$$\begin{aligned} a = {} & [(q_{15}|_2 - q_{15}|_1)(\kappa_{11}|_1 \mu_{11}|_1 - a_{11}^2|_1) - (a_{11}|_2 - a_{11}|_1) \\ & \times (e_{15}|_1 \mu_{11}|_1 - a_{11}|_1 q_{15}|_1) - (\mu_{11}|_2 - \mu_{11}|_1) \\ & \times (q_{15}|_1 \kappa_{11}|_1 - a_{11}|_1 e_{15}|_1)]c_1 \end{aligned}$$

$$\begin{aligned} b = {} & [(e_{15}|_2 - e_{15}|_1)(\kappa_{11}|_1 \mu_{11}|_1 - a_{11}^2|_1) - (a_{11}|_2 - a_{11}|_1) \\ & \times (q_{15}|_1 \kappa_{11}|_1 - a_{11}|_1 e_{15}|_1) - (\kappa_{11}|_2 - \kappa_{11}|_1) \\ & \times (e_{15}|_1 \mu_{11}|_1 - a_{11}|_1 q_{15}|_1)]c_1 \end{aligned}$$

$$\begin{aligned} c = {} & [(C_{44}|_2 - C_{44}|_1)(\kappa_{11}|_1 \mu_{11}|_1 - a_{11}^2|_1) + (e_{15}|_2 - e_{15}|_1) \\ & \times (e_{15}|_1 \mu_{11}|_1 - a_{11}|_1 q_{15}|_1) + (q_{15}|_2 - q_{15}|_1) \\ & \times (q_{15}|_1 \kappa_{11}|_1 - a_{11}|_1 e_{15}|_1)]c_1 \\ & + \frac{j}{c_2}[h(ic - gb) + e(fb - ia) + d(ga - fc)] \end{aligned}$$

$$\begin{aligned} d = {} & [-(a_{11}|_2 - a_{11}|_1)(a_{11}|_1 C_{44}|_1 + e_{15}|_1 q_{15}|_1) \\ & + (e_{15}|_2 - e_{15}|_1)(e_{15}|_1 \mu_{11}|_1 - a_{11}|_1 q_{15}|_1) \\ & + (\kappa_{11}|_2 - \kappa_{11}|_1)(q_{15}^2|_1 + C_{44}|_1 \mu_{11}|_1)]c_1 \\ & + \frac{j}{c_2}[h(ic - gb) + e(fb - ia) + d(ga - fc)] \end{aligned}$$

$$\begin{aligned} e = {} & [(q_{15}|_2 - q_{15}|_1)(a_{11}|_1 C_{44}|_1 + e_{15}|_1 q_{15}|_1) + (C_{44}|_2 - C_{44}|_1) \\ & \times (e_{15}|_1 \mu_{11}|_1 - a_{11}|_1 q_{15}|_1) - (e_{15}|_2 - e_{15}|_1) \\ & \times (q_{15}^2|_1 + C_{44}|_1 \mu_{11}|_1)]c_1 \end{aligned}$$

$$
\begin{aligned}
f = {} & [-(a_{11}|_2 - a_{11}|_1)(a_{11}|_1\, C_{44}|_1 + e_{15}|_1\, q_{15}|_1) \\
& + (q_{15}|_2 - q_{15}|_1)(q_{15}|_1 \kappa_{11}|_1 - a_{11}|_1 e_{15}|_1) \\
& + (\mu_{11}|_2 - \mu_{11}|_1)(e_{15}^2|_1 + C_{44}|_1 \kappa_{11}|_1)]c_1 \\
& + \frac{j}{c_2}[h(ic - gb) + e(fb - ia) + d(ga - fc)]
\end{aligned}
$$

$$
\begin{aligned}
g = {} & [(e_{15}|_2 - e_{15}|_1)(a_{11}|_1 C_{44}|_1 + e_{15}|_1 q_{15}|_1) + (C_{44}|_2 - C_{44}|_1) \\
& \times (q_{15}|_1 \kappa_{11}|_1 - a_{11}|_1 e_{15}|_1) - (q_{15}|_2 - q_{15}|_1) \\
& \times (e_{15}^2|_1 + C_{44}|_1 \kappa_{11}|_1)]c_1
\end{aligned}
$$

$$
\begin{aligned}
h = {} & [(q_{15}|_2 - q_{15}|_1)(\mu_{11}|_1 e_{15}|_1 - a_{11}|_1 q_{15}|_1) - (\mu_{11}|_2 - \mu_{11}|_1) \\
& \times (C_{44}|_1 a_{11}|_1 - e_{15}|_1 q_{15}|_1) + (a_{11}|_2 - a_{11}|_1) \\
& \times (q_{15}^2|_1 + C_{44}|_1 \mu_{11}|_1)]c_1
\end{aligned}
$$

$$
\begin{aligned}
i = {} & [(e_{15}|_2 - e_{15}|_1)(\kappa_{11}|_1\, q_{15}|_1 - a_{11}|_1\, e_{15}|_1 - (\kappa_{11}|_2 - \kappa_{11}|_1) \\
& \times (C_{44}|_1 a_{11}|_1 + e_{15}|_1 q_{15}|_1) + (a_{11}|_2 - a_{11}|_1) \\
& \times (e_{15}^2|_1 + C_{44}|_1 \kappa_{11}|_1)]c_1
\end{aligned}
$$

Constants for the Laminate Composite

$$
r = \frac{c_2}{l(bh - df) + c(nf - jh) + g(jd - nb)}
$$

$$
t = \frac{c_1}{\begin{aligned} & C_{33}|_1\, \kappa_{33}|_1\, \mu_{33}|_1 + e_{33}^2|_1\, \mu_{33}|_1 \\ & - 2a_{33}|_1\, e_{33}|_1\, q_{33}|_1 - a_{33}^2|_1 C_{33}|_1 + \kappa_{33}|_1\, q_{33}^2|_1 \end{aligned}}
$$

$$
\begin{aligned}
a = {} & (C_{31}|_2 - C_{31}|_1)(\kappa_{33}|_1 \mu_{33}|_1 - a_{33}^2|_1) + (e_{31}|_2 - e_{31}|_1) \\
& \times (e_{33}|_1 \mu_{33}|_1 - a_{33}|_1 q_{33}|_1) \\
& + (q_{31}|_2 - q_{31}|_1)(\kappa_{33}|_1 q_{33}|_1 - a_{33}|_1 e_{33}|_1)
\end{aligned}
$$

$$
\begin{aligned}
b = {} & \frac{1}{t} + (C_{33}|_2 - C_{33}|_1)(\kappa_{33}|_1 \mu_{33}|_1 - a_{33}^2|_1) \\
& + (e_{33}|_2 - e_{33}|_1)(e_{33}|_1 \mu_{33}|_1 - a_{33}|_1 q_{33}|_1) \\
& + (q_{33}|_2 - q_{33}|_1)(\kappa_{33}|_1 q_{33}|_1 - a_{33}|_1 e_{33}|_1)
\end{aligned}
$$

$$
\begin{aligned}
c = {} & (e_{33}|_2 - e_{33}|_1)(\kappa_{33}|_1 \mu_{33}|_1 - a_{33}^2|_1) - (a_{33}|_2 - a_{33}|_1) \\
& \times (\kappa_{33}|_1 q_{33}|_1 - e_{33}|_1\, a_{33}|_1) \\
& - (\kappa_{33}|_2 - \kappa_{33}|_1)(e_{33}|_1 \mu_{33}|_1 - a_{33}|_1 q_{33}|_1)
\end{aligned}
$$

$$
\begin{aligned}
d = {} & (q_{33}|_2 - q_{33}|_1)(\kappa_{33}|_1 \mu_{33}|_1 - a_{33}^2|_1) - (\mu_{33}|_2 - \mu_{33}|_1) \\
& \times (\kappa_{33}|_1 q_{33}|_1 - e_{33}|_1 a_{33}|_1) \\
& - (a_{33}|_2 - a_{31}|_1)(e_{33}|_1 \mu_{33}|_1 - a_{33}|_1 q_{33}|_1)
\end{aligned}
$$

$$
\begin{aligned}
e = {} & (q_{31}|_2 - q_{31}|_1)(a_{33}|_1 C_{33}|_1 + e_{33}|_1 q_{33}|_1) \\
& + (C_{13}|_2 - C_{13}|_1)(e_{33}|_1 \mu_{33}|_1 - a_{33}|_1 q_{33}|_1) \\
& - (e_{31}|_2 - e_{31}|_1)(C_{33}|_1 \mu_{33}|_1 + q_{33}^2|_1)
\end{aligned}
$$

$$
\begin{aligned}
f = {} & (q_{33}|_2 - q_{33}|_1)(a_{33}|_1 C_{33}|_1 + e_{33}|_1 q_{33}|_1) \\
& + (C_{33}|_2 - C_{33}|_1)(e_{33}|_1 \mu_{33}|_1 - a_{33}|_1 q_{33}|_1) \\
& - (e_{33}|_2 - e_{33}|_1)(C_{33}|_1 \mu_{33}|_1 + q_{33}^2|_1)
\end{aligned}
$$

$$
\begin{aligned}
g = {} & \frac{1}{t} - (a_{33}|_2 - a_{33}|_1)(a_{33}|_1 C_{33}|_1 + e_{33}|_1 q_{33}|_1) \\
& + (e_{33}|_2 - e_{33}|_1)(e_{33}|_1 \mu_{33}|_1 - a_{33}|_1 q_{33}|_1) \\
& + (\kappa_{33}|_2 - \kappa_{33}|_1)(C_{33}|_1 \mu_{33}|_1 + q_{33}^2|_1)
\end{aligned}
$$

$$
\begin{aligned}
h = {} & (q_{33}|_2 - q_{33}|_1)(e_{33}|_1 \mu_{33}|_1 - a_{33}|_1 q_{33}|_1) \\
& - (\mu_{33}|_2 - \mu_{33}|_1)(a_{33}|_1 C_{33}|_1 + e_{33}|_1 q_{33}|_1) \\
& + (a_{33}|_2 - a_{33}|_1)(C_{33}|_1 \mu_{33}|_1 + q_{33}^2|_1)
\end{aligned}
$$

$$
\begin{aligned}
i = {} & (e_{31}|_2 - e_{31}|_1)(a_{33}|_1 C_{33}|_1 + e_{33}|_1 q_{33}|_1) \\
& + (C_{13}|_2 - C_{13}|_1)(\kappa_{33}|_1 q_{33}|_1 - a_{33}|_1 e_{33}|_1) \\
& - (q_{31}|_2 - q_{31}|_1)(C_{33}|_1 \kappa_{33}|_1 + e_{33}^2|_1)
\end{aligned}
$$

$$
\begin{aligned}
j = {} & (e_{33}|_2 - e_{33}|_1)(a_{33}|_1 C_{33}|_1 + e_{33}|_1 q_{33}|_1) \\
& + (C_{33}|_2 - C_{33}|_1)(\kappa_{33}|_1 q_{33}|_1 - a_{33}|_1 e_{33}|_1) \\
& - (q_{33}|_2 - q_{33}|_1)(C_{33}|_1 \kappa_{33}|_1 + e_{33}^2|_1)
\end{aligned}
$$

$$
\begin{aligned}
l = {} & (e_{33}|_2 - e_{33}|_1)(\kappa_{33}|_1 q_{33}|_1 - e_{33}|_1 a_{33}|_1) \\
& - (\kappa_{33}|_2 - \kappa_{33}|_1)(a_{33}|_1 C_{33}|_1 + q_{33}|_1 e_{33}|_1) \\
& + (a_{33}|_2 - a_{33}|_1)(C_{33}|_1 \kappa_{33}|_1 + e_{33}^2|_1)
\end{aligned}
$$

$$
\begin{aligned}
n = {} & \frac{1}{t} - (a_{33}|_2 - a_{33}|_1)(a_{33}|_1 C_{33}|_1 + e_{33}|_1 q_{33}|_1) \\
& + (q_{33}|_2 - q_{33}|_1)(\kappa_{33}|_1 q_{33}|_1 - a_{33}|_1 e_{33}|_1) \\
& + (\mu_{33}|_2 - \mu_{33}|_1)(C_{33}|_1\, \kappa_{33}|_1 + e_{33}^2|_1)
\end{aligned}
$$

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