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On micromechanics approximation for the effective thermoelastic moduli of multi-phase composite materials

Jiang Yu Li [1]

Center of Excellence for Advanced Materials, University of California, 9500 Gilman Drive, La Jolla, San Diego, CA 92093-0416, USA

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Abstract

It has been shown that for multi-phase composite materials, the Mori–Tanaka and self-consistent approaches may give non-symmetric effective moduli, may violate some exact connections between the effective moduli, may violate the Hashin–Shtrikman variational bounds, and may exhibit incorrect behavior at the unitary (100%) reinforcement concentration limit. An effective-medium-field micromechanics approximation using normalized concentration factors is proposed and is shown to overcome these difficulties. Such a normalization is necessary to satisfy the consistency relationship.

Keywords: Micromechanics; Composite; Effective moduli; Diagonal symmetry

1. Introduction

Over the last forty years or so, substantial progress has been made in the micromechanics of heterogeneous materials. By micromechanics, we are referring to the analysis of mechanics at the microstructure level. Typically, the objective of such an analysis is to estimate the effective properties of heterogeneous materials in terms of the properties of the constituents and their microstructure. The most common properties are the elastic constants, usually anisotropic. Other properties of interest include the dielectric constants, thermal properties, and more recently, the coupling coefficients between the elastic, electric, and magnetic fields. The various micromechanics approaches for evaluating the effective moduli of heterogeneous materials fall into three broad categories: (1) those that obtain internal-consistency relationships between the effective moduli; (2) those that obtain upper and lower bounds on the effective moduli; and (3) those that obtain direct estimates of the effective moduli. The advances in this field can be split into two categories: (1) development of a rigorous theoretical framework for the analysis and (2) development of simplifying assumptions that are required for a tractable analysis. The development of internal-consistency relationships and upper and lower bounds for the effective moduli fall into the first category since they are rigorous for the microstructure considered. The direct estimations of the effective moduli, on the other hand, involve both rigorous development and simplifying assumptions; the latter is the topic of this work.

In the rigorous theoretical framework, the well-known stress and strain concentration factors

[1] E-mail: jjli@starlite.ucsd.edu.

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obtained through the fundamental solution of the case of a single reinforcement embedded in an infinite medium are used, and the effective moduli are expressed rigorously in term of the concentration factors. To this end, Eshelby's (Eshelby, 1957) solution for the stress and strain fields in an ellipsoidal inclusion is usually used for three reasons: (1) under a uniform eigenstrain, the stress and strain in the ellipsoidal inclusion are uniform, thus trivializing the otherwise complicated problem of determining the volume-averaged fields; (2) under a uniform load, the stress and strain fields in an ellipsoidal inhomogeneity can be obtained using Eshelby's equivalent-inclusion concept; and (3) the ellipsoidal shape enables the simulation of a wide range of microstructural geometries, ranging from thin flakes to continuous fibers.

In the simplifying assumption, various micromechanics models have been developed to estimate the concentration factors and, thus, the effective moduli of heterogeneous materials. In general, these models are based on the assumption of statistical homogeneity, in which recourse is made to the ergodic assumption that, in an ensemble of specimens, local details occur in any single specimen with the same frequency that they occur in a single neighborhood (solids with periodical microstructure will not be discussed here, and can be referred to in Nemat-Nasser and Hori, 1993). Subject to this assumption, ensemble averages are replaced by volume averages over some representative volume that is small relative to the specimen size but large relative to the microscale and, thus, on average, is typical of the entire heterogeneous material. Those micromechanics theories that have received the most attention and use are the dilute, self-consistent, Mori–Tanaka, and differential approaches. These methods have been applied with great success to a number of problems in the uncoupled mechanical and electrical behavior of heterogeneous materials. The fundamentals of the methods can be referred to in Mura (1987), Nemat-Nasser and Hori (1993), and Taya and Arsenault (1989). More recently, they are applied successfully to heterogeneous materials with coupled behaviors between the elastic, electric, and magnetic fields (Dunn and Taya, 1993; Hori and Nemat-Nasser, 1998; and Li and Dunn, 1998a).

It is well known that the effective stiffness and compliance tensors of heterogeneous materials must be diagonally symmetric from an energy argument. The various micromechanics approximations, however, do not guarantee the diagonal symmetry of the estimated effective moduli. Benveniste et al. (1991) showed that the Mori–Tanaka and self-consistent approaches yield a diagonally symmetric effective stiffness tensor only for two-phase composites or multi-phase composites where the reinforcements have a similar shape and alignment. They further showed that two equivalent evaluations on the effective thermal-stress tensor give different values for multi-phase composites with differently shaped reinforcements under the Mori–Tanaka approach. Similar observations on diagonal symmetry were made by Qiu and Weng (1990). These observations cast doubt on the theoretical rigor of these micromechanics approximations in multi-phase composites. Nemat-Nasser and Hori (1993) discussed the symmetry of the effective stiffness and compliance tensors in detail. They showed that by the energy-based definition, the effective moduli are the symmetric part of the effective moduli defined by the average stress and strain in the composites, and thus, are always symmetric. However, as they pointed out, the effective stiffness and compliance tensors estimated from the energy-based definition may not be each other's inverse. Norris (1989) and Qiu and Weng (1990) noticed that the Mori–Tanaka approach in multi-phase composites may violate Hashin-Shtrikman bounds (Hashin and Shtrikman, 1962, 1963). Ferrari (1991) showed that the effective moduli predicted by the Mori-Tanaka approach may depend on the matrix properties at the unitary (100%) reinforcement concentration limit and, thus, is physically unacceptable. Summarizing all these observations in the literature, it can be concluded that five criteria should be satisfied by all approximation schemes: (1) the effective moduli should be diagonally symmetric; (2) the internal-consistency relationships between the effective moduli should be satisfied; (3) at a dilute reinforcement concentration limit the effective moduli should recover Eshelby's exact solution; (4) at the unitary reinforcement concentration limit, the effective moduli should be independent of the properties of the matrix; and (5) the effective moduli should comply with the variational bounds. The requirement on internal consistency implies that the effective stiffness and compliance tensors should be each other's inverse; equivalent evaluations on the effective thermal moduli should give identical results; and the exact connections between different components of the effective moduli should be satisfied. Actually the self-consistent approach got its name because it guarantees that the estimated stiffness and compliance tensors are each other's inverse, which is self-consistent (Hill, 1965). Detailed discussions on the diagonal symmetry, dilute behavior, internal consistency, and variational bounds of the effective moduli can be found in Nemat-Nasser and Hori (1993). Norris (1989), Qiu and Weng (1990), Benveniste et al. (1991), and Ferrari (1991) showed that the application of the Mori-Tanaka approach in multi-phase composites may violate requirements (1), (2), (4) and (5), and thus is questionable. With the rapid development and increasing application of multi-phase composites, especially the short-fiber reinforced composites, a micromechanics approximation which can overcome these difficulties is highly desirable.

It should be pointed out that the original Mori and Tanaka (1973) approach was proposed for composites with reinforcement of similar shape, based on the Tanaka and Mori (1972) observation that the average disturbance in an annulus (between the inclusion and a larger similar surface) vanishes. In such a case, the average field in the double inclusion is exactly known, and the only approximation is using the effective properties of the double inclusion to represent the effective properties of the composite by ignoring the interaction between double inclusions; see Nemat-Nasser and Hori (1993, pp. 340). All requirements are satisfied in this class of composites. This is also true for composite materials consisting of coated reinforcements embedded in a continuous matrix (Benveniste et al., 1991 noticed that the Mori–Tanaka approach returns a diagonally symmetric effective stiffness tensor for this class of composites). When reinforcements of different shapes are involved in the composite, such an approximation is no longer possible (since fields in different annuluses around dissimilar inclusions are different). Therefore, it is the extension from two-phase composites to multi-phase composites with differently shaped reinforcements, not the original Mori and Tanaka assumption, that produces conflicting results. Based on the Tanaka and Mori observation, an elegant double-inclusion method as well as the multi-inclusion method were proposed by Nemat-Nasser and Hori (1993), Hori and Nemat-Nasser (1994) using the average field in a multi inclusion embedded in an infinite medium with arbitrary assigned elastic moduli. When the stiffness tensor of the matrix is assigned to the infinite medium, the Mori–Tanaka approach is obtained; if the unknown effective stiffness tensor is assigned instead, the self-consistent approach is obtained. Thus the Mori–Tanaka and self-consistent approaches are under a uniform framework and are special cases of the more general scheme. The multi-inclusion method was also proven to comply with variational bounds for a composite with similar reinforcements. Its application to multi-phase composites containing reinforcements with different shapes or alignments remains to be evaluated.

It is worth mentioning a closely related problem, the internal field analysis of heterogeneous materials. Kreher and Pompe (1989) and Li and Dunn (1998b) showed that the Mori–Tanaka approach leads to zero field variations in constituents of two-phase elastic and piezoelectric composites, which is unrealistic. A more reasonable approximation is also desirable to accurately model the internal field distribution in composite materials.

To overcome the theoretical difficulties of various micromechanics approaches, an effective-medium-field approximation scheme is proposed, which is based on both effective-medium and effective-field assumptions. The proposed approximation is especially designed for multi-phase composite containing reinforcements of different shapes and alignments, where the conventional approximations fail. At the unitary concentration limit where the matrix phase disappears, the formalism of the self-consistent approach for polycrystalline aggregates is obtained (Walpole, 1969), except that multi-phase grains are allowed in the current formalism. It is shown that the proposed approximation satisfies the requirements (1), (2), (3), and (4). Since no variational bounds for the multi-phase composite containing reinforcements of different shapes and alignments exist yet, to the best of the author's knowledge, requirement (5) cannot be evaluated. The paper is organized in the following manner: the rigorous framework is introduced in Section 2, followed by an introduction of various micromechanics approximations in Section 3; the micromechanics approximations are then examined and discussed in Section 4, along with various requirements; finally, a conclusion is drawn from the discussion.

2. The rigorously framework

We consider linear thermoelastic media. The constitutive equations for the stationary linear response can be expressed as

$$\sigma = C\varepsilon + \lambda\theta, \tag{1a}$$

or in inverse form

$$\varepsilon = D\sigma + \alpha\theta, \tag{1b}$$

where σ, ε, and θ are the stress and strain tensors and the temperature change with respect to a reference temperature, respectively; C and D are the elastic stiffness and compliance tensors, respectively; they are each other's inverse; λ and α are the thermal stress and strain tensors, respectively; they satisfy the relationship $\lambda = -C\alpha$.

Let us consider a piece-wise uniform material described by moduli that vary as

$$C(\mathbf{x}) = \sum_r C_r \Phi_r(\mathbf{x}) \tag{2a}$$

and

$$\lambda(\mathbf{x}) = \sum_r \lambda_r \Phi_r(\mathbf{x}), \tag{2b}$$

where the subscript r is used to denote a quantity of phase r which is regarded to be uniform over phase r, and $\mathbf{x}$ is used to denote a position in the heterogeneous material. Similar equations hold for compliance tensor D and thermal strain tensor α. $\Phi_r(\mathbf{x})$ is the characteristic function that describes the topology of the microstructure of the heterogeneous material:

$$\Phi_r(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in r, \\ 0, & \mathbf{x} \notin r. \end{cases} \tag{3}$$

An important property of $\Phi_r(\mathbf{x})$ is the relationship

$$\langle \Phi_r(\mathbf{x}) P(\mathbf{x}) \rangle = c_r P_r, \tag{4}$$

where $\langle \bullet \rangle = 1/V \int_V (\bullet)\,\mathrm{d}V$ denotes the average over the volume of a heterogeneous material V; $P(\mathbf{x})$ is any integrable material property varying over the microstructure; and c_r is the volume fraction of phase r and satisfies $\sum_r c_r = 1$.

For the heterogeneous material considered subjected to uniform linear displacement $u = \mathbf{x}\varepsilon^0$ and temperature change θ at the boundary, we define the effective thermoelastic constitutive equation in terms of the average stress and strain in a statistical sense, under the assumption of macroscopic homogeneity:

$$\langle \sigma \rangle = C^* \langle \varepsilon \rangle + \lambda^* \theta, \tag{5}$$

where C^* and λ^* are the effective stiffness and thermal stress tensors, respectively. Note that the temperature, unlike the stress and strain fields, is assumed to be uniform in the entire heterogeneous material. Taking into account the average strain theorem (see, e.g., Nemat-Nasser and Hori, 1993)

$$\langle \varepsilon \rangle = \varepsilon^0 \tag{6}$$

and the constitutive equations for phase r

$$\sigma_r = C_r \varepsilon_r + \lambda_r \theta, \tag{7}$$

the effective stiffness and thermal stress tensor can be expressed as

$$C^* = \sum_r c_r C_r A_r \tag{8}$$

and

$$\lambda^* = \sum_r c_r (C_r a_r + \lambda_r), \tag{9}$$

respectively, where the elastic and thermal strain concentration factors A_r and a_r are defined by

$$\varepsilon_r = A_r \varepsilon^0 + a_r \theta, \tag{10}$$

based on linearity. From the average strain theorem, it can be verified that $\sum_r c_r A_r = I$ and $\sum_r c_r a_r = 0$, where I is the fourth-order unit tensor.

An analogous development for the applied uniform stress boundary conditions, $t = n\sigma^0$, yields the effective compliance and thermal strain tensors

$$D^* = \sum_r c_r D_r B_r \tag{11}$$

and

$$\alpha^* = \sum_r c_r (D_r b_r + \alpha_r), \tag{12}$$

respectively, where the elastic and thermal stress concentration factors B_r and b_r are defined by

$$\sigma_r = B_r \sigma^0 + b_r \theta \tag{13}$$

and satisfy $\sum_r c_r B_r = I$ and $\sum_r c_r b_r = 0$.

Up to this point the development is completely rigorous and the expressions for the effective thermoelastic moduli Eqs. (8)–(12) are exact. In order to estimate the effective thermoelastic moduli of heterogeneous materials, assumptions must be made for the concentration factors A_r and a_r, or B_r and b_r, which have been the focus of various micromechanics approximations. It is those assumptions that lead to the violation of requirements (1) to (5) in certain cases, as discussed in the introduction. The objective of this work is to propose an effective-medium-field approximation that satisfies the various requirements of the approximation schemes.

The estimate of the effective thermal stress tensor deserves more discussion. Levin (1967) and Rosen and Hashin (1970) found that the effective thermal stress tensor can be expressed rigorously as

$$\lambda^* = \sum_r c_r A_r^{\mathrm{T}} \lambda_r, \tag{14}$$

where the superscript T is used to denote the tensor transpose. On the other hand, Benveniste and Dvorak (1990) have recently established an exact relationship between the concentration factors A_r and a_r (r ≠ 1),

$$a_r = (I - A_r)(C_1 - C_r)^{-1}(\lambda_r - \lambda_1), \tag{15}$$

where the superscript -1 is used to denote the tensor inversion. Therefore, the effective thermal stress tensor can be evaluated by two different but equivalent methods. One method would employ the estimated A_r in Eq. (14). Alternatively, a_r can be found directly from the estimated A_r, and then utilized in Eq. (9). Since Eqs. (14), (9) and (15) are rigorous, a valid micromechanics approximation should give identical results. Benveniste et al. (1991) showed, however, that the Mori–Tanaka approach gives different results for the effective thermal stress tensor, using Eqs. (14) and (9) for multi-phase composite with differently shaped reinforcements. This observation casts doubt on the application of the Mori–Tanaka approach to multi-phase composites.

3. Micromechanics approximations

From the rigorous framework, it is clear that both the estimate of the effective stiffness tensor and the estimate of the effective thermal stress tensor depend on the estimate of concentration factor A_r, directly or indirectly. Various micromechanics approximations have been proposed to estimate A_r in composite materials, among them, the most popular ones are perhaps the Mori–Tanaka mean-field approach and the self-consistent effective-medium approach. The basic assumptions and equations of these two approaches will be reviewed here. An effective-medium-field approximation will also be proposed to overcome the theoretical difficulties faced by the two approaches in multi-phase composites. We restrict ourselves to the matrix-based composites, where phase 1 is reserved for the matrix, while phases 2 to n are for the reinforcements.

3.1. Mori–Tanaka mean-field approach

The basic assumption of the Mori–Tanaka approach is that the average strain in reinforcement r equals the strain in a single reinforcement with elastic stiffness tensor C_r embedded in an infinite matrix with elastic stiffness tensor C_1, and subjected to a uniform strain at the boundary, which equals the yet unknown average strain ε_1 in the matrix (Mori and Tanaka, 1973; Benveniste, 1987). In light of this assumption, the average strain in phase r is

$$\varepsilon_r = A_r^{\mathrm{dil}} \varepsilon_1 + a_r \theta, \tag{16}$$

where A_r^{dil} can be obtained from Eshelby's (1957) solution for reinforcements 2 to n,

$$A_r^{\mathrm{dil}} = [I + S_r C_1^{-1}(C_r - C_1)]^{-1}; \tag{17}$$

this is the unit tensor for matrix. S_r is the Eshelby tensor, which is a function of the stiffness tensor C_1 of the matrix and the shape of reinforcement r. From the average strain theorem and the definition of the concentration factor, it can be shown that

$$A_r^{\mathrm{MT}} = A_r^{\mathrm{dil}}[c_1 I + \sum_{r=2}^{n} c_r A_r^{\mathrm{dil}}]^{-1}. \tag{18}$$

The Mori–Tanaka approach has been applied successfully to two-phase composite materials. Its application to multi-phase composites containing reinforcements of different shapes or alignments, however, is questionable, since several criteria are violated, as shown by Benveniste et al. (1991), Norris (1989), Qiu and Weng (1990), and Ferrari (1991).

3.2. Self-consistent effective-medium approach

The basic assumption of the self-consistent approach is that the average strain in reinforcement r equals the strain in a single reinforcement with elastic stiffness tensor C_r embedded in an infinite matrix with the yet to be determined effective stiffness tensor C^*, and subjected to an applied uniform strain ε^0 at the boundary. In light of this assumption, the strain concentration factor can be expressed as

$$A_r^{\mathrm{sc}} = [I + S_r C^{*-1}(C_r - C^*)]^{-1}. \tag{19}$$

Now Eshelby's tensor S_r is a function of the effective stiffness tensor C^* instead of C_1 of the matrix. Since the effective stiffness tensor C^* appears on both sides, Eq. (8) must be solved numerically, in general. The self-consistent approach is usually regarded unsuitable for a matrix-based composite when the reinforcement volume fraction is high.

3.3. The effective-medium-field approximation

To overcome several theoretical difficulties of the Mori–Tanaka and the self-consistent approaches in multi-phase composites, we propose an effective-medium-field approximation here. For the applied uniform strain boundary condition, we assume that the average strain in reinforcement r equals the strain in a single reinforcement with elastic stiffness tensor C_r embedded in an infinite matrix with the yet to be determined effective elastic stiffness tensor C^* (effective medium assumption), and subjected to a uniform strain at the boundary, which equals the yet unknown average strain ε_1 in the matrix (effective field assumption). In light of this assumption, the average strain in phase r is

$$\varepsilon_r = A_r^{\mathrm{sc}} \varepsilon_1 + a_r \theta, \tag{20}$$

where A_r^{sc} is given by Eq. (19) for reinforcements 2 to n, and is the unit tensor for the matrix. Taking into account the average strain theorem and the definition of the concentration factor, the elastic strain concentration factor of the proposed approximation can be written as

$$A_r^{\mathrm{emf}} = A_r^{\mathrm{sc}}[c_1 I + \sum_{r=2}^{n} c_r A_r^{\mathrm{sc}}]^{-1}. \tag{21}$$

The thermal strain concentration factor can be obtained by the combination of Eqs. (21) and (15). Again, since the effective stiffness tensor C^* and the effective thermal stress tensor λ^* appear on both sides, Eqs. (8) and (9) must be solved numerically, in general.

Analogously, when the uniform stress is applied at the boundary, we assume that the average stress in phase r equals the stress in a single reinforcement with elastic stiffness tensor C_r embedded in an infinite matrix with the yet to be determined effective elastic stiffness tensor C^*, and subjected to a uniform stress at the boundary, which equals the yet unknown average stress σ_1 in the matrix. In light of this assumption, the average stress in phase r is

$$\sigma_r = B_r^{\mathrm{sc}} \sigma_1 + b_r \theta, \tag{22}$$

where

$$B_r^{\mathrm{sc}} = [I + C^*(I - S_r)(C_r^{-1} - C^{*-1})]^{-1} \tag{23}$$

for reinforcements 2 to n according to the solution of the auxiliary inhomogeneity problem; it is the unit tensor for the matrix. Taking into account the average stress theorem and the definition of the concentration factor, the stress concentration factor of the proposed approximation can be written as

$$B_r^{\mathrm{emf}} = B_r^{\mathrm{sc}}[c_1 I + \sum_{r=2}^{n} c_r B_r^{\mathrm{sc}}]^{-1}. \tag{24}$$

From the derivation, it is clear that the proposed approximation has both effective medium and effective-field assumptions. Actually this work is motivated by the success of the self-consistent approach in polycrystalline aggregates (Walpole, 1969). To demonstrate this, let us consider a two-phase composite with a certain reinforcement orientation distribution. In such a case, the volume averages over all reinforcements reduce to the orientational averages. At the unitary reinforcement concentration limit where the volume fraction of the matrix goes to zero, Eq. (21) becomes $A_r^{\mathrm{emf}} = A_r^{\mathrm{sc}} \langle A_r^{\mathrm{sc}} \rangle^{-1}$, which corresponds to Walpole's original equation for polycrystals. Interestingly, Walpole proposed such a normalization to guarantee that the consistency relationship $\langle A_r \rangle = I$ is satisfied. Such a requirement is not satisfied by Eq. (19) when the grains cannot be taken on average as spheres or aligned ellipsoids. Iwakuma and Nemat-Nasser (1984) and Nemat-Nasser and Obata (1986) examined different normalization schemes in their studies of elasto-plastic deformation of polycrystals, and they found that Walpole's normalization leads to a more suitable numerical iterative calculation. It was concluded by Nemat-Nasser and Agah-Tehrani (1992) from a perturbation argument that normalization of the concentration tensor is an integral part of the self-consistent approach, otherwise the consistency relationship may be violated. Although their work focused on polycrystals, the observation is also enlightening when dealing with multi-phase composites. In composite materials, only the concentration factors of the reinforcements are calculated directly, and the concentration factor of the matrix is calculated from $\sum_r c_r A_r = I$; thus the consistency constraint is satisfied automatically. However, imagine that at the unitary reinforcement limit where the matrix phase disappears, the consistency relationship will be no longer satisfied by Eq. (19) for a multi-phase composite with general geometry, following a similar argument of Nemat-Nasser and Agah-Tehrani. It can then be concluded that the normalization of the concentration factor is also necessary for composite materials. Unfortunately, such a normalization has long been ignored, probably because most of the work in this area concerns two-phase materials with aligned reinforcements, and $\sum_r c_r A_r = I$ is always satisfied in such cases. Interestingly, such a normalization can be derived directly from the effective-medium-field approximation instead of being a constraint imposed by the consistency relationship.

With the elastic strain and stress concentration factors given by Eqs. (21) and (24), the effective stiffness and compliance tensors can be estimated by Eqs. (8) and (11), while the effective thermal stress tensors can be estimated by Eqs. (9) and (14). The question remaining is whether or not such estimates are self-consistent, i.e., whether or not the estimated stiffness and compliance tensors are each other's inverse, and the estimated thermal stress tensors given by Eqs. (9) and (14) are identical. Also important is whether the estimated effective moduli are diagonally symmetric. These issues are discussed in the next section. The correspondence with the self-consistent approach of polycrystalline aggregates has already ensured the correct behavior at the unitary reinforcement concentration limit.

4. Discussion

As we discussed, all micromechanics approaches should satisfy certain criteria. In this section, we examine the Mori–Tanaka, the self-consistent, and the proposed approach under various criteria.

4.1. The diagonal symmetry of the effective stiffness tensor

Benveniste et al. (1991) have examined the application of the Mori–Tanaka and the self-consistent approach in a three-phase composite consisting of a Ti_3Al matrix (phase 1), a carbon circular disc (phase 2) with the normal of the plane face of the disc in the direction of x_3 of a Cartesian coordinate system, and continuous SiC fibers (phase 3) of circular cross section aligned with x_3.

The phase moduli and volume fractions they used are listed in Table 1, where E, G, and α are the Young modulus, shear modulus, and thermal expansion coefficients, respectively. They found that these two approaches yield a non-symmetric stiffness tensor. We have calculated the effective stiffness tensor of the composite using the Mori–Tanaka, the self-consistent, and the proposed approach. The results are listed in Table 2. It is found that the Mori–Tanaka and the self-consistent approaches give different values for C_{13} and C_{31}, violating criterion (1). The proposed approach, however, gives identical C_{13} and C_{31}, so that the estimated effective stiffness tensor is diagonally symmetric. As we show later, the estimated effective stiffness and compliance tensors are each other's inverse in the proposed approximation, so that the estimated effective compliance tensor is also diagonally symmetric. It is also found that the numerical difference between the Mori–Tanaka and the proposed approach is small. It is clear that this calculation favors the proposed approach over the Mori–Tanaka approach in multi-phase composites with differently shaped reinforcements. Our calculated values by the Mori–Tanaka and the self-consistent approaches are slightly different from those of Benveniste et al. (1991), which is believed to be due to numerical error.

4.2. The equivalence of the effective thermal stress tensors

In the same paper, Benveniste et al. (1991) also showed that the Mori–Tanaka approach gives different values for the effective thermal stress tensor of the three-phase composite via two different, but equivalent methods. Both methods are rigorous except for the Mori–Tanaka mean-field assumption. This observation also casts some doubt on the application of the Mori–Tanaka approach to multi-phase composites. We have calculated the effective thermal stress tensor from Eqs. (14) and (9), respectively, using the Mori–Tanaka, the self-consistent, and the proposed approach. The results are listed in Table 3. It is found that both the Mori–Tanaka and the self-consistent approach give different values for the effective thermal stress tensor using the two methods, violating criterion (2). The proposed approach, however, gives identical values of the effective thermal stress tensor by these two methods, and thus is self-consistent. The numerical difference between the Mori–Tanaka and the proposed approach is small. Again, the calculation favors the proposed approach over the Mori–Tanaka approach in multi-phase composites with differently shaped reinforcements.

4.3. Internal-consistency relationships between the effective moduli

For the internal consistency relationship, it is noted that for a finite sample the effective stiffness and compliance tensors predicted by Eqs. (8) and (11) need not be each other's inverse, since different boundary conditions are imposed. Such a consistency, however, should be retained under statistical homogeneity. Weng (1990) showed that

Table 1
Phase moduli and volume fraction of constituents in a three-phase composite

Material	E (GPa)	G (GPa)	α($\times 10^{-6}$/°C)	c
Ti_3Al	95.6	37.1	9.25	0.55
C	34.4	14.3	3.33	0.25
SiC	431.0	172	4.86	0.2

Table 2
The effective stiffness moduli (GPa) of a three-phase composite by different approaches

Approach	C_{11}	C_{12}	C_{13}	C_{31}	C_{33}	C_{44}	C_{66}
Mori–Tanaka	123.7	43.98	36.14	22.20	124.22	32.99	39.88
Self-consistent	124.0	44.25	36.22	22.34	124.26	32.69	39.86
Proposed	128.1	47.24	35.10	35.10	125.47	33.32	40.45

Table 3
The effective thermal stress tensor (10^{-2} GPa/°C) of a three-phase composite by different approaches. (Method I is by Eq. (14), method II is by Eq. (9))

Approach	λ_{11} (I)	λ_{33} (I)	λ_{11} (II)	λ_{33} (II)
Mori–Tanaka	−0.1422	−0.1293	−0.1501	−0.0824
Self-consistent	−0.1425	−0.1294	−0.1502	−0.0823
Proposed	−0.1588	−0.1304	−0.1588	−0.1304

this is the case for the Mori–Tanaka approach. The self-consistent approach also satisfies this requirement; the energy-based definition may not (see Nemat-Nasser and Hori, 1993). For the proposed approximation, it can be deduced from Eqs. (20) and (22) that

$$C_r A_r^{\mathrm{sc}} = B_r^{\mathrm{sc}} C_1 \tag{25a}$$

and

$$D_r B_r^{\mathrm{sc}} = A_r^{\mathrm{sc}} D_1, \tag{25b}$$

so that the effective stiffness and compliance tensors can be written as

$$C^* = (c_1 I + \sum_{r=2}^{n} c_r B_r^{\mathrm{sc}}) C_1 [c_1 I + \sum_{r=2}^{n} c_r A_r^{\mathrm{sc}}]^{-1} \tag{26a}$$

and

$$D^* = (c_1 I + \sum_{r=2}^{n} c_r A_r^{\mathrm{sc}}) D_1 [c_1 I + \sum_{r=2}^{n} c_r B_r^{\mathrm{sc}}]^{-1}, \tag{26b}$$

respectively, according Eqs. (8) and (11). Then it is easy to verify that C^* and D^* are indeed each other's inverse.

Hill (1964) obtained two exact connections between different components of the effective stiffness tensor of a two-phase composite, expressed as

$$\frac{k-k_1}{l-l_1} = \frac{k-k_2}{l-l_2} = \frac{l-c_1 l_1 - c_2 l_2}{n - c_1 n_1 - c_2 n_2} = \frac{k_1 - k_1}{l_2 - l_2} \tag{27}$$

and

$$\frac{1/k - /k_1}{v - v_1} = \frac{1/k - /k_2}{v - v_2} = \frac{v - c_1 v_1 - c_2 v_2}{-(E - c_1 E_1 - c_2 E_2)/4} = \frac{1/k_1 - 1/k_2}{v_1 - v_2}, \tag{28}$$

where k is the plane-strain bulk modulus for lateral dilation without longitudinal extension; n is the modulus for longitudinal uniaxial straining, and l is the associated cross-modulus; and v and E are the Poisson ratio and Young's modulus under the longitudinal loading, respectively (Hill, 1964). We have examined the Mori–Tanaka, the self-consistent, and the proposed approach in a SiC fiber-reinforced Ti_3Al composite using these exact connections. The results are listed in Table 4. It is found that all three approaches satisfy these exact connections.

4.4. Dilute and unitary reinforcement concentration limits

Ferrari (1991) pointed out that micromechanics approximations should have correct behaviors at dilute and unitary concentration limits. At the dilute limit, Eshelby's (Eshelby, 1957) exact solution should be obtained, while at the unitary limit, the estimated effective moduli should not depend on the matrix properties. He further showed that the Mori–Tanaka approach does not have the correct behavior at the unitary limit, where the effective stiffness tensor becomes

$$C^* = \langle C_r A_r^{\mathrm{dil}} \rangle \langle A_r^{\mathrm{dil}} \rangle^{-1}. \tag{29}$$

Since the concentration factor A_r^{dil} is a function of the stiffness tensor of the matrix, in general, the effective stiffness tensor given by Eq. (29) is a function of the stiffness tensor of the matrix even at the unitary reinforcement concentration unless the reinforcements are isotropic or aligned. The proposed approach, however, avoids such an undesirable behavior by the effective-medium assumption, in which the effective stiffness tensor becomes

$$C^* = \langle C_r A_r^{\mathrm{sc}} \rangle \langle A_r^{\mathrm{sc}} \rangle^{-1} \tag{30}$$

at the unitary limit. Now the concentration factor A_r^{sc} is a function of the effective stiffness tensor of the composite instead of the matrix, and it will not depend on the matrix properties. This observation favors the application of the proposed approach over the Mori–Tanaka approach in multi-phase

Table 4
Examination of different approaches using the exact connections between the different components of the effective stiffness tensor

Approach	$\frac{k-k_1}{l-l_1}$	$\frac{k-k_2}{l-l_2}$	$\frac{l-c_1 l_1-c_2 l_2}{n-c_1 n_1-c_2 n_2}$	$\frac{k_1-k_2}{l_1-l_2}$
Mori–Tanaka	2.1229	2.1229	2.1229	2.1229
Self-consistent	2.1229	2.1229	2.1229	2.1229
Proposed	2.1229	2.1229	2.1229	2.1229
Approach	$\frac{1/k-1/k_1}{\nu-\nu_1}$	$\frac{1/k-1/k_2}{\nu-\nu_2}$	$\frac{\nu-c_1\nu_2-c_2\nu_2}{-(E-c_1E_1-c_2E_2)/4}$	$\frac{1/k_1-1/k_2}{\nu_1-\nu_2}$
Mori–Tanaka	1.654×10^{-10}	1.654×10^{-10}	1.654×10^{-10}	1.654×10^{-10}
Self-consistent	1.654×10^{-10}	1.654×10^{-10}	1.654×10^{-10}	1.654×10^{-10}
Proposed	1.654×10^{-10}	1.654×10^{-10}	1.654×10^{-10}	1.654×10^{-10}

composites with different alignments. At the dilute limit, the concentration factor recovers that of the self-consistent approach to the first degree of approximation,

$$A_r^{\text{emf}} = A_r^{\text{sc}}, \quad (31)$$

and it is known that the self-consistent approach recovers Eshelby's exact solution to the first degree of approximation at the dilute concentration limit. So criterion (3) is also satisfied by the proposed approach.

5. Conclusions

An effective-medium-field approximation based on normalized concentration factors is proposed to estimate the effective thermoelastic moduli of multi-phase composites. Such normalization is necessary to ensure the satisfaction of the consistency relationship. It is shown that the proposed approach gives a diagonally symmetric stiffness tensor, gives identical thermal stress tensors using two equivalent methods, satisfies the internal-consistency relationships between the effective moduli, and exhibits the correct behaviors at dilute and unitary concentration limits. These observations favor the application of the proposed approach over the Mori–Tanaka approach to multiphase composites with differently shaped and aligned reinforcements.

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