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Anisotropic coupled-field inclusion and inhomogeneity problems

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Abstract

We study the fields in and around inclusions and inhomogeneities in anisotropic solids exhibiting full coupled-field behaviour using Eshelby's pioneering approach. Explicit expressions are obtained for the generalized Eshelby tensors, as are a class of shape-independent relations for the Eshelby tensors. These can be used in the same manner as Eshelby's tensor for elastic inclusions.

§ 1. Introduction

Eshelby's (1957, 1959) classical analyses of the stress and strain fields in elastic solids containing ellipsoidal inclusions and inhomogeneities are widely recognized for both their elegance and wide-ranging applicability. In micromechanics analysis of heterogeneous solids, a theoretical framework has emerged where exact expressions for effective properties and internal fields can be obtained in terms of concentration factors that relate the average fields in the phases to uniform applied fields. It is at this point that an exact analysis typically ceases and approximate schemes must be introduced to estimate the concentration factors. In most micromechanics schemes, the reinforcement is modelled as ellipsoidal and recourse is made to Eshelby's solution. Eshelby's solution is used for at least three reasons: (i) the general ellipsoidal shape can be used to model a wide range of microstructural geometries; (ii) explicit, closed-form expressions exist for the stresses and strains in the ellipsoidal inhomogeneity; (iii) the elastic fields in the ellipsoidal inhomogeneity have the remarkable property that they are uniform, rendering the computation of average fields in the inhomogeneity trivial. Numerous examples of, and references to, such applications can be found in the authoritative texts of Mura (1987) and Nemat-Nasser and Hori (1993).

Perhaps the most widely used result of Eshelby's analyses is his closed-form expression for what is now known as Eshelby's tensor: a fourth-order tensor that is a function only of the elastic moduli of the matrix and the shape of the inclusion. With explicit expressions for Eshelby's tensor in hand, solutions to many problems involving inclusions and inhomogeneities are reduced to algebraic tensor manipulation. Although he only provided explicit results for inclusions in isotropic solids, Eshelby laid the groundwork for the study of inclusions in anisotropic solids which was taken up by subsequent researchers (Hill 1961, Willis 1964, Walpole 1967, 1977, Kinoshita and Mura 1971, Lin and Mura 1973, Asaro and Barnett 1975, Bacon *et al.* 1978, Withers, 1989). In addition, Eshelby's basic ideas have been extended to treat other uncoupled-field problems such as heat conduction, electrostatics and magnetostatics (Hatta and Taya 1988).

The emphasis of this work is on partial and full coupled-field problems pertaining to inclusions and inhomogeneities in anisotropic media. By partial coupling we mean phenomena such as uncoupled thermoelasticity where elastic and thermal phenomena are coupled, but modelled in such a way that a temperature change induces a change in strain, but a change in strain does not induce a temperature change. Thus, temperature enters the mechanical problem as a parameter through the constitutive equations and the temperature distribution can be computed simply by the theory of heat conduction. Static poroelasticity also falls into this category when one recognizes the mathematical analogy between poroelasticity and thermoelasticity (Norris 1992). Inclusion and inhomogeneity problems in media exhibiting partial coupling between elastic and other fields can be treated using Eshelby's results (or the corresponding results for anisotropic media) directly, and treating the coupling term as a transformation strain (in Eshelby's terminology) or an eigenstrain (in Mura's terminology). Such examples in the context of thermoelasticity are discussed in detail by Mura. Full coupling, on the other hand, presents more of a challenge. Here we are referring to phenomena such as piezoelectricity where not only does a change in the electric field induce strain, but also strain induces a change in the electric field. In this case, Eshelby's results cannot be directly applied, but his ideas can be. Specifically, one can solve the inclusion problem for full coupling between two or more fields using a generalization of his elegant cutting–straining–welding approach.

The motivation for this work is the study of heterogeneous media with coupled fields, and the belief that inclusion and inhomogeneity problems play a key role in the analysis of such materials. In particular, the phases can be modelled as inhomogeneities. When combined with a rigorous micromechanical framework, substantial progress can be made toward understanding the complicated microstructural-level fields in heterogeneous media with coupled fields. This has been demonstrated for piezoelectric media by Deeg (1980), Wang (1992), Benveniste (1992), Dunn and Taya (1993), Chen (1993a,b), Dunn and Wienecke (1996, 1997), and the many references contained therein. Interesting composite materials consisting of combinations of piezoelectric and piezomagnetic phases have already been studied both experimentally and theoretically (Harshe *et al.* 1993a, b Avelleneda and Harshe 1994, Nan 1994, Benveniste 1995). These materials have been shown to exhibit the phenomena of magnetoelectricity, a phenomena that does not appear in the individual phases themselves.

In this study we consider the general case of inclusions and inhomogeneities in media exhibiting full coupling between multiple fields. The key to the solution is the generalized Eshelby tensor. We show that the fully coupled problem can be treated in a very general way, completely analogous to the treatment of anisotropic elasticity. This realization emerges naturally when the problem is cast using a convenient notation introduced in the context of piezoelectricity by Barnett and Lothe (1975), extended to piezoelectromagnetic media by Alshits *et al.* (1992), and used by Alshits *et al.* (1995), Chung and Ting (1995) and Kirchner and Alshits (1996). The focus here is on static behaviour; the corresponding dynamic case is more involved as each field reacts differently to dynamic effects. While we emphasize the treatment of general coupled fields, we give explicit results for the special case where full coupling exists between elastic, electric and magnetic fields. The generalized Eshelby tensor is expressed in terms of a surface integral over a unit sphere, which can be evaluated numerically. The integrals are evaluated in closed form for special shapes when the medium is transversely isotropic. The results can be used immediately in the micromechanics analysis of heterogeneous media. We also show that when full coupling exists between two or more fields and, in addition, partial coupling exists, the treatment of the partial coupling proceeds in a manner analogous to that for elastic inclusions.

§ 2. Basic equations

A three-dimensional Cartesian coordinate system is employed where position is denoted by the vector $\mathbf{x}$ or x_i. We consider materials that exhibit linear, static, anisotropic coupled-field phenomena. These phenomena are described by three sets of equations: constitutive equations, divergence equations and gradient equations. To clearly illustrate, we explicitly consider piezoelectromagnetic media, but we emphasize that the formalism is valid for more general coupled-field problems where the basic equations have the same structure. In full index form, the field equations (constitutive, divergence and gradient) are:

$$\begin{aligned} \sigma_{ij} &= C_{ijkl}u_{k,l} + e_{ijl}\phi_{,l} + q_{ijl}\varphi_{,l}, \\ D_i &= e_{ikl}u_{k,l} - \kappa_{il}\phi_{,l} - a_{il}\varphi_{,l}, \\ B_i &= q_{ikl}u_{k,l} - a_{il}\phi_{,l} - \mu_{il}\varphi_{,l}, \end{aligned} \tag{1}$$

$$\begin{aligned} \sigma_{ij,j} &= 0, \\ D_{i,i} &= 0, \\ B_{i,i} &= 0, \end{aligned} \tag{2}$$

$$\begin{aligned} \varepsilon_{ij} &= \tfrac{1}{2}(u_{i,j} + u_{j,i}), \\ E_i &= -\phi_{,i}, \\ H_i &= -\varphi_{,i}. \end{aligned} \tag{3}$$

In equations (1)–(3), σ_{ij}, ε_{ij} and u_i are the elastic stress, strain and displacement respectively; D_i, E_i and ϕ are the electric displacement, field, and potential respectively; B_i, H_i and φ are the magnetic flux, field, and potential respectively. C_{ijkl}, κ_{il} and μ_{il} are the elastic stiffness, the dielectric, and the magnetic permeability tensors. They directly connect like fields, e.g., stresses to strains. Elastic fields are coupled to the electric and magnetic fields through the piezoelectric e_{ikl} and piezomagnetic q_{ikl} coefficients respectively. Finally, electric and magnetic fields are coupled through the magnetoelectric coefficients a_{il}. The symmetry conditions satisfied by the moduli are given by Nye (1957).

In the analysis to follow, it is convenient to treat the elastic, electric, and magnetic variables on an equal footing. To this end, the notation introduced by Barnett and Lothe (1975) for piezoelectric analysis and generalized to incorporate magnetic coupling by Alshits *et al.* (1992) is utilized. This notation is identical to conventional indicial notation with the exception that lowercase subscripts take on the range 1–3, while uppercase subscripts take on the range 1–5 and repeated uppercase subscripts are summed over 1–5. With this notation, the field variables take the following forms:

$$U_M = \begin{cases} u_m \\ \phi \\ \varphi \end{cases} \qquad Z_{Mn} = \begin{cases} \varepsilon_{mn} = \frac{1}{2}(u_{m,n} + u_{n,m}) \\ -E_n = \phi_{,n} \\ -H_n = \varphi_{,n} \end{cases} \qquad \Sigma_{nM} = \begin{cases} \sigma_{nm} & M = 1,2,3 \\ D_n & M = 4 \\ B_n & M = 5. \end{cases} \tag{4}$$

The moduli are expressed as:

$$E_{iJMn} = \begin{cases} C_{ijmn} & J, M = 1,2,3 \\ e_{ijn} & M = 4, J = 1,2,3 \\ q_{ijn} & M = 5, J = 1,2,3 \\ e_{imn} & J = 4, M = 1,2,3 \\ -\kappa_{in} & J = 4, M = 4 \\ -a_{in} & J = 4, M = 5 \\ q_{imn} & J = 5, M = 1,2,3 \\ -a_{in} & J = 5, M = 4 \\ -\mu_{in} & J = 5, M = 5. \end{cases} \tag{5}$$

With this shorthand notation, the divergence, gradient, and constitutive equations, can be written as:

$$\Sigma_{iJ,i} = 0, \qquad Z_{Mn} = U_{M,n}, \qquad \Sigma_{iJ} = E_{iJMn} Z_{Mn}. \tag{6}$$

§ 3. Inclusion and inhomogeneity problems

Consider an infinite solid D containing an ellipsoidal inclusion, the volume of which is denoted by Ω with a surface denoted by $|\Omega|$. The inclusion has the same moduli, E_{iJMn}, as the matrix, but is allowed to undergo a uniform transformation. We denote by Z^*_{Mn} the uniform transformation that would occur if Ω were unconstrained by $D - \Omega$. In the elastic case, the physical interpretation of the transformation strain is well known. In a more general setting it is not so clear. For example, in the case of magnetostatics, Z^*_{Mn} can be interpreted as a volume distribution of magnetic dipoles in Ω. The same interpretation can be attached to analogous field problems such as heat conduction and electrostatics. Despite the different physical interpretations of Z^*_{Mn}, we can still appeal to the imaginary cutting, straining, and welding operations of Eshelby (1957) to calculate the actual (constrained) fields, and this leads to:

$$\begin{aligned} U_M(\mathbf{x}) &= \iint_{|\Omega|} G_{MJ}(\mathbf{x} - \mathbf{x}')\Sigma^*_{iJ} n_i \, dS(\mathbf{x}') - \iiint_{\Omega} G_{MJ}(\mathbf{x} - \mathbf{x}')\Sigma^*_{iJ,i} \, dV(\mathbf{x}') \\ &= -E_{iJAb} Z^*_{Ab} \iiint_{\Omega} G_{MJ,i}(\mathbf{x} - \mathbf{x}') \, dV(\mathbf{x}'). \end{aligned} \tag{7}$$

In equation (7) $G_{MJ}(\mathbf{x})$ are generalized Green's functions, $\Sigma^*_{iJ} = E_{iJMn} Z^*_{Mn}$, and the differentiation is with respect to $\mathbf{x}$. When we differentiate U_M with respect to $\mathbf{x}$, after substantial manipulation we can write $U_{M,n}$ inside the inclusion as:

$$U_{M,n}(\mathbf{x}) = -E_{iJAb} Z^*_{Ab} \iiint_{\Omega} G_{MJ,in}(\mathbf{x} - \mathbf{x}') \, dV(\mathbf{x}'). \tag{8}$$

The manipulations required to get from equation (7) to equation (8) follow exactly those outlined by Bacon *et al.* (1978) in the context of anisotropic elasticity and Deeg (1980) in piezoelectricity. Following Eshelby (1957) we write:

$$Z_{Mn} = S_{MnAb} Z^*_{Ab}, \tag{9}$$

where:

$$S_{MnAb} = \frac{1}{8\pi} E_{iJAb} \begin{cases} \displaystyle\int_{-1}^{1}\int_{0}^{2\pi} \left[G_{mJin}(\mathbf{z}) + G_{nJim}(\mathbf{z})\right] d\theta\, d\xi_3 & M = 1,2,3 \\ \displaystyle 2\int_{-1}^{1}\int_{0}^{2\pi} G_{4Jin}(\mathbf{z})\, d\theta\, d\xi_3 & M = 4 \\ \displaystyle 2\int_{-1}^{1}\int_{0}^{2\pi} G_{5Jin}(\mathbf{z})\, d\theta\, d\xi_3 & M = 5. \end{cases} \tag{10}$$

In equation (10) $z_i = \xi_i / a_i$ (no sum on i), and ξ_1 and ξ_2 can be expressed in terms of ξ_3 and θ by $\xi_1 = (1 - \xi_3^2)^{1/2} \cos\theta$ and $\xi_2 = (1 - \xi_3^2)^{1/2} \sin\theta$. In addition, $G_{MJin} = z_i z_n K^{-1}_{MJ}(\mathbf{z})$ where K^{-1}_{MR} is the inverse of $K_{JR} = z_i z_n E_{iJRn}$.

Formally S_{MnAb} is a generalization of Eshelby's tensor. In the case of piezo-magnetoelectricity, S_{MnAb} is a collection of nine tensors: one fourth-order, four second-order and four third-order. S_{MnAb} is a function only of the moduli E_{iJKl} and the shape of the inclusion. If the Eshelby tensors S_{MnAb} are known, then for prescribed eigenfields Z^*_{Mn}, the constrained fields in the inclusion can be expressed as:

$$\Sigma_{iJ} = E_{iJMn}\left[Z_{Mn} - Z^*_{Mn}\right]. \tag{11}$$

As shown by Deeg (1980), Benveniste (1992) and Dunn and Taya (1993), the fields in the ellipsoidal inclusion are uniform.

Once the solution for the ellipsoidal inclusion (a transformed region with the same moduli as the matrix) is obtained, the solution for the ellipsoidal inhomogeneity (a region with different moduli from the matrix) easily follows. As shown by Eshelby (1957) in the elastic case, the inhomogeneity can be simulated by an equivalent inclusion. To fix ideas, consider the infinite solid D with moduli E_{iJMn} which contains an ellipsoidal inhomogeneity Ω with moduli E^*_{iJMn}. In the absence of an applied load, the fields in both the inhomogeneity and matrix are zero. When subjected to a far-field uniform load Σ^0_{iJ}, the fields $\Sigma^0_{iJ} + \Sigma_{iJ}$ in the inhomogeneity can be written as:

$$\Sigma^0_{iJ} + \Sigma_{iJ} = E^*_{iJMn}\left[Z^0_{Mn} + Z_{Mn}\right] = E_{iJMn}\left[Z^0_{Mn} + Z_{Mn} - Z^*_{Mn}\right]. \tag{12}$$

In (12), Z^0_{Mn} is the uniform field that would exist in the absence of the inhomogeneity ($\Sigma^0_{iJ} = E_{iJMn} Z^0_{Mn}$) and Z_{Mn} is the disturbance of the uniform fields due to the presence of the inhomogeneity. The first right-hand side of equation (12) represents the fields in the actual inhomogeneity while the second one represents the fields in an inclusion of the same shape and orientation as the inhomogeneity and with eigenfields Z^*_{Mn}, i.e. an equivalent inclusion. The simulation of the inhomogeneity by the equivalent inclusion is possible if an appropriate Z^*_{Mn} can be found to enforce the second equality of equation (12) (where equation (9) holds in the equivalent inclusion). Substituting equation (9) into (12) and solving for Z^*_{Mn} gives

$$Z^*_{Pq} = -A^{-1}_{PqiJ}\left[E^*_{iJMn} - E_{iJMn}\right] Z^0_{Mn}, \tag{13}$$

where $A_{iJAb} = \left[E^*_{iJMn} - E_{iJMn}\right]S_{MnAb} + E_{iJAb}$. Once $Z^*_{Mn}(Z^0_{Mn})$ is obtained from equation (13), it can be used with equations (9) and (12) to obtain the fields in the inhomogeneity due to the applied load.

An inhomogeneous inclusion is an inhomogeneity with prescribed eigenfields Z^T_{Pq}. Consider the infinite solid D with moduli E_{iJMn} that contains an ellipsoidal inhomogeneity Ω with moduli E^*_{iJMn} and eigenfields Z^T_{Pq}. The fields in the inhomogeneous inclusion are:

$$\Sigma^0_{iJ} + \Sigma_{iJ} = E^*_{iJMn}\left[Z_{Mn} - Z^T_{Mn}\right] = E_{iJMn}\left[Z_{Mn} - Z^T_{Mn} - Z^{**}_{Mn}\right] = E_{iJMn}\left[Z_{Mn} - Z^*_{Mn}\right]. \tag{14}$$

In equation (14) $Z^*_{Mn} = Z^T_{Mn} + Z^{**}_{Mn}$, where Z^{**}_{Mn} are fictitious eigenfields and $Z_{Mn} = S_{MnAb}Z^*_{Ab}$.

The above results for the interior fields can be used to obtain the fields just outside an inclusion (and thus of course also an inhomogeneity) by making use of the continuity conditions on Z_{Mn} and the jump conditions on U_M at the inclusion–matrix interface. The fields just outside the inclusion can be expressed as:

$$\Sigma^{out}_{iJ} = \Sigma^{in}_{iJ} + E_{iJKl}\left[- E_{pQMn}Z^*_{Mn}K^{-1}_{QK}n_p n_l + Z^*_{Kl}\right]. \tag{15}$$

In equation (15) the interior fields Σ^{in}_{iJ} are obtained by the approach discussed above and K^{-1}_{QK} is the inverse of $K_{JK} = K_{KJ} = n_i n_l E_{iJKl}$ where n_i is the outward normal from the inclusion surface.

Also important are energy calculations for heterogeneous systems. Consider a solid containing an inhomogeneity subjected to far-field loads $\Sigma^0_{iJ} n_i$. These loads would result in a uniform fields Σ^0_{iJ} in a homogeneous solid. The total free energy of the inhomogeneous solid can be expressed as:

$$W = \frac{1}{2}\int_D \Sigma^0_{iJ} U^0_{J,i}\, dV + \frac{1}{2}\int_\Omega \Sigma^0_{iJ} Z^*_{Ji}\, dV - \int_S \Sigma^0_{iJ} n_i U^0_J\, dS \tag{16}$$

where V and S denote the volume and surface of the solid respectively and Ω denotes the volume of the inhomogeneity. The interaction energy between $\Sigma^0_{iJ} n_i$ and the inhomogeneity is then:

$$\Delta W = W - W^0 = \frac{1}{2}\int_\Omega \Sigma^0_{iJ} Z^*_{Ji}\, dV - \int_S \Sigma^0_{iJ} n_i U_J\, dS = -\frac{1}{2}\Sigma^0_{iJ} Z^*_{Ji} V_\Omega \tag{17}$$

where the volume of the ellipsoid is $V_\Omega = \frac{4}{3}\pi a_1 a_2 a_3$. Other energy expressions can be readily calculated from these results.

We close this section by presenting two interesting and useful results regarding the fields in and around an ellipsoidal inclusion. First, consider two ellipsoidal domains V_1 and V_2 that are of the same shape and orientation as V_Ω and surround V_Ω so that $V_\Omega \subseteq V_1 \subseteq V_2$. If we integrate Z_{Mn} of equation (9) over the annular ellipsoidal region $V_2 - V_1$ we find:

$$\int_{V_2 - V_1} Z_{Mn}\, dV = 0. \tag{18}$$

This result is a generalization of the Tanaka–Mori (1972) theorem regarding volume integrals of elastic fields around ellipsoidal inclusions. This result has been used advantageously by Hori and Nemat-Nasser (1993) to analyse the complicated elastic fields in double inclusions. It has also been used by Dunn and Ledbetter (1995) to model the elastic moduli of composites reinforced by multiphase particles.

The second result concerns invariant shape-independent relations for the generalized Eshelby tensors. We can couple equations (8) and (9) with the formal definition of the coupled Green's functions G_{MJ} to obtain:

$$S_{MnMn} = 5, \qquad S_{mnmn} = 3, \qquad S_{4n4n} = S_{5n5n} = 1. \tag{19}$$

These relations are independent of the shape of the ellipsoidal inclusion. They extend the results of Chen (1994) for ellipsoidal inclusions in piezoelectric media.

§ 4. Explicit results

For a general ellipsoidal shape and general material anisotropy, the Eshelby tensors can be obtained by evaluating the integrals in equations (10) by numerical integration. A convenient algorithm for this purpose is given by Gavazzi and Lagoudas (1990) in the context of elasticity. The extension to coupled-field material behaviour is straightforward. For certain material symmetries and inclusion shapes, the integrals in (10) can be evaluated in closed form. In the context of piezomagnetoelectricity, we have carried out such calculations for two inclusion shapes in transversely isotropic media: a thin disc and an elliptical cylinder. In both cases the a_3 semi-axis of the inclusion is aligned with the unique material axis. For transversely isotropic piezoelectromagnetic solids, the independent components of the moduli consist of: five elastic coefficients, C_{11}, C_{12}, C_{13}, C_{33} and C_{44}; three piezoelectric coefficients, e_{31}, e_{33}, e_{15}; three piezomagnetic coefficients, q_{31}, q_{33}, q_{15}; two dielectric coefficients, κ_{11}, κ_{33}; two magnetoelectric coefficients, a_{11}, a_{33}; and two magnetic permeability coefficients, μ_{11}, μ_{33}. Below we list explicitly the nonzero components of S_{MnAb} for two cases of technological importance.

4.1. *Elliptical cylindrical inclusion*

For an elliptical cylindrical inclusion, we have $a_3 \to \infty$ and we define the in-plane aspect ratio $\alpha = a_2/a_1$. The nonzero S_{MnAb} are:

$$S_{1111} = \frac{[(3+2\alpha)C_{11} + C_{12}]\alpha}{2(1+\alpha)^2 C_{11}},$$

$$S_{1122} = \frac{[(1+2\alpha)C_{12} - C_{11}]\alpha}{2(1+\alpha)^2 C_{11}},$$

$$S_{1133} = \frac{\alpha C_{13}}{(1+\alpha)C_{11}},$$

$$S_{2211} = \frac{(2+1\alpha)C_{12} - \alpha C_{11}}{2(1+\alpha)^2 C_{11}},$$

$$S_{2222} = \frac{(2+3\alpha)C_{11} + \alpha C_{12}}{2(1+\alpha)^2 C_{11}},$$

$$S_{2233} = \frac{C_{13}}{(1+\alpha)C_{11}},$$

$$S_{2323} = S_{2332} = S_{3223} = S_{3232} = \frac{1}{2(1+\alpha)}, \tag{20}$$

$$S_{1313} = S_{1331} = S_{3113} = S_{3131} = \frac{\alpha}{2(1+\alpha)},$$

$$S_{1212} = S_{1221} = S_{2112} = S_{2121} = \frac{(1+\alpha+\alpha^2)C_{11} - \alpha C_{12}}{2(1+\alpha)^2 C_{11}},$$

$$S_{1143} = \frac{\alpha e_{31}}{(1+\alpha)C_{11}},$$

$$S_{2243} = \frac{e_{31}}{(1+\alpha)C_{11}},$$

$$S_{1153} = \frac{\alpha q_{31}}{(1+\alpha)C_{11}},$$

$$S_{2253} = \frac{q_{31}}{(1+\alpha)C_{11}},$$

$$S_{4141} = \frac{\alpha}{(1+\alpha)},$$

$$S_{4242} = \frac{1}{1+\alpha},$$

$$S_{5151} = \frac{\alpha}{(1+\alpha)},$$

$$S_{5252} = \frac{1}{1+\alpha}.$$

Two extreme cases of the elliptical inclusion are particularly important: a circular cylindrical inclusion ($\alpha = 1$) and thin-disc inclusions ($\alpha \to 0$ or $\alpha \to \infty$). Explicit results for these cases immediately follow from equations (20). In addition, one can use these results to model slit cracks by expanding S_{MnAb} in a Taylor series in α and then retaining the lowest order terms. We mention this because one cannot just set $\alpha = 0$, but in general must retain the leading terms in α in order to proceed with the analysis.

4.2. *Thin-disc inclusion*

For a thin-disc inclusion we have $a_1 = a_2$ and $\alpha = a_3/a_1 \to 0$. The nonzero S_{MnAb} are:

$$S_{3311} = S_{3322} = \frac{\kappa_{33}q_{31}q_{33} + \mu_{33}e_{31}e_{33} + C_{13}\kappa_{33}\mu_{33} - a_{33}(a_{33}C_{13} + e_{33}q_{31} + e_{31}q_{33})}{\kappa_{33}q_{33}^2 + \mu_{33}e_{33}^2 + C_{33}\kappa_{33}\mu_{33} - a_{33}^2C_{33} - 2a_{33}e_{33}q_{33}},$$

$$S_{2323} = S_{2332} = S_{3223} = S_{3232} = S_{1313} = S_{1331} = S_{3113} = S_{3131} = \tfrac{1}{2},$$

$$S_{2342} = S_{3242} = S_{1341} = S_{3141} = \frac{e_{15}}{2C_{44}},$$

$$S_{2352} = S_{3252} = S_{1351} = S_{3151} = \frac{q_{15}}{2C_{44}}, \tag{21}$$

$$S_{4311} = S_{4322} = \frac{a_{33}(C_{33}q_{31} - C_{13}q_{33}) + q_{33}(e_{33}q_{31} - e_{31}q_{33}) - \mu_{33}(C_{33}e_{31} - C_{13}e_{33})}{\mu_{33}(e_{33}^2 + C_{33}\kappa_{33}) - a_{33}(a_{33}C_{33} + e_{33}q_{33}) - q_{33}(a_{33}e_{33} - \kappa_{33}q_{33})},$$

$$S_{5311} = S_{5322} = \frac{a_{33}(C_{33}e_{31} - C_{13}e_{33}) - e_{33}(e_{33}q_{31} - e_{31}q_{33}) - \kappa_{33}(C_{33}q_{31} - C_{13}q_{33})}{\mu_{33}(e_{33}^2 + C_{33}\kappa_{33}) - a_{33}(a_{33}C_{33} + e_{33}q_{33}) - q_{33}(a_{33}e_{33} - \kappa_{33}q_{33})},$$

$$S_{3333} = S_{4343} = S_{5353} = 1.$$

Note that for both elliptical cylinder and thin-disc inclusion shapes, there are no coupling terms for the Eshelby tensor between the electric and magnetic fields, i.e. S_{4151}, S_{4252}, S_{4353}, S_{5141}, S_{5242} and S_{5343} are identically zero. This is despite the fact that constitutive coupling exists, i.e. a_{11} and a_{33} are nonzero.

§ 5. Conclusion

We have used Eshelby's (1957) approach to analyse the fields in and around inclusions and inhomogeneities in anisotropic solids exhibiting full coupled-field behaviour. Explicit results are given for coupling between elastic, electric and magnetic fields. The key results of this study are the expressions for the generalized Eshelby tensors that can in general be evaluated by a simple numerical integration over the surface of a unit sphere. For certain inclusion shapes in transversely isotropic media, closed-form expressions were obtained for the Eshelby tensors. The explicit expressions for the generalized Eshelby tensors can be used in the same manner as Eshelby's tensor for elastic inclusions to solve a wide range of problems in the mechanics and physics of heterogeneous media. These include the study of internal fields, effective moduli of heterogeneous media, cracks and phase transformations. Furthermore, the results presented here can serve as the backbone for micromechanically based nonlinear constitutive modelling of ferroelectric and ferromagnetic media.

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