UNIQUENESS AND RECIPROCITY THEOREMS FOR LINEAR THERMO-ELECTRO-MAGNETO-ELASTICITY

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[Received 21 September 2001. Revise 15 January 2002]

Summary

We present uniqueness and reciprocity theorems for thermo-electro-magneto-elasticity without making restrictions on the positive definiteness of the elastic moduli. The application of the reciprocity theorem is also discussed.

1. Introduction

With the development of active material systems, there is significant interest in the coupling effects between elastic, electric, magnetic and thermal fields, for their applications in sensing and actuation. For example, piezoelectric materials (electric–elastic coupling) have been used as ultrasonic transducers and micro-actuators; pyroelectric materials (thermal–electric coupling) have been applied in thermal-imaging devices; and piezomagnetic materials (elastic–magnetic coupling) are pursued for health monitoring of civil structures. Among these coupling effects, piezoelectricity is probably the most studied; the equations governing small vibration of piezoelectric crystals have been developed by Mindlin (**1**).

Although natural materials rarely show full coupling between elastic, electric, magnetic and thermal fields, some artificial materials do. Van Run *et al.* reported the fabrication of $\mathrm{BaTiO_3-CoFe_2O_4}$ composite which had the magnetoelectric effect not existing in either constituent (**2**). Bracke and Van Vliet reported a broad-band magnetoelectric transducer with a flat frequency response using a piezoelectric–piezomagnetic composite material (**3**). Since then, numerous researchers have studied this class of composites both theoretically and experimentally (**4** to **12**). Li and Dunn quantitatively explained the magnetoelectric coupling created through the interaction between piezoelectric and piezomagnetic phases; they further pointed out that the pyromagnetic effect would be created through the interaction between piezomagnetic effect and thermal expansion (**10**, **11**).

Motivated by these studies, we present a uniqueness theorem and a reciprocity theorem governing the coupled behaviour between elastic, electric, magnetic and thermal fields, generalizing the uniqueness theorem of Dhaliwal and Wang (**13**) and the reciprocity theorem of Nowacki (**14**) in piezoelectricity. Related work on thermoelasticity, electromagnetism and thermoelectromagnetism can be found in (**15** to **19**).

2. Preliminaries

We consider a thermo-electro-magneto-elastic medium occupying a finite domain V in three-dimensional Euclidean space bounded by piecewise smooth boundary ∂V at time $t = 0$, which exhibits full linear coupling between the elastic, electric, magnetic and thermal fields, with the constitutive equations given by

$$\begin{aligned} \sigma_{ij} &= C_{ijkl}\varepsilon_{kl} - e_{ijk}E_k - q_{ijk}H_k - \lambda_{ij}\theta, \\ D_i &= e_{ikl}\varepsilon_{kl} + \kappa_{ik}E_k + a_{ik}H_k + p_i\theta, \\ B_i &= q_{ikl}\varepsilon_{kl} + a_{ik}E_k + \mu_{ik}H_k + m_i\theta, \\ s &= \lambda_{kl}\varepsilon_{kl} + p_kE_k + m_kH_k + \gamma\theta, \end{aligned} \tag{1}$$

where σ_{ij}, D_i, B_i and s are stress tensor, electric displacement, magnetic intensity and entropy density, respectively; ε_{kl}, E_k, H_k and θ are strain tensor, electric field, magnetic field and temperature change with respect to a reference temperature, respectively. The C_{ijkl}, κ_{ik}, μ_{ik} and γ are constitutive moduli which directly connect like fields (for example, stress to strain). On the other hand e_{kij}, q_{kij}, a_{ik}, λ_{ij}, p_i and m_i are coupling coefficients connecting dissimilar fields (for example, stress to electric field). The constitutive parameters satisfy the following symmetry conditions (**20**):

$$\begin{aligned} &C_{ijkl} = C_{jikl} = C_{klij},\ e_{kij} = e_{kji} = e_{ijk}, \\ &q_{kij} = q_{kji} = q_{ijk},\ \lambda_{ij} = \lambda_{ji},\ \kappa_{ij} = \kappa_{ji},\ a_{ij} = a_{ji},\ \mu_{ij} = \mu_{ji}. \end{aligned} \tag{2}$$

We notice that other sets of constitutive equations can be obtained by choosing different sets of independent variables.

Under quasi-static conditions, where the rate of change of magnetic field is small (electric field is curl free) and there is no electric current (magnetic field is curl free), the stress, electric displacement, magnetic intensity and temperature change in the medium need to satisfy the following equations:

$$\sigma_{ij,j} + F_i = \rho\ddot{u}_i,\ D_{i,i} = \rho_e,\ B_{i,i} = 0,\ \eta_{ij}\theta_{,ij} = T_0\dot{s} - Q, \tag{3}$$

where F_i, ρ, ρ_e and Q are body force, mass density, electric charge density and heat source intensity, respectively; η_{ij} is the symmetric heat conduction coefficient, and T_0 is the reference temperature; the overhead dot is used to denote the time derivative. Meanwhile it is necessary that strain, electric field and magnetic field satisfy gradient equations, that is, they can be derived from the gradient of elastic displacement u_i, electric potential ϕ and magnetic potential ψ,

$$\varepsilon_{ij} = u_{(i,j)},\ E_k = -\phi_{,k},\ H_k = -\psi_{,k}, \tag{4}$$

where the subscript $_{,i}$ is used to denote the partial differentiation with respect to x_i and subscript $_{()}$ is used to indicate symmetrization operation.

To complete the problem the boundary condition and initial condition need to be specified. For convenience we introduce the following notation:

$$T_i = \sigma_{ij}n_j, \quad d = D_in_i, \quad b = B_in_i, \quad k = -\eta_{ij}\theta_{,i}n_j, \tag{5}$$

where n_i is the outward normal of the surface. We assume the medium is subjected to the following mixed boundary conditions:

$$\begin{aligned} u_i = u_i^0(\mathbf{x}, t) \text{ on } \partial V_1, &\qquad T_i = T_i^0(\mathbf{x}, t) \text{ on } \partial V_1^c, \\ \phi = \phi^0(\mathbf{x}, t) \text{ on } \partial V_2, &\qquad d = d^0(\mathbf{x}, t) \text{ on } \partial V_2^c, \\ \psi = \psi^0(\mathbf{x}, t) \text{ on } \partial V_3, &\qquad b = b^0(\mathbf{x}, t) \text{ on } \partial V_3^c, \\ \theta = \theta^0(\mathbf{x}, t) \text{ on } \partial V_4, &\qquad k = k^0(\mathbf{x}, t) \text{ on } \partial V_4^c, \end{aligned} \tag{6}$$

and homogeneous initial conditions

$$u_i(\mathbf{x},0)=0,\ \dot{u}_i(\mathbf{x},0)=0,\ \phi(\mathbf{x},0)=0,\ \psi(\mathbf{x},0)=0,\ \theta(\mathbf{x},0)=0, \tag{7}$$

where ∂V_i and ∂V_i^c are the partial surface of V and the corresponding complementary surface. All the causes producing the coupled fields start to act at the moment when $t=0^+$.

The problem is to determine the elastic displacement u_i, electric potential ϕ, magnetic potential ψ and temperature change θ of C^2 in the medium, governed by (1) to (5) and subjected to initial-boundary conditions (6) and (7). Other field variables can then be determined from the gradient equations and the constitutive equations. The problem can be cast equivalently into a variational framework where the minimization of potential functionals is concerned. In the following sections, we will establish uniqueness and reciprocity theorems for this coupled field problem, and discuss some of their applications.

3. A uniqueness theorem

In this section we present a uniqueness theorem. A lemma will be established first, and then used to prove the uniqueness theorem. The medium is heterogeneous with arbitrary anisotropy, but the functional dependence of field variables on the position $\mathbf{x}$ will be suppressed to simplify the notation, so that $u_i(\mathbf{x},t)=u_i(t)$, etc.

LEMMA 3.1. *Let $\{u_i,\phi,\psi,\theta\}$ in C^2 be the solution for the mixed initial-boundary value problem defined by* (1) to (7)*, with F_i, ρ_e and Q all being zero. Let*

$$L(a,b)=\int_{\partial V}[\dot{u}_i(a)\sigma_{ij}(b)n_j-\dot{D}_i(a)\phi(b)n_i-\dot{B}_i(a)\psi(b)n_i+T_0^{-1}\eta_{ij}\theta_{,j}(a)\theta(b)n_i]\,dS, \tag{8}$$

so that $L(a,a)$ represents power flow into the considered medium. Then

$$\begin{aligned}L(a,b)=\int_V[&\rho\dot{u}_i(a)\,\ddot{u}_i(b)+C_{ijkl}\dot{\varepsilon}_{ij}(a)\,\varepsilon_{kl}(b)+\kappa_{ik}\dot{E}_k(a)\,E_i(b)+\mu_{ik}\dot{H}_k(a)\,H_i(b)\\&+\gamma\dot{\theta}(a)\,\theta(b)+T_0^{-1}\eta_{ij}\theta_{,j}(a)\,\theta_{,i}(b)+a_{ik}\dot{H}_k(a)\,E_i(b)+a_{ik}\dot{E}_k(a)\,H_i(b)\\&+p_i\dot{\theta}(a)\,E_i(b)+p_k\dot{E}_k(a)\,\theta(b)+m_i\dot{\theta}(a)\,H_i(b)+m_k\dot{H}_k(a)\,\theta(b)]\,dV.\end{aligned} \tag{9}$$

Proof. Applying the divergence theorem to (8), we obtain

$$\begin{aligned}L(a,b)=\int_V[&\sigma_{ij,j}(b)\,\dot{u}_i(a)+\dot{u}_{i,j}(a)\,\sigma_{ij}(b)-\dot{D}_{i,i}(a)\,\phi(b)-\dot{D}_i(a)\,\phi_{,i}(b)\\&-\dot{B}_{i,i}(a)\,\psi(b)-\dot{B}_i(a)\,\psi_{,i}(b)+T_0^{-1}\eta_{ij}\theta_{,ij}(a)\,\theta(b)+T_0^{-1}\eta_{ij}\theta_{,j}(a)\,\theta_{,i}(b)]\,dV.\end{aligned}$$

Taking into account the equilibrium equations (3) and gradient equations (4), we obtain

$$\begin{aligned}L(a,b)=\int_V[&\rho\ddot{u}_i(b)\,\dot{u}_i(a)+\sigma_{ij}(b)\,\dot{\varepsilon}_{ij}(a)+\dot{D}_i(a)\,E_i(b)\\&+\dot{B}_i(a)\,H_i(b)+\dot{s}(a)\,\theta(b)+T_0^{-1}\eta_{ij}\theta_{,j}(a)\,\theta_{,i}(b)]\,dV.\end{aligned}$$

Finally making use of the constitutive equations (1) and symmetry properties (2), we obtain (9), the desired result. This completes the proof.

THEOREM 3.1. *If ρ, γ and T_0 are positive constants, and the* 3×3 *matrix* η_{ij} *and* 6×6 *matrix* $\Gamma = \begin{pmatrix} \boldsymbol{\kappa} & \mathbf{a} \\ \mathbf{a} & \boldsymbol{\mu} \end{pmatrix}$ *are positive definite, then the coupled initial-boundary-value problem defined by* (1) to (7) *has at most one solution* $\{u_i, \phi, \psi, \theta\}$.

Proof. It suffices to prove that for $F_i = 0$, $\rho_e = 0$, $Q = 0$ and homogeneous boundary conditions, the solution is trivial. From a straightforward calculation on (9), we have

$$\int_0^t [L(t+r, t-r) - L(t-r, t+r)]dr = \int_0^t \int_V \Phi \, dV \, dr, \tag{10}$$

where

$$\begin{aligned}\Phi = {} & \rho\dot{u}_i(t+r)\,\ddot{u}_i(t-r) - \rho\dot{u}_i(t-r)\,\ddot{u}_i(t+r) \\ & + C_{ijkl}\dot{\varepsilon}_{ij}(t+r)\,\varepsilon_{kl}(t-r) - C_{ijkl}\dot{\varepsilon}_{ij}(t-r)\,\varepsilon_{kl}(t+r) \\ & + \kappa_{ij}\dot{E}_j(t+r)\,E_i(t-r) - \kappa_{ij}\dot{E}_j(t-r)\,E_i(t+r) + \mu_{ij}\dot{H}_j(t+r)\,H_i(t-r) \\ & - \mu_{ij}\dot{H}_j(t-r)\,H_i(t+r) + \gamma\dot{\theta}(t+r)\,\theta(t-r) - \gamma\dot{\theta}(t-r)\,\theta(t+r) \\ & + a_{ik}\dot{H}_k(t+r)\,E_i(t-r) - a_{ik}\dot{H}_k(t-r)\,E_i(t+r) + a_{ik}\dot{E}_k(t+r)\,H_i(t-r) \\ & - a_{ik}\dot{E}_k(t-r)\,H_i(t+r) + p_i\dot{\theta}(t+r)\,E_i(t-r) - p_i\dot{\theta}(t-r)\,E_i(t+r) \\ & + p_i\dot{E}_i(t+r)\,\theta(t-r) - p_i\dot{E}_i(t-r)\,\theta(t+r) + m_i\dot{\theta}(t+r)\,H_i(t-r) \\ & - m_i\dot{\theta}(t-r)\,H_i(t+r) + m_i\dot{H}_i(t+r)\,\theta(t-r) - m_i\dot{H}_i(t-r)\,\theta(t+r).\end{aligned}$$

Integrating (10) by parts and taking into account the homogeneous initial conditions and symmetry properties, we obtain

$$\begin{aligned}\int_0^t [L(t+r, t-r) - L(t-r, t+r)]\,dr = {} & \int_V (\rho\dot{u}_i\dot{u}_i - C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - \kappa_{ij}E_iE_j - \mu_{ij}H_iH_j)\,dV \\ & - \int_V (\gamma\theta^2 + 2a_{ik}E_iH_k + 2p_i\theta E_i + 2m_i\theta H_i)\,dV.\end{aligned} \tag{11}$$

On the other hand, from (9) we have

$$\begin{aligned}\int_0^t 2L(r, r)\,dr = {} & \int_V (\rho\dot{u}_i\dot{u}_i + C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \kappa_{ij}E_iE_j + \mu_{ij}H_iH_j + \gamma\theta^2)\,dV \\ & + \int_V (2a_{ik}H_kE_i + 2p_iE_i\theta + 2m_iH_i\theta)\,dV + \int_0^t \int_V \frac{2}{T_0}\eta_{ij}\theta_{,i}\theta_{,j} dV\,dr.\end{aligned} \tag{12}$$

For homogeneous boundary conditions, it is clear from (8) that $L(a, b) = 0$, so that adding (11) and (12) yields

$$\int_V \rho\dot{u}_i\dot{u}_i\,dV + \int_0^t \int_V \frac{1}{T_0}\eta_{ij}\theta_{,i}\theta_{,j}\,dV\,dr = 0.$$

Because $\rho > 0$, $T_0 > 0$, and η_{ij} is positive definite, it is necessary that $\dot{u}_i = 0$ and $\theta_{,i} = 0$ for $\mathbf{x} \in V$ and $t > 0$. From the homogeneous boundary conditions and initial conditions, we then conclude that $u_i = 0$ and $\theta = 0$ for $\mathbf{x} \in V$ and $t > 0$, with which (12) becomes

$$\int_V (\kappa_{ij}E_iE_j + \mu_{ij}H_iH_j + 2a_{ik}H_kE_i)\,dV = 0.$$

From the positive definiteness of Γ, it is concluded that $E_i = 0$ and $H_i = 0$ for $\mathbf{x} \in V$ and $t > 0$. As a result for the homogeneous boundary condition with $F_i = 0$, $\rho_e = 0$ and $Q = 0$, the solution for the coupled initial-boundary-value problem is trivial; for the general boundary conditions, it is unique provided it exists. This completes the proof.

Finally, we notice that in proving this uniqueness theorem we do not require the elastic moduli to be positive definite. There is also no restriction on piezoelectric moduli, piezomagnetic moduli, and thermal coupling coefficients other than symmetry conditions.

4. A reciprocity theorem

In this section we present a reciprocity theorem for the coupled field problem, where two sets of causes and responses are considered. The proof of the theorem will be established first, and its application will then be discussed. In the following we use $*$ to indicate the time convolution: $f * h = \int_0^t f(\mathbf{x}, t-\tau)h(\mathbf{x},\tau)\,d\tau$.

THEOREM 4.1. *Let* $\{u_i, \phi, \psi, \theta\}$ *in* C^2 *be the solution to the coupled initial-boundary-value problem* (1) to (7) *caused by* $\{F_i, u_i^0, T_i^0, Q, \theta^0, k^0, \rho_e, \phi^0, d^0, \psi^0, b^0\}$*, and let* $\{u_i', \phi', \psi', \theta'\}$ *be the solution caused by* $\{F_i', u_i^{0'}, T_i^{0'}, Q', \theta^{0'}, k^{0'}, \rho_e', \phi^{0'}, d^{0'}, \psi^{0'}, b^{0'}\}$*. Then, we have*

$$\begin{aligned}\int_V \left(F_i * \frac{\partial u_i'}{\partial\tau} - F_i' * \frac{\partial u_i}{\partial\tau}\right) dV + \int_{\partial V}\left(T_i * \frac{\partial u_i'}{\partial\tau} - T_i' * \frac{\partial u_i}{\partial\tau}\right) dS + \int_{\partial V}(k*\theta' - k'*\theta)\,\frac{dS}{T_0}\\ - \frac{1}{T_0}\int_V (Q*\theta' - Q'*\theta)\,dV - \int_V\left(\rho_e * \frac{\partial\phi'}{\partial\tau} - \rho_e' * \frac{\partial\phi}{\partial\tau}\right) dV\\ + \int_{\partial V}\left(d * \frac{\partial\phi'}{\partial\tau} - d' * \frac{\partial\phi}{\partial\tau}\right) dS + \int_{\partial V}\left(b * \frac{\partial\psi'}{\partial\tau} - b' * \frac{\partial\psi}{\partial\tau}\right) dS = 0, \quad (13)\end{aligned}$$

where T, b, d *and* k *are specified by boundary conditions* (6) *on* ∂V_i^c *or determined from solution by* (5) *on* ∂V_i.

Proof. Applying the Laplace transform $\bar{f}(\mathbf{x}, g) = \int_0^\infty e^{-gt} f(\mathbf{x}, t)\,dt$ to $(1)_1$ and using the symmetry properties of C_{ijkl}, we obtain

$$(\bar\sigma_{ij} + \lambda_{ij}\bar\theta + e_{ijk}\bar E_k + q_{ijk}\bar H_k)\bar\varepsilon_{ij}' = (\bar\sigma_{ij}' + \lambda_{ij}\bar\theta' + e_{ijk}\bar E_k' + q_{ijk}\bar H_k')\bar\varepsilon_{ij}.$$

Integrating this equation yields

$$\begin{aligned}\int_V (\bar\sigma_{ij}\bar\varepsilon_{ij}' - \bar\sigma_{ij}'\bar\varepsilon_{ij})\,dV = -\int_V [\lambda_{ij}(\bar\theta\bar\varepsilon_{ij}' - \bar\theta'\bar\varepsilon_{ij})\\ + e_{ijk}(\bar E_k\bar\varepsilon_{ij}' - \bar E_k'\bar\varepsilon_{ij}) + q_{ijk}(\bar H_k\bar\varepsilon_{ij}' - \bar H_k'\bar\varepsilon_{ij})]\,dV.\end{aligned}$$

Using the divergence theorem and equilibrium equation (3), we obtain

$$\begin{aligned}\int_V (\bar\sigma_{ij}\bar\varepsilon_{ij}' - \bar\sigma_{ij}'\bar\varepsilon_{ij})\,dV &= \int_V (\bar\sigma_{ij}\bar u_{i,j}' - \bar\sigma_{ij}'\bar u_{i,j})\,dV\\ &= \int_V [(\bar\sigma_{ij}\bar u_i')_{,j} - (\bar\sigma_{ij}'\bar u_i)_{,j}]\,dV - \int_V (\bar\sigma_{ij,j}\bar u_i' - \bar\sigma_{ij,j}'\bar u_i)\,dV\\ &= \int_{\partial V}(\bar T_i\bar u_i' - \bar T_i'\bar u_i)\,dS + \int_V (\bar F_i\bar u_i' - \bar F_i'\bar u_i)\,dV.\end{aligned}$$

Finally we have

$$\int_{\partial V}(\bar{T}_i\bar{u}_i' - \bar{T}_i'\bar{u}_i)\,dS + \int_V(\bar{F}_i\bar{u}_i' - \bar{F}_i'\bar{u}_i)\,dV$$
$$= -\int_V[\lambda_{ij}(\bar{\theta}\bar{\varepsilon}_{ij}' - \bar{\theta}'\bar{\varepsilon}_{ij}) + e_{ijk}(\bar{E}_k\bar{\varepsilon}_{ij}' - \bar{E}_k'\bar{\varepsilon}_{ij}) + q_{ijk}(\bar{H}_k\bar{\varepsilon}_{ij}' - \bar{H}_k'\bar{\varepsilon}_{ij})]\,dV.$$

In a similar manner, we obtain

$$\frac{1}{gT_0}\int_{\partial V}(\bar{k}\bar{\theta}' - \bar{k}'\bar{\theta})\,dS - \frac{1}{gT_0}\int_V(\bar{Q}\bar{\theta}' - \bar{Q}'\bar{\theta})\,dV$$
$$= \int_V[\lambda_{ij}(\bar{u}_{i,j}'\bar{\theta} - \bar{u}_{i,j}\bar{\theta}') + p_i(\bar{\phi}_{,i}\bar{\theta}' - \bar{\phi}_{,i}'\bar{\theta}) + m_i(\bar{\psi}_{,i}\bar{\theta}' - \bar{\psi}_{,i}'\bar{\theta})]\,dV,$$

$$\int_{\partial V}(\bar{d}\bar{\phi}' - \bar{d}'\bar{\phi})\,dS - \int_V(\bar{\rho}_e\bar{\phi}' - \bar{\rho}_e'\bar{\phi})\,dV$$
$$= \int_V[e_{ikl}(\bar{u}_{k,l}\bar{\phi}_{,i}' - \bar{u}_{k,l}'\bar{\phi}_{,i}) - a_{ik}(\bar{\psi}_{,k}\bar{\phi}_{,i}' - \bar{\psi}_{,k}'\bar{\phi}_{,i}) + p_i(\bar{\theta}\bar{\phi}_{,i}' - \bar{\theta}'\bar{\phi}_{,i})]\,dV,$$

$$\int_{\partial V}(\bar{b}\bar{\psi}' - \bar{b}'\bar{\psi})\,dS = \int_V[q_{ikl}(\bar{u}_{k,l}\bar{\psi}_{,i}' - \bar{u}_{k,l}'\bar{\psi}_{,i}) - a_{ik}(\bar{\phi}_{,k}\bar{\psi}_{,i}' - \bar{\phi}_{,k}'\bar{\psi}_{,i}) + m_i(\bar{\theta}\bar{\psi}_{,i}' - \bar{\theta}'\bar{\psi}_{,i})]\,dV.$$

Combining the last four equations yields

$$\int_{\partial V}(\bar{T}_i\bar{u}_i' - \bar{T}_i'\bar{u}_i)\,dS + \int_V(\bar{F}_i\bar{u}_i' - \bar{F}_i'\bar{u}_i)\,dV + \frac{1}{gT_0}\int_{\partial V}(\bar{k}\bar{\theta}' - \bar{k}'\bar{\theta})\,dS$$
$$- \frac{1}{gT_0}\int_V(\bar{Q}\bar{\theta}' - \bar{Q}'\bar{\theta})\,dV + \int_{\partial V}(\bar{d}\bar{\phi}' - \bar{d}'\bar{\phi})\,dS$$
$$- \int_V(\bar{\rho}_e\bar{\phi}' - \bar{\rho}_e'\bar{\phi})\,dV + \int_{\partial V}(\bar{b}\bar{\psi}' - \bar{b}'\bar{\psi})\,dS = 0, \tag{14}$$

the reciprocity theorem in the transformed domain. Inverting the Laplace transform yields the reciprocity theorem in the desired form. This completes the proof.

This is the reciprocity theorem of time-convolution type. Notice that we assume the existence of the Laplace transform of the involved functions.

COROLLARY 4.1. *With homogeneous boundary conditions, the two sets of causes and responses in a coupled medium governed by* (1) to (7) *satisfy*

$$\int_V\left[\left(F_i * \frac{\partial u_i'}{\partial\tau} - F_i' * \frac{\partial u_i}{\partial\tau}\right) - (Q * \theta' - Q' * \theta) - \left(\rho_e * \frac{\partial\phi'}{\partial\tau} - \rho_e' * \frac{\partial\phi}{\partial\tau}\right)\right]dV = 0. \tag{15}$$

This result follows easily from Theorem 4.1 since the surface integrals vanish for homogeneous boundary conditions. One particular example is the infinite medium occupying $\mathbb{R}^3$.

Next we demonstrate some applications of the reciprocity theorem. To this end, let us introduce the Green's functions for the coupled medium (**9**, **11**, **21**)

$$G_{IJ}(\mathbf{x}, \boldsymbol{\xi}, t) = \begin{cases} u_i, & I = 1, 2, 3, \\ \phi, & I = 4, \\ \psi, & I = 5, \\ \theta, & I = 6, \end{cases} \tag{16}$$

which represents the Ith component of the response of the medium at $\mathbf{x}$ due to the Jth component of an instantaneous concentrated stimulus S_J applied at $\boldsymbol{\xi}$,

$$S_J(\mathbf{x}, t) = \begin{cases} F_j(\mathbf{x}, t) = \delta(\mathbf{x} - \boldsymbol{\xi})\delta(t)\delta_{ji}, & J = 1, 2, 3, \\ \rho_e(\mathbf{x}, t) = \delta(\mathbf{x} - \boldsymbol{\xi})\delta(t), & J = 4, \\ \rho_m(\mathbf{x}, t) = \delta(\mathbf{x} - \boldsymbol{\xi})\delta(t), & J = 5, \\ Q(\mathbf{x}, t) = \delta(\mathbf{x} - \boldsymbol{\xi})\delta(t), & J = 6. \end{cases} \tag{17}$$

This is a generalization of the shorthand notation introduced by Barnett and Lothe (**22**). Both uppercase and lowercase subscripts are used, where the lowercase subscript ranges from 1 to 3 and the uppercase subscript ranges from 1 to 6. Repeated subscripts are summed from 1 to 3 or 1 to 6, depending on whether it is lowercase or uppercase. We also note that magnetic monopole or charge ρ_m is introduced for mathematical convenience.

Now let us assume an instantaneous concentrated force applied at point $\boldsymbol{\xi}$ of the medium, directed in the x_j direction, $F_i' = \delta(\mathbf{x} - \boldsymbol{\xi})\delta(t)\delta_{ij}$, for the primed cause. This force will produce in the medium a displacement $u_i' = G_{ij}(\mathbf{x}, \boldsymbol{\xi}, t)$, an electric potential $\phi' = G_{4j}(\mathbf{x}, \boldsymbol{\xi}, t)$, a magnetic potential $\psi' = G_{5j}(\mathbf{x}, \boldsymbol{\xi}, t)$, and a temperature change $\theta' = G_{6j}(\mathbf{x}, \boldsymbol{\xi}, t)$. Substituting these into (14), we obtain

$$\begin{aligned} \bar{u}_j(\boldsymbol{\xi}, g) = & \int_V \bar{F}_i(\mathbf{x}, g)\, \bar{G}_{ij}(\mathbf{x}, \boldsymbol{\xi}, g)\, dV(\mathbf{x}) - \frac{1}{gT_0} \int_V \bar{Q}(\mathbf{x}, g)\, G_{6j}(\mathbf{x}, \boldsymbol{\xi}, g)\, dV(\mathbf{x}) \\ & + \int_{\partial V_1^c} \bar{T}_i^0(\mathbf{x}, g)\, \bar{G}_{ij}(\mathbf{x}, \boldsymbol{\xi}, g)\, dS(\mathbf{x}) - \int_{\partial V_1} \bar{T}_i^j(\mathbf{x}, \boldsymbol{\xi}, g)\, \bar{u}_i^0(\mathbf{x}, g)\, dS(\mathbf{x}) \\ & + \frac{1}{gT_0} \int_{\partial V_4^c} \bar{k}^0(\mathbf{x}, g)\, \bar{G}_{6j}(\mathbf{x}, \boldsymbol{\xi}, g)\, dS(\mathbf{x}) - \frac{1}{gT_0} \int_{\partial V_4} \bar{k}^j(\mathbf{x}, \boldsymbol{\xi}, g)\, \bar{\theta}^0(\mathbf{x}, g)\, dS(\mathbf{x}) \\ & - \int_V \bar{\rho}_e(\mathbf{x}, g)\, G_{4j}(\mathbf{x}, \boldsymbol{\xi}, g)\, dV(\mathbf{x}) \\ & + \int_{\partial V_2^c} \bar{d}^0(\mathbf{x}, g)\, G_{4j}(\mathbf{x}, \boldsymbol{\xi}, g)\, dS(\mathbf{x}) - \int_{\partial V_2} \bar{d}^j(\mathbf{x}, \boldsymbol{\xi}, g)\, \bar{\phi}^0(\mathbf{x}, g)\, dS(\mathbf{x}) \\ & + \int_{\partial V_3^c} \bar{b}^0(\mathbf{x}, g)\, G_{5j}(\mathbf{x}, \boldsymbol{\xi}, g)\, dS(\mathbf{x}) - \int_{\partial V_3} \bar{b}^j(\mathbf{x}, \boldsymbol{\xi}, g)\, \bar{\psi}^0(\mathbf{x}, g) dS\,(\mathbf{x}). \end{aligned} \tag{18}$$

In this equation, $T_i^j = \sigma_{ik}^j n_k$, $d^j = D_i^j n_i$, $b^j = B_i^j n_i$, and $k^j = -\eta_{ik} G_{6j,i} n_k$, while σ_{ik}^j, D_i^j, and B_i^j are obtained from the Green's function $G_{Ij}(\mathbf{x}, \boldsymbol{\xi}, t)$ using the constitutive equations. Equation (18) enables us to determine the displacement at $\boldsymbol{\xi}$ with a known distribution of body force, heat source, electric charge, and the prescribed boundary values.

Let us then assume that the primed set is a product of the action of an instantaneous concentrated source of heat in a medium, $Q' = \delta(\mathbf{x}-\boldsymbol{\xi})\delta(t)$, which produces $u_i' = G_{i6}(\mathbf{x}, \boldsymbol{\xi}, t)$, $\phi' = G_{46}(\mathbf{x}, \boldsymbol{\xi}, t)$,

$\psi' = G_{56}(\mathbf{x}, \boldsymbol{\xi}, t)$ and $\theta' = G_{66}(\mathbf{x}, \boldsymbol{\xi}, t)$. Substituting into (14), we obtain

$$\begin{aligned}
-\frac{\bar{\theta}(\boldsymbol{\xi}, g)}{gT_0} &= \int_V \bar{F}_i(\mathbf{x}, g)\, \bar{G}_{i6}(\mathbf{x}, \boldsymbol{\xi}, g)\, dV(\mathbf{x}) - \frac{1}{gT_0}\int_V \bar{Q}(\mathbf{x}, g)\, \bar{G}_{66}(\mathbf{x}, \boldsymbol{\xi}, g)\, dV(\mathbf{x}) \\
&\quad + \int_{\partial V_1^c} \bar{T}_i^0(\mathbf{x}, g)\, \bar{G}_{i6}(\mathbf{x}, \boldsymbol{\xi}, g)\, dS(\mathbf{x}) - \int_{\partial V_1} \bar{T}_i^Q(\mathbf{x}, \boldsymbol{\xi}, g)\, \bar{u}_i^0(\mathbf{x}, g)\, dS(\mathbf{x}) \\
&\quad + \frac{1}{gT_0}\int_{\partial V_4^c} \bar{k}^0(\mathbf{x}, g)\, \bar{G}_{66}(\mathbf{x}, \boldsymbol{\xi}, g)\, dS(\mathbf{x}) - \frac{1}{gT_0}\int_{\partial V_4} \bar{k}^Q(\mathbf{x}, \boldsymbol{\xi}, g)\, \bar{\theta}^0(\mathbf{x}, g)\, dS(\mathbf{x}) \\
&\quad - \int_V \bar{\rho}_e(\mathbf{x}, g)\, \bar{G}_{46}(\mathbf{x}, \boldsymbol{\xi}, g)\, dV(\mathbf{x}) \\
&\quad + \int_{\partial V_2^c} \bar{d}^0(\mathbf{x}, g)\, \bar{G}_{46}(\mathbf{x}, \boldsymbol{\xi}, g)\, dS(\mathbf{x}) - \int_{\partial V_2} \bar{d}^Q(\mathbf{x}, \boldsymbol{\xi}, g)\, \bar{\phi}^0(\mathbf{x}, g)\, dS(\mathbf{x}) \\
&\quad + \int_{\partial V_3^c} \bar{b}^0(\mathbf{x}, g)\, \bar{G}_{56}(\mathbf{x}, \boldsymbol{\xi}, g)\, dS(\mathbf{x}) - \int_{\partial V_3} \bar{b}^Q(\mathbf{x}, \boldsymbol{\xi}, g)\, \bar{\psi}^0(\mathbf{x}, g)\, dS(\mathbf{x}). \qquad (19)
\end{aligned}$$

The quantities with superscript Q are obtained from G_{IJ} as with (18). Equation (19) enables us to determine the temperature inside the body for a known distribution of the body forces, heat sources and electric charge, and prescribed boundary values.

Finally, let us assume that the primed set is the result of action of an instantaneous concentrated electric charge at the point $\boldsymbol{\xi}$ of the medium, $\rho_e' = \delta(\mathbf{x} - \boldsymbol{\xi})\delta(t)$, which produces the displacement $G_{i4}(\mathbf{x}, \boldsymbol{\xi}, t)$, electric potential $G_{44}(\mathbf{x}, \boldsymbol{\xi}, t)$, magnetic potential $G_{54}(\mathbf{x}, \boldsymbol{\xi}, t)$, and temperature $G_{64}(\mathbf{x}, \boldsymbol{\xi}, t)$. Substituting them into (14), we obtain

$$\begin{aligned}
-\bar{\phi}(\boldsymbol{\xi}, g) &= \int_{\partial V_1^c} \bar{T}_i^0(\mathbf{x}, g)\, \bar{G}_{i4}(\mathbf{x}, \boldsymbol{\xi}, g)\, dS(\mathbf{x}) - \int_{\partial V_1} \bar{T}_i^e(\mathbf{x}, \boldsymbol{\xi}, g)\, \bar{u}_i^0(\mathbf{x}, g)\, dS(\mathbf{x}) \\
&\quad + \int_V \bar{F}_i(\mathbf{x}, g)\, \bar{G}_{i4}(\mathbf{x}, \boldsymbol{\xi}, g)\, dV(\mathbf{x}) \\
&\quad - \frac{1}{gT_0}\int_V \bar{Q}(\mathbf{x}, g)\, \bar{G}_{64}(\mathbf{x}, \boldsymbol{\xi}, g)\, dV(\mathbf{x}) - \int_V \bar{\rho}_e(\mathbf{x}, g)\, \bar{G}_{44}(\mathbf{x}, \boldsymbol{\xi}, g)\, dV(\mathbf{x}) \\
&\quad + \frac{1}{gT_0}\int_{\partial V_4^c} \bar{k}^0(\mathbf{x}, g)\, \bar{G}_{64}(\mathbf{x}, \boldsymbol{\xi}, g)\, dS(\mathbf{x}) - \frac{1}{gT_0}\int_{\partial V_4} \bar{k}^e(\mathbf{x}, \boldsymbol{\xi}, g)\, \bar{\theta}^0(\mathbf{x}, g)\, dS(\mathbf{x}) \\
&\quad + \int_{\partial V_2^c} \bar{d}^0(\mathbf{x}, g)\, \bar{G}_{44}(\mathbf{x}, \boldsymbol{\xi}, g)\, dS(\mathbf{x}) - \int_{\partial V_2} \bar{d}^e(\mathbf{x}, \boldsymbol{\xi}, g)\, \bar{\phi}^0(\mathbf{x}, g)\, dS(\mathbf{x}) \\
&\quad + \int_{\partial V_3^c} \bar{b}^0(\mathbf{x}, g)\, \bar{G}_{54}(\mathbf{x}, \boldsymbol{\xi}, g)\, dS(\mathbf{x}) - \int_{\partial V_3} \bar{b}^e(\mathbf{x}, \boldsymbol{\xi}, g)\, \bar{\psi}^0(\mathbf{x}, g)\, dS(\mathbf{x}). \qquad (20)
\end{aligned}$$

Again, the quantities with superscript e are obtained from G_{IJ} as with (18). Equation (20) allows us to determine the electric potential in the solid due to a known distribution of body force, heat source, and electric charge, and prescribed boundary values.

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