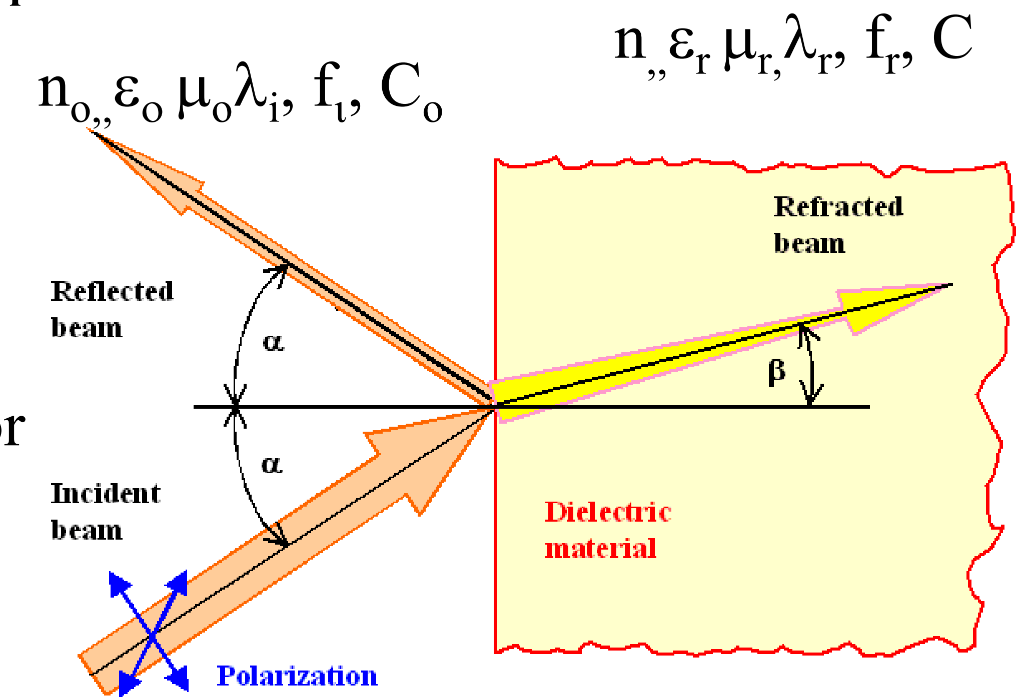
Electromagnetic Wave Theory

Wei-Chih Wang
ME557
Department of Mechanical Engineering
University of Washington

Refraction and reflection (wave equation)

The *incident beam* is characterized by its wavelength $\lambda_{\mathbf{i}}$, its frequency $\nu_{\mathbf{i}}$ and its velocity $\mathbf{c_0}$ **and refracted beam is** characterized by its wavelength $\lambda_{\mathbf{r}}$, its frequency $\nu_{\mathbf{r}}$ and its velocity **c** , the simple **dispersion relation** for vacuum.



$$C_o = f_i\ \ \lambda_i$$

$$C = f_r\ \ \lambda_r$$

The speed of light in a medium is related to the electric and magnetic properties of the medium, and the speed of light in vacuum can be expressed as

$$c_o = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

ε_0 = electric permittivity

μ_0 = magnetic permeability

The speed of light in a material to the material "constants" ε_r and the corresponding magnetic permeability μ_0 of vacuum and μ_r of the material is

$$c = \frac{1}{\sqrt{\mu_r \mu_o \varepsilon_r \varepsilon_o}}$$

The **index of refraction *n*** of a non-magnetic material $\mu_r = 1$ is linked to the dielectric constant $\boldsymbol{\varepsilon_r}$ via a simple relation, which is a rather direct result of the Maxwell equations.

$$\frac{c_o}{c} = \frac{1/\sqrt{\mu_o \varepsilon_o}}{1/\sqrt{\mu_r \mu_o \varepsilon_r \varepsilon_o}} = \sqrt{\varepsilon_r} = n$$

Plug back into dispersion relation,

$$\frac{c_o}{c} = \frac{\lambda_i f_i}{\lambda_r f_r} = n$$

Since $f_i = f_r,$

$$n = \frac{\lambda_i}{\lambda_r}$$

Maxwell Equations

Integral form in the absence of magnetic or polarized media:

I. Faraday's law of induction $\oint \vec{E} \cdot d\vec{l} = -\dfrac{d\Phi_B}{dt}$

II. Ampere's law $\oint \vec{B} \cdot d\vec{s} = \mu_0 i + \dfrac{1}{c^2}\dfrac{\partial}{\partial t}\int \vec{E} \cdot d\vec{A}$

III. Gauss' law for magnetism $\oint \vec{B} \cdot d\vec{A} = 0$

IV. Gauss' law for electricity $\oint \vec{E} \cdot d\vec{A} = \dfrac{q}{\varepsilon_0}$

E = Electric Field (V/m) | ρ *= charge density (c/m³)* | *i = electric current (A)*

B = Magnetic flux density(Web/m², T) | ε_0 *= permittivity* | *J = current density(A/m²)*

D = Electric flux density (c/m²) or electric displacement field | μ_0 *= permeability* | *c = speed of light*

H = Magnetic Field (A/m) | Φ_B *= Magnetic flux (Web)* | *P = Polarization*

q = charge, 1.6×10^{-19} coulombs, $\mu_o = 1.26\times10^{-6}$ *H/m,* $\varepsilon_o = 8.85\times10^{-12}$ *F/m*

Electric flux

For instance, Gauss's law states that the flux of the electric field out of a closed surface is proportional to the electric charge enclosed in the surface (regardless of how that charge is distributed). The constant of proportionality is the reciprocal of the permittivity of free space. Its integral form is:

$$\oint_A \epsilon_0 \mathbf{E} \cdot d\mathbf{A} = Q_A$$

The electric flux in an unclosed surface: $\phi_E = \int E \cdot dA$

Sometimes electric flux appears in terms of flux density D as: $\phi_E = \int D \cdot dA = \int \varepsilon E \cdot dA$

The electric elasticity equation

(Displacement field) $D = \varepsilon E$

Where E = electric field

ε = permittivity (dielectric constant)

in air $\varepsilon_o = 8.85x10^{-12}\ F/m$

Magnetic flux

We know from Gauss's law for magnetism that in a close surface,

$$\oint \vec{B} \cdot d\vec{A} = 0$$

Normally, the magnetic flux in an unclosed surface

$$\phi_B = \int B \cdot dA$$

Where B = magnetic flux density

But when the generated fields pass through magnetic materials which themselves contribute internal magnetic fields, ambiguities can arise about what part of the field comes from the external currents and what comes from the material itself. It has been common practice to define another magnetic field quantity, usually called the "magnetic field strength" designated by H. It can be defined by the relationship

$$B = \mu H + M$$

M = magnetization. Normally, the $M = 0$ for nonmagnetic material

If in air, $\mu_o = 1.26x10^{-6} H/m$

Faraday's Law of Induction

Integral Form

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

Differential Form

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

This line integral is equal to the generated voltage or emf in the loop, so Faraday's law is the basis for electric generators. It also forms the basis for inductors and transformers.

Ampere's Law

Integral form

$$\oint B \cdot ds = \mu_0 i + \frac{1}{c^2}\frac{\partial}{\partial t}\int E \cdot dA$$

Differential form

$$\nabla x B = \frac{4\pi k}{c^2} J + \frac{1}{c^2}\frac{\partial E}{\partial t}$$

$$\nabla x B = \frac{J}{\varepsilon_0 c^2} + \frac{1}{c^2}\frac{\partial E}{\partial t}$$

$$k = \frac{1}{4\pi\varepsilon_0} \qquad c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

In the case of static electric field, the line integral of the magnetic field around a closed loop is proportional to the electric current flowing through the loop. This is useful for the calculation of magnetic field for simple geometries.

Gauss’s Law for Magnetism

Integral Form

$$\oint \vec{B} \cdot d\vec{A} = 0$$

Differential Form

$$\nabla \cdot B = 0$$

The net magnetic flux out of any closed surface is zero. This amounts to a statement about the sources of magnetic field. For a magnetic dipole, any closed surface the magnetic flux directed inward toward the south pole will equal the flux outward from the north pole. The net flux will always be zero for dipole sources. If there were a magnetic monopole source, this would give a non-zero area integral. The divergence of a vector field is proportional to the point source density, so the form of Gauss' law for magnetic fields is then a statement that there are no magnetic monopoles.

Gauss’s Law for Electricity

Integral Form

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q}{\varepsilon_0}$$

Differential

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$$

<u>The electric flux out of any closed surface is proportional to the total charge enclosed within the surface.</u>

The integral form of Gauss' Law finds application in calculating electric fields around charged objects.

In applying Gauss' law to the electric field of a point charge, one can show that it is consistent with Coulomb's law.

While the area integral of the electric field gives a measure of the net charge enclosed, the divergence of the electric field gives a measure of the density of sources. It also has implications for the conservation of charge.

Maxwell Equations

Faraday's Law
$$\nabla \times E = \frac{-\partial B}{\partial t}$$

Ampere's Law
$$\nabla \times H = J + \frac{\partial D}{\partial t}$$

Gauss's Law for Magnetism
$$\nabla \bullet B = 0$$

Gauss's Law for Electricity
$$\nabla \bullet D = \rho$$

E = Electric Field (V/m)	ρ = *charge density* (c/m^3)	*i = electric current (A)*
B = Magnetic flux density(Web/m^2, *T*)	ε_0 = *permittivity*	*J = current density*(A/m^2)
D = Electric flux density (c/m^2)	μ_0 = *permeability*	*c = speed of light*
H = Magnetic Field (A/m)	Φ_B = *Magnetic flux (Web)*	*P = Polarization*

Wave equation

Maxwell's Equations contain the wave equation for electromagnetic waves. One approach to obtaining the wave equation:

1. Take the curl of Faraday's law:

$$\nabla \times (\nabla \times E) = -\frac{\partial(\nabla \times B)}{\partial t}$$

2. Substitute Ampere's law for a charge and current-free region:

$$\nabla \times (\nabla \times E) = -\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

This is the three-dimensional wave equation in vector form. It looks more familiar when reduced a plane wave with field in the x-direction only:

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

Curl

The curl of a vector function is the vector product of the del operator with a vector function:

$$\nabla \times E = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)i + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)k$$

where i,j,k are unit vectors in the x, y, z directions. It can also be expressed in determinant form:

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

Curl in Cylindrical Polar Coordinates

The curl in cylindrical polar coordinates, expressed in determinant form is:

$$\nabla \times E = \begin{vmatrix} \frac{1_r}{r} & 1_\theta & \frac{k}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ E_r & rE_\theta & E_z \end{vmatrix}$$

Curl in Spherical Polar Coordinates

The curl in spherical polar coordinates, expressed in determinant form is:

$$\nabla \times E = \begin{vmatrix} \dfrac{1_r}{r^2 \sin\theta} & \dfrac{1_\theta}{r \sin\theta} & \dfrac{1_\phi}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ E_r & rE_\theta & r \sin\theta \, E_\phi \end{vmatrix}$$

Use $\nabla \times (\nabla \times E) = \nabla(\nabla \bullet E) - \nabla^2 E$ Wave equation

becomes $\nabla^2 E + \omega^2 \mu_o \varepsilon_o E = 0$

We consider the simple solution where E field is parallel to the x axis and its function of z coordinate only, the wave equation then becomes,

$$\frac{\partial^2 E_x}{\partial z^2} + \omega^2 \mu_o \varepsilon_o E_x = 0$$

A solution to the above differential equation is

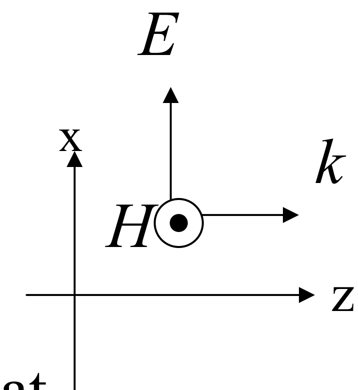
$$E = \hat{x} E_o e^{-jkz}$$

Substitute above equation into wave equation yields,

$$(-k^2 + \omega^2 \mu\varepsilon) E = 0 \Rightarrow \boxed{k^2 = \omega^2 \mu\varepsilon}$$ (dispersion relation)

Let's transform the solution for the wave equation into real space and time, (assume time harmonic field)

$$E(z,t) = \mathrm{Re}\{Ee^{j\omega t}\} = \hat{x}E_o \cos(\omega t - kz)$$



$k = 2\pi/\lambda$, where k = wave number

Image we riding along with the wave, we asked what Velocity shall we move in order to keep up with the wave, The answer is phase of the wave to be constant

$$\omega t - kz = \text{a constant}$$

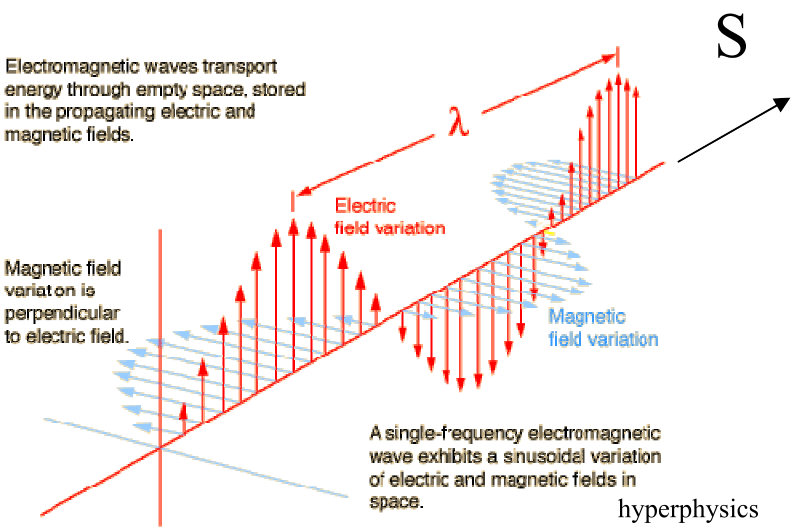
The velocity of propagation is therefore given by,

$$\frac{dz}{dt} = v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_o \varepsilon_o}} \qquad \text{(phase velocity)}$$

Poynting's Theorem

For a time –harmonic electromagnetic wave, the power density Per unit area associate with the wave is defined in complex Representation by vector S,

$$S = E \times H^* \ (\mathrm{W/m^2})$$



hyperphysics

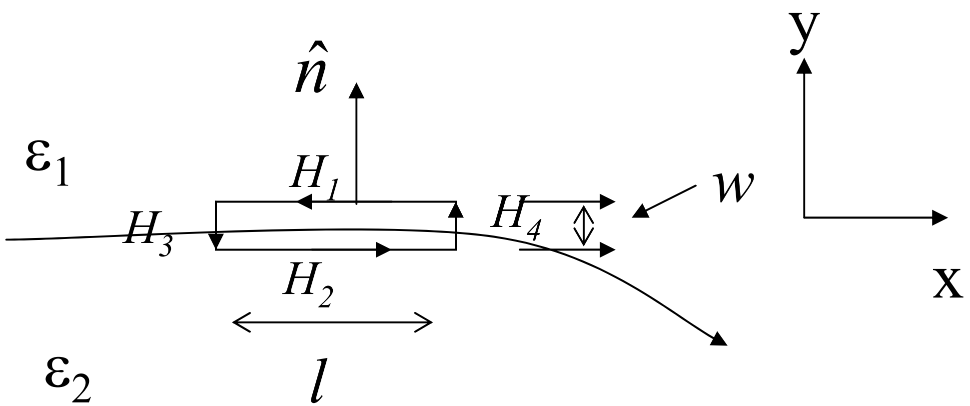
Time average pontying vector <s> is defined as average of the Time domain Poynting vector S over a period T=2π/ω.

$$< S >= \frac{1}{2\pi}\int_0^{2\pi} d(\omega t) E(x,y,z,t) \times H(x,y,z,t)$$

(or)

$$< S >= \frac{1}{2}\mathrm{Re}\{E \times H\}$$

Boundary Conditions



At an interface between two media, the file quantities must satisfy Certain conditions. Consider an interface between two dielectric media With dielectric constants ε_1 and ε_2, in the z component Ampere's Law, we have,

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z + jwD_z$$

or

$$\frac{H_4 - H_3}{l} - \frac{H_1 - H_2}{w} = J_z + jwD_z$$

Now let area shrink to a point where *w* goes to zero before *l* does. So $J_z = J_s \sim J_v w$, then

$$H_2 - H_1 = J_z$$

Or in general,

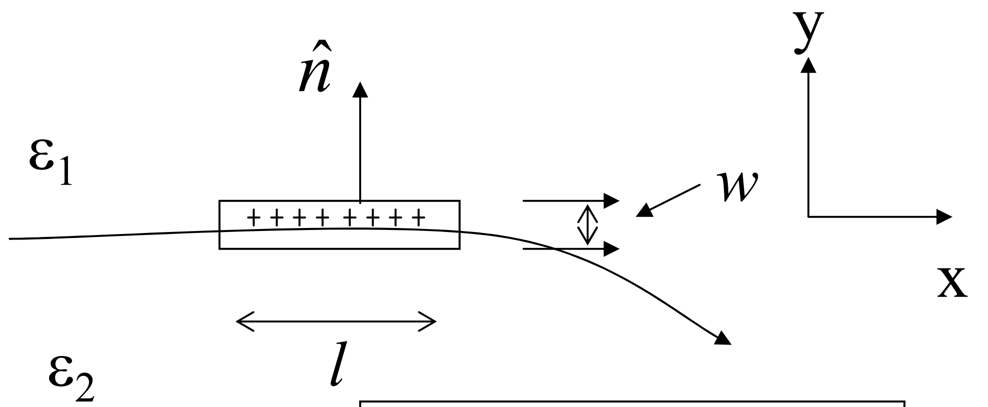
$$\boxed{\hat{n} \times (H_2 - H_1) = J_s}$$

Applying the same above argument to Faraday's Law and we get,

$$\boxed{\hat{n} \times (E_1 - E_2) = 0}$$

<u>The tangential electric field *E* is continuous across the boundary surface.The discontinuity in the tangential component of *H* is equal to the surface current density J_s.</u>

Apply the divergence theory $\nabla \bullet B = 0$ and $\nabla \bullet D = \rho$ for The pillbox volume shown



As w -> 0, we get

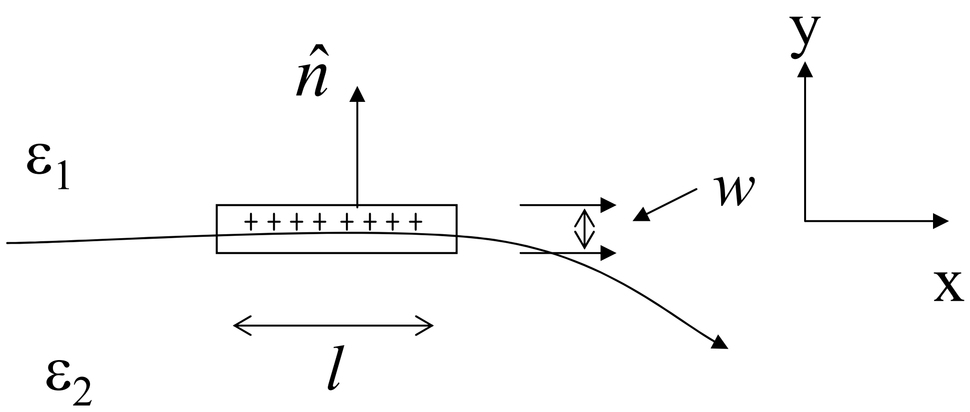
$$(B_1 - B_2) \bullet \hat{n} = 0$$

$$(D_1 - D_2) \bullet \hat{n} = \rho$$

The normal component of B is continuous across the boundary surface.The discontinuity in the normal component of D is equal to the surface charge density ρ

Boundary Condition for Perfect Conductor

On the surface of a perfect conductor, $E_2 = 0$ *and* $H_2 = 0$



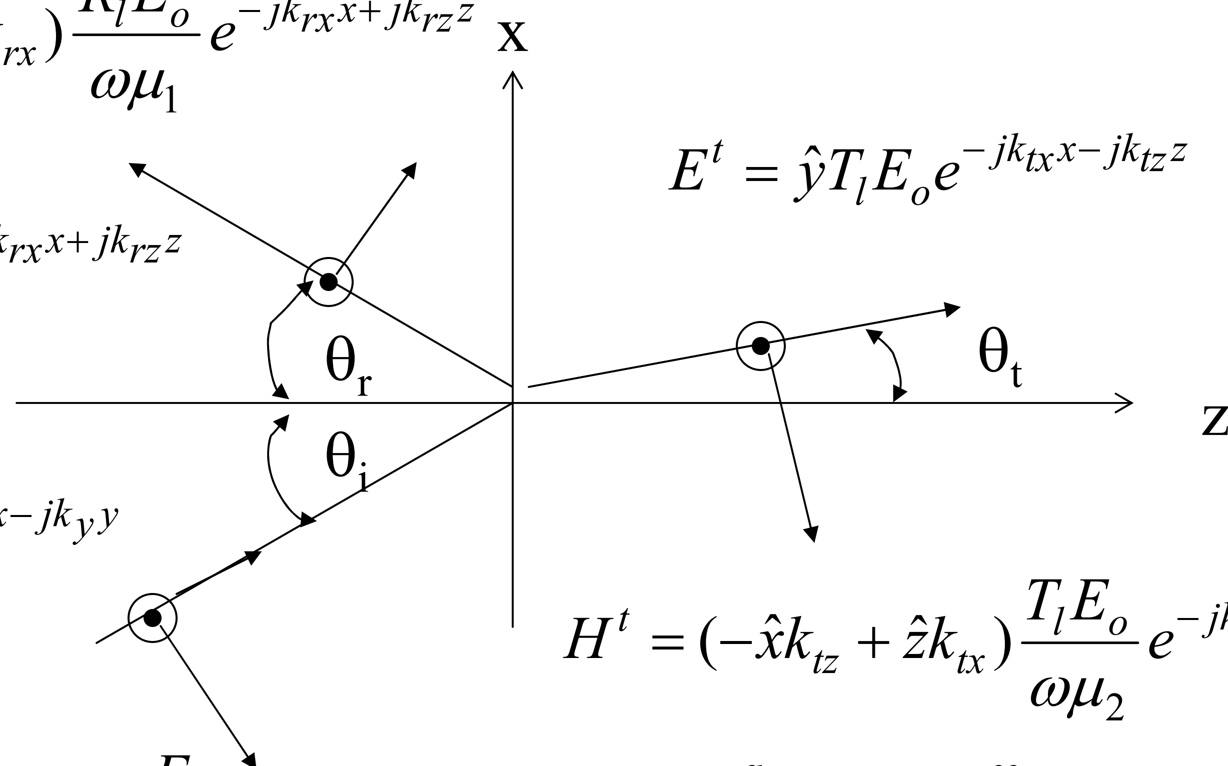
Reflection and Transmission (TE, S wave)

$\mu_1, \varepsilon_1, n_1$ $\qquad\qquad$ $\mu_2, \varepsilon_2, n_2$

$$H^r = (\hat{x}k_{rz} + \hat{z}k_{rx})\frac{R_l E_o}{\omega\mu_1} e^{-jk_{rx}x + jk_{rz}z}$$

$$E^t = \hat{y}T_l E_o e^{-jk_{tx}x - jk_{tz}z}$$

$$E^r = \hat{y}R_l E_o e^{-jk_{rx}x + jk_{rz}z}$$



$$E^i = \hat{y}E_o e^{-jk_x x - jk_y y}$$

$$H^t = (-\hat{x}k_{tz} + \hat{z}k_{tx})\frac{T_l E_o}{\omega\mu_2} e^{-jk_{tx}x - jk_{tz}z}$$

$$H^i = (-\hat{x}k_z + \hat{z}k_x)\frac{E_o}{\omega\mu_1} e^{-jk_x x - jk_z z}$$

R_l*=reflection coefficient*

T_l*=transmission coefficient*

TE = transverse electric, perpendicularly polarized (E perpendicular to plan of incident)

If neither two are perfect conductors, J_s=0, then boundary conditions requires both the tangential electric-filed and magnetic-field components be continuous at z =0 thus,

$$e^{-jk_x x} + R_l e^{-jk_{rx} x} = T_l e^{-jk_{tx} x} \qquad \text{(E component)}$$

$$\frac{-k_z}{\omega\mu_1} e^{-jk_x x} + \frac{k_{rz}}{\omega\mu_1} R_l e^{-jk_{rx} x} = \frac{-k_{tz}}{\omega\mu_2} T_l e^{-jk_{tx} x} \qquad \text{(B component)}$$

For the above equations to hold at all x, all components must be the same, thus we get the phase matching condition:

$$k_x = k\sin\theta_i = k_{rx} = k_r \sin\theta_r = k_{tx} = k_t \sin\theta_t$$

From this we obtain law of reflection:

$$\boxed{\theta_i = \theta_r}$$

Since $k = k_r$ *because* $k^2 = k_r^2 = \omega^2 \mu_1 \varepsilon_1 = k_1$

And Snell's Law:

$$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2}$$

$$\left\{ \begin{array}{l} n_1 = c\sqrt{\mu_1 \varepsilon_1} = \dfrac{c}{\omega} k_1 \\ n_2 = c\sqrt{\mu_2 \varepsilon_2} = \dfrac{c}{\omega} k_2 \end{array} \right.$$

Substitute solution for E^i, E^r, E^t, into wave equation

$$\nabla^2 E^i + \omega^2 \mu_1 \varepsilon_1 E^i = 0$$

$$\nabla^2 E_r + \omega^2 \mu_1 \varepsilon_1 E_r = 0$$

$$\nabla^2 E^t + \omega^2 \mu_2 \varepsilon_2 E^t = 0$$

We find,

$$k_x^2 + k_z^2 = k_1^2 = k_{rx}^2 + k_{rz}^2$$

$$k_{tx}^2 + k_{tz}^2 = k_2^2$$

Using Phase matching condition, we get,

$$1 + R_l = T_l$$

$$1 - R_l = \frac{\mu_1 k_{tz}}{\mu_2 k_z} T_l$$

$$\Longrightarrow$$

$$R_l = \frac{\mu_2 k_z - \mu_1 k_{tz}}{\mu_2 k_z + \mu_1 k_{tz}}$$

$$T_l = \frac{2\mu_2 k_z}{\mu_2 k_z + \mu_1 k_{tz}}$$

Reflection and Transmission (TM, P wave)

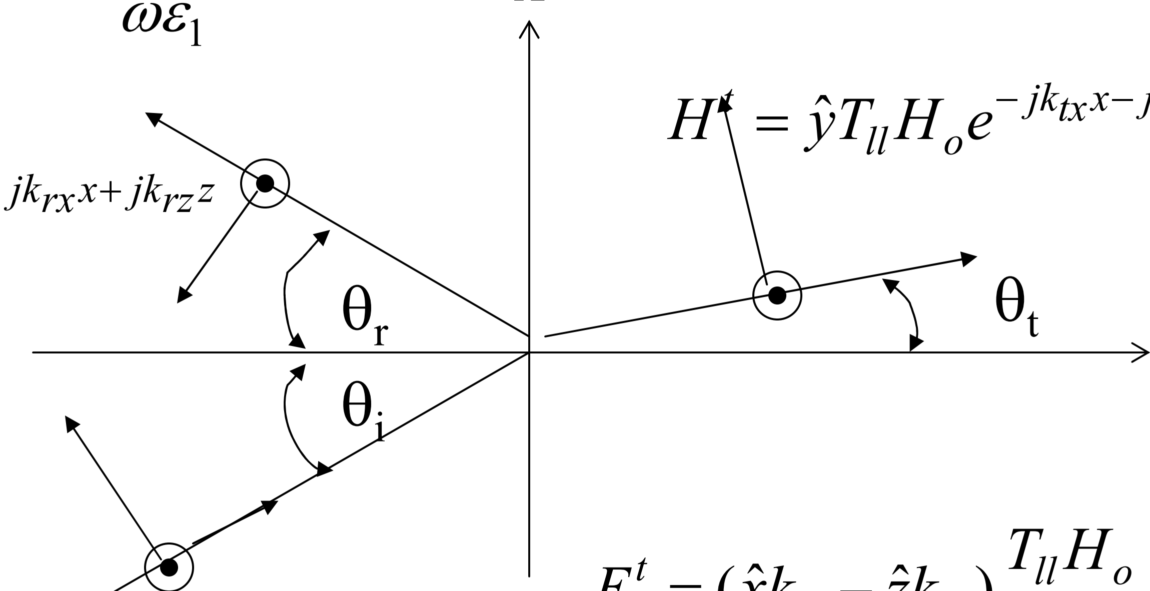
$\mu_1, \varepsilon_1, n_1$

$\mu_2, \varepsilon_2, n_2$

$$E^r = (-\hat{x}k_{rz} - \hat{z}k_{rx})\frac{R_{ll}H_o}{\omega\varepsilon_1}e^{-jk_{rx}x + jk_{rz}z}$$

$$H^r = \hat{y}R_{ll}H_o e^{-jk_{rx}x + jk_{rz}z}$$

$$H^t = \hat{y}T_{ll}H_o e^{-jk_{tx}x - jk_{tz}z}$$



$$E^t = (\hat{x}k_{tz} - \hat{z}k_{tx})\frac{T_{ll}H_o}{\omega\varepsilon_2}e^{-jk_{tx}x - jk_{tz}z}$$

$$E^i = (\hat{x}k_z - \hat{z}k_x)\frac{H_o}{\omega\varepsilon_1}e^{-jk_x x - jk_z z}$$

$$H^i = \hat{y}H_o e^{-jk_x x - jk_y y}$$

R_{ll}= reflection coefficient
T_{ll}= transmission coefficient

TM = transverse magnetic, parallel polarized (E parallel to plan of incident)

Substitute solution for E^i, E^r, E^t, into wave equation

We get,

$$1 + R_{ll} = T_{ll}$$

$$1 - R_{ll} = \frac{\varepsilon_1 k_{tz}}{\varepsilon_2 k_z} T_{ll}$$

$$\Rightarrow$$

$$R_{ll} = \frac{\varepsilon_2 k_z - \varepsilon_1 k_{tz}}{\varepsilon_2 k_z + \varepsilon_1 k_{tz}}$$

$$T_{ll} = \frac{2\varepsilon_2 k_z}{\varepsilon_2 k_z + \varepsilon_1 k_{tz}}$$

Critical Angle

In case of $n_1 > n_2$, when incident angle is greater than critical angle θ_c, k_x is larger than the magnitude of k_2

$$k_{tz}^2 = k_2^2 - k_x^2 < 0 \quad \Rightarrow \quad k_{tz}^2 = -j\alpha$$

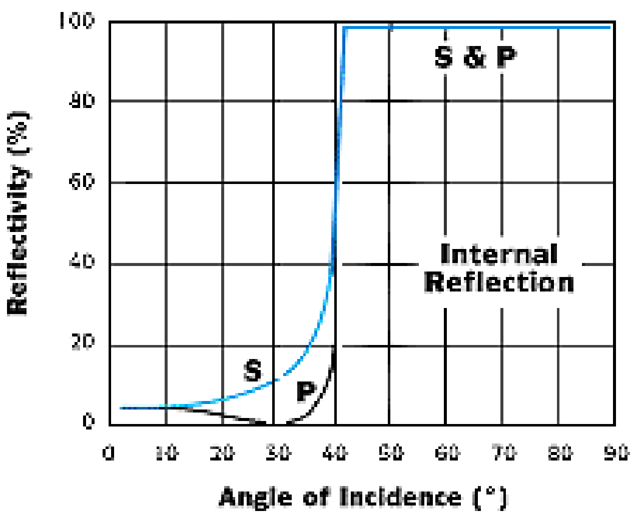
$$E^t = \hat{y} T_l E_o e^{-jk_{tx}x - j\alpha z} \Rightarrow E^t = \hat{y} T_l E_o e^{-j\alpha z} \cos(\omega t - k_x x)$$

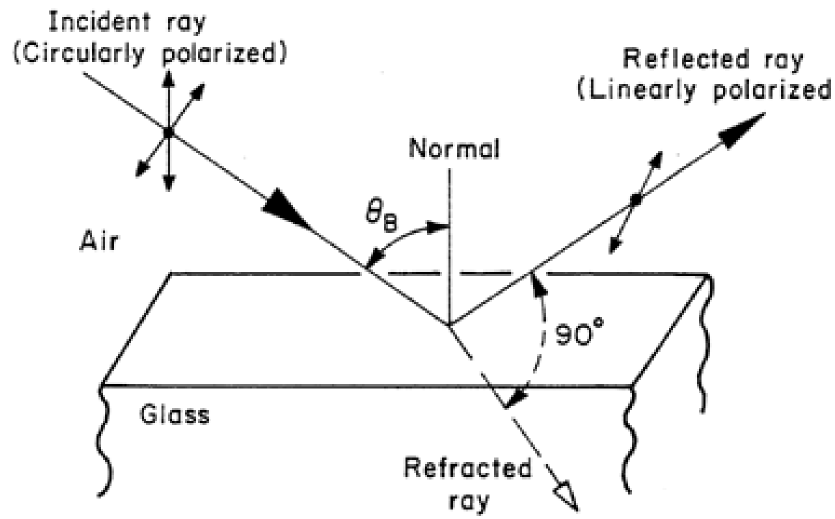
Because it decays away from interface and because the wave propagating along the interface, the wave is also called surface wave.

Critical angle is defined as

$$\theta_c = \theta_1 = \sin^{-1}\frac{k_2}{k_1}$$

$$\theta_c = \theta_1 = \sin^{-1}\frac{n_2}{n_1}$$





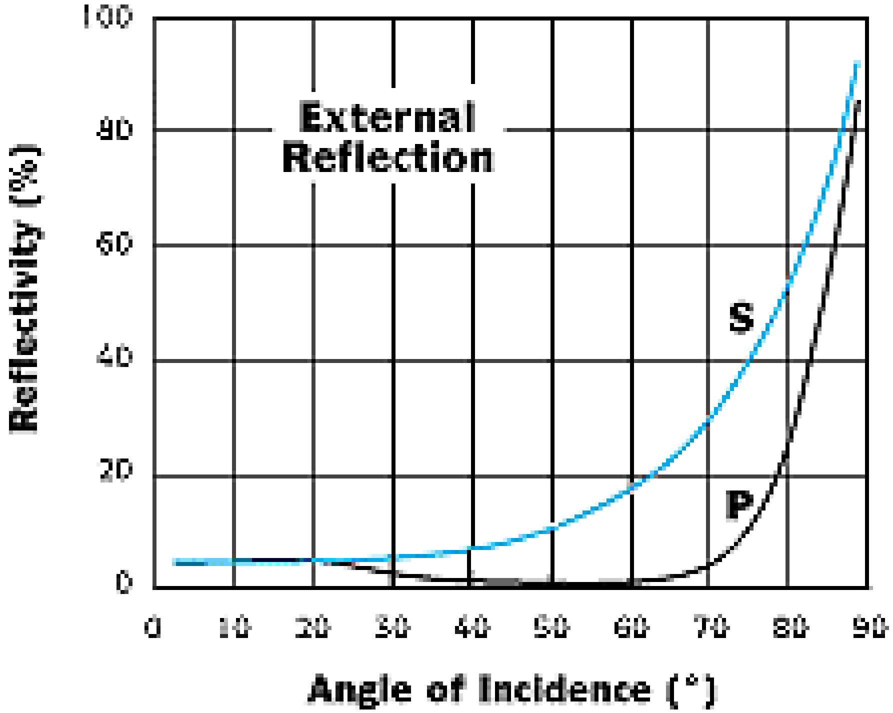
For a case where $\mu_1 = \mu_2$ and parallel polarization, there is always a angle θ_b such that wave is totally transmitted and the reflection coefficient is zero $R_{ll} = 0$, $\omega\sqrt{\mu_1\varepsilon_2}\cos\theta_1 = \omega\sqrt{\mu_1\varepsilon_1}\cos\theta_2$

Phase matching conduction gives $\omega\sqrt{\mu_1\varepsilon_1}\sin\theta_1 = \omega\sqrt{\mu_1\varepsilon_2}\sin\theta_2$

$$\theta_b = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$ (Brewster Angle)

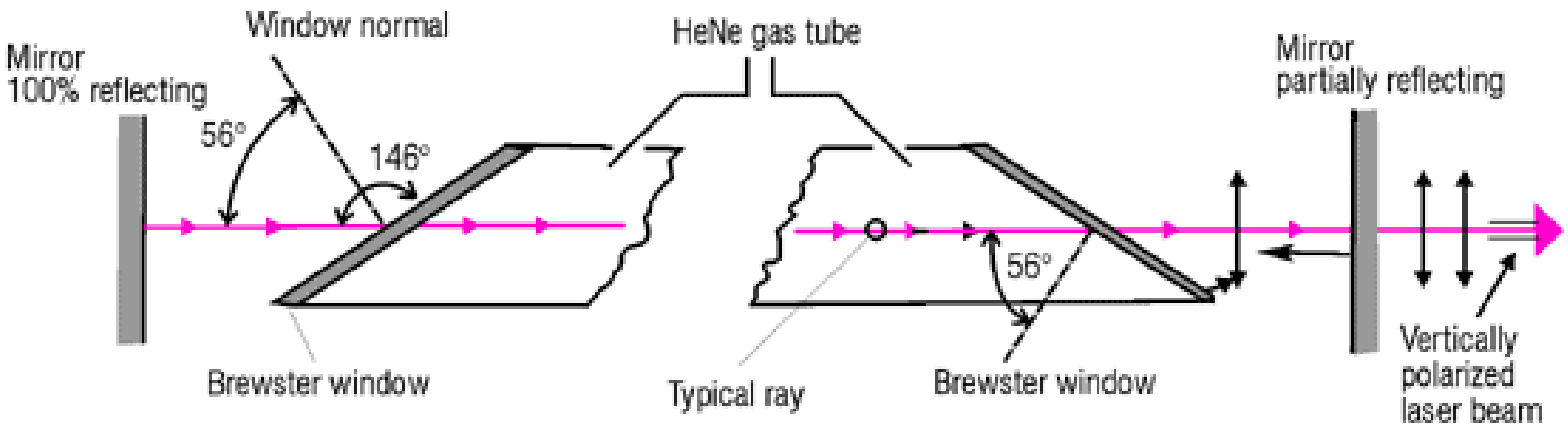
$$\theta_b = \tan^{-1}\sqrt{\frac{n_2}{n_1}}$$

Reflection of Unpolarized Light from Dielectrics

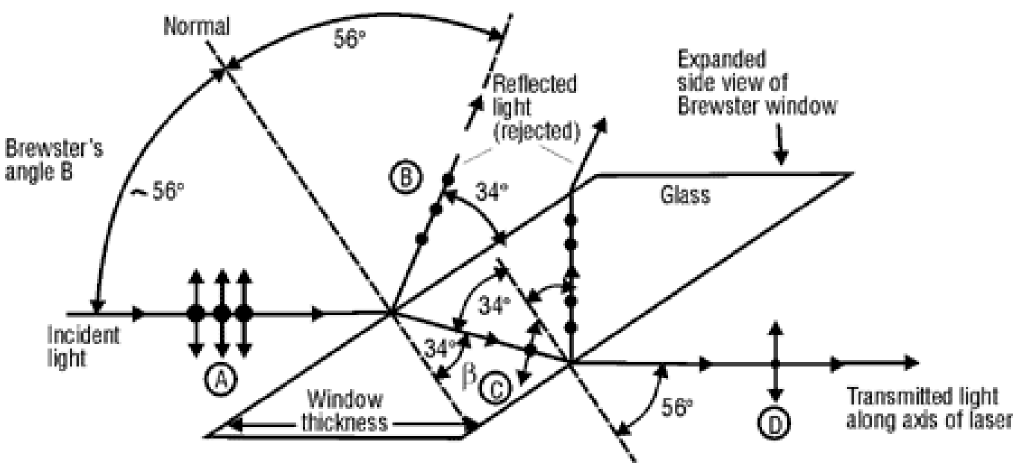


Polarizing unpolarized wave

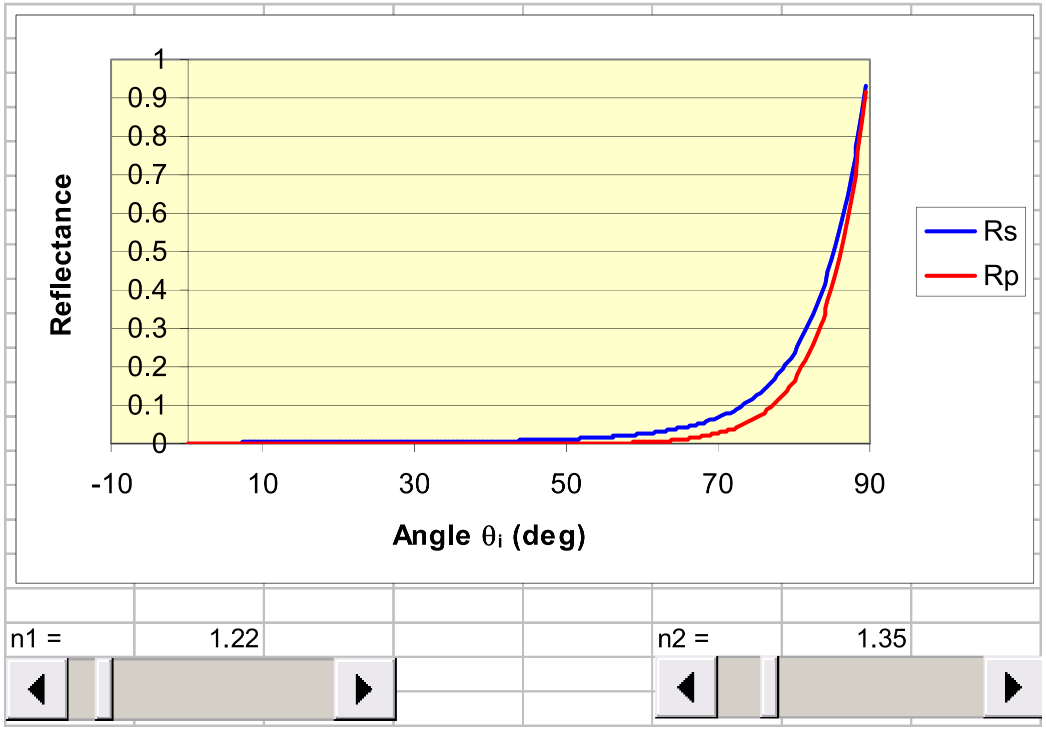
Brewster windows in a laser cavity



Brewster windows are used in laser cavities to ensure that the laser light after bouncing back and forth between the cavity mirrors emerges as linearly polarized light.

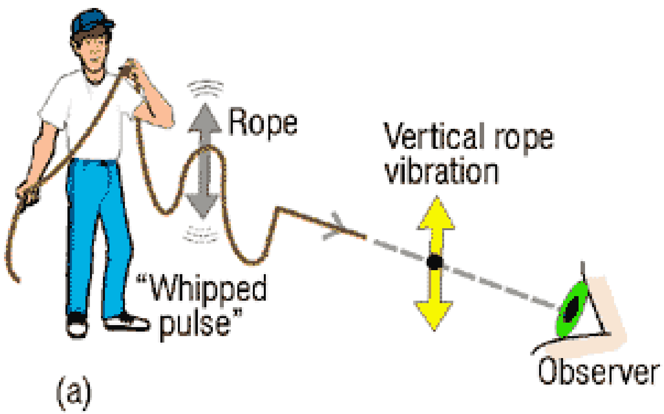


Unpolarized light passing through both faces at a Brewster angle

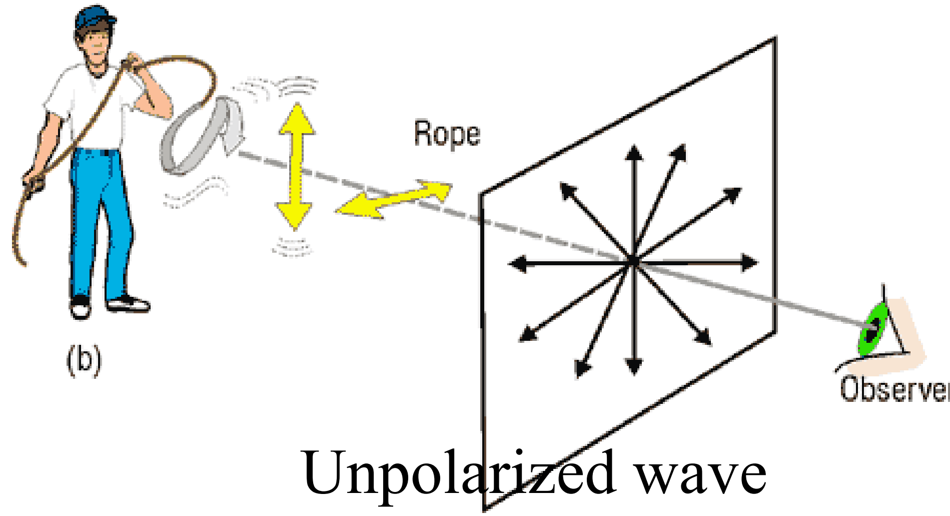


graph of the reflectance R for s- and p-polarized light as a function of n1, n2, and q1

Polarization

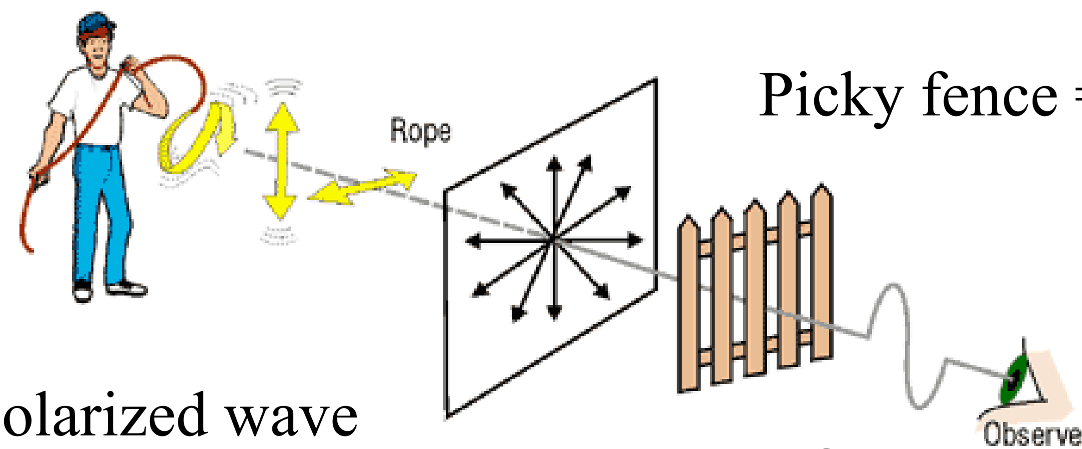


Polarized wave



Unpolarized wave

Imagine a "magic" rope that you can whip up and down at one end, thereby sending a *transverse* "whipped pulse" (vibration) out along the rope.



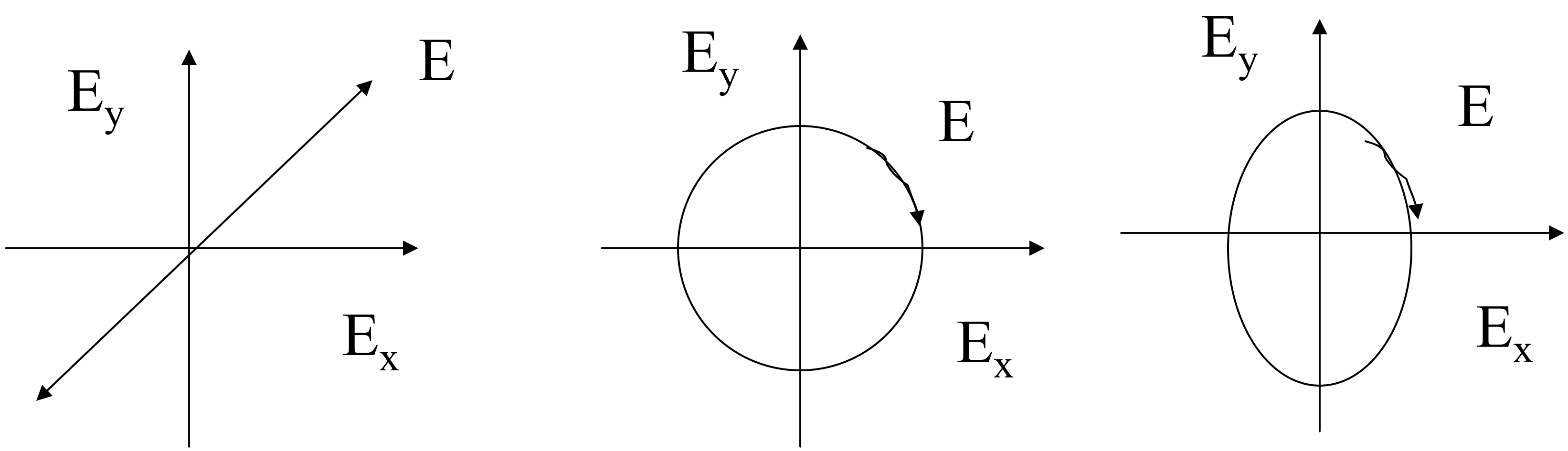
Picky fence = polarizer

Input put = Unpolarized wave

Output wave = polarized

Polarization

<u>A fixed point in space</u>, E vector of a time-harmonic electromagnetic wave varies sinusoidally with time. The polarization of the wave is described by the locus of the tip of the E vector as time progress. when the locus is a straight line, the wave is said to be linearly polarized. If the locus is a circle then the wave is said to be circularly polarized and if locus is elliptical then the wave is elliptically polarized.



Let's assume the real time-space E vector has x and y components:

$$E(z,t) = a\cos(\omega t - kz + \phi_a)\hat{x} + b\cos(\omega t - kz + \phi_b)\hat{y}$$

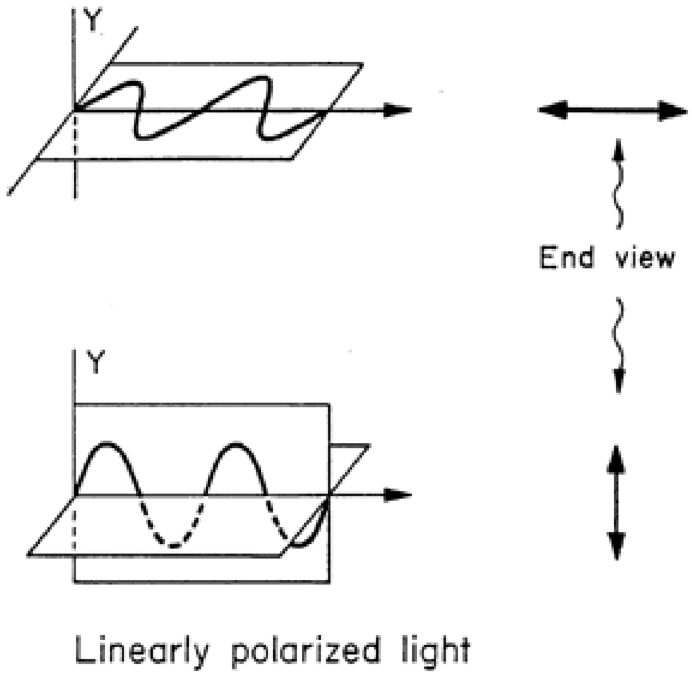
$E_y/E_x = Ae^{j\phi}$

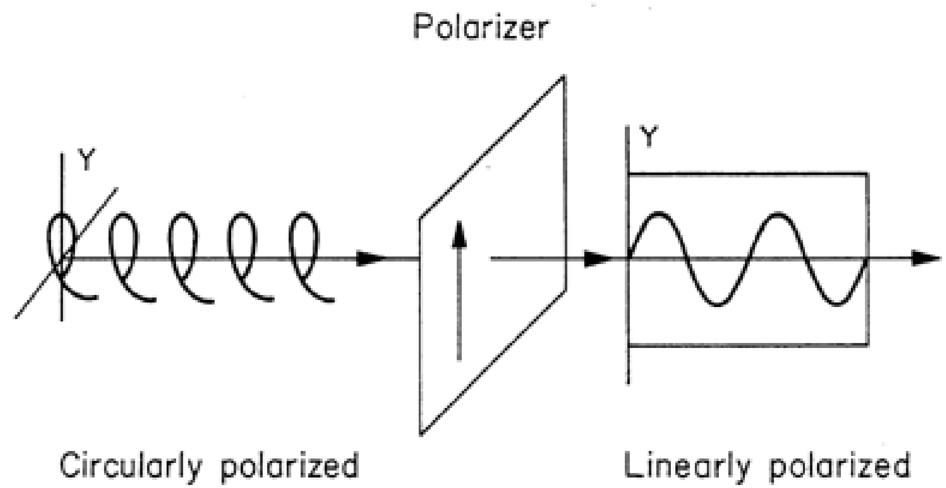
linearly polarized: $\phi_b - \phi_a = 0..or\,\pi$ $\quad E_y = \pm(\frac{b}{a})E_x$

circularly polarized: $\phi_b - \phi_a = \pm\frac{\pi}{2}$ $\quad \frac{E_y}{E_x} = \frac{b}{a} = 1$

Elliptically polarized: $\phi_b - \phi_a = anything..except..0, \pi, \pm\frac{\pi}{2}$

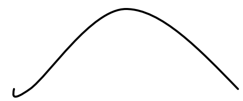
$$\frac{E_y}{E_x} = \frac{b}{a} = anything$$





hyperphysics

Plane, Spherical and Cylindrical Wave



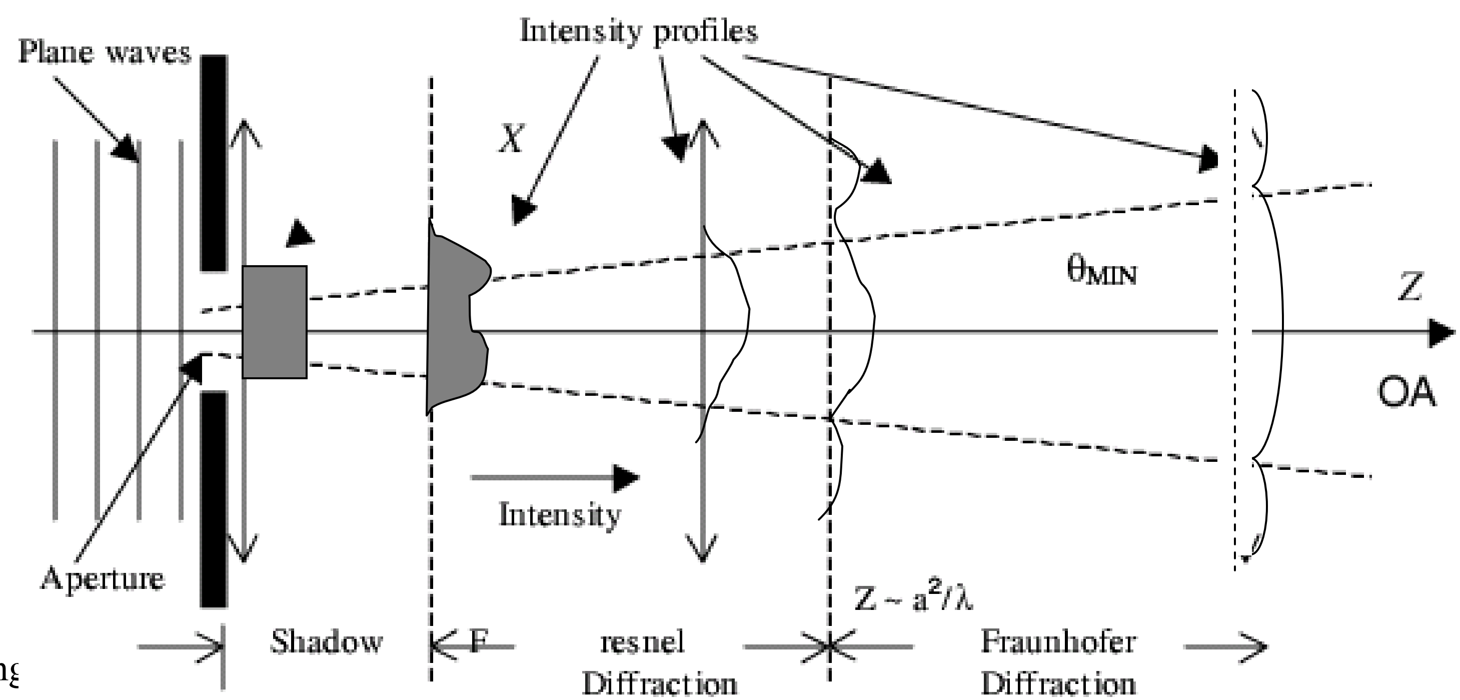
$$\psi = \psi_0 \cos(\omega t - \mathbf{k} \cdot \mathbf{r} + \phi),$$

$$\psi(r,t) \approx \frac{A}{\sqrt{r}} \cos(kr \pm \omega t).$$

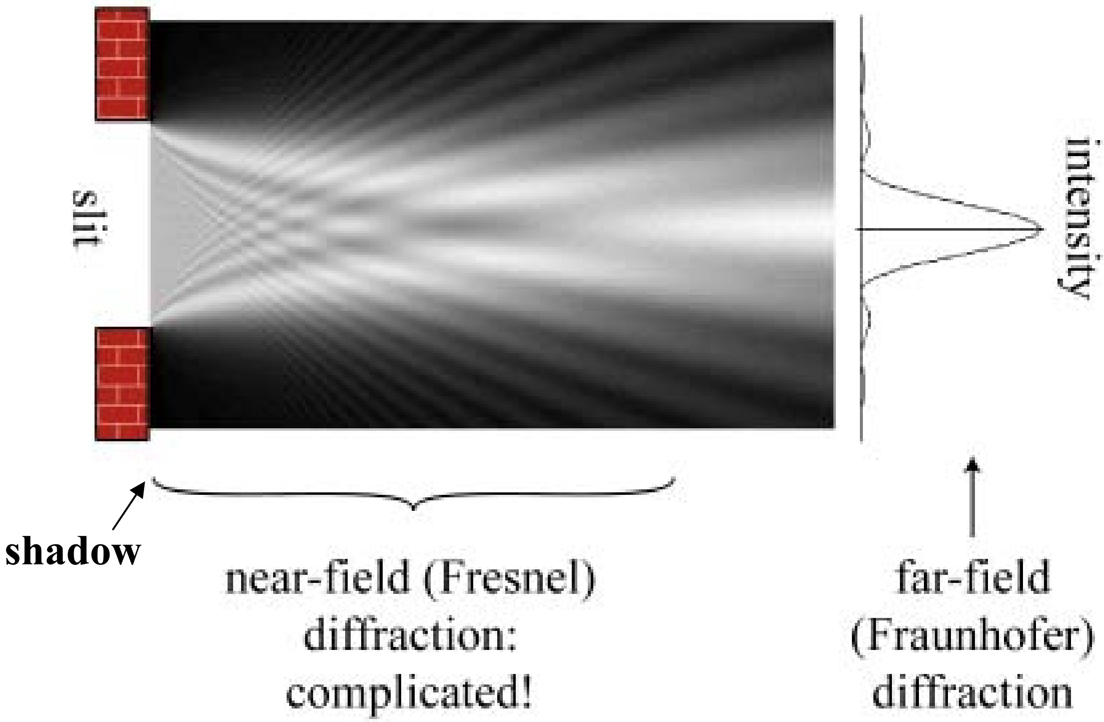
$$\psi(r,t) = \frac{\psi_0}{r} \cos(\omega t - kr + \phi).$$

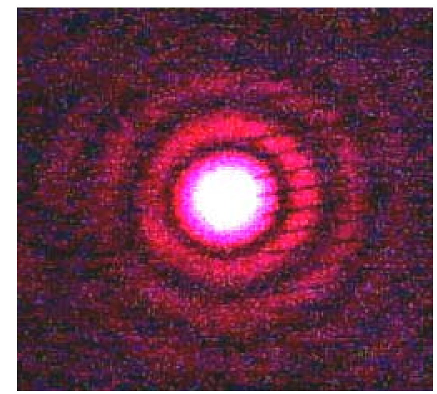
Diffraction

Diffraction refers to the spread of waves and appearance of fringes that occur when a wave front is constricted by an aperture in a a screen that is otherwise opaque. The light pattern changes as you move away from the aperture, being characterized by three regions



The intensity pattern behind a narrow single slit under uniform monochromatic illumination looks something like this:





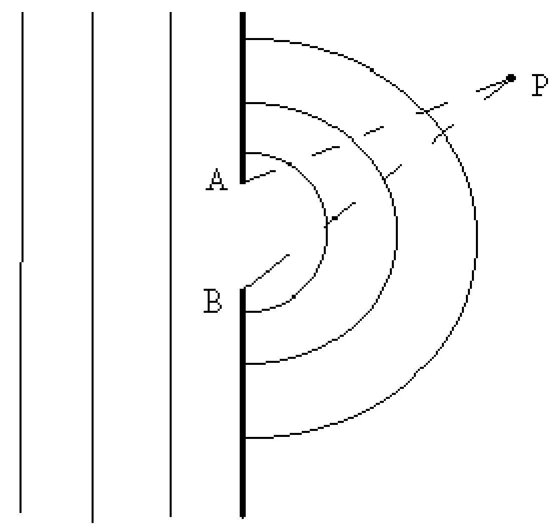
Derive an quantities expression for the diffraction can be done using

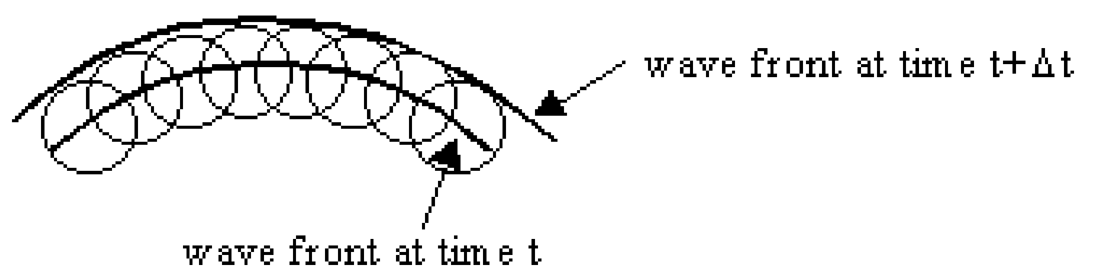
* Kirchhoff Fresnel– Derivation of diffraction from wave equation

* Fourier Optics

Huygens-Fresnel principle

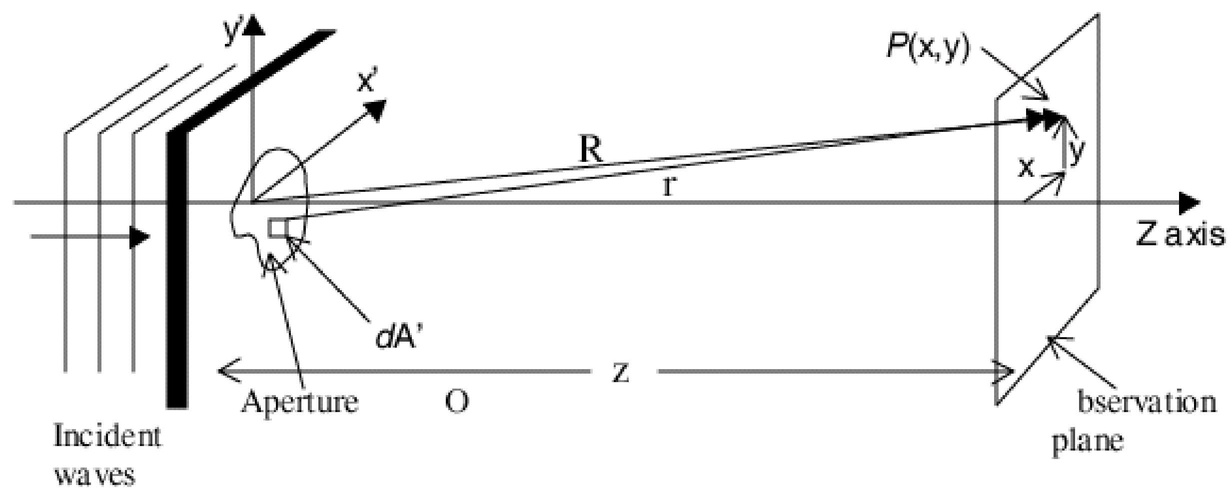
Huygens-Fresnel principle, states that <u>**every unobstructed point of a wavefront, at a given instant in time, serves as a source of spherical secondary wavelets, with the same frequency as that of the primary wave. The amplitude of the optical field at any point beyond is the superposition of all these wavelets, taking into consideration their amplitudes and relative phases**</u>.





Using Kirchoff-Fresnel Diffraction Integral, we can derive an quantitative expression for the irradiating field of a finite aperture.

straightforward than in region 2.



Consider plane wave incident on an aperture, the incident filed is described as

$$E_{INC}(z,t) = E_o\, e^{j(kz-\omega t)}$$

At z = 0 => $E_{INC}(z,t) = E_o\, e^{j(-\omega t)}$. A typical element of the wave fron of the area *dA'* and at position *r' (x', y', 0)* then act as a source of Huygens wavelets. Assume we are interested in detecting light at point *P*, the distance from element *dA' to P* is given by $r = \left|\vec{R} - \vec{r}\,'\right|$

The field at P due to the element dA' is then equal to

$$dE(P) = \left[\frac{E_o e^{-j\omega t}}{\lambda} dA' \right] \times \left[\frac{e^{jkr}}{r} \right]$$

= (Source strength) x (Huygens spherical wave)
(point source)

The field at P due to the entire aperture is then a superposition of the wavelets from all elements areas,

$$E(P) = \iint_{aperture \quad area} \left[\frac{E_o e^{-j\omega t}}{\lambda} dA' \right] \times \left[\frac{e^{jkr}}{r} \right]$$

Since the detector measures the light intensity at P, E field is covert to intensity using the time averaged Polynting vector

$$\vec{S} = \frac{\vec{E} x \vec{B}}{2\mu_o} = \frac{|E|^2}{2Z_o}\hat{z} \quad \Longrightarrow \quad I(P) = \frac{|E|^2}{2Z_o}$$

where z_o = air impedance
= $\mu\varepsilon$

Usually Fraunhofer condtion applied when **z>> a²/ λ**. The parallel rays is adequately assume at a ditance of **z ~ 10 a²/ λ**

Single Slit Diffraction Intensity

Under the Fraunhofer conditions, the wave arrives at the single slit as a plane wave. Divided into segments, each of which can be regarded as a point source, the amplitudes of the segments will have a constant phase displacement from each other, and will form segments of a circular arc when added as vectors. The resulting relative intensity will depend upon the total phase displacement according to the relationship:

$$I = I_o \frac{\sin^2(\frac{\delta}{2})}{(\frac{\delta}{2})^2}$$

Where total phase angle
Relate to derivation of θ

$$\delta = \frac{2\pi a \sin\theta}{\lambda}$$

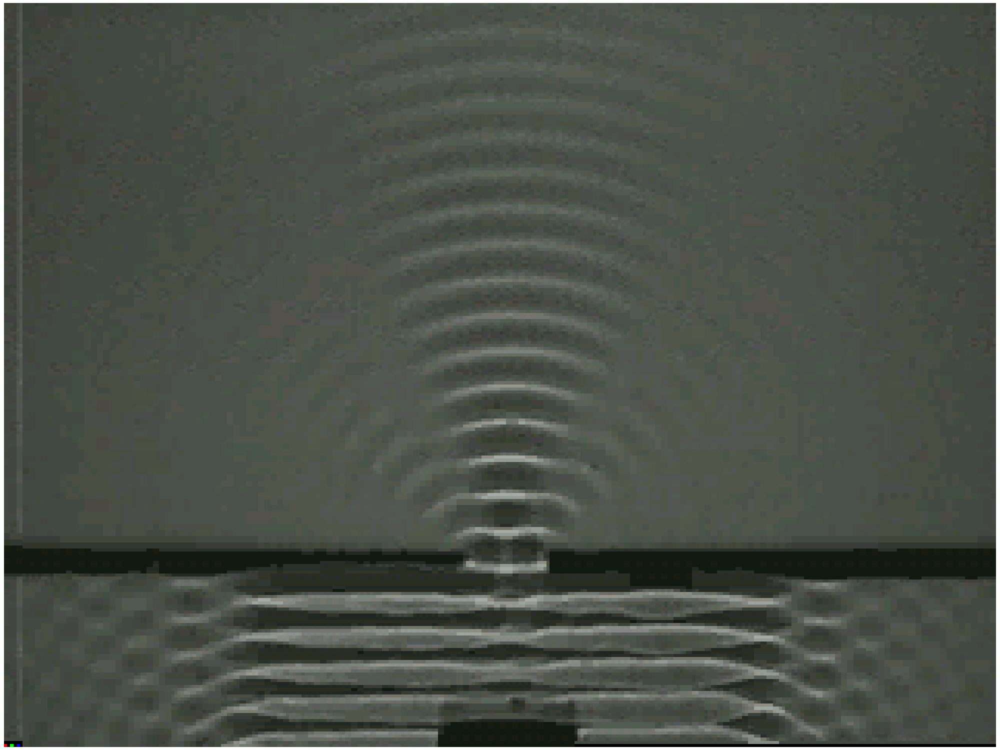
Intensity as a function of θ

$$I = I_o \frac{\sin^2(\frac{\pi a \sin\theta}{\lambda})}{(\frac{\pi a \sin\theta}{\lambda})^2}$$

Intensity as a function of y

$$I = I_o \frac{\sin^2(\frac{\pi a y}{\lambda D})}{(\frac{\pi a y}{\lambda D})^2}$$

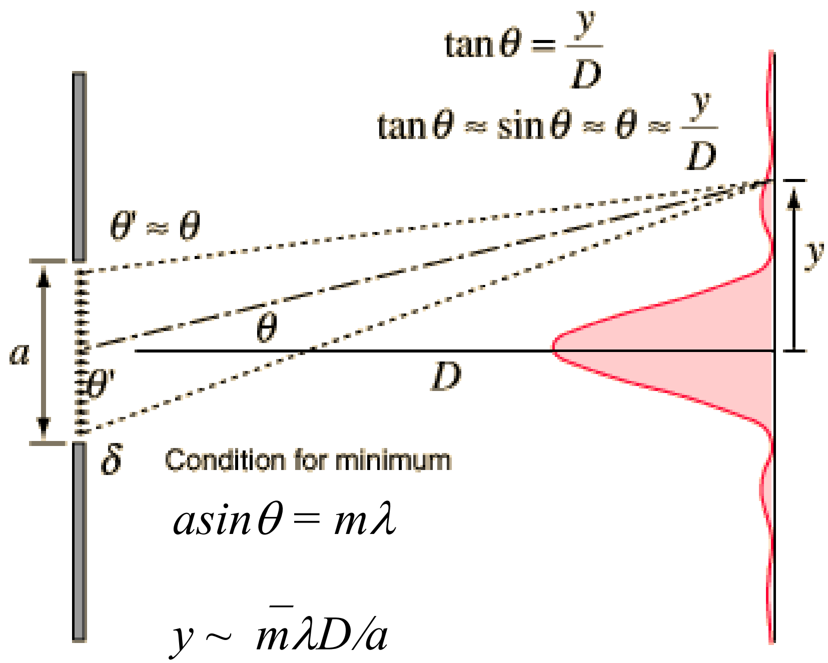
Single Slit Ripple Tank Experiment

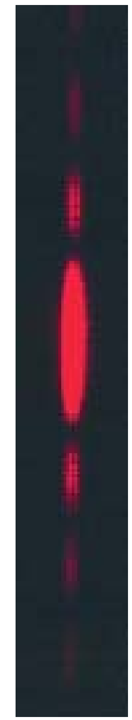


http://www.phy.davidson.edu/introlabs/labs220-230/html/lab10diffract.htm

Fraunhofer diffraction

Single slit

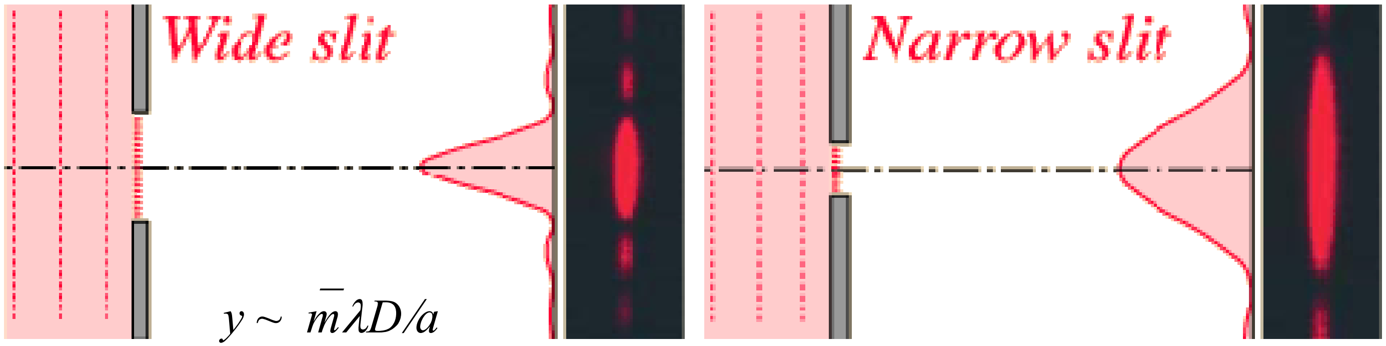




hyperphysics

The diffraction pattern at the right is taken with a helium-neon laser and a narrow single slit. To obtain the expression for the displacement y above, the small angle approximation was used.

Fraunhofer diffraction



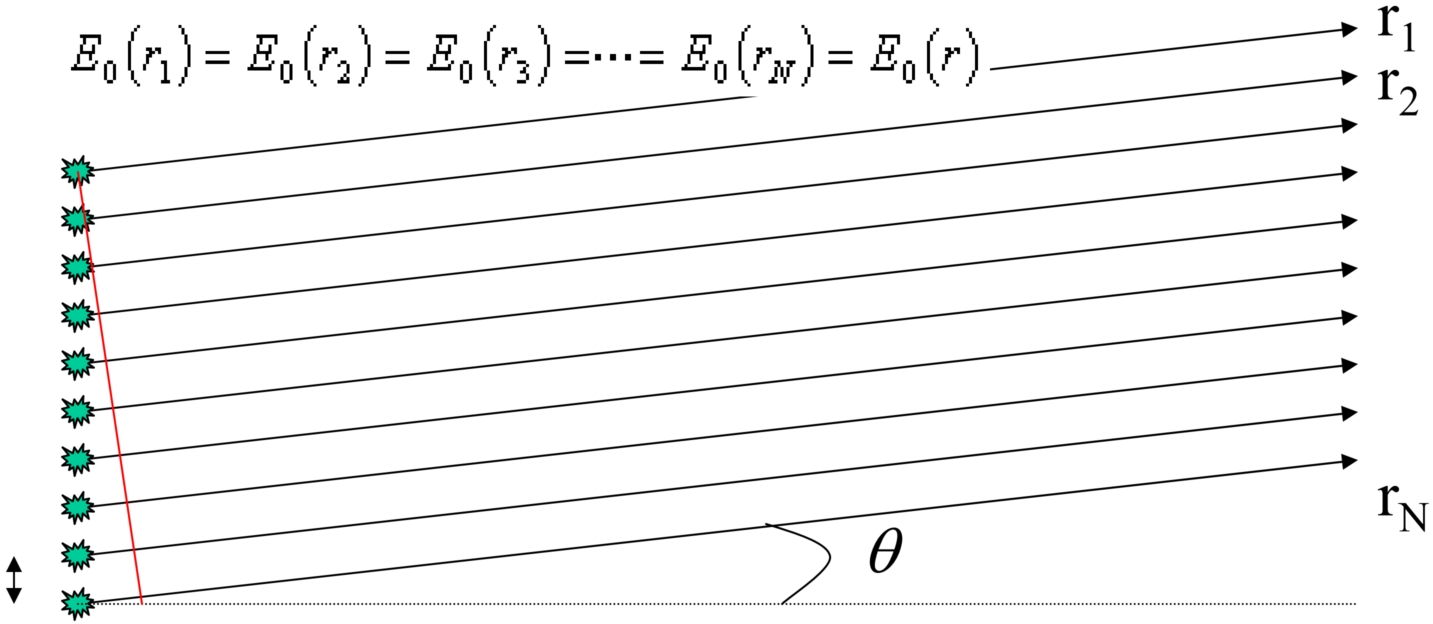
hyperphysics

The diffraction patterns were taken with a helium-neon laser and a narrow single slit. The slit widths used were on the order of 100 micrometers, so their widths were 100 times the laser wavelength or more. A slit width equal to the wavelength of the laser light would spread the first minimum out to 90° so that no minima would be observed. The relationships between slit width and the minima and maxima of diffraction can be explored in the single slit calculation.

Line of point sources (pinholes), all in phase with same amplitude

If the spatial extent of the oscillator array is small compared to the wavelength of the radiation, then the amplitudes of the separate waves arriving at some observation point *P* will be essentially equal,

$$E_0(r_1) = E_0(r_2) = E_0(r_3) = \cdots = E_0(r_N) = E_0(r)$$



d=distance between sources

Note that:

$$r_2 - r_1 = d\sin\theta \qquad r_3 - r_1 = 2d\sin\theta$$

$$r_4 - r_1 = 3d\sin\theta \qquad r_N - r_1 = (N-1)d\sin\theta$$

The sum of the interfering spherical wavelets yields a composite electric field at *P* that is the real part of

$$E = E_0(r)e^{i(kr_1 - \omega t)} + E_0(r)e^{i(kr_2 - \omega t)} + \cdots + E_0(r)e^{i(kr_N - \omega t)}$$

Rearrange to get

$$E = E_0(r)e^{-i\omega t}e^{ikr_1}\left[1 + e^{ik(r_2 - r_1)} + e^{ik(r_3 - r_1)} + \cdots + e^{ik(r_N - r_1)}\right]$$

The phase difference between adjacent sources is obtained from the expression $\delta = k_0\Lambda$ where the maximum optical-path length difference is $\Lambda = nd\sin\theta$ in a medium with an index of refraction *n*.

But, since d is the distance between two adjacent oscillators, it can be easily seen that $d\ sin\theta = r_2 - r_1$. Thus, the field at P becomes

$$E = E_0(r)e^{-i\omega t}e^{ikr_1}\left[1+\left(e^{i\delta}\right)+\left(e^{i\delta}\right)^2+\left(e^{i\delta}\right)^3+\cdots+\left(e^{i\delta}\right)^{N-1}\right]$$

$$= E_0(r)e^{-i\omega t}e^{ikr_1}\frac{e^{iN\delta}-1}{e^{i\delta}-1}$$

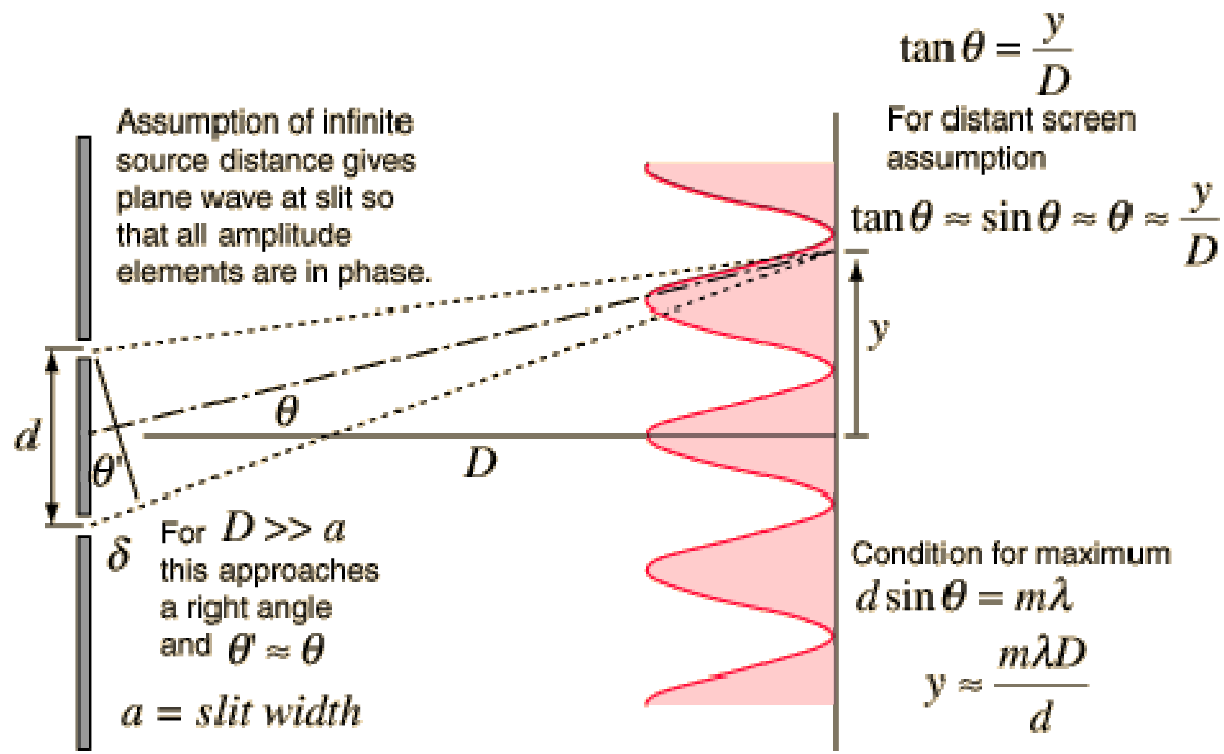
$$= E_0(r)e^{-i\omega t}e^{ikr_1}\frac{e^{iN\delta/2}\left(e^{iN\delta/2}-e^{-iN\delta/2}\right)}{e^{i\delta/2}\left(e^{i\delta/2}-e^{-i\delta/2}\right)}$$

$$= E_0(r)e^{-i\omega t}e^{ikr_1}e^{i(N-1)\delta/2}\frac{\sin(N\delta/2)}{\sin(\delta/2)}$$

$$= E_0(r)e^{i(kR-\omega t)}\frac{\sin(N\delta/2)}{\sin(\delta/2)}$$

where $R = r_1 + \frac{1}{2}(N-1)d\sin\theta$ is the distance from the center of the line of oscillators to the point P.

Double Slits Interference



hyperphysics

$$d\sin\theta = m\lambda$$

$$y \sim \bar{m}\lambda D/d$$

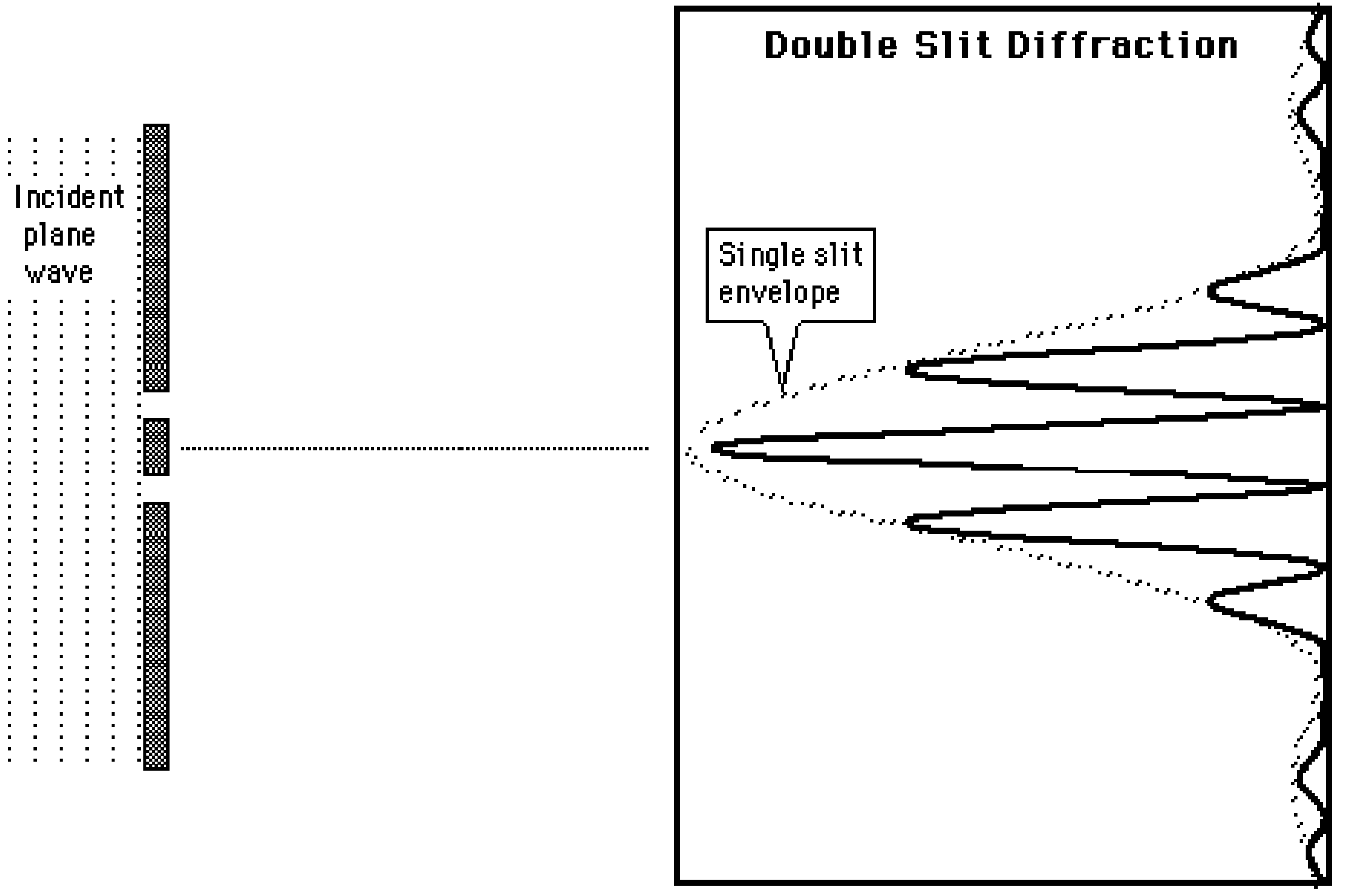
$$I = I_0 \left[\frac{\sin\left(\frac{Nkd}{2}\sin\theta\right)}{\sin\left(\frac{kd}{2}\sin\theta\right)} \right]^2$$

Where $N = 2$

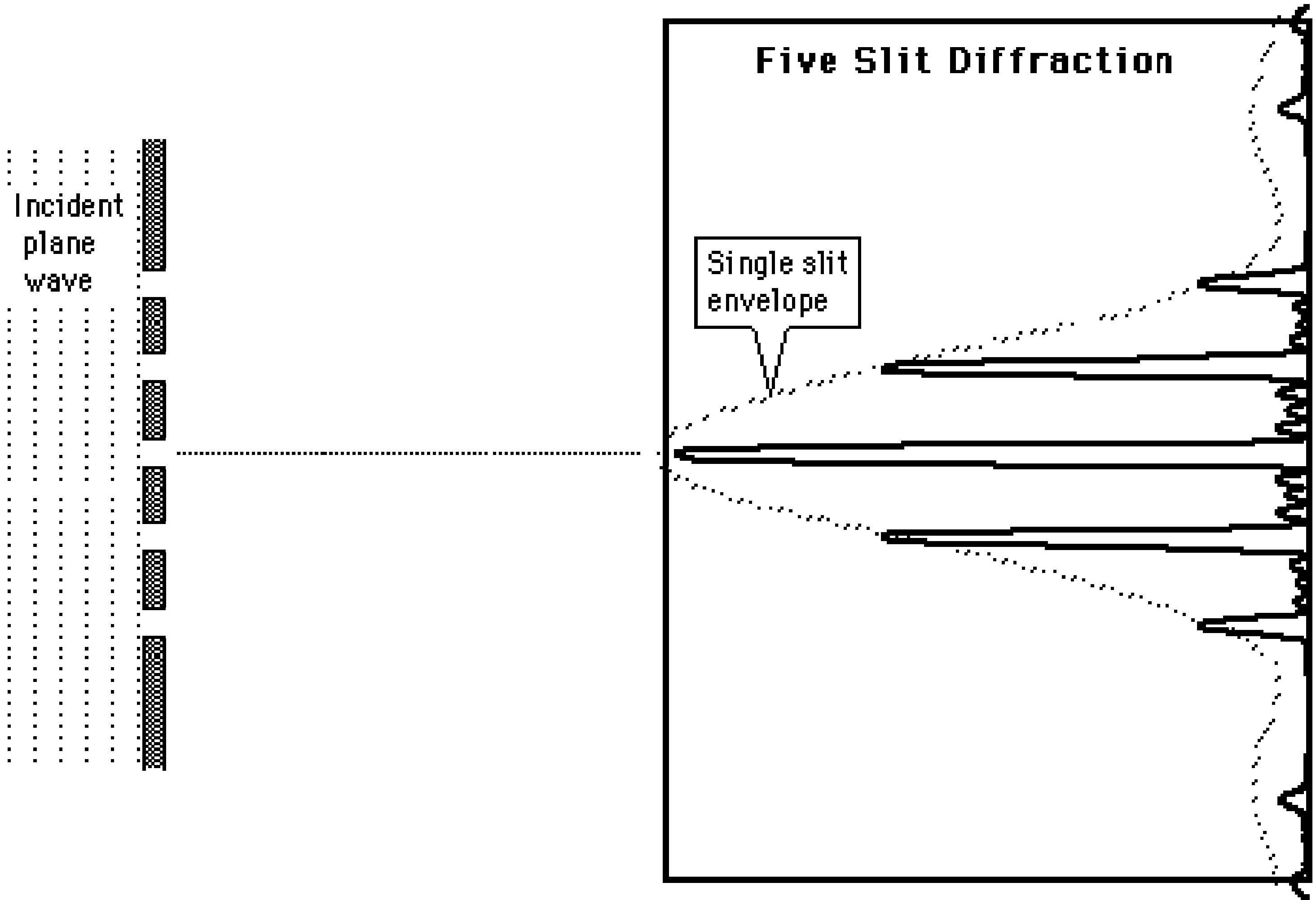
Double slits Ripple Tank Experiment

http://www.phy.davidson.edu/introlabs/labs220-230/html/lab10diffract.htm

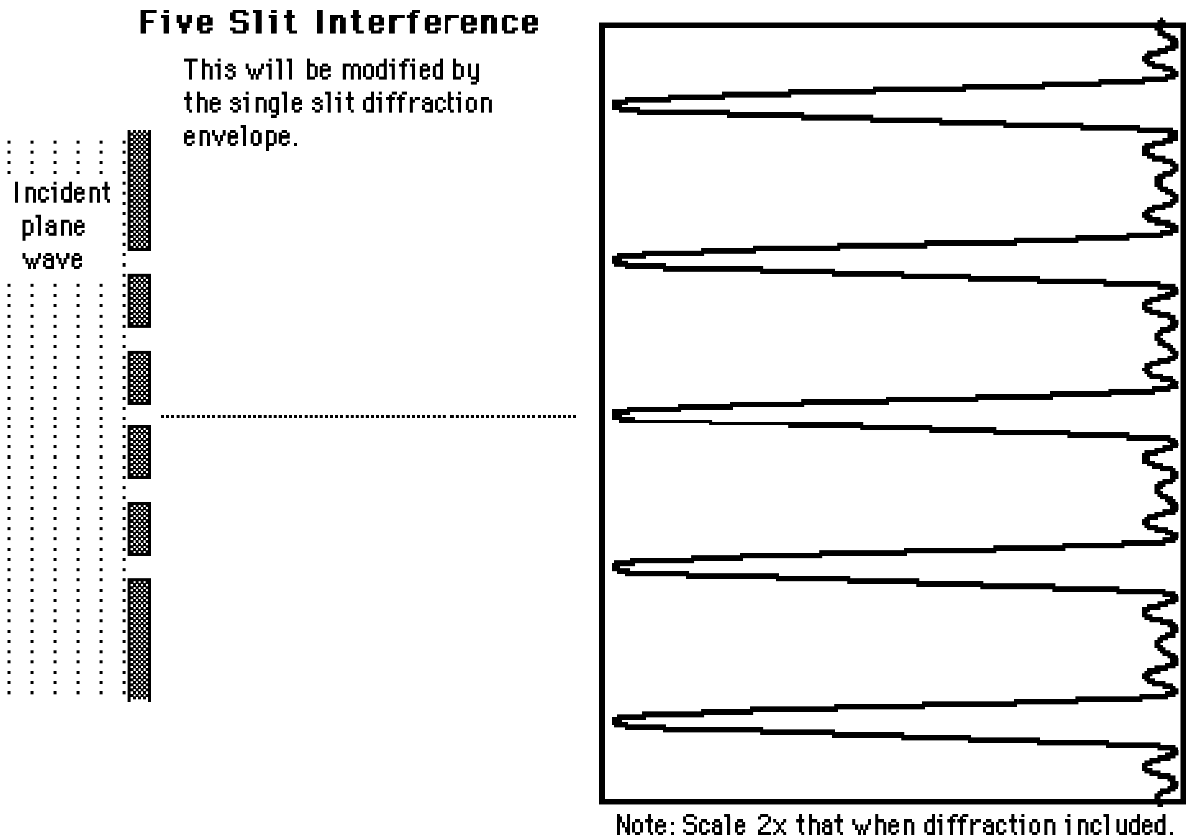
$$d >> a$$



Slit separation ~ slit width


Incident
plane
wave
Five Slit Diffraction
Single slit
envelope

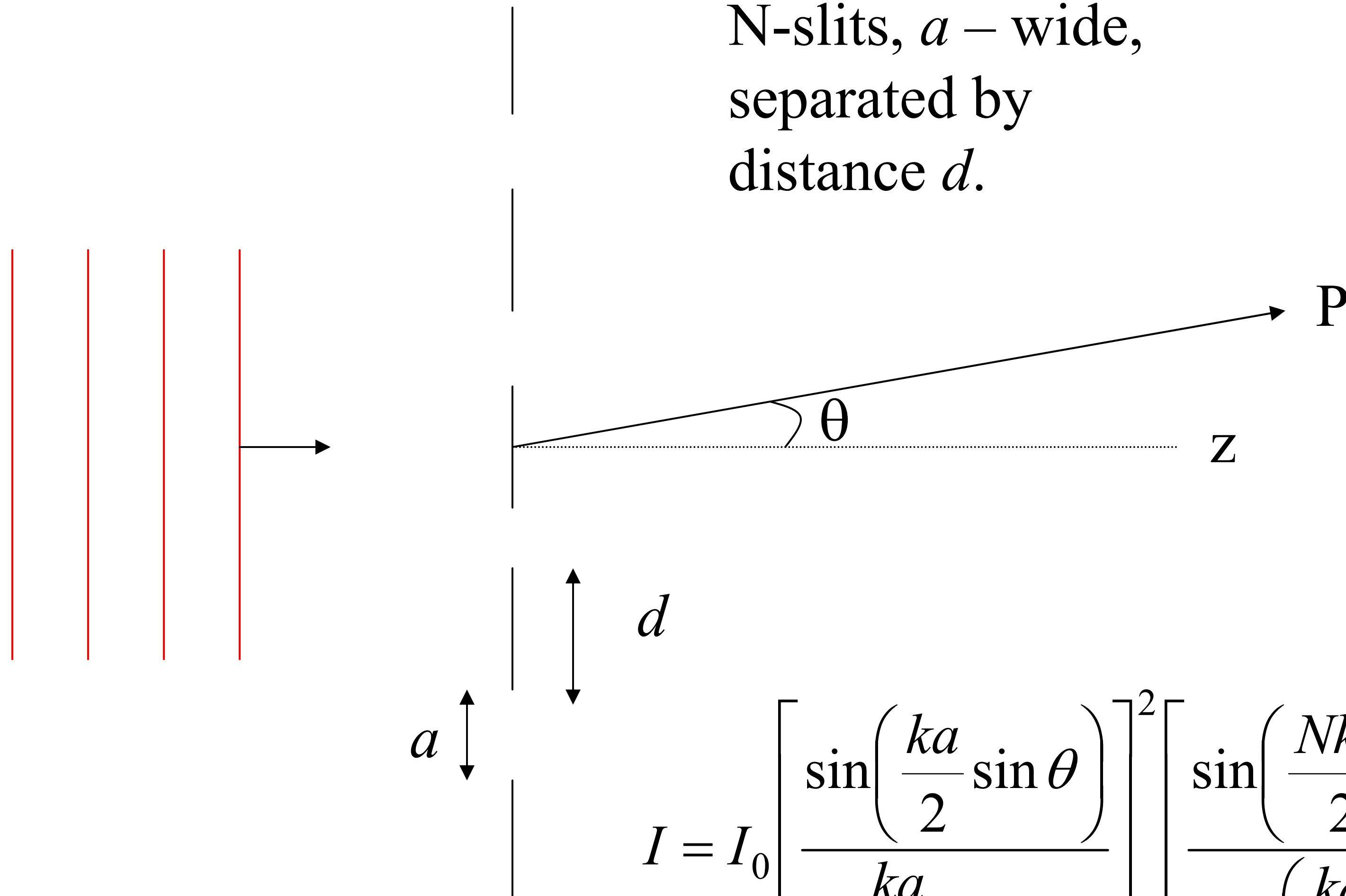
Multiple Slits



Under the Fraunhofer conditions, the light curve of a multiple slit arrangement will be the interference pattern multiplied by the single slit diffraction envelope. This assumes that all the slits are identical. In this case $a <<< d$.

Use Fraunhofer to model a transmission grating of N-slits

N-slits, a – wide, separated by distance d.



$$I = I_0\left[\frac{\sin\left(\dfrac{ka}{2}\sin\theta\right)}{\dfrac{ka}{2}\sin\theta}\right]^2\left[\frac{\sin\left(\dfrac{Nkd}{2}\sin\theta\right)}{\sin\left(\dfrac{kd}{2}\sin\theta\right)}\right]^2$$

Grating Intensity

The intensity is given by the interference intensity expression

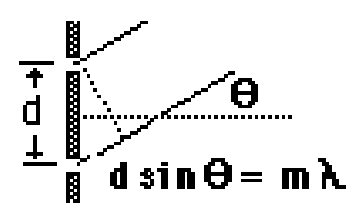
$$I = I_0 \left[\frac{\sin\left(\frac{Nkd}{2} \sin \theta \right)}{\sin\left(\frac{kd}{2} \sin \theta \right)} \right]^2$$

Modulated by the single slit diffraction envelope for the slits which make up the grating:

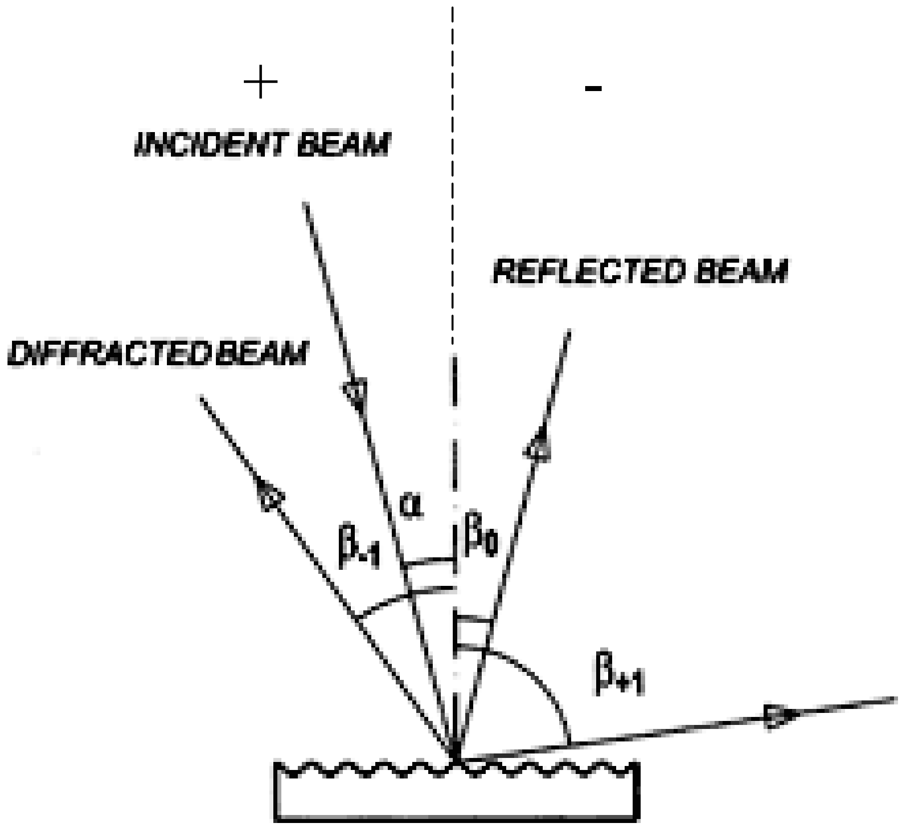
$$I = I_0 \left[\frac{\sin\left(\frac{ka}{2} \sin \theta \right)}{\frac{ka}{2} \sin \theta} \right]^2$$

The given total intensity expression,

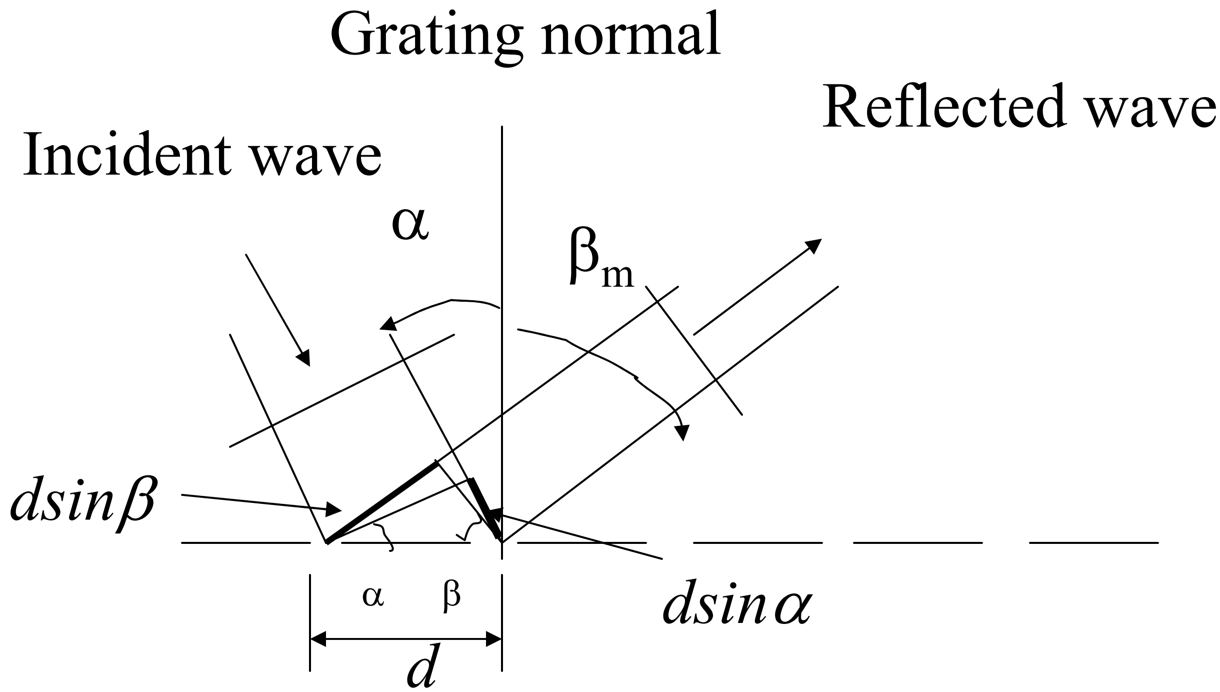
$$I = I_0 \left[\frac{\sin\left(\frac{ka}{2} \sin \theta \right)}{\frac{ka}{2} \sin \theta} \right]^2 \left[\frac{\sin\left(\frac{Nkd}{2} \sin \theta \right)}{\sin\left(\frac{kd}{2} \sin \theta \right)} \right]^2$$



Reflective Grating



Grating can be made into reflective type and diffractive
Grating theory still hold.



The geometrical path difference between light from adjacent grooves is seen to be $dsin\alpha$ + $dsin\beta$. The principle of interference dictates that only when this difference equals the wavelength l of the light, or some integral multiple thereof, will the light from adjacent grooves be in phase (lead to constructive interference)

Path length difference creates constructive interference:

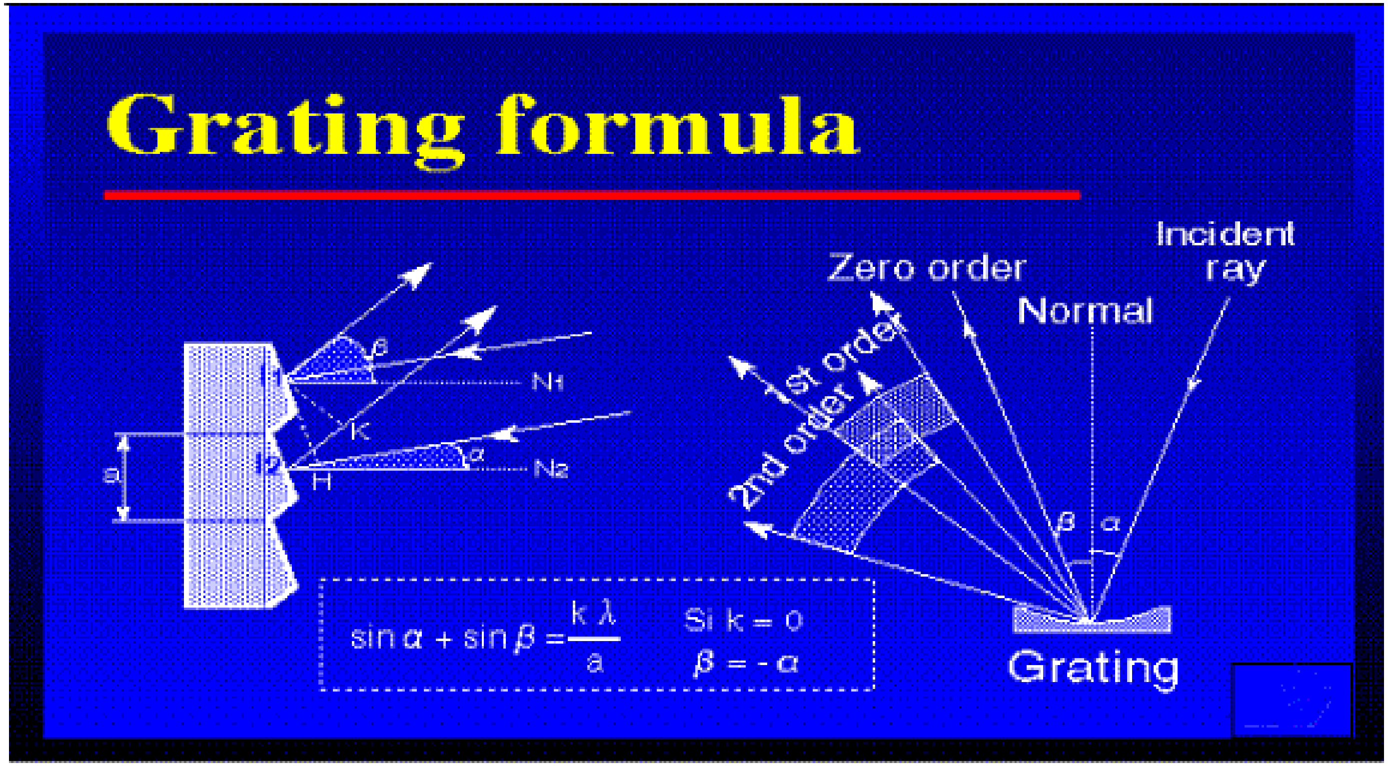
$$d sin\alpha + d sin\beta_m = m\lambda$$

Where m = diffraction order

For a ray arriving with an angle of incidence α, the angle β under which it will be diffracted by a grating of *N* lines per millimetre depends on the wavelength λ by the grating equation:

$$sin\alpha + sin\beta_m = Nm\lambda$$

Frequency of the grating structure is defined N

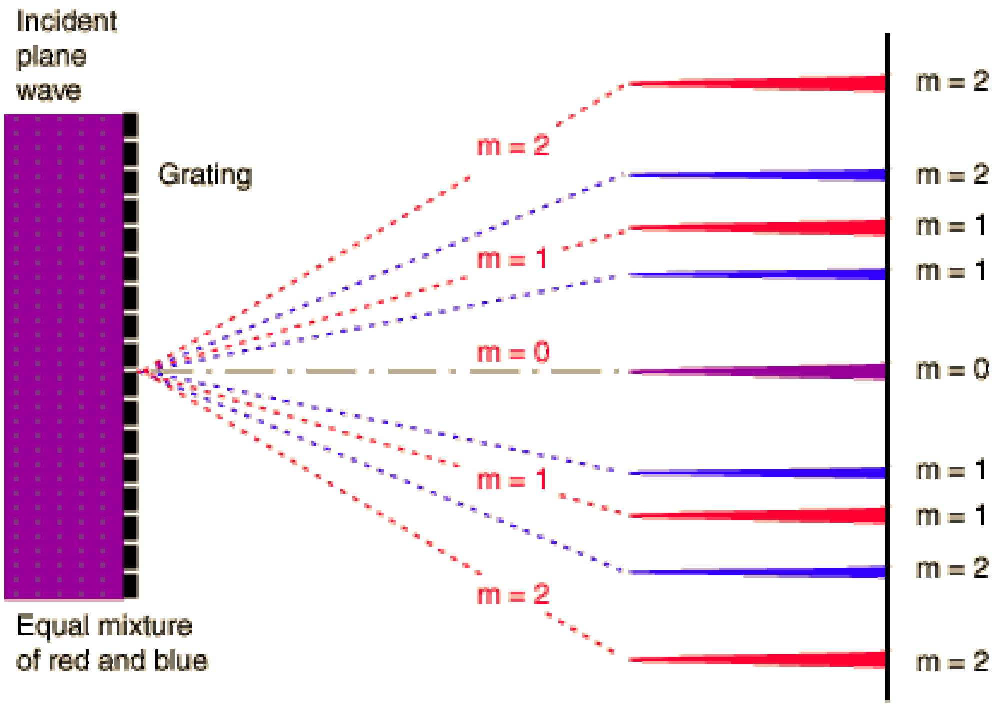


Surface Analytical

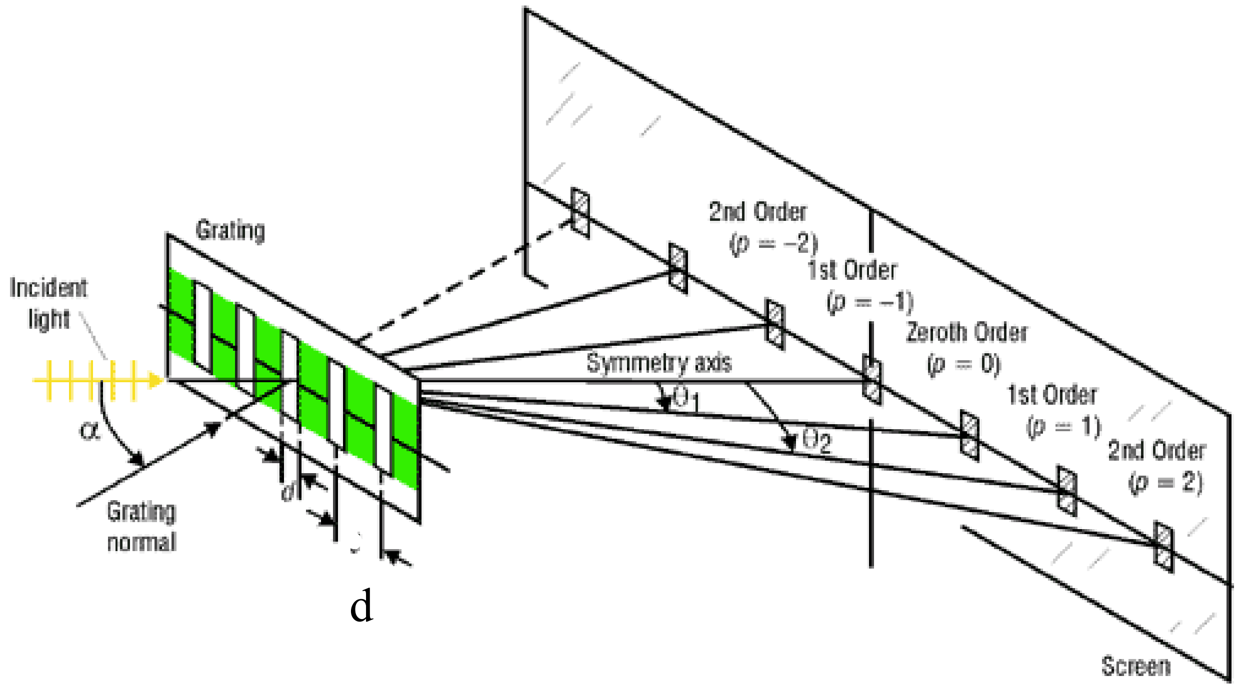
Order zero represents about 40% of the total energy.The rest of the energy is distributed amongst the various orders. Generally, the higher the order, the lower the brightness of its spectrum. The highest orders carry almost no energy. In practice, only the first and second orders are usable.

Diffraction Grating

A diffraction grating is an optical component that serves to periodically modulate the phase or the amplitude of the incident wave. It can be made of a transparent plate with periodically varying thickness or periodically graded refractive index



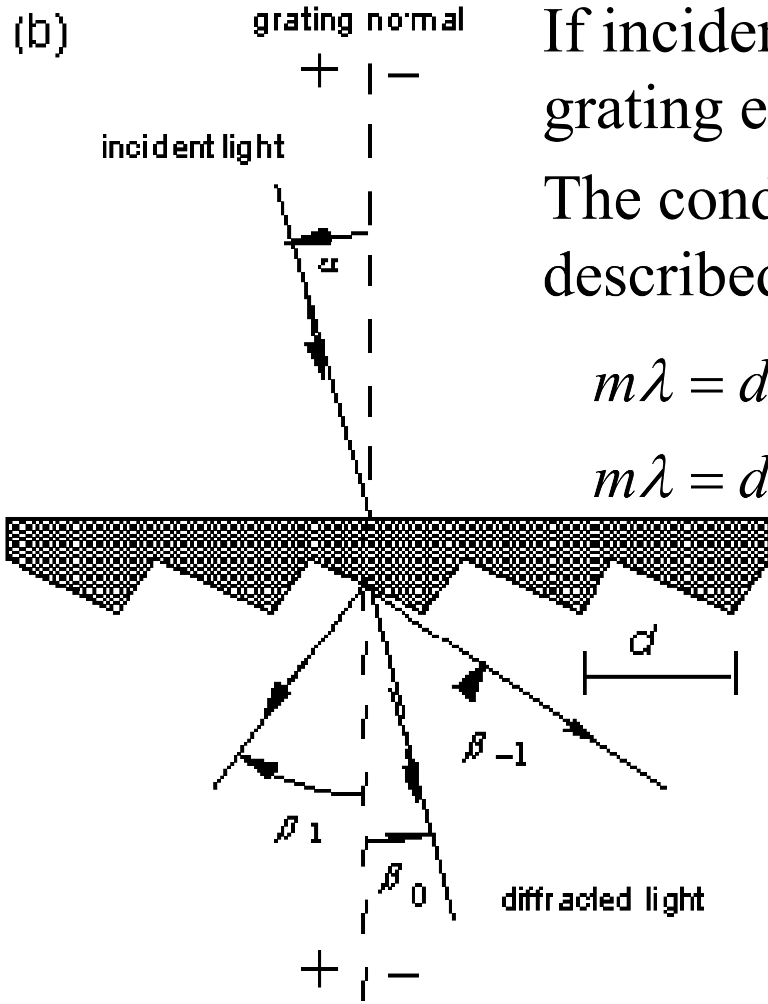
Diffraction Grating



The light is incident on the grating along the grating normal ($\alpha \neq 0$), the grating equation,

$$d(\sin\alpha + \sin\theta_p) = p\lambda \text{ where } p = 0, \pm 1, \pm 2 \ldots\ldots$$

Diffractive Grating



If incident angle is not normal, the grating equation $m\lambda = d\sin(\beta_m)$

The conditions of diffraction are described by two equations:

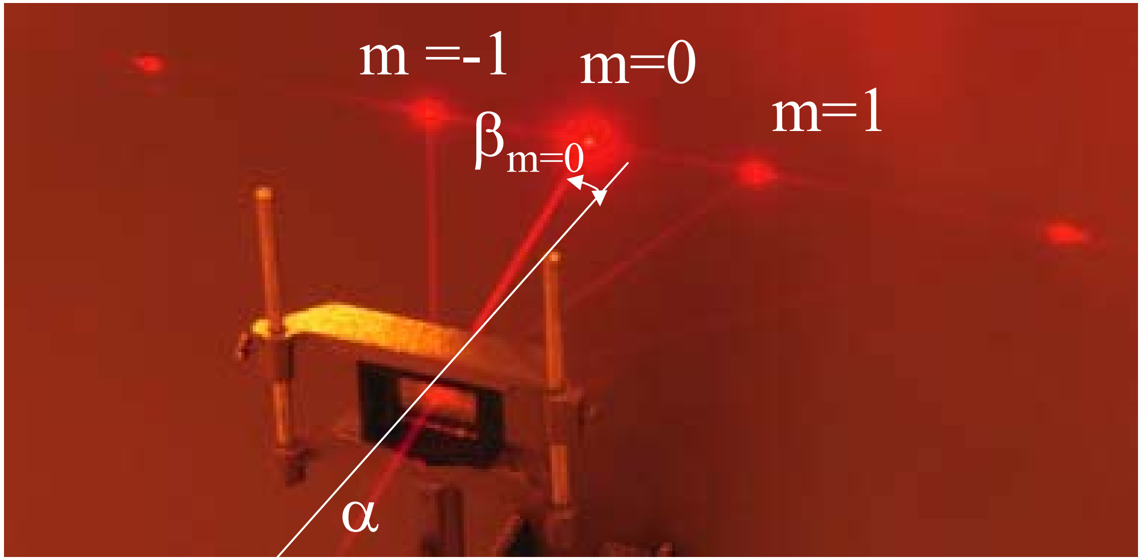
$$m\lambda = d[\sin(\beta_m + \alpha) - \sin\alpha] \qquad m = +$$

$$m\lambda = d[\sin(\beta_m - \alpha) + \sin\alpha] \qquad m = -$$

The condition for maximum intensity is the same as that for the double slit or multiple slits, but with a large number of slits the intensity maximum is very sharp and narrow, providing the high resolution for spectroscopic applications. The peak intensities are also much higher for the grating than for the double slit.

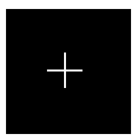
When light of a single wavelength , like the 632.8nm red light from a helium-neon laser at left, strikes a diffraction grating it is diffracted to each side in multiple orders. Orders 1 and 2 are shown to each side of the direct beam. Different wavelengths are diffracted at different angles, according to the grating relationship.

Diffraction Grating and Helium-Neon Laser

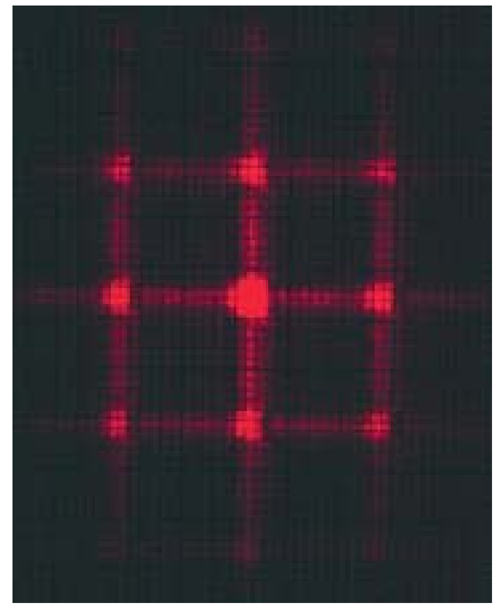


While directing the 632.8 nm red beam of a helium-neon laser through a 600 lines/mm diffraction grating, a cloud was formed using liquid nitrogen. You can see the direct beam plus the first and second orders of the diffraction.

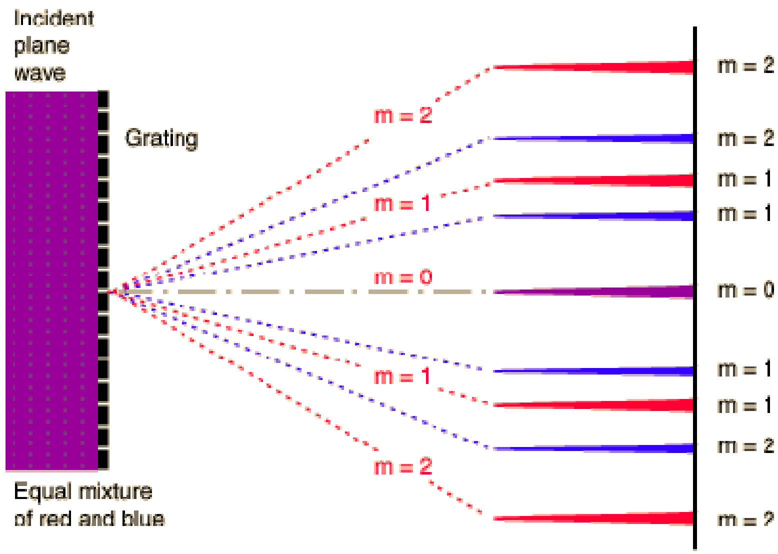
Diffraction from Crossed Slits



Appearance of object.



Angular Dispersion



hperphysics

A diffraction grating is the tool of choice for separating the colors in incident light. This is dispersion effect similar to prism. The angular dispersion is the amount of change of diffraction angle per unit change of the wavelength. It is a measure of the angular separation between beams of adjacent wavelengths. An expression for the angular dispersion can be derived from earlier equation by differentiating, keeping the angle fixed.

$$D = \frac{d\beta_m}{d\lambda} = \frac{-m}{d\cos\beta_m} \qquad (2)$$

D is measure of the angular separation produced between two incident monochromatic waves whose wavelengths differ by a small wavelength interval

Resolvance and wavelength resolution

To distinguish light waves whose wavelengths are close together, the maxima of these wavelengths formed by the grating should be as narrow as possible. Express otherwise, resolvance or "chromatic resolving power" for a device used to separate the wavelengths of light is defined as

$$R = \lambda/\Delta\lambda = mN$$

where $\Delta\lambda$ = smallest resolvable wavelength difference
m = order number
N = grating frequency

Using the limit of resolution is determined by the Raleigh criterion as applied to the diffraction maxima, i.e., two wavelengths are just resolved when the maximum of one lies at the first minimum of the other, the above R = mN can be derived.

The resolvance of such a grating depends upon how many slits are actually covered by the incident light source; i.e., if you can cover more slits, you get a higher resolution in the projected spectrum

Examples of Resolvance

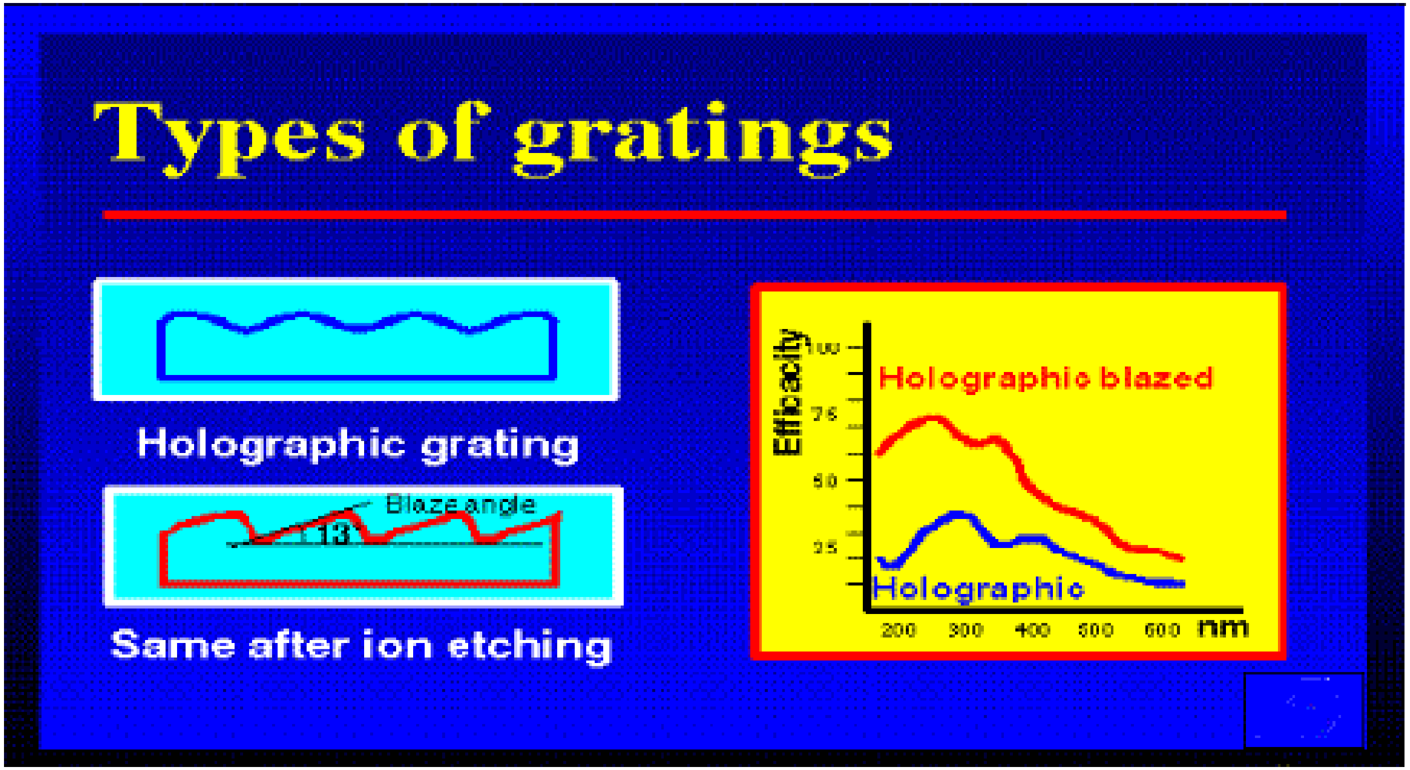
A standard benchmark for the resolvance of a grating or other spectroscopic instrument is the resolution of the sodium doublet. The two sodium "D-lines" are at 589.00 nm and 589.59 nm. Resolving them corresponds to resolvance

$$R=\lambda/\Delta\lambda = 0.589/.59 = 1000$$

Use R and assume a m you want to use and find out what N is needed to resolve these two wavelengths

$$R = NM = 1000$$

Blazed versus Sinusoidal

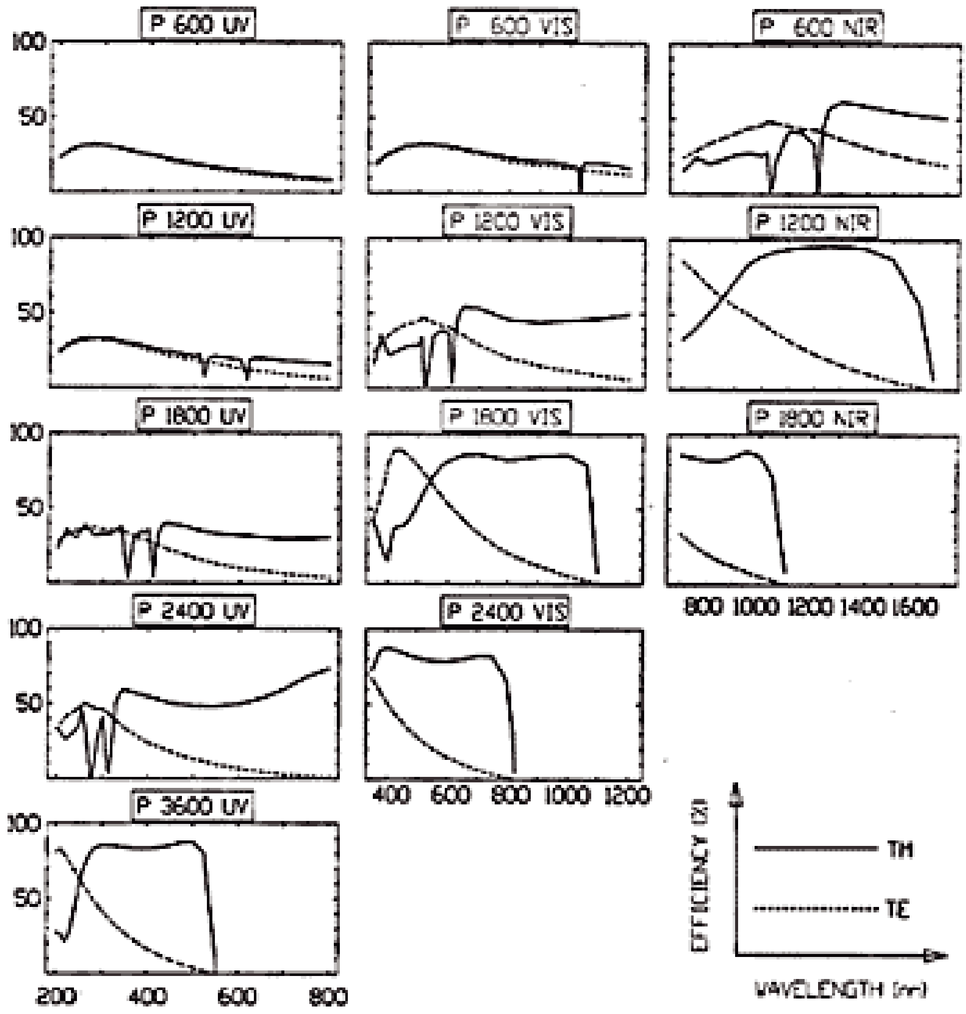


Surface Analytical

Sinusoidal gratings

- Holographically manufactured
- Gratings of standard type have a sinusoidal groove profile.
- The efficiency curve is rather smooth and flatter than for ruled gratings. The efficiency is optimized for specific spectral regions by varying the groove depth, and it may still be high, especially for gratings with high frequency.
- When the groove spacing is less than about 1.25 times the wavelength, only the -1 and 0 orders exist, and if the grating has an appropriate groove depth, most of the diffracted light goes into the -1 order. In this region, holographically recorded gratings give well over 50 % absolute efficiency.

Efficiency Curve



http://www.spectrogon.com/gratpropert.html

The absolute efficiency is defined as the amount of the incident flux that is diffracted into a given diffraction order. The relative efficiency is related to the reflectance of a mirror, coated with the same material as the grating, and it should be noted that the relative efficiency is always higher than the absolute efficiency.

Efficiency curves for the most common holographic grating types. Each grating is denoted P XXXX YY, where P stands for Plane holographic grating, XXXX is the groove frequency, and YY is the spectral range where the efficiency is highest.

Littrow Condition

Blazed grating groove profiles are calculated for the Littrow condition where the incident and diffracted rays are in auto collimation (i.e., $\alpha = \beta$). The input and output rays, therefore, propagate along the same axis. In this case at the "blaze" wavelength λ_B.

$$\sin\alpha + \sin\beta = mN\lambda_B$$

$$\omega = \alpha = \beta\,,\ \omega = \text{blazed angle}$$

$$2\sin\omega = mN\lambda_B$$

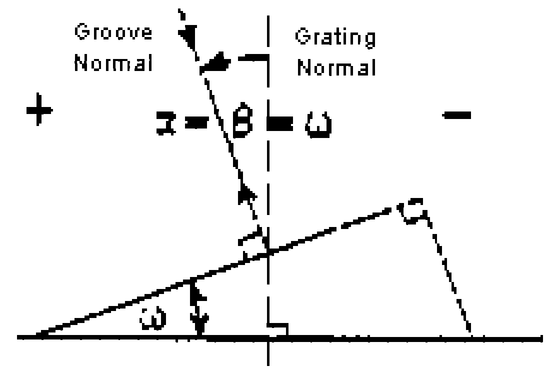


Figure 4 - Littrow Condition for a Single Groove of a Blazed Grating

For example, the blaze angle (ω) for a 1200 g/mm grating blazed at 250 nm is 8.63° in first order (m = 1).

Blaze: The concentration of a limited region of the spectrum into any order other than the zero order. Blazed gratings are manufactured to produce maximum efficiency at designated wavelengths. A grating may, therefore, be described as "blazed at 250 nm" or "blazed at 1 micron" etc. by appropriate selection of groove geometry.

A blazed grating is one in which the grooves of the diffraction grating are controlled to form right triangles with a "blaze angle, ω," as shown in Fig. 4. However, apex angles up to 110° may be present especially in blazed holographic gratings. The selection of the peak angle of the triangular groove offers opportunity to optimize the overall efficiency profile of the grating.

Blazed grating usually formed by dry etching (Reactive ion etching) with a tilted bottom electrode.

Beam spot size

$$\frac{W_s}{W_o} = \frac{\lambda R_o}{\pi W_o^2}$$

Where Ws = beam spot size at focus, Wo = beam spot size,
L = operating wavelength, Ro = radius of curvature