Recap from last time

Nuclear Decay Occurs….

…when a nucleus is unstable (lower open energy levels)

- An unstable nucleus metamorphoses ("decays") into a more stable (more tightly bound) nucleus
- Difference in binding energy ==>

  mass and kinetic energy of the decay products
- Mass is converted into energy ==> radiation $E = mc^2$

Summary from last time

Nuclear Decay Characteristics

- Type of decay (fission, alpha, beta, electron capture, etc.)
- Lifetime (transformation rate)
  - $N = N_0 e^{-T/\tau}$ Half-life, $T_{1/2} = 0.693\ \tau$
- Radiation type (β^+, β^-, α, fission fragments, etc.)
- Emission energy -- if continuum, then express as maximum energy or mean (average) energy
- Associated gamma (γ) or x rays
- “Daughter nucleus”
  - is it stable?
  - Produced in “ground state” or “excited state”?
  - With what probabilities (“branching ratios”)?

Decay of Radioactivity

or: The Math of Radioactive Decay

- Activity, Lifetime, etc
- Exponential Decay Function
- Determining Decay Factors
- Activity corrections
- Parent-Daughter Decay

Slides @: **http://depts.washington.edu/nucmed/IRL/education.shtml**

Disclaimer

Radioactive decay

- is **spontaneous** process

- can **not** be predicted exactly for any single nucleus

- can only be described statistically and probabilistically

  i.e. can give averages and probabilities

The description of the mathematical aspects of radioactive decay is today's topic

Activity

- Average number of radioactive decays per unit time (rate)
- or - Change in number of radioactive nuclei present:

$$A = -dN/dt$$

- Depends on number of nuclei present. During decay of a given sample, *A* will decrease with time.
- Measured in Becquerel (*Bq*):

  1 Bq = 1 disintegration per second (dps)

- { Traditionally measured in Curies (*Ci*):

  $1\ Ci = 3.7 \times 10^{10}\ Bq \quad (1mCi = 37\ MBq)$

  Traditionally: 1Ci is the activity of 1g ^{226}Ra. }

Decay Constant

- Fraction of nuclei that will decay per unit time:

$$\lambda = -(dN/dt) / N(t) = A(t) / N(t)$$

- Constant in time, characteristic of each nuclide
- Related to activity: $A = \lambda \cdot N$
- Measured in $(\text{time})^{-1}$

Example: Tc-99m has $\lambda = 0.1151\ \text{hr}^{-1}$, i.e. 11.5% decay per hour

Mo-99 has $\lambda = 0.252\ \text{day}^{-1}$, i.e. 25.2% decay per day

- If nuclide has several decay modes, each has its own λ_i.

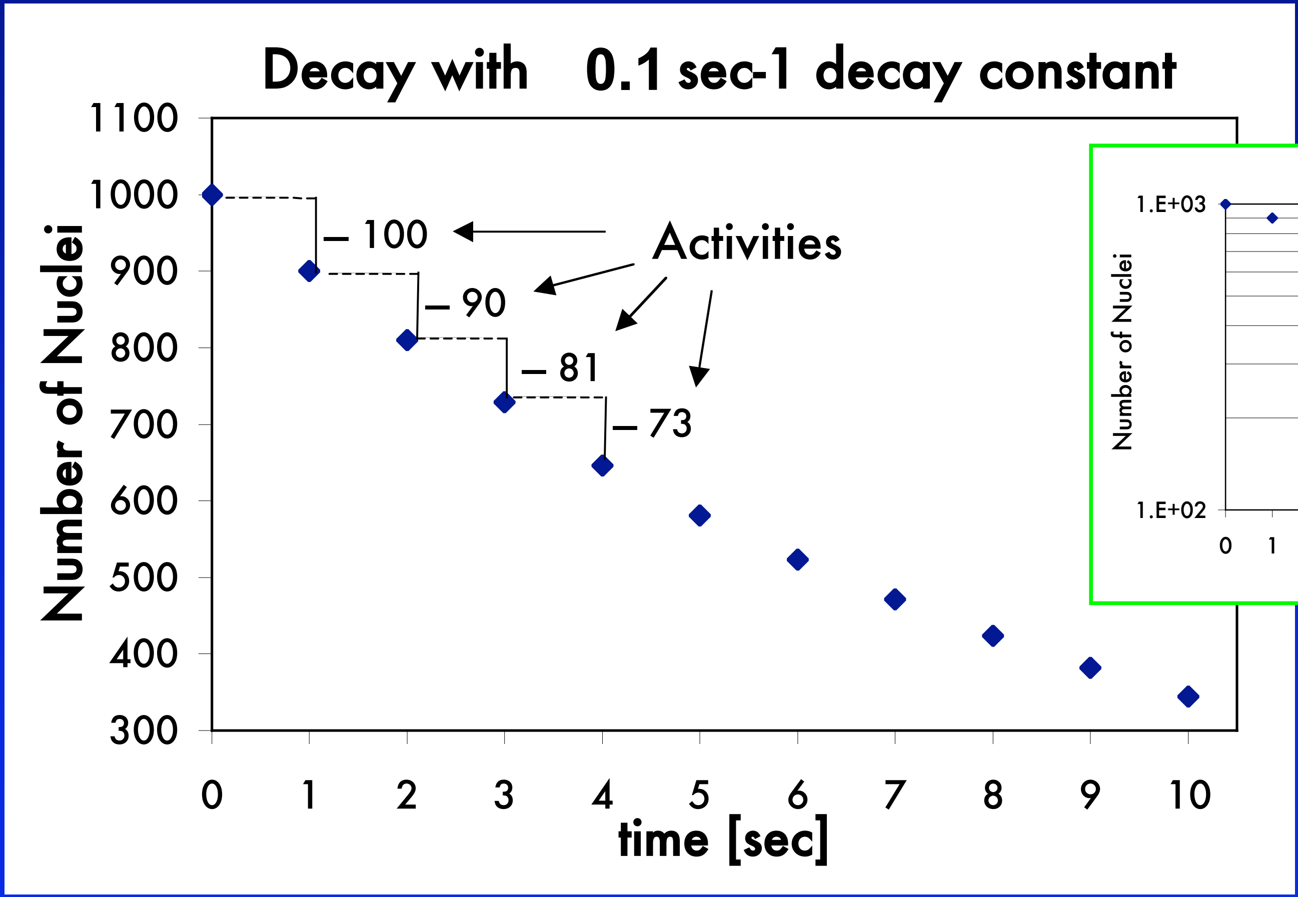
Total decay constant is sum of modes: $\lambda = \lambda_1 + \lambda_2 + \lambda_3 + \ldots$

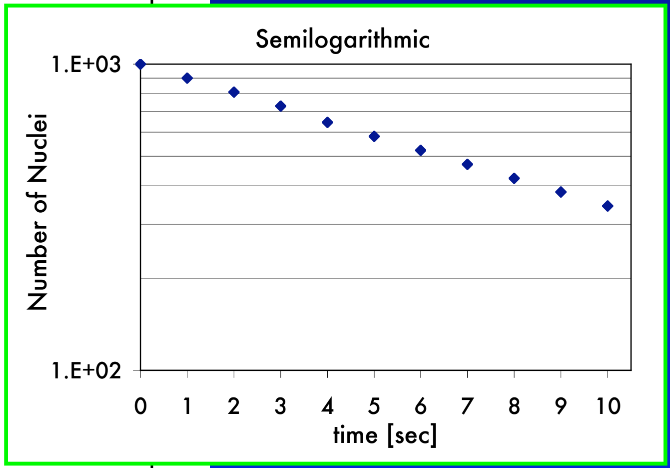
- Branching ratio = fraction in particular mode: $BR_i = \lambda_i / \lambda$

Example: ^{18}F: 97% β^+, 3% Electron Capture (EC)

Radioactive Decay

Example: N(0) = 1000, λ = 0.1 sec^{-1}





Both the number of nuclei and activity decrease by λ with time

Exponential Decay Equation

The number of decaying <u>and remaining</u> nuclei is proportional to the original number: $dN/dt = -\lambda \cdot N$

$$\Rightarrow^{*} \boxed{N(t) = N(0) \cdot e^{-\lambda \cdot t}}$$

This also holds for the activity (number of decaying nuclei):

$$\boxed{A(t) = A(0) \cdot e^{-\lambda \cdot t}}$$

<u>Decay Factor</u>: $\boxed{e^{-\lambda \cdot t}}$

= N(t)/N(0): fraction of nuclei remaining

or = A(t)/A(0): fraction of activity remaining

* Mathematical derivation: $\Rightarrow$ $dN/N = -\lambda \cdot dt \Rightarrow \int 1/N \, dN = \int -\lambda \, dt$

Exponential Equations

- Can describe growth or decrease of an initial sample.
- Describe the disappearance/addition of a constant fraction of the present quantity in a sample in any given time interval.
- Rate of change is greater than linear.
- The number of decays is proportional to the number of undecayed nuclei.

General Math for exponentials:

- inverse is natural logarithm: ln: $\ln(e^{-\lambda t}) = -\lambda t$
- additive exponents yield multiplicative exponentials:
  $e^{-(\lambda_1 t + \lambda_2 t + \lambda_3 t + \ldots)} = e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} \cdot e^{-\lambda_3 t} \cdot \ldots$
- for small exponents ($-\lambda t$): approximate $e^{-\lambda t} \approx 1 - \lambda t$

Half-Life

Radioactive decay shows disappearance of a constant fraction of activity per unit time

Half-life: time required to decay a sample to 50% of its initial activity: $1/2 = \exp(-\lambda \cdot T_{1/2})$

$$\Rightarrow \boxed{\begin{aligned} T_{1/2} &= \ln 2 / \lambda \\ \lambda &= \ln 2 / T_{1/2} \end{aligned}} \qquad (\ln 2 \approx 0.693)$$

Constant in time, characteristic for each nuclide

$\Rightarrow$ Convenient to calculate the decay factor in multiples of $T_{1/2}$:

$$A(t)/A(0) = \exp(-\ln 2 \cdot t/T_{1/2}) = (1/2)^{t/T_{1/2}}$$

Some Clinical Examples

Radionuclide	$T_{1/2}$	λ
Fluorine 18	110 min	0.0063 min^{-1}
Technetium 99m	6.2 hr	0.1152 hr^{-1}
Iodine 123	13.3 hr	0.0522 hr^{-1}
Molybdenum 99	2.75 d	0.2522 d^{-1}
Iodine 131	8.02 d	0.0864 d^{-1}

Average Lifetime

Average time until decay varies between nuclides from 'almost zero' to 'almost infinity'.

This average lifetime τ is characteristic for each nuclide and doesn't change over time:

$$\tau = 1/\lambda = T_{1/2} / \ln 2$$

τ is longer than the half-life by a factor of 1/ln 2 ($\approx$ 1.44)

Average lifetime important in radiation dosimetry calculations

Determining Decay Factors ($e^{-\lambda t}$)

Table of Decay factors for Tc-99m

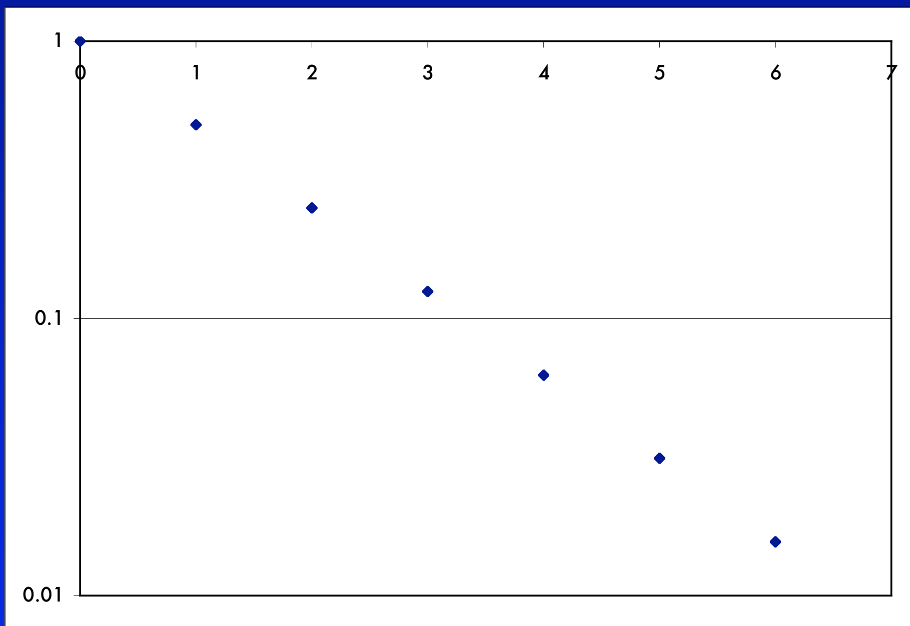
Hours \ Minutes	0	15	30	45
0	1.000	0.972	0.944	0.917
1	0.891	0.866	0.844	0.817
3	0.707	0.687	0.667	0.648
5	0.561	0.545	0.530	0.515
7	0.445	0.433	0.420	0.408
9	0.354	0.343	0.334	0.324

Read off DF directly or combine for wider time coverage:

$DF(t_1+t_2+..) = DF(t_1) \times DF(t_2) \times ...$

Other Methods to determine DF

- From a graph of activity fraction vs. time in half-lives: A(t)/A(0) vs.



Determine the time at which the DF is needed as multiples of $T_{1/2}$.
Read the corresponding DF number off the chart.

- Pocket calculator with exponential/logarithm functions

Examples

Tc-99m:

Minutes / Hours	0	15	30	45
0	1.000	0.972	0.944	0.917
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What is the DF after 6 hrs? (Remember, exponentials are multiplicative)

DF(6hrs) = DF(5hrs) x DF(1hr) = DF(3hrs) x DF(3hrs) (any combination)

DF(6hrs) = 0.5. So, $T_{1/2}$ is 6 hours!

Another Example

C-11 has a half-life of 20 minutes. Initial sample has 1000 nuclei.

Q: How many are left after 40 minutes?
After 80 minutes?
When is less than 1 left?

A: After 2 half-lives (40 min) $1/2^2 = 1/4$ of the initial activity is left (25%).
After 4 half-lives (80 minutes) it's 1/16th left (6.25%).
To have less than one left, one needs a DF $\geq 1/1000$.
This happens after 10 half lives, because $(1/2)^{10} = 1/1024$, so after 200 minutes (3hrs 20min).

More Examples

Now consider ^{18}F (positron decay) $T_{1/2} = 110$min.

Q: Start with 1000, when is less than one left?

A: Again after 10 half-lives, but here this is 1100 minutes (18.3 hours)

Patient injected with 10 mCi F-18 FDG, scan started 60 min later. F-18 has $\lambda = 0.0063\ \text{min}^{-1}$, $T_{1/2} = 110$ min.

Q: How much activity is present in the scan?

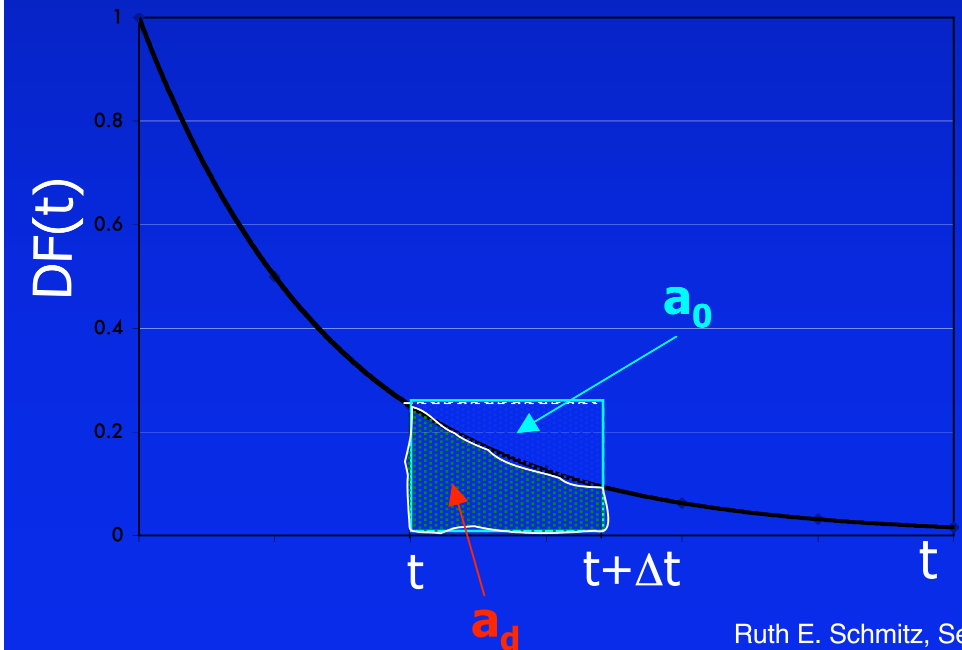
$$\Rightarrow \text{A:} \quad A(60\text{min}) = A_0 \times e^{-\lambda t} = 10\text{mCi} \times e^{-(60\text{min}*0.0063/\text{min})}$$
$$= 10\ \text{mCi} \times 0.685 = 6.85\ \text{mCi}$$

Image Frame Decay Correction

If the acquisition time of an image frame Δt is not short compared to the half-life of the used nuclide, then the nuclide is decaying measurably during the acquisition time.

To calculate the number of decays in this frame, need to correct the decay factor for it's change during Δt, because $a_0 = DF(t)*\Delta t \neq a_d$.

$\Rightarrow DF_{eff}(t,\Delta t) = DF(t) * a_d/a_0 = DF(t) * [(1\text{-}exp(-x))/x], \quad x = ln\ 2 * \Delta t/T_{1/2}$



Approximations for small x:

$DF_{eff} \approx 1/2 * [DF(t)+DF(t+\Delta t)]$

$DF_{eff} \approx DF(t+(\Delta t/2))$

$DF_{eff} \approx DF(t) * [1 - (x/2)]$

Example - Effective Decay Factor

^{15}O study – image frame is 30 - 45 sec after injection.

How much difference does it make to use or not to use the exact effective DF, and is approximation ok?

$T_{1/2}(^{15}O)=124$ sec.

$\Rightarrow$ $DF(30sec) = \exp(-\ln 2 * t / T_{1/2}) = 0.846$

Params: $x = \ln 2*(\Delta t/ T_{1/2}) = 0.084$, $[1-\exp(-x)]/x \approx 0.959$

- Exact Solution:

$\Rightarrow$ $\mathbf{DF_{eff} =}$ DF(30 sec) * [1-exp(-x)]/x **= 0.811**

Decay correction factor for this frame: 1/0.811 = 1.233

- With approximation: $\mathbf{DF_{eff} =}$ DF(t) * [1 - (x/2)] **= 0.810**

$\Rightarrow$ Approximation is very good in this case!

Specific Activity

- Ratio of total radioisotope activity to total mass of the *element* present
- Measured in Bq/g or Bq/mole
- Useful if radioactive and non-radioactive isotopes of the same element are in mixture (sample *with carrier*)
- Maximum Specific Activity in *carrier-free* samples:

Carrier Free Specific Activity (CFSA)

CFSA [Bq/g] ≈ $4.8 \times 10^{18} / (A \times T_{1/2})$ for $T_{1/2}$ given in days, A: atomic weight

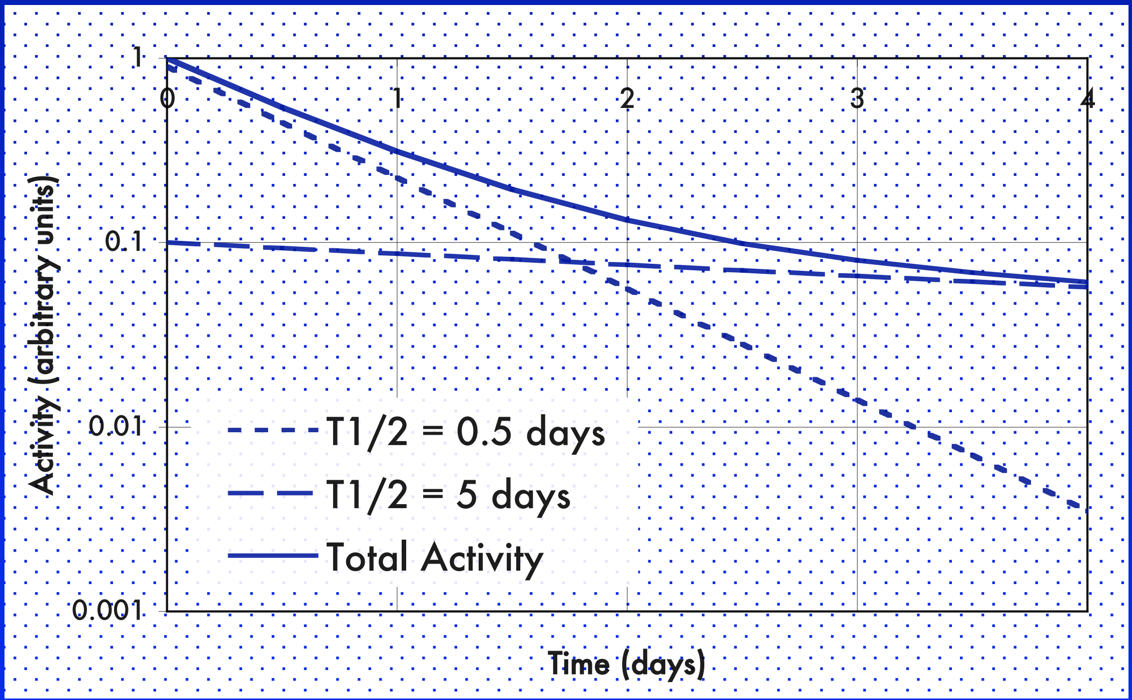
CFSA(Tc-99m) = $4.8 \times 10^{18} / (0.99 * 10^{2} * 0.25)$ = 1.9×10^{17} Bq/g

= 5.3×10^{6} Ci/g

Samples of Mixed Radionuclides

For unrelated radionuclides in a mixture, the total Activity is the sum of the individual activities:

$A_t(t)=A_1(0)*\exp(-\ln 2 * t/T_{1/2,1}) + A_2(0)*\exp(-\ln 2 * t/T_{1/2,2}) + \ldots$



At(t) always eventually follows the slope of the longest half-life nuclide. This can be used to extrapolate back to the contributors to the mix.

Parent-Daughter Decays

Special Case: a sample contains both a radionuclide parent and its radioactive daughter.

Parent decay equation as usual, daughter more involved: it is being formed from parent while decaying itself:

<u>Bateman equation</u>:

$$A_d(t) = \{[\, A_p(0)\, \lambda_d/(\lambda_d - \lambda_p) * \{exp(-\lambda_p * t) - exp(-\lambda_d * t)\}\,] * B.R.\}$$
$$+ A_d(0) * exp(-\lambda_d * t)$$

-- Typically $A_d(0)=0.$

From: $dN_p/dt = -\lambda_p N_p$

$$dN_d/dt = -\lambda_d N_d + \lambda_p N_p$$

$$N_d(t) = N_p(0)\, \lambda_p/(\lambda_d - \lambda_p) * \{exp(-\lambda_p * t) - exp(-\lambda_d * t)\}\,] +$$
$$+ N_d(0) * exp(-\lambda_d * t)$$

Secular Equilibrium

Parent-Daughter Decay Special Case:

Half-life of parent very long, decrease of A_p negligible during observation, $\lambda_p \approx 0$.

Example: Ra-226 ($T_p \approx$ 1620yr) → Rn-222 ($T_d \approx$ 4.8 days)

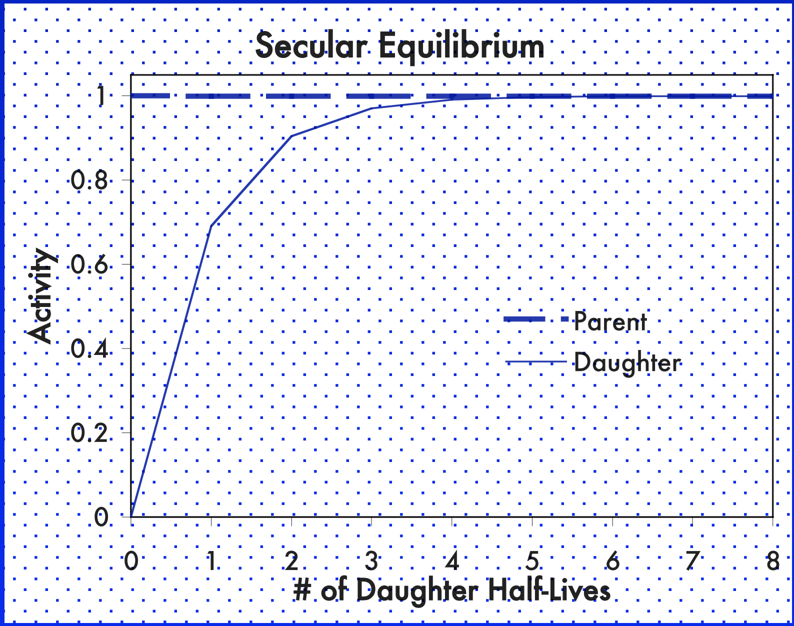
Bateman Approximation: $A_d(t) \approx A_p(0)\,(1 - \exp(-\lambda_d t)) * BR$

After one daughter half-life T_d:

$$A_d(t) \approx 1/2\, A_p(t)$$

After $2{*}T_d$: $A_d(t) \approx 3/4\, A_p(t)$

Eventually $A_d \approx A_p$, asymptotically approaches equilibrium



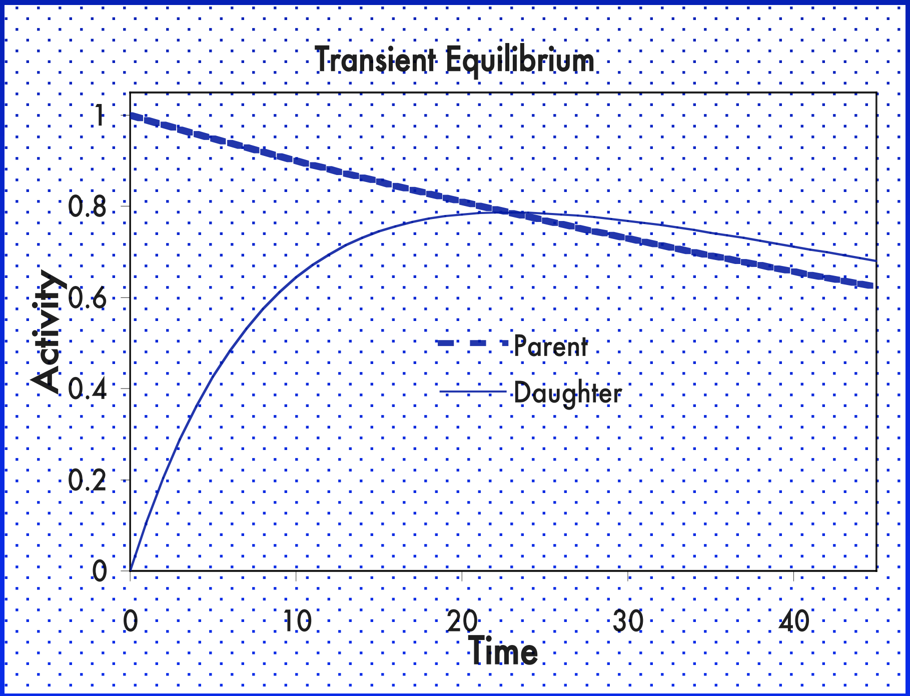
Transient Equilibrium

Parent Half-life longer than Daughter's, but not 'infinite'
Example: ^{99}Mo ($T_{1/2}$ = 66hr) $\rightarrow$ ^{99m}Tc ($T_{1/2}$ = 6 hr)

A_p decreases significantly over the observation time, so $\lambda_p \not\approx 0$.

The daughter activity increases, exceeds that of the parent, reaches a maximum value, then decreases and follows the decay of the parent. Then the *ratio of daughter to parent is constant*: transient equilibrium.

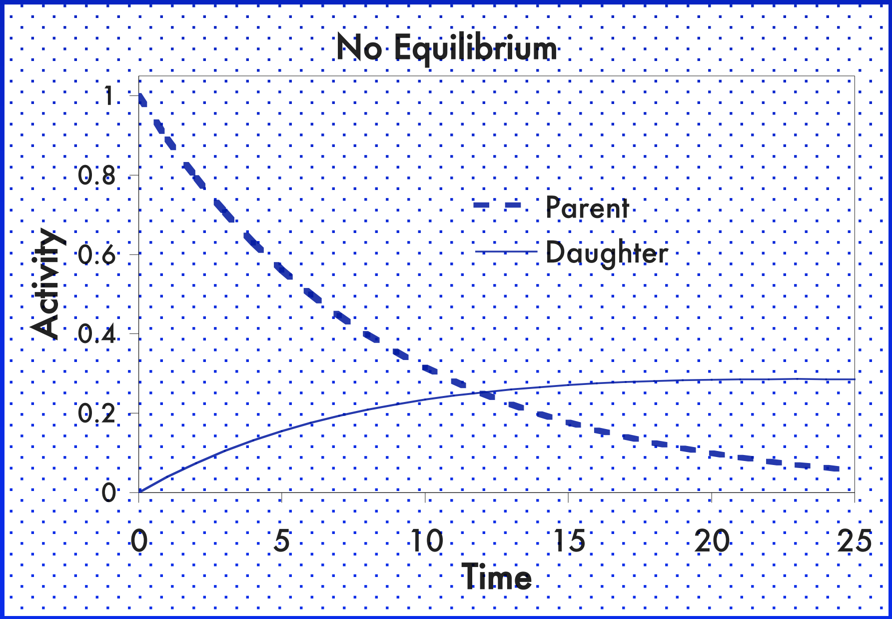
$$A_d/A_p = [T_p/(T_p - T_d)] * BR$$



No Equilibrium

If $T_d > T_p$, no equilibrium is possible.
Example: ^{131m}Te ($T_{1/2}$ = 30hr) ^{131}I ($T_{1/2}$ = 8 days)

Buildup and decay of daughter shows maximum again. When parent distribution gone, further daughter decay is according to the daughter's decay equation only.



Summary

Mathematical description of decay only yields probabilities and averages.

Decay equation: $N(t) = N(0) \cdot e^{-\lambda \cdot t} = N(0) \cdot e^{-\ln 2 / T_{1/2} \cdot t}$

λ: decay constant, $T_{1/2} = \ln 2 / \lambda$: half-life,
$\tau = 1 / \lambda$: average lifetime

Decay factor (DF) = $e^{-\lambda \cdot t}$

Image frame decay correction - effective decay factors

Mixed, and parent-daughter decays

NEXT WEEK: Dr. Miyaoka - Interactions with Matter