Singular Value Decomposition and Empirical Orthogonal Functions (EOFs)

For any matrix **A**, one can always define matrices **U**, **L** and **V** such that:

$$\mathbf{A}(M \times N) = \mathbf{U}(M \times M)\mathbf{L}(M \times N)\mathbf{V}^T(N \times N).$$

We can write **U** and **V** as:

$\mathbf{U}=[\mathbf{u}_1\ \mathbf{u}_2\ \ldots\ \mathbf{u}_M]$, where $\mathbf{u}_i$ are $M \times 1$ column vectors, and

$\mathbf{V}=[\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_N]$, where $\mathbf{v}_i$ are $N \times 1$ column vectors.

L has the form:

$$\mathbf{L} = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ldots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_M & 0 & \ldots & 0 \end{bmatrix} \quad \text{for } M<N$$

or

$$\mathbf{L} = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_M \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} \quad \text{for } M>N$$

To match the situation in Assignment #3 and for simplicity, we will now just work with the case $M < N$.

This decomposition of a matrix is called the **singular value decomposition (SVD)**. The λ_i are called the **singular values**. (Sometimes the columns of **U** are called the **left singular vectors** and the columns of **V** the **right singular vectors**, sometimes "singular vectors" is used more casually to refer to them both.) The $\mathbf{u}_i$ are orthogonal to each other, as are the $\mathbf{v}_i$

The matrix **A** can be reconstructed by multiplying each column and row vector from, respectively, **U** and **V** with the corresponding singular value and adding them all together:

$$\mathbf{A} = \mathbf{u}_1(M \times 1)\lambda_1\mathbf{v}_1^T(1 \times N) + \mathbf{u}_2\lambda_2\mathbf{v}_2^T + \ldots$$

The decomposition is arranged so that $\sigma_i > 0$ and $\sigma_1 > \sigma_2 > ... > \sigma_M$.

We can geometrically interpret the column and row vectors that correspond to the largest singular value:

> $\mathbf{u}_1$ is the vector most parallel to columns of A
> $\mathbf{v}_1$ is the vector most parallel to rows of A
> σ_1 is the best fitting scale factor

In applying SVD to determine the dominant modes of climate variability, each row of the matrix **A** will represent a spatial map of a field at a given time. *M* is the number of observation points in time. *N* is the number of observation points in space. (We assume that the data has already been interpolated onto a regular grid in space and time, and the mean of each time-series has been removed.)

$$\mathbf{A} = \begin{bmatrix} T(\mathbf{x}_1,t_1) & T(\mathbf{x}_2,t_1) & \dots & T(\mathbf{x}_N,t_1) \\ T(\mathbf{x}_1,t_2) & T(\mathbf{x}_2,t_2) & \dots & T(\mathbf{x}_N,t_2) \\ \vdots & \vdots & \ddots & \vdots \\ T(\mathbf{x}_1,t_M) & T(\mathbf{x}_2,t_M) & \dots & T(\mathbf{x}_N,t_M) \end{bmatrix}$$

We can then interpret the singular vectors:

> $\mathbf{v}_1$ is the vector that captures most of the space-dependence of the field. It is called the first **Empirical Orthogonal Function** (EOF) of the field. $\mathbf{v}_2$ is called the 2nd EOF and so on. The EOF is non-dimensional.
> $\mathbf{u}_1\sigma_1$ is describes the evolution in time of this spatial function. It is often referred to as the **Principal Component** (PC). The time-series has the units of the original data.

(The matrix of observations can be defined the other way with M space points, N time points and the physical interpretations reversed.)

The fraction of total variance explained by the *i*th EOF is given by:

$$\frac{\sigma_i^2}{\sum_{j=1}^{M} \sigma_j^2}$$

Notes: Since we represent two dimensions of space with one dimension across the rows of matrix A, the "map" has to be carefully constructed. Usually, to transform a 2-D map into a 1-D row of the matrix, one goes from left to write, top to bottom as if one were "reading" the data at each grid-point. Keep this in mind when you do the exercise.

Glossary

There are a confusing number of different terms for the same thing that have evolved in different fields.

The process	$\mathbf{v}_i$	$\square_i$	$\mathbf{u}_i \square_i$
EOF analysis	EOF	Singlar value	Amplitude time-series
Eigenvector analysis	Eigenvector	Eigenvalue	Principal Component
Principal Component Analysis (PCA)	Principal Component loading pattern		Expansion coefficients
			Expansion coefficient time-series

There are also a number of variations on the method:

Joint EOF analysis. Applies the method to two different fields. Given data matrices, as above, for two fields A and B, form the cross-covariance matrix, $C = A^T B$. A SVD is made on matrix C to find matrices U, L and V as above. Now the $\mathbf{u}_i$ and $\mathbf{v}_i$ are interpreted as spatial patterns of co-variability modes for the two fields. The columns of AU and BV contain the expansion coefficients of each mode.

Complex EOF analysis (CEOF). Used to analyze velocity records, as $u + iv$, where u is east-west velocity and v is north-south velocity. This allows for rotation of the field as observed in velocity for various waves.

Rotated EOF analysis, *Varimax rotation of EOFs*. Takes linear combinations of the EOFs and projects them onto the original data matrix to determine the corresponding expansion coefficient time-series. It is an attempt to fix problems with dependence of the results on the domain being analyzed.

Extended EOF analysis (EEOF). An extension of EOF analysis that allows for time-lagged covariance.