Potential vorticity = $(\zeta+f)/H$

$$\frac{D}{Dt}\left(\frac{f+\zeta}{H}\right)=0$$ (PV is conserved unless direct forcing or friction)

What is the relative importance of f and ζ?

$PV = (f+\zeta)/H$. Sometimes ζ can be neglected relative to f. When is this appropriate? $\zeta \sim du/dy$, so depends on size of velocity gradients.

$$\frac{\varsigma}{f}\approx\frac{U/L}{f}=\frac{U}{fL}=Rossby\ number$$

When the Rossby number is small, relative vorticity can be neglected.

$$f\sim 5\times 10^{-5}\ s^{-1}$$

In the interior, velocity gradients are small, something like:

$$\frac{U}{L}\sim\frac{10\ cm/s=10^{-1}\ m/s}{1000\ km=10^{6}m}\approx 10^{-7}\ s^{-1}$$

For a western boundary current, however, ζ might be something like

$$\frac{U}{L}\sim\frac{1\ m/s}{100\ km=10^{5}m}\approx 10^{-5}\ s^{-1}$$

So relative vorticity can be important in strong flows with relatively short length scales, but is not important in large-scale flows like the broad ocean gyres.

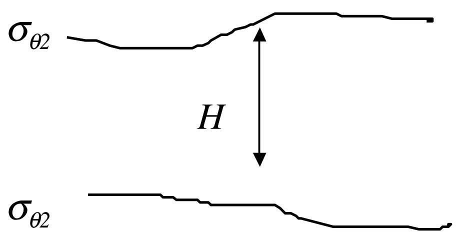
Large-scale potential vorticity PV = f/H = const.

Gradients in f as dynamical barrier. X flow vs. y flow.

What is *H*?

Homogeneous (barotropic) : *H* = total fluid depth, determined by topography (this tends to happen more at high latitudes where the ocean is more weakly stratified and f is larger). In this case, the motion tends to be in the same direction from the top to the bottom of the fluid.

Stratified (baroclinic) H = thickness of a layer between isopycnals. (this happens more in the subtropics). In this case, the flow can change directions between layers.



For a stratified ocean, we think of a fluid parcel being contained between two isopycnal surfaces and it is this thickness that is relevant to the stretching or squashing of a fluid column.

In the limit of continuous stratification,

$$N^2 = -\frac{g}{\rho_0}\frac{\partial \rho}{\partial z} \approx \frac{g}{\rho_0}\frac{\Delta \rho}{H}$$

So: large-scale PV = f/H ~ fN2

Where does conservation of potential vorticity break down?

Boundaries (friction)
Surface -- is directly torqued or stratification is changed by heating and cooling.
Can be changed by interior mixing.
Hydrothermal vents. -- act as a source or sink (by changing directly changing stratification through buoyancy forcing.

(ii) Ekman Pumping and PV conservation

Vertical velocity induced by Ekman transport divergence causes vortex stretching and drives flow across lines of planetary vorticity to conserve potential vorticity.

$\frac{f}{H} = const$ implies flow towards the equator under regions of Ekman pumping.

$H \downarrow \quad so \quad f \downarrow$

Formally:

To derive the vorticity balance for the large scale ocean circulation, we start with mass conservation. The ocean is nearly incompressible, and mass conservation is equivalent to the statement of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

If we assume that in the ocean interior away from the surface, the flow is geostrophic, we have

$$fu_g = -\frac{1}{\rho}\frac{\partial P}{\partial y}$$

$$fv_g = \frac{1}{\rho}\frac{\partial P}{\partial y}$$

If we plug in these expressions into the continuity equation, we have

$$\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$-\frac{\partial}{\partial x}\frac{1}{\rho f}\frac{\partial P}{\partial y} + \frac{\partial}{\partial y}\frac{1}{\rho f}\frac{\partial P}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial}{\partial y}\left(\frac{1}{\rho f}\right)\frac{\partial P}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\frac{-\beta}{f^2 \rho}\frac{\partial P}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\frac{-\beta}{f} v_g + \frac{\partial w}{\partial z} = 0$$

or

$$\beta v_g = f\frac{\partial w}{\partial z}$$

To find the total geostrophic transport, we integrate to the base of the wind-driven ocean, which we think of as the base of the thermocline, about 300-500m. Then we have

$$\beta \int_{z=-D_W}^{z=-D_E} v\partial z = f\left[w_E - w(-D_W)\right] \qquad \textit{KEY ASSUMPTION}$$

$$= f w_E$$

$$\int_{z=-D_W}^{z=-D_E} v\partial z = \frac{1}{\rho\beta}\left(\frac{\partial \tau_W^y}{\partial x} - \frac{\partial \tau_W^x}{\partial y}\right) + \frac{1}{\rho f}\tau_W^x$$

$$T_{GEO}^y = \frac{1}{\rho\beta}\nabla \times \underline{\tau} - T_{Ek}^y$$

$$T_{GEO}^y + T_{Ek}^y = \frac{1}{\rho\beta}\nabla \times \underline{\tau}$$

Interpretation:

Sverdrup transport = geostrophic transport + Ekman transport

$$T_{Sverdrup}^y = \frac{1}{\rho\beta}\left(\frac{\partial \tau_W^y}{\partial x} - \frac{\partial \tau_W^x}{\partial y}\right)$$

Again, these are transports per unit meter of horizontal distance, in units of m2/s. To get transport across the basin, integrate, or multiply the average value by the width in meters.

Get zonal transport predicted by this theory by integrating continuity equation from condition of u=0 at eastern boundary. $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$$T^x_{GEO} - \cancel{T^{x=x_E}_{GEO}} = \int_x^{x_E} \frac{\partial T^y_{GEO}}{\partial y} dx$$

Summary:

1. Ekman transport 90° from the direction of the wind.
2. Curl in the wind field implies convergence or divergence of the Ekman transport.
3. To conserve mass there must be upwelling or downwelling at the base of the Ekman layer.
4. How will the layer below the Ekman layer respond? Stretched or squashed…will change latitude to conserve f/H.