Team A: $\tanh(x)$ on $[-2\pi, 2\pi]$.
Team B: $1/(1 + \cos^2 x)$ on $[-3\pi, 3\pi]$.
Team C: $e^{-x^2}$ on $[-\pi, \pi]$.
Team D: $(1 - x^2) \ln(1 - x^2)$ on $[-1, 1]$.
Team E: $1/(e^x + 1)$ on $[-1, 1]$.
Team F: $e^{-1/(1-x^2)}$ on $[-1, 1]$.
Team G: $\sin(x^2)$ on $[-3\sqrt{\pi}, 3\sqrt{\pi}]$.
Team H: $\sqrt{4 - x}$ on $[-2, 2]$.

For each function $f(x)$ defined on an interval $[-\ell, \ell]$:

1. Approximate the function with a truncated power series, $f_{\text{Power}}^{(N)}(x) = \sum_{k=0}^{N} a_k (x/\ell)^k$, constructed to agree with the given function at $N+1$ equally spaced between $-\ell$ and $\ell$ (including the two endpoints), so the spacing between points is $\Delta x = 2\ell/N$.

2. Approximate the function by a truncated Fourier series, $f_{\text{Fourier}}^{(N)}(x) = \sum_{k=-N/2}^{N/2} b_k e^{i\pi kx/\ell}$, constructed to agree with the given function at $N+1$ equally spaced points within the interval with spacing $\Delta x = 2\ell/(N+1)$.

3. Approximate the function by a truncated Chebyshev series, $f_{\text{Chebyshev}}^{(N)} = \sum_{k=0}^{N} c_k T_k(x/\ell)$, constructed to agree with the given function on the “Gauss-Lobatto” grid points, $x_k^{(N)} \equiv -\ell \cos(k\pi/N)$, $k = 0, \cdots, N$.

Examine the convergence and accuracy of these different approximations as $N$ increases. For each approximation, estimate the maximum error, $\epsilon(N) \equiv \max_{-\ell \leq x \leq \ell} |f(x) - f^{(N)}(x)|$. The goal is to find the maximum error correct to one significant digit. Highly accurate maximization is not needed — it is sufficient to test the deviation on a grid of points which is significantly finer than the starting $N+1$ point grid.

What can you learn from these results? In each case, how fast do the coefficients decrease with increasing $k$? What controls this behavior? Which expansion is most efficient (fewest terms required to obtain good accuracy) for your given test function? Any general conclusions?