Contents

1 Lyapunov spectrum: stadium

Expected difficulty: 2

The *Lyapunov spectrum* of a classical dynamical system characterizes the rates of divergence of neighboring trajectories. Let $\mathbf{z}(t;\mathbf{z}^0)$ denote the trajectory (in phase space, for Hamiltonian systems) originating at the point $\mathbf{z}^0$ at time $t = 0$. If this is an N-dimensional dynamical system, then $\mathbf{z}$ is an N-dimensional vector with components z_i, $i = 1, \cdots, N$. The Jacobian matrix

$$J(t) = \|J(t)_{ij}\| \equiv \left\| \frac{\partial z(t;\mathbf{z}^0)_i}{\partial z_j^0} \right\|$$

measures the sensitivity of the location at time t to infinitesimal changes in initial conditions at time 0. The Lyapunov spectrum is defined as the set of eigenvalues of the matrix

$$\Lambda \equiv \lim_{t\to\infty} \frac{1}{2t} \ln(J(t)J(t)^T)\,.$$

Positive eigenvalues indicate exponential divergence (with time) of neighboring trajectories. Systems with one or more positive Lyapunov exponents are called *chaotic*.

Problem: A point particle is confined to a *stadium*, defined as the region which is the union of a square of size L plus two semicircular disks (of diameter L) which cap opposite sides of the square. (So the boundary of the stadium has two straight segments and two semicircular arcs, connected in a smooth continuous manner.) The particle moves freely (i.e., along straight lines) inside the stadium, and reflects off the boundary of the stadium, with the angle of reflection equal to the angle of incidence. For a variety of initial positions and directions of motion, calculate (and visualize) trajectories and evaluate the Lyapunov spectrum. Does it depend on the initial conditions? What is the maximal Lyapunov exponent? Is there more than one positive exponent? What is the sum of all exponents? Is the system chaotic?

Suggestions: To evaluate the Lyapunov matrix Λ one must well-approximate both the long time limit, and the infinitesimal variation implied by the derivative with respect to initial conditions. If $\mathbf{z}(t;\mathbf{z}^0{+}\boldsymbol{\epsilon})$ is the trajectory resulting from a small but finite displacement $\boldsymbol{\epsilon}$ in initial conditions, then the deviation $\Delta\mathbf{z}(t) = \mathbf{z}(t;\mathbf{z}^0{+}\boldsymbol{\epsilon}) - \mathbf{z}(t;\mathbf{z}^0)$ may grow with time and eventually no longer be (nearly) linear in $\boldsymbol{\epsilon}$. Which means that $\Delta\mathbf{z}(t)_i/\boldsymbol{\epsilon}_j$ no longer provides a good approximation to $J(t)_{ij}$. To avoid this breakdown, one needs to monitor the size of $\Delta\mathbf{z}(t)$ as the time integration proceeds, and switch from a perturbed trajectory $\mathbf{z}(t;\mathbf{z}^0{+}\boldsymbol{\epsilon})$ to a nearby "parallel" but less perturbed trajectory $\mathbf{z}(t;\mathbf{z}^0{+}\boldsymbol{\epsilon}/s)$, for some $s > 1$, whenever the deviation gets too big (all the while both remaining in the regime of linearized deviations, and avoiding bad precision loss in the difference).

2 Quantum density of states: 1D quartic well

Expected difficulty: 3

A non-relativistic particle of mass m moves in one dimensions subject to a quartic potential,

$$V(x) \equiv \lambda\, x^4\,,$$

with $\lambda > 0$. Let $N(E)$ denote the total number of eigenstates of the Hamiltonian with energy less then or equal to E. This is the *integrated density of states.* At sufficiently high energies, the integrated density $N(E)$ should be well approximated by the quasi-classical result,

$$N(E)_{\text{quasiclassical}} \equiv \int \frac{dp\,dx}{2\pi\hbar}\,\Theta(E - h(p,x))\,,$$

where $\Theta(z)$ is a unit step function and $h(p,x) \equiv \frac{1}{2m}\,p^2 + V(x)$ is the classical Hamiltonian. The integral $N(E)_{\text{quasiclassical}}$ is the volume in phase space (divided by $2\pi\hbar$) in which the classical energy is less than E.

Warm-up: Calculate $N(E)_{\text{quasiclassical}}$. Explain why $N(E) = f(m^{2/3}\lambda^{-1/3}E)$ (using units where $\hbar \equiv 1$), with f some univariate function of the indicated argument.

Problem: Accurately compute $N(E)$ from low energies up to sufficiently high energies to see the approach to the quasiclassical result. Examine the difference $N(E) - N(E)_{\text{quasiclassical}}$ and characterize how fast the quantum result converges to the quasiclassical approximation. (I.e., what are subleading terms in the large E asymptotic behavior?) How accurately can you extract the large E asymptotic form of $N(E)$ from your data?

Suggestions: Perform a (pretty big) finite basis calculation, using a good choice of basis set (or spectral representation). Carefully study the stability of the resulting approximate energy eigenvalues as the size of the basis set is increased and determine, for a given size calculation, how many of the resulting eigenvalues are reliable (at a given level of precision), and how many are numerical garbage.

3 Double well: ground state splitting

Expected difficulty: 3

A non-relativistic particle of mass m moves in one dimension subject to the potential

$$V(x) \equiv \lambda\,(x^2 - a^2)^2\,,$$

with $\lambda > 0$. Only one (independent) dimensionless quantity may be constructed from the parameters of the theory. Using units in which $\hbar = 1$, this may be chosen to be $\eta \equiv \sqrt{m\lambda}\,a^3$. Energy levels in this theory may be written in the form $E_n = (\lambda a^2/m)^{1/2}\epsilon_n(\eta)$, where ϵ_n is some univariate function. Justify these assertions.

Problem: Accurately calculate the functions ϵ_0, ϵ_1, and their difference $\delta\epsilon \equiv \epsilon_1 - \epsilon_0$, for a wide range of values of η. Consider the asymptotic behavior of these functions for both large and small values of η. One may show that

$$\epsilon_n(\eta) \sim c_n^{(0)}\,\eta^{-1/3} + c_n^{(1)}\,\eta^{1/3} + c_n^{(2)}\,\eta + c_n^{(3)}\,\eta^{5/3} + \cdots\,,$$

as $\eta \to 0$. And

$$\epsilon_n(\eta) \sim b_n^{(0)} + b_n^{(1)}\,\eta^{-1} + b_n^{(2)}\,\eta^{-2} + b_n^{(3)}\,\eta^{-3} + \cdots\,,$$

as $\eta \to \infty$. But the difference $\delta\epsilon$ vanishes faster than any inverse power of η as $\eta \to \infty$,

$$\delta\epsilon \sim A\,\eta^B\,e^{-K\eta}\,.$$

(Can you justify these asymptotic forms?) Using your numerical results, what are the best estimates you can produce for the values of the coefficients in these asymptotic forms?

Suggestions: You will need highly accurate results, for widely varying values of η, to extract good asymptotic forms. You will have to carefully study the stability of your extraction of asymptotic coefficients, and make sure that inaccuracies in the computed energies are not significantly corrupting your estimates of the asymptotic coefficients.

Related reading:

S. Coleman, *The uses of instantons*, in *Aspects of Symmetry*, Cambridge, 1985.

4 Anharmonic oscillator analyticity

Expected difficulty: 4

A non-relativistic particle of mass m moves in an oscillator potential with quartic anharmonicity,

$$V(x) \equiv \tfrac{1}{2} m\Omega^2 x^2 + \lambda\, x^4 \,.$$

Explain why the ground state energy must have the form $E_0 = \Omega\, \epsilon(\lambda m^{-2} \Omega^{-3})$, for some dimensionless function ϵ.

Standard Raleigh-Schrodinger perturbation theory shows that ϵ has an expansion of the form

$$\epsilon(z) \sim c^{(0)} + c^{(1)}\, z + c^{(2)}\, z^2 + c^{(3)}\, z^3 + \cdots \,,$$

but this expansion is only asymptotic, not convergent. Consequently, $\epsilon(z)$ is not analytic in a neighborhood of the origin; in fact it has a branch point at $z = 0$ and (may be defined to have) a branch cut running along the negative real axis. One may show that the large order behavior of perturbation theory (i.e., the above expansion in powers of z) is related to the behavior of the discontinuity of $\epsilon(z)$ across this branch cut for small negative values of z.

Warm-up: Consider the Schrodinger equation when λ is an arbitrary complex number with $-\pi < \arg\lambda < \pi$. Any solution must behave as

$$\psi(x) \sim C_+ \,\exp(\tfrac{1}{3}\sqrt{2\lambda}\, x^3) + C_- \,\exp(-\tfrac{1}{3}\sqrt{2\lambda}\, x^3)$$

as $x \to +\infty$, and as

$$\psi(x) \sim C'_+ \,\exp(\tfrac{1}{3}\sqrt{2\lambda}\, x^3) + C'_- \,\exp(-\tfrac{1}{3}\sqrt{2\lambda}\, x^3)$$

as $x \to -\infty$, for some values of $C_\pm$ and $C'_\pm$. The solution is normalizable (i.e., does not blow up as $x \to \pm\infty$) only if $C_+ = C'_- = 0$. Only for discrete (complex) values of the energy E will a solution exist which satisfies both asymptotic conditions. These are the analytic continuations of the energies of the usual bound states with λ real and positive. When $\arg\lambda \to \pm\pi$, the potential $V(x)$ becomes real but is not longer bounded below, and the above-described solutions describe *resonances* with complex energies and purely outgoing (or incoming) behavior at large distance. Justify these assertions.

Problem: Let $\lambda = |\lambda|\, e^{i\theta}$. Calculate, for a wide variety of values of $|\lambda|$, the ground state energy as a function of θ, and extract the discontinuity $\Delta E_0 = E_0(|\lambda| e^{i\theta})\big|_{\theta=-\pi}^{\theta=\pi}$. Characterize the behavior of this discontinuity as $|\lambda| \to 0$ and $|\lambda| \to \infty$.

Related reading:

G. Parisi, *Asymptotic estimates in perturbation theory*, Phys. Lett. 67B:167 (1977).

R. de la Madrid, *A pedestrian introduction to Gamow vectors.*

5 Lyapunov spectrum: 2D x^2y^2 potential

Expected difficulty: 4

The *Lyapunov spectrum* of a classical dynamical system characterizes the rates of divergence of neighboring trajectories. Let $\mathbf{z}(t;\mathbf{z}^0)$ denote the trajectory (in phase space, for Hamiltonian systems) originating at the point $\mathbf{z}^0$ at time $t=0$. If this is an N-dimensional dynamical system, then $\mathbf{z}$ is an N-dimensional vector with components z_i, $i=1,\cdots,N$. The Jacobian matrix

$$J(t) = \|J(t)_{ij}\| \equiv \left\| \frac{\partial z(t;\mathbf{z}^0)_i}{\partial z_j^0} \right\|$$

measures the sensitivity of the location at time t to infinitesimal changes in initial conditions at time 0. The Lyapunov spectrum is defined as the set of eigenvalues of the matrix

$$\Lambda \equiv \lim_{t\to\infty} \frac{1}{2t} \ln(J(t)J(t)^T)\,.$$

Positive eigenvalues indicate exponential divergence (with time) of neighboring trajectories. Systems with one or more positive Lyapunov exponents are called *chaotic*.

Problem: A point particle moves in two dimensions subject to the potential $V(\mathbf{x}) \equiv \lambda x^2 y^2$. For a variety of initial positions and velocities, calculate (and visualize) trajectories and evaluate the Lyapunov spectrum. Does it depend on the initial conditions? What is the maximal Lyapunov exponent? Is there more than one positive exponent? What is the sum of all exponents? Is the system chaotic?

Suggestions: To evaluate the Lyapunov matrix Λ one must well-approximate both the long time limit, and the infinitesimal variation implied by the derivative with respect to initial conditions. If $\mathbf{z}(t;\mathbf{z}^0{+}\boldsymbol{\epsilon})$ is the trajectory resulting from a small but finite displacement $\boldsymbol{\epsilon}$ in initial conditions, then the deviation $\Delta\mathbf{z}(t) = \mathbf{z}(t;\mathbf{z}^0{+}\boldsymbol{\epsilon}) - \mathbf{z}(t;\mathbf{z}^0)$ may grow with time and eventually no longer be (nearly) linear in $\boldsymbol{\epsilon}$. Which means that $\Delta\mathbf{z}(t)_i/\boldsymbol{\epsilon}_j$ no longer provides a good approximation to $J(t)_{ij}$. To avoid this breakdown, one needs to monitor the size of $\Delta\mathbf{z}(t)$ as the time integration proceeds, and switch from a perturbed trajectory $\mathbf{z}(t;\mathbf{z}^0{+}\boldsymbol{\epsilon})$ to a nearby "parallel" but less perturbed trajectory $\mathbf{z}(t;\mathbf{z}^0{+}\boldsymbol{\epsilon}/s)$, for some $s>1$, whenever the deviation gets too big (all the while both remaining in the regime of linearized deviations, and avoiding bad precision loss in the difference).

6 Particle in stadium

Expected difficulty: 5

A non-relativistic point particle of mass m is confined to a *stadium*, defined as the region which is the union of a square of size L plus two semicircular disks (of diameter L) which cap opposite sides of the square. (So the boundary of the stadium has two straight segments and two semicircular arcs, connected in a smooth continuous manner.) The particle moves in a potential which is zero inside the stadium and infinite outside.

Warmup: What is the dependence of energy levels on the parameters m and L?

Problem: Accurately calculate as many energy levels, and associated wavefunctions, as you can. (A few hundred is a good goal.) Construct the distribution of spacings between adjacent energy levels. How does it compare to the corresponding distribution of a particle confined to a square region? Examine the spatial probability distributions for various energy levels, and show than many (but not all) energy levels have probability distributions which reflect the presence of periodic (but unstable) trajectories in the underlying classical system.

Suggestions: See, for example, this paper which uses finite difference approximations. Try to do better using suitable spectral approximations. To visualize the spatial probability distribution, one nice approach is to use the Metropolis Monte Carlo method to generate hundreds to thousands of points drawn from the desired distribution, and then just plot the positions of all of these points.

7 Quantum density of states: 2D x^2y^2 potential

Expected difficulty: 5

A non-relativistic particle of mass m moves in two dimensions subject to the potential

$$V(\mathbf{x}) \equiv \lambda\, x^2\, y^2 \,.$$

Consider the integrated density of states $N(E)$, defined as the total number of eigenstates of the Hamiltonian with energy less then or equal to E.

Warm-up: What does dimensional analysis imply about the dependence of $N(E)$ on m and λ? Explain why one may, without loss of generality, set $m = \lambda = 1$. Make a picture of the potential (a density or contour plot) which nicely illustrates the region in space in which the potential is less than a given value. Notice that this region extends arbitrarily far away from the origin. Consider the quasi-classical approximation to the integrated density,

$$N(E)_{\text{quasiclassical}} \equiv \int \frac{d^2p\, d^2x}{(2\pi\hbar)^2}\, \Theta(E - h(\mathbf{p}, \mathbf{x}))\,,$$

where $\Theta(z)$ is a unit step function and $h(\mathbf{p}, \mathbf{x}) \equiv \frac{1}{2m}\,\mathbf{p}^2 + V(\mathbf{x})$ is the classical Hamiltonian. This is the volume in phase space (divided by $(2\pi\hbar)^2$) in which the classical energy is less than E. Compute $N(E)_{\text{quasiclassical}}$ and show that it diverges, $N(E)_{\text{quasiclassical}} = \infty$ for all $E > 0$. Nevertheless, the quantum theory has only normalizable bound states and discrete energy levels. Can you explain why?

Problem: *Accurately* compute $N(E)$ from low energies up to sufficiently high energies where $N(E) \gg 1$. What are the values of the ground state energy and the first few excited states? Show that some energy levels are doubly degenerate. Can you explain why? Characterize the behavior of $N(E)$ as $E \to \infty$. (In other words, based on your numerical results, attempt to deduce the form of the appropriate large E asymptotic expansion.) Can you predict the leading behavior without using numerics?

Suggestions: You will need to perform a pretty big finite basis diagonalizations using well-chosen 2D basis sets (or spectral approximations). Carefully study the stability of the resulting approximate energy eigenvalues as the size of the basis set is increased and determine, for a given size calculation, how many of the resulting eigenvalues are reliable (at a given level of precision), and how many are numerical garbage. You will have to make sure that inaccuracies in your computed energies do not significantly corrupt your estimation of asymptotic behavior.

8 Quasinormal modes: charged black brane

Expected difficulty: 6

Reissner-Nordström (RN) black branes are static solutions of Einstein-Maxwell theory having an unbounded planar event horizon. 5D asymptotically anti-de Sitter solutions are related, via gauge/gravity duality, to finite temperature equilibrium states of 4D maximally supersymmetric $SU(N)$ Yang-Mill theory with a non-zero chemical potential. Small perturbations of the RN black brane geometry, with infalling boundary conditions, are directly related to linear response of perturbations away from equilibrium in the dual field theory. Perturbations with exponential time dependence are known as *quasi-normal modes*.

The 5D Einstein-Maxwell theory action $S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g}\, \left(R - 2\Lambda - L^2 F_{MN}F^{MN}\right)$, with $G_5 \equiv \frac{\pi}{2} L^3/N^2$ the 5D Newton gravitational constant, $\Lambda = -6/L^2$ the cosmological constant, R the Ricci curvature scalar, g the determinant of the metric g_{MN}, and L the AdS curvature scale. Setting to zero the variation of the action with respect to the metric gives the Einstein equation $R_{KL} + (\Lambda - \frac{1}{2}R)\, G_{KL} = 2L^2\, (F_{KM}F_L{}^M - \frac{1}{4}G_{KL}F_{MN}F^{MN})$, while varying the 5D gauge field $\mathcal{A}_M$ (with $F_{MN} \equiv \nabla_M \mathcal{A}_N - \nabla_N \mathcal{A}_M$) gives the Maxwell equation $\nabla_K F^{KL} = 0$.

In convenient infalling coordinates, the RN black brane metric may be written as

$$ds^2 = 2dt(dr - A(r)\, dr) + (r^2/L^2)(dx^2 + dy^2 + dz^2)\,,$$

with $A(r) = \frac{1}{2}r^2/L^2 - \frac{1}{2}mL^2/r^2 + \frac{1}{6}\rho^2 L^{10}/r^4$. The only non-zero component of the Maxwell field strength is a radial electric field, $F_{0r} = -\rho L^4/r^3$. The parameter m is related to the energy density (or temperature), while ρ is the charge density in the dual field theory. The charge density has a maximal value given by $(\rho_{\max} L^3)^4 = \frac{4}{3}m^3$; is it convenient to write ρ in terms of the fraction x of the maximal value, $\rho = x\rho_{\max}$. The horizon radius r_h is the outermost positive root of $A(r)$.

<u>Warm-up</u>:

Verify that the above metric and field strength give a solution of Einstein-Maxwell theory. Consider perturbations to the solution with exponential dependence on t, x, y, and z; without loss of generality one may fix the direction of the wavevector and take perturbations proportional to $e^{-i\omega t + qz}$. Assume that q is real, but ω may be complex. Perturbations may be decomposed into symmetry channels with definite helicity under rotations about the z axis. The easiest case to consider is helicity ± 2, for which it is sufficient to only perturb the metric component $g_{xy} = g_{yx}$. Derive the linearized equations of motion satisfied by infinitesimal perturbations of this form. Find the local behavior of solutions at the horizon, $r = r_h$, and at the boundary, $r = \infty$. How do these change precisely at $\rho = \rho_{\max}$?

<u>Problem</u>: Compute the spectrum of (low-lying) helicity ± 2 quasinormal mode frequencies by solving the linearized perturbation equation, on the domain outside the horizon, with boundary conditions that perturbations be regular at the horizon and vanish at $r = \infty$. Study how the spectrum changes as the background charge density increases. Can you reach the maximal value $\rho_{\max}$? Can you do a separate calculation precisely at $\rho_{\max}$?

<u>Suggestions</u>: Learn how to use the EDCRGTC Mathematica package for analytic work.

9 Wannier-Stark resonances

Expected difficulty: 7

Consider a particle in a one-dimensional periodic potential, with Hamiltonian

$$H = \frac{p^2}{2m} + U \cos(x/a)\,.$$

Warm-up: Show that a single (independent) dimensionless quantity can be constructed from the parameters of the theory. Using units in which $\hbar \equiv 1$, this may be chosen to be $\eta \equiv mUa^2$. The periodicity of the potential implies that eigenstates $|k;n\rangle$ may be labeled by a continuous *crystal wavevector* $k \in [-\pi/a, \pi/a]$ together an integer *band index* $n = 1$, 2, 3,... and the corresponding energy eigenvalue may be written $\epsilon_n(k) = U\, f_n(ka, \eta)$ where f_n is some dimensionless function of the indicated arguments. The state $|k;n\rangle$ and its energy $\epsilon_n(k)$ are *periodic* functions of the wavevector k with periodicity $2\pi/a$. Justify these assertions. Calculate, numerically, $\epsilon_1(k)$ for a bunch of different values of η ranging from quite small $\eta \ll 1$, to quite large, $\eta \gg 1$. What accuracy can you achieve? Judge this both by examining the internal consistency of your numerics as you improve your approximation, and by comparison with the exact answer involving Mathieu functions.

Problem: Now add to the theory a constant electric field E, assuming that the particle has electric charge q, so that $H \to H - qEx$. Suppose $E > 0$. Sketch the resulting combined potential $V(x)$ in which the particle moves. Show that two different dimensionless quantities can now be constructed from the parameters of the theory. Any non-zero field E significantly changes the spectral properties of the theory, producing *resonances* which have complex energies and wavefunctions with purely *outgoing* behavior.

Consider solutions of the (time-independent) Schrodinger equation having some complex energy ϵ. What are the possible asymptotic forms of the wavefunction as $x \to \pm\infty$? For a discrete set of complex energies ϵ_n solutions exist which have exponential decay as $\mathbf{x} \to -\infty$ together with purely outgoing behavior as $x \to +\infty$. Solve for the resonance energies for various representative values of the dimensionless parameters of the theory. What can you deduce about the behavior of the resonance energies in the limit of small, or large, electric field? (What does small or large field mean — compared to what?)

Related reading:

R. de la Madrid, *A pedestrian introduction to Gamow vectors.*
J. Avron, *Model calculation of Stark ladder resonacnes*, PRL 37, 1568 (1976).

10 Quantum time evolution

Expected difficulty: 7

Consider a one-dimensional quartic oscillator with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{\lambda}{4}\, x^4 \,.$$

With no loss of generality, one may choose units in which $\hbar = m = \lambda = 1$. (Justify this.)

Problem: *Accurately* compute the time evolution of an initial state whose wavefunction is a Gaussian wavepacket with mean energy large compared to the ground state energy (comparable or larger than, say, the thirtieth energy level) and a width which is small compared to the size of the classically allowed region at this energy. Evolve this state though a time which is large compared to the classical oscillation period (at the mean energy). Evaluate and examine the time dependence of the position expectation value $\bar{x}(t) \equiv \langle x(t) \rangle$ and its rms deviation $\Delta x(t) \equiv [\langle x(t)^2 \rangle - \langle x(t) \rangle^2]^{1/2}$. Is there a time beyond which the wavefunction no longer resembles a wavepacket whose mean position and momentum follow the classical evolution? Is there a time beyond which the mean position remains small (compared to the initial position, or to the size of the allowed region), throughout a classical period? Is there a time beyond which the rms deviation is comparable to the size of the allowed region? Is there a time beyond which your calculation loses accuracy? How do the answers to these questions depend on the energy and width of the initial state?

Suggestions: One approach is to time evolve in the basis of energy eigenstates. For this approach, you will need to compute highly accurate stationary state energies and eigenfunctions from the ground state up to energies which are substantially larger than your mean energy, project the initial state onto these eigenstates, and then time evolve in this basis: $|\psi(t)\rangle = \sum_n e^{-iE_n t/\hbar} |n\rangle\langle n|\psi(0)\rangle$. The challenge will be to control the errors in this highly oscillatory sum.

Alternatively, one may work directly with the position space wavefunction and view this as a 2D PDE problem: $i\hbar \frac{d}{dt}\, \Psi(x,t) = (-\frac{1}{2m}\frac{d^2}{dx^2} + \frac{\lambda}{4}\, x^2)\Psi(x,t)$. Discretize space and use spectral methods to approximate the spatial derivative, thereby converting the PDE to a system of coupled ODEs. Use a good time integration scheme (RK4, predictor-corrector, or "symplectic" integrators) to solve these ODEs. Carefully study dependence of your results on the spatial discretization and time step.

11 Infalling geodesic congruence

Expected difficulty: 8

The metric

$$ds^2 = r^2 \left[-dt^2 + dx^2 + dy^2 + dz^2\right] + r^{-2} \left[dr^2 + h(z{-}t)\,(dz - dt)^2\right]$$

describes a colliding planar shock propagating in asymptotically anti-de Sitter (AdS) spacetime, with $h(z)$ an arbitrary function characterizing the longitudinal profile of the shock. Here, r is a radial variable which runs from 0 to ∞, which is the boundary of the spacetime. The other coordinates (t, x, y, z) may be regarded as ordinary Minkowski coordinates and label events on the AdS boundary. The above coordinates (called Fefferman-Graham coordinates) are very inconvenient for performing numerical calculations of collisions of planar shocks. For the later purpose, infalling (or Eddington-Finkelstein) coordinates in which the metric has the general form $ds^2 = \tilde{r}^2 g_{\mu\nu}(\tilde{x}, \tilde{r})\, d\tilde{x}^\mu\, d\tilde{x}^\nu + 2d\tilde{r}\, d\tilde{x}^0$, (with $\mu, \nu = 0, \cdots, 3$) are far superior. To transform the geometry into infalling coordinates, one must solve for the infalling radial *null geodesic congruence*. Events on the null (future directed) geodesic which begins on the AdS boundary at boundary coordinates $\tilde{x}^\mu$ may be labeled by an affine parameter $\tilde{r}$. Consider "radial" geodesics whose tangents, at the boundary, lie in the t-r plane. Let $Y(\tilde{r})^M \equiv \{t(\tilde{r}), x(\tilde{r}), y(\tilde{r}), z(\tilde{r}), r(\tilde{r})\}$ denote the Fefferman-Graham coordinates of events along this geodesic; the infalling coordinates of these events are $\widetilde{X}^N \equiv \{\tilde{x}, \tilde{r}\}$. In other words, the infalling coordinates of an event specify the boundary position of the geodesic on which the event lies, together with the affine parameter labeling position along the geodesic.

Warm-up: Evaluate the (Fefferman-Graham) Christoffel symbols $\Gamma^M{}_{PQ}$ in the above given geometry. To transform the above geometry into infalling coordinates, it is sufficient to compute, explicitly, only those geodesics which begin on the boundary at $t = x = y = 0$, and some arbitrary value of z. Why is this?

Problem: Choose a Gaussian profile function, $h(z) \equiv \mu^3 (2\pi w^2)^{-1/2} e^{-frac12 z^2/w^2}$, with some width w. Here μ is a characteristic energy scale (which by a suitable choice of units may be sent to unity). Solve the geodesic equation $\frac{d^2 Y^M}{ds^2} + \Gamma^M{}_{PQ} \frac{dY^P}{ds} \frac{dY^Q}{ds} = 0$ for infalling null geodesics which begin on the AdS boundary at $t = x = y = 0$ and all values of z (i.e., a fine grid). Do this for as small a width w as possible (try to get to down to $w = 0.1$ or less). Evaluate the derivatives $\partial Y^M / \partial \widetilde{X}^P$ (these are what is needed to transform the metric components) and examine how far down into the geometry you can integrate (i.e., to how small a value of r) before clear numerical artifacts appear in these derivatives.

Suggestions: Learn how to use the EDCRGTC Mathematica package for analytic work. Change to inverted radial variables $s = 1/r$ and $u = 1/\tilde{r}$, to move the starting point of geodesics to a finite coordinate location. Use spectral methods, combined with Newton iteration, to handle the singularity at $u = 0$ and solve the geodesic equation in an interval near the $u = 0$ boundary, and then switch to an adaptive ODE integrator to move deeper into the geometry. Resample onto a spectral grid and use spectral methods to evaluate $\partial Y^M / \partial \widetilde{X}^P$. Carefully monitor errors.

12 Two dimensional turbulence

Expected difficulty: 8

Fluid flow is described by the Navier-Stokes equation,

$$\frac{\partial}{\partial t}\mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v} - \nu\nabla^2\mathbf{v} = \frac{1}{\rho}\,(\mathbf{f} - \boldsymbol{\nabla}p)\,,$$

where $\mathbf{v}$ is the *fluid velocity*, p is the pressure, ρ is the mass density, $\mathbf{f}$ is an external force density, and ν is the kinematic viscosity. The *vorticity* is the curl of the velocity, $\boldsymbol{\omega} \equiv \boldsymbol{\nabla}\times\mathbf{v}$. For an *incompressible* fluid, $\rho = \rho_0 = \text{const.}$, and $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$. Taking the curl of the Navier-Stokes equation, assuming incompressibility plus the absence of external forces, eliminates the pressure gradient and produces the *vorticity equation*,

$$\frac{\partial}{\partial t}\boldsymbol{\omega} + (\mathbf{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\,\mathbf{v} - \nu\nabla^2\boldsymbol{\omega} = 0\,.$$

For incompressible flow in two dimensions (relevant for thin fluid films), one may write the flow velocity as the curl of a scalar *stream function*, $\mathbf{v} = \boldsymbol{\nabla}\times\psi$. (That is, $v_i = \epsilon_{ij}\,\partial_j\,\psi$, with ϵ_{ij} the 2D antisymmetric symbol, or explicitly $v_x = \partial_y\,\psi$ and $v_y = -\partial_x\,\psi$.) The vorticity has a single component, $\boldsymbol{\omega} = \omega\,\hat{z}$, with $\omega = -\nabla^2\psi$. Hence, for 2D impressible flow without external forces, the Navier-Stokes equations reduce to a single (non-linear) scalar equation,

$$-\frac{\partial}{\partial t}\,\nabla^2\psi + \epsilon_{ij}\,(\partial_i\psi)\nabla^2(\partial_j\psi) + \nu\,(\nabla^2)^2\psi = 0\,.$$

This may also be written as an equation for the vorticity,

$$\frac{\partial}{\partial t}\,\omega - \epsilon_{ij}\,(\partial_i\psi)(\partial_j\omega) - \nu\,\nabla^2\omega = 0\,,$$

with the stream function viewed as the solution of a Laplace equation, $\psi \equiv (-\nabla^2)^{-1}\omega$.

Problem: Study turbulent, freely decaying solutions of 2D incompressible fluid flow. For convenience, impose periodic boundary conditions (in space) with some period L. Choose initial data in which the vorticity has sinusoidal variation with ≈ 10 wavelengths across the box of size L, plus small random perturbations. The viscosity should be sufficiently small, so that the Reynolds number $\mathrm{R}e \equiv v_{\max}L/\nu$ is large — at least several thousand. Examine how the vorticity evolves. (Make nice movies.) Do large eddies break up into smaller eddies? Do small eddies merge and form larger eddies? Do eddies with opposite circulation annihilate? How does the kinetic energy $\mathcal{K} \equiv \frac{1}{2}\rho_0 \int d^2x\,\mathbf{v}^2$ evolve with time? How does the *enstrophy* $\mathcal{E} \equiv \frac{1}{2}\int d^2x\,\boldsymbol{\omega}^2$ evolve with time? How accurate are your results? How do inviscid ($\nu = 0$) results differ from results with non-zero viscosity?

Suggestions: Working in real space, with Fourier basis spectral differential matrices, or working directly in momentum (Fourier) space and using FFTs to transform back and forth, are both reasonable approaches. You'll want to pick one or the other.

13 Helium in any dimension

Expected difficulty: 9

A non-relativistic Helium atom, in the limit of infinite nuclear mass, is described by the Hamiltonian

$$H = \frac{\mathbf{p}_1^2}{2m} + \frac{\mathbf{p}_2^2}{2m} - \frac{2e^2}{|\mathbf{x}_1|} - \frac{2e^2}{|\mathbf{x}_2|} + \frac{e^2}{|\mathbf{x}_1 - \mathbf{x}_2|}\,,$$

(in a unit system in which $4\pi\epsilon_0 \equiv 1$). For the real world, the spatial vectors $\mathbf{p}_1$, $\mathbf{x}_1$, etc. are, of course, three-dimensional. But it is interesting to consider this theory in an arbitrary number D of spatial dimensions. The expectation value of the energy in a (normalized) state with wavefunction $\Psi(\mathbf{x}_1, \mathbf{x}_2)$ is given by the integral

$$\langle H \rangle = \int d^D\mathbf{x}_1\, d^D\mathbf{x}_2 \left[\tfrac{\hbar^2}{2m} |\boldsymbol{\nabla}_1 \Psi|^2 + \tfrac{\hbar^2}{2m} |\boldsymbol{\nabla}_2 \Psi|^2 + e^2 \left(\tfrac{1}{|\mathbf{x}_1 - \mathbf{x}_2|} - \tfrac{2}{|\mathbf{x}_1|} - \tfrac{2}{|\mathbf{x}_2|} \right) |\Psi|^2 \right],$$

where $\boldsymbol{\nabla}_1$ denotes the gradient with respect to $\mathbf{x}_1$, etc.

Warm-up: The ground state wavefunction is rotationally invariant, and hence can only depend on three scalar variables: the radial positions of the two electrons, $r_1 \equiv |\mathbf{x}_1|$ and $r_2 \equiv |\mathbf{x}_2|$, and the angle θ between $\mathbf{x}_1$ and $\mathbf{x}_2$, defined by $\mathbf{x}_1 \cdot \mathbf{x}_2 = r_1 r_2 \cos\theta$. Justify this assertion, and then reexpress $\langle H \rangle$ as a three-dimensional integral over r_1, r_2, and θ.

Problem: Accurately determine the ground state energy (as a function of D) by minimizing $\langle H \rangle$ subject to the constraint that Ψ be normalized. Find the most accurate $D = 3$ result you can in the literature; how close is your best answer? Generate numerical results spanning a wide interval in D and study how the ground state energy varies as D increases or decreases from 3. Extract from your results the best estimate you can generate for the large D asymptotic behavior of the ground state energy.

Related reading:

L. Yaffe, *Large-N quantum mechanics and classical limits*, Physics Today, Aug. 1983.

14 Squeezed baryons

Expected difficulty: 9

One may study QCD (the theory of strong interactions) in a hypothetical world in which one spatial dimension is made compact and periodic with a circumference L which is tiny compared to the normal size of hadrons. In this regime (when using certain cleverly chosen boundary conditions in the periodic dimension), one may show that mesons and baryons continue to exist, and their properties become easier to calculate. In particular, the lightest baryons, composed of three equal mass quarks, are described by the non-relativistic Hamiltonian

$$H = \sum_{i=1}^{3} \frac{\mathbf{p}_i^2}{2m} + \eta \sum_{i\langle j=1}^{3} \ln(\mu|\mathbf{x}_i - \mathbf{x}_j|)\,,$$

where $\eta \equiv g^2/(6\pi L)$ (with g^2 the QCD coupling evaluated as the scale of $1/L$), and μ an arbitrary inverse spatial scale. Here, the position and momentum of each quark, $\mathbf{x}_i$ and $\mathbf{p}_i$ for $i = 1, 2, 3$, are two-component spatial vectors in the uncompactified dimensions. The expectation value of the energy in a (normalized) state with wavefunction $\Psi(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ is given by the 6D integral

$$\langle H \rangle = \int d^2\mathbf{x}_1\, d^2\mathbf{x}_2\, d^2\mathbf{x}_3 \left[\sum_{i=1}^{3} \tfrac{\hbar^2}{2m} |\boldsymbol{\nabla}_i \Psi|^2 + \eta \sum_{i\langle j=1}^{3} \ln(\mu|\mathbf{x}_i - \mathbf{x}_j|) |\Psi|^2 \right],$$

where $\boldsymbol{\nabla}_i$ denotes the gradient with respect to $\mathbf{x}_i$.

Warm-up: One may, without loss of generality, choose units in which $m = \eta = \hbar = 1$. The ground state wavefunction is rotationally invariant and may be written as the product of a constant center-of-mass wavefunction (i.e., having zero total momentum) and a relative motion wavefunction which only depends on the quark positions relative to the center of mass. This implies that the ground state wavefunction depends on three scalar variables which may be chosen to be the radial positions of the two of the quarks, r_1 and r_2, and the angle θ between these two quarks, all relative to the center of mass. Justify these assertions, and then reexpress $\langle H \rangle$ as a three-dimensional integral over r_1, r_2, and θ.

Problem: Accurately determine the ground state baryon energy by minimizing $\langle H \rangle$ subject to the constraint that Ψ be normalized.

Related reading:

Aitken, Cherman, Poppitz & Yaffe, *QCD on a small circle*, section 6.3.3.