Nearby-fluids equilibria. I. Formalism and transition to single-fluid magnetohydrodynamics

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The *nearby-fluids model* is an improvement of the basic two-fluid flowing equilibrium system developed elsewhere [L.C. Steinhauer, Phys. Plasmas **6**, 2734 (1999)]. The two-fluid system is a singular perturbation problem in which the usually small ion-inertia length (ion skin depth) gives rise to a small parameter. This has frustrated attempts to find practical equilibria. The *nearby-fluids ordering* assumes that the ion and electron flow surfaces are close to ("nearby") each other but do not coincide exactly. This eliminates the singularity and "softens" the stiff differential equations, thus facilitating numerical solution. Previous treatments of two-fluid equilibria did not find a smooth transition to the single-fluid magnetodynamics (MHD) model in the small ion-inertia-length limit. This difficulty is only partly eliminated by recognizing that the electric field is ordered "large" in MHD. A small but nonzero ion-inertia length produces several important effects, including the appearance of a transition layer between the "closed" and "open" field regions of a field-reversed configuration plasma. © *2006 American Institute of Physics*. [DOI: 10.1063/1.2200610]

I. INTRODUCTION

In general, static plasma equilibria are well understood. Efficient tools have been constructed both for computing equilibria and fitting them to experimental observations. Static equilibria are governed by the familiar Grad-Shafranov (GS) equation, a second-order partial differential equation for ψ, and the magnetic flux function. On the other hand flowing plasma equilibria, despite their increasing importance, are much less understood. Toroidal rotation velocities comparable to the ion thermal speed have been observed in experiments. Even flows at more modest speeds can profoundly alter the plasma structure and with it, the stability.

Theoretical studies of flowing equilibria began early[1,2] and have been revisited several times.[3–5] These employ the standard magnetohydrodynamics (MHD) model, which assumes the ideal Ohm's law, i.e., a *single* fluid where the electron and ion fluid motions are locked together, except in an ignorable coordinate. A second order partial differential equation still governs ψ, but it is more complicated than the GS equation.[1,3,4] This equation has been used to explore flowing MHD stability.[5] In MHD equilibria, the flow and magnetic surfaces (ψ=const) are identical, in accordance with the single-fluid paradigm. More recently, flowing equilibria have been extended to add the Hall term to Ohm's law,[6] which gives the electrons some freedom of motion independent of the ions, but only so to provide a quasineutral background. Using this "Hall MHD" model, some equilibria have been investigated.[7,8]

Several years ago the general formalism for axisymmetric, multifluid equilibria was developed.[9] This model includes the pressure, inertia, and full Lorentz force for each species. It is more general than the Hall MHD in that it accounts for the electron pressure, p_e and inertia. The former may makes an important difference. For example, Hall MHD (p_e=0) has uniform electrical potential on magnetic surfaces ψ=const. With flow, $p_e \neq 0$ breaks this uniformity, causing parallel electric fields that may strongly affect stability.[10]

The key parameter distinguishing a "two-fluid" from a "single-fluid" is the ion-inertia-length parameter, the ratio of the ion-inertia length $\ell_i \equiv c/\omega_{pi}$ (c=speed of light; ω_{pi}=ion plasma frequency) to the length scale of interest L,

$$\varepsilon \equiv \ell_i / L. \tag{1}$$

Here $\ell_i = (m_i c^2/4\pi e^2 n)^{1/2}$, where m_i, e, and n are the ion mass, charge, and density, respectively. Note that the ion-inertia length has often be called the ion skin depth. It is called the inertia length here because $\ell_i \propto m_i^{1/2}$ has an obvious connection to the ion inertia. The parameter ε appeared naturally in a two-fluid stability analysis[11] and later in studies of relaxed[12,13] or "natural" states[14] of a two-fluid. The ion-inertia-length parameter ε should not be confused with the Hall parameter.[15] The typical size of ε is discussed in Appendix A, with values ranging from $\varepsilon \sim 0.04$ to 0.2. Thus ε can be regarded as a small parameter. Of course, localized regions may appear with smaller L, in which $\ell_i \sim L$. Examples include the scrape-off layer, transport barriers, and layers with high velocity shear.

Two difficulties emerged for two-fluid flowing equilibria with small but nonzero ε. (1) The transition from a two-fluid (or the more restrictive Hall MHD) to the MHD limit (ε=0), proved unexpectedly troublesome. An attempt to formulate the small ε system for Hall MHD led to equations with $1/\varepsilon$ singularities.[8] This reflects the fact that two-fluid equilibrium is a singular perturbation problem.[14] (2) Numerical computation of realistic two-fluid equilibria has proved

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surprisingly difficult. Examples were computed in simplified geometries[13] or assuming uniform density,[16] but attempts to compute actual experiments proved difficult.[17] This obstacle was sidestepped in Ref. 8 where the equilibria actually computed used the MHD model.

Here we develop a flowing equilibrium system that allows small but nonzero ε yet avoids $1/\varepsilon$ singularities. Several important results emerge: (1) *MHD ordering*. The MHD model retains the $O(\varepsilon)$ electrical field while dropping two other terms of the same order. This irregularity must be accounted for in making the transition from a two-fluid to MHD. (2) *Stiffness*. For small-but-finite ε, the electron equation of motion is a *stiff* partial differential equation, i.e., the highest-order derivative term is multiplied by the small parameter ε. This frustrates attempts to compute equilibria. (3) *Nearby-fluids ordering*. The stiff system can be "softened" by expressing the ion stream function so as to assure a small, $O(\varepsilon)$, separation between the ion and electron flow surfaces. The proximity of the two sets of surfaces suggests the term *nearby-fluids equilibria*. (4) A small but nonzero ε has a profound effect on the equilibrium, especially in the edge region, with likely effects the confinement and stability. (5) In a companion paper[18] the nearby-fluids system is applied to a field-reversed configuration to interpret field and flow measurements.

The outline of the paper is as follows: Sec. II presents the basic equations of flowing two-fluid equilibria in a convenient dimensionless form. Attention is focused on axisymmetric equilibria so that the equations can be expressed in terms of surface variables. Section III develops the nearby-fluids system. Section IV considers the reduction to a single-fluid and a comparison with MHD. Section V concludes the paper with a discussion.

II. TWO-FLUID EQUILIBRIUM SYSTEM

A. Two-fluid equations

The analysis is illuminated by introducing dimensionless variables (see Appendix B). The primary scales are (1) L the length scale of the plasma, (2) B_R, a representative magnetic field, and (3) n_R, a representative density. These give rise to derived scales for velocities $V_{AR}=B_R/(4\pi m_i n_R)^{1/2}$ (Alfvén speed), and the electrical field $E_R=B_R^2/4\pi e n_R L$. The ion-inertia-length parameter ε, Eq. (1), is the only parameter appearing in the system. Note that the reference velocity is the full Alfvén speed V_{AR}, with drift speeds of order εV_{AR}. This dimensionless variable scheme differs from the *drift-ordering* scheme in which the drifts appear to leading order.

Using the dimensionless variable, equilibria $(\partial/\partial t=0)$ of a flowing, quasineutral two-fluid with massless electrons, and singly charged ions, are governed by the following equations. The equations of motion for the ion and electron fluids are

$$\mathbf{u}_i\cdot\nabla\mathbf{u}_i=-\nabla p_i/n+\mathbf{E}+(1/\varepsilon)\mathbf{u}_i\times\mathbf{B}, \tag{2a}$$

$$0=-\nabla p_e/n-\mathbf{E}-(1/\varepsilon)\mathbf{u}_e\times\mathbf{B}. \tag{2b}$$

The continuity, entropy conservation, and state equations for each species $(\alpha=i,e)$ are

$$\nabla\cdot(n\mathbf{u}_\alpha)=0, \tag{3a}$$

$$\mathbf{u}_\alpha\cdot\nabla s_\alpha=0, \tag{3b}$$

$$p_\alpha=n^\gamma\exp[(\gamma-1)s_\alpha]. \tag{3c}$$

The variables $\mathbf{B},\mathbf{E},n,\mathbf{u}_\alpha,p_\alpha,s_\alpha,\gamma$, are the dimensionless magnetic field, electric field, density, species fluid velocity, pressure, entropy, and adiabatic constant, respectively. The caloric state equation (3c) assumes an ideal fluid. A single γ is adopted here, but separate values for the two species are allowed. The true thermodynamic entropy density s is used instead of the entropy-like variable S used elsewhere.[19] The temperatures follow *a posteriori* from the thermal equation of state $T_\alpha=p_\alpha/n$. The system is completed by the steady forms of Faraday's law $\nabla\times\mathbf{E}=0$, and Ampere's law $n(\mathbf{u}_i-\mathbf{u}_e)=\varepsilon\nabla\times\mathbf{B}$.

It is conventional to (a) eliminate $\mathbf{u}_e$ using Ampere's law, (b) use a combined motion equation, [the sum of Eqs. (2a) and (2b)], and (c) relabel Eq. (2b) as Ohm's law. Henceforth we adopt the simplified notation $\mathbf{u}_i\to\mathbf{u}$, which is the density-weighted bulk fluid velocity in the case of massless electrons. The motion and Ohm's equations then become

$$n\mathbf{u}\cdot\nabla\mathbf{u}=-\nabla(p_i+p_e)+(\nabla\times\mathbf{B})\times\mathbf{B}, \tag{4}$$

$$\mathbf{u}\times\mathbf{B}=-\varepsilon\left(\frac{\nabla p_e}{n}+\mathbf{E}-\frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{n}\right). \tag{5}$$

The only place the parameter ε appears is in Ohm's law Eq. (5) where it multiplies the electric field, electron pressure, and Hall terms. These $O(\varepsilon)$ terms give rise to three drifts, all of order εV_A. The "single-fluid" limit $\varepsilon\to 0$ reduces the Ohm's law to $\mathbf{u}\times\mathbf{B}=0$. This differs from the MHD model in which the Ohm's law is $\mathbf{E}+\mathbf{u}\times\mathbf{B}=0$. This difference arises because MHD employs a different electric field ordering. It complicates the transition from the two-fluid to the MHD. This difficulty is addressed in Sec. IV.

B. Introduction of surface functions

Since continuity equations govern the magnetic field $\nabla\cdot\mathbf{B}=0$, and flow, Eq. (3a), the most general axisymmetric field and flow can be expressed in the forms

$$\mathbf{B}=\hat{\boldsymbol{\theta}}B_\theta+\nabla\psi\times\hat{\boldsymbol{\theta}}/r, \tag{6a}$$

$$\mathbf{u}=\hat{\boldsymbol{\theta}}u_\theta+(\nabla\psi_i/n)\times\hat{\boldsymbol{\theta}}/r \tag{6b}$$

using cylindrical coordinates (r,θ,z), B_θ, u_θ, ψ, and ψ_i are scalar functions of (r,z). In common fluid-mechanical parlance ψ_i is the "stream function" for the poloidal ion flow. From Eq. (6b) the poloidal flow speed is $u_p=|\nabla\psi_i|/nr$. The electron flow is analogous to Eq. (6b) with $\mathbf{u}\to\mathbf{u}_e u_\theta\to u_{e\theta}$, and $\psi_i\to\psi_e$, where ψ_e is the stream function for the electron fluid. Ampere's law, $n(\mathbf{u}-\mathbf{u}_e)=\varepsilon\nabla\times\mathbf{B}$, with Eq. (6) separates into azimuthal and poloidal parts,

$$r^2\nabla\cdot(\nabla\psi/r^2)=-(1/\varepsilon)nr(u_\theta-u_{e\theta}), \tag{7a}$$

$$\nabla(\psi_i - \psi_e - \varepsilon r B_\theta) \times \hat{\boldsymbol{\theta}}/r = 0, \tag{7b}$$

respectively. The left-hand side of Eq. (7a) is the familiar GS operator $\Delta^*\psi = rj_\theta/\varepsilon$ (j_θ=azimuthal current density). Evidently the quantity in parentheses in Eq. (7b) has no (r,z) dependence; with $\partial/\partial\theta=0$ this leads to $(\psi_i-\psi_e-\varepsilon rB_\theta)$ =const. The ψ_i and ψ_e, which are the stream functions for the poloidal flows, have an arbitrary additive constant. Absorbing the constant into, say ψ_e, gives an expression for the toroidal field

$$B_\theta = (1/\varepsilon)(\psi_i - \psi_e)/r. \tag{8}$$

In constructing the equations of motion, it is useful to introduce the canonical fluid momenta: $\mathbf{P}=\mathbf{A}+\varepsilon\mathbf{u}$ for the ions, and $\mathbf{P}_e=\mathbf{A}$ for the massless electrons, where $\mathbf{A}$ is the vector potential. The curl of the canonical momenta gives the generalized vorticities, $\boldsymbol{\Omega}=\nabla\times\mathbf{P}$, $\boldsymbol{\Omega}_e=\mathbf{B}$. Observe that both $\mathbf{P}$ and $\boldsymbol{\Omega}$ have both electromagnetic and mechanical parts: the latter, which is proportional to $\varepsilon \propto m_i^{1/2}$, is the ion inertia effect. Using the scalar variables in Eq. (6), the generalized vorticity is

$$\boldsymbol{\Omega} = \left[rB_\theta - \varepsilon r^2 \nabla \cdot \left(\frac{\nabla\psi_i}{nr^2} \right) \right] \frac{\hat{\boldsymbol{\theta}}}{r} + \nabla \mathrm{Y} \times \frac{\hat{\boldsymbol{\theta}}}{r}; \tag{9a}$$

$$\mathrm{Y} \equiv \psi + \varepsilon r u_\theta \tag{9b}$$

where the *ion* surface variable $\mathrm{Y}(r,z)$ is the sum of electromagnetic and mechanical elements. The *electron* surface variable is $Y_e=\psi$, i.e., the magnetic and massless electron flow surfaces coincide. Hereafter ψ will be used as the electron surface variable. The second term in the square brackets of Eq. (9a) is a density-weighted form of the GS operator and is $\propto(\nabla\times\mathbf{u})_\theta$.

Separate Eqs. (4) and (5) into equations of motion using both $\mathbf{u}$ and $\mathbf{u}_e$. Then, using Eq. (9a) for Ω, the equations of motion for each species take compact and symmetric forms,

$$\nabla(h_i + u^2/2 + \phi) - T_i \nabla s_i = (1/\varepsilon)\mathbf{u} \times \boldsymbol{\Omega}, \tag{10}$$

$$\nabla(h_e - \phi) - T_e \nabla s_e = -(1/\varepsilon)\mathbf{u}_e \times \mathbf{B}, \tag{11}$$

where the species enthalpies and temperatures are given by

$$T_\alpha = n^{\gamma-1} \exp[(\gamma-1)s_\alpha], \tag{12a}$$

$$h_\alpha = \gamma T_\alpha/(\gamma - 1). \tag{12b}$$

The electrical potential ϕ in Eqs. (10) and (11) springs from Faraday's law $\nabla\times\mathbf{E}=0$, which implies $\mathbf{E}=-\nabla\phi$. The left-hand sides of Eqs. (10) and (11) include three effects: pressure gradient, $\nabla p/n=\nabla h-T\nabla s$, the electric potential ϕ, and the ion flow energy effect $u^2/2$.

C. Three components of the equations of motion

Consider three components of the vector equations of motion (10) and (11): *azimuthal* ($\hat{\boldsymbol{\theta}}$); *parallel* (to the flow $\mathbf{u}$ or $\mathbf{u}_e$); and *perpendicular* (∇Y for ions, $\nabla\psi$ for electrons). The azimuthal component gives $\hat{\boldsymbol{\theta}}\cdot(\nabla\psi_i\times\nabla\mathrm{Y})=0$ and $\hat{\boldsymbol{\theta}}\cdot(\nabla\psi_e\times\nabla\psi)=0$, which implies that the surfaces ψ_i=const and Y=const coincide, and the surfaces ψ_e=const and ψ=const coincide. Thus one can be expressed in terms of the other using the *surface functions*:

$$\psi_i = \bar{\psi}_i(\mathrm{Y}), \quad \psi_e = \bar{\psi}_e(\psi). \tag{13}$$

The overbar notation distinguishes the "arbitrary" function $\bar{\psi}_i(\ldots)$ from the ion stream function $\psi_i(r,z)$, as in Eqs. (6)–(9), and similarly for the electron fluid. From Eq. (6b) the poloidal flows are $\mathbf{u}_p=(\bar{\psi}_i'/n)\boldsymbol{\Omega}_p$ and $\mathbf{u}_{ep}=(\bar{\psi}_e'/n)\mathbf{B}_p$. Here and elsewhere, the prime denotes the derivative of a function with respect to its argument.

Entropy conservation Eq. (3b) implies that the entropies convect with their respective fluids. Substituting Eq. (6b) and its electron equivalent into Eq. (3b) gives $\hat{\boldsymbol{\theta}}\cdot(\nabla\psi_i\times\nabla s_i)=0$ and $\hat{\boldsymbol{\theta}}\cdot(\nabla\psi_e\times\nabla s_e)=0$. Thus the entropy "surfaces" (e.g., s_i=const) and flow surfaces coincide, giving rise to the third and fourth "arbitrary" functions,

$$s_i = S_i(Y), \quad s_e = S_e(\psi). \tag{14}$$

Again the entropy density variable, e.g., $s_i(r,z)$, must be distinguished from the associated function expressed in capital letters, e.g., $S_i(Y)$.

Next consider the *parallel* components of the equations of motion. These follow by taking dot products with $\mathbf{u}$ and $\mathbf{u}_e$. Using Eq. (3b), the only terms that survive are $\mathbf{u}\cdot\nabla(h_i+u^2/2+\phi)=0$ and $\mathbf{u}_e\cdot\nabla(h_e-\phi)=0$. Thus the *total enthalpies* (to borrow a term from gas dynamics) must also be surface functions:

$$h_i + u^2/2 + \phi = H_i(\mathrm{Y}); \tag{15a}$$

$$h_e - \phi = H_e(\psi). \tag{15b}$$

Again capital letters are used to distinguish the functional forms $H_i(\ldots)$ and $H_e(\ldots)$. Equations (15) are the generalized Bernoulli equations for each species. In summary, Eqs. (13)–(15) introduce *six* arbitrary functions for two-fluid equilibria.

Finally, consider the *perpendicular* components of the motion equations. These are found by taking dot products with ∇Y and $\nabla\psi$, respectively. Use the surface properties Eqs. (13)–(15) and Eq. (9) for $\boldsymbol{\Omega}$:

$$\bar{\psi}_i' r^2 \nabla \cdot \left(\frac{\bar{\psi}_i'}{n} \frac{\nabla Y}{r^2} \right) = \frac{r}{\varepsilon}(B_\theta \bar{\psi}_i' - nu_\theta) + nr^2(H_i' - T_i S_i'), \tag{16}$$

$$r^2 \nabla \cdot \left(\frac{\nabla\psi}{r^2} \right) = \frac{r}{\varepsilon}(B_\theta \bar{\psi}_e' - nu_\theta) - nr^2(H_e' - T_e S_e'). \tag{17}$$

Equation (17) is in "Ohm's law" format, i.e., $\mathbf{u}_e$ was eliminated using Ampere's law. Here, the primes denote the derivative with the respective surface variable: $\bar{\psi}_i' \equiv d\bar{\psi}_i/dY$ and likewise for H_i', S_i'; and $\bar{\psi}_e' \equiv d\bar{\psi}_e/d\psi$ and likewise for H_e', S_e'.

In the foregoing the ion inertia appears three times. (1) The mechanical part of Y, namely $\varepsilon r u_\theta$, is inertial since $\varepsilon \propto \sqrt{m_i}$; this implies a centrifugal effect that is balanced by a "radial" force. (2) The left-hand side of Eq. (16) is the inertia of poloidal flow along a curved poloidal surface; this gives an acceleration that is balanced by a "perpendicular" force. (3) The $u^2/2$ in Eq. (10) is actually the ion flow energy; it is the usual Bernoulli effect—increased flow speed reduces the enthalpy.

III. NEARBY-FLUIDS EQUILIBRIA

A. Singular nature of the equilibrium system

Inspection of Eqs. (16) and (17) shows a difficulty with the two-fluid system. Both have terms of order $1/\varepsilon$ on the right-hand side. In effect, the highest-order derivative terms (left-hand sides) are multiplied by a small parameter ε, i.e., *stiff* differential equations. This singularity was noted by Mahajan and Yoshida[14] who identified two-fluid equilibria as a "singular perturbation" problem. Ilgisonis[8] also recognized this in an attempt to make the transition from a two-fluid to MHD for $\varepsilon \to 0$. The difficulty was exacerbated[20] by the fact that the MHD model orders the electric field as "large." (Section IV addresses the relationship between the two-fluid and MHD models in more detail.)

This complication is partly eliminated by a uniform treatment of $O(\varepsilon)$ terms, which removes the false singularity in the electric field. However, the stiffness problem remains and requires a more subtle treatment. Ilgisonis[8] proposed a promising approach, suggesting the use of ψ and ru_θ as the primary variables rather than ψ and Y. However, this still did not achieve regular behavior for $\varepsilon \ll 1$. In what follows we develop a method to remove the singularity.

B. Two postulates and the nearby-fluids ordering

Two postulates guide the procedure to overcome the singularity:

$$(1)\ B_\theta = O(1), \tag{18a}$$

$$(2)\ u_\theta = O(1). \tag{18b}$$

In view of Eq. (9b), the second postulate assures that the ion surfaces Y=const and electron (magnetic) surfaces ψ=const differ only to $O(\varepsilon)$. The first postulate is more difficult: in view of Eq. (8), it requires that the "arbitrary" functions $\bar{\psi}_e(\ldots)$ and $\bar{\psi}_i(\ldots)$ must differ only to $O(\varepsilon)$. This is more difficult because these appear in the equations as functions of different variables $\bar{\psi}_e(\psi)$ and $\bar{\psi}_i(Y)$. Thus, to satisfy the first postulate, these functions cannot be completely arbitrary. This difficulty is resolved by replacing these two "arbitrary" functions with an alternate pair of arbitrary functions $F(\ldots)$ and $G(\ldots)$ expressed as follows:

$$\bar{\psi}_e(x) = \int_0^x F(x')dx', \tag{19a}$$

$$\bar{\psi}_i(x) = \int_0^x F(x')dx' + \varepsilon G(x). \tag{19b}$$

These forms become a restrictive assumption when ε is recognized as a small parameter while $|G(\ldots)| = O(1)$. This restriction on the arbitrary functions is called the *nearby-fluids ordering*. As will be shown, this restriction will guarantee $B_\theta \sim O(1)$. Equilibria associated with the nearby fluids ordering Eq. (19) are called *nearby-fluids equilibria*.

C. Extended-GS equation

Apply the nearby-fluids ordering Eq. (19) to the poloidal component of Ampere's law, Eq. (8). Use a Taylor expansion, regarding ε as a small parameter. Then the toroidal field becomes

$$B_\theta = \frac{G(Y)}{r} + u_\theta\left[F(\psi) + \varepsilon F'(\psi)\frac{ru_\theta}{2} + O(\varepsilon^2)\right]. \tag{20}$$

Evidently, the nearby-fluids ordering guarantees that the first postulate, Eq. (18a), is satisfied.

There remains the issue of the stiffness of the equations of motion, Eqs. (16) and (17). In terms of the F and G functions, these become

$$-\bar{\psi}_i' r^2 \nabla \cdot \left(\frac{\bar{\psi}_i'}{n}\frac{\nabla Y}{r^2}\right) = -\frac{1}{\varepsilon}[rB_\theta F(Y) - rnu_\theta] - rB_\theta G'(Y) - nr^2[H_i'(Y) - T_i S_i'(Y)], \tag{21}$$

$$r^2 \nabla \cdot \left(\frac{\nabla\psi}{r^2}\right) = \frac{1}{\varepsilon}[rB_\theta F(\psi) - rnu_\theta] - nr^2[H_e'(\psi) - T_e S_e'(\psi)]. \tag{22}$$

The stiffness is handled as follows: (1) Construct the combined equation of motion by adding the equations of motion of the two species. (2) Regard the electron motion equation [Ohm's law, Eq. (22)], as an algebraic equation for u_θ. This is inspired by the suggestion of Ilgisonis, namely, to regard u_θ as a primary variable. The combined equation of motion is a second-order partial differential equation for ψ as such it is the extended GS equation:

$$\begin{aligned} r^2 \nabla \cdot \left(\frac{\nabla\psi}{r^2}\right) &- [F(Y) + \varepsilon G'(Y)]r^2 \nabla \cdot \left(\frac{F(Y) + \varepsilon G'(Y)}{n}\frac{\nabla Y}{r^2}\right) \\ &= -rB_\theta\left[\frac{F(Y) - F(\psi)}{\varepsilon} + G'(Y)\right] - nr^2[H_i'(Y) + H_e'(\psi) - T_i S_i'(Y) - T_e S_e'(\psi)]. \end{aligned} \tag{23}$$

This equation has no $1/\varepsilon$ singularity. The quantity $[F(Y) - F(\psi)]/\varepsilon$ is nonsingular since Y only differs from ψ by $O(\varepsilon)$.

Now consider the Ohm's law. Solve Eq. (22) for u_θ:

$$u_\theta = \frac{F(\psi)}{n}B_\theta - \varepsilon r[H_e'(\psi) - T_e S_e'(\psi)] - \varepsilon\frac{r}{n}\nabla \cdot \left(\frac{\nabla\psi}{r^2}\right). \tag{24}$$

This too contains no explicit singularity. Observe that this form assures that the second postulate, $u_\theta = O(1)$, is satisfied. The terms contributing to the toroidal flow, Eq. (24), are the MHD term (parallel flow), the drifts effect (term with square brackets), and the Hall effect. The second and third of these are $O(\varepsilon)$. The drifts include both the electron diamagnetic drift (electron pressure) and the electric drift since $H_e = h_e - \phi$ and $\nabla h_e = \nabla p_e - T_e \nabla s_e$.

With only one primary second-order equation, the extended GS equation, Eq. (23), it might seem like there is only one set of characteristic surfaces instead of two. This is not the case, as is proven in Appendix C.

D. Auxiliary equations

The nearby-fluids system is closed by auxiliary equations. Equation (20) gives B_θ and Eq. (24) gives u_θ. The *combined* Bernoulli equation is the sum of electron and ion parts of Eq. (15),

$$h_i + h_e + u^2/2 = H_i(Y) + H_e(\psi), \tag{25}$$

where $u^2 = u_p^2 + u_\theta^2$. Recall that the enthalpies are given by Eq. (11). The poloidal flow velocity (denoted by the subscript "p") is

$$\mathbf{u}_p = [F(Y) + \varepsilon G'(Y)] \nabla Y \times \hat{\boldsymbol{\theta}}/nr. \tag{26}$$

This closes the nearby-fluids system in which there are five variables ψ, Y, u_θ, u_p, n, governed by the extended GS equation (23), and the auxiliary Eqs. (9b), (11), and (24)–(26).

Other important variables are found *a posteriori*. The poloidal field is $B_p = |\nabla \psi|/r$. The electrical potential follows from the *electron* Bernoulli equation (15b)

$$\phi = \gamma T_e/(\gamma - 1) - H_e(\psi). \tag{27}$$

Note that the electrical potential is *not* a magnetic surface function. This property is completely missed by the Hall MHD model because it ignores the electron pressure, which is equivalent to dropping the first term in Eq. (27).

The toroidal flow in Eq. (24) is the sum of "parallel" flow along field lines and *drift* across them. The drift speed is the difference between u_θ and the "parallel" part, $u_D \equiv u_\theta - u_{\theta\|}$. For the ion fluid the parallel part is $u_{\theta\|} = u_{p\|}\Omega_\theta/\Omega_p$ (where $u_{p\|} = \mathbf{u}\cdot\boldsymbol{\Omega}_p/\Omega_p$), and for the electrons $u_{e\theta\|} = u_{ep\|}B_\theta/B_p$ (where and $u_{ep\|} = \mathbf{u}_e \cdot \mathbf{B}_p/B_p$). Then the two drifts are

$$u_{iD} = \varepsilon r(H_i' - T_i S_i'); \quad u_{eD} = \varepsilon r(H_e' - T_e S_e'). \tag{28}$$

Clearly the drift speed is $O(\varepsilon)$. This is in accord with the remark in Sec. II A that the reference velocity is the full Alfvén speed V_{AR}, while the drifts, Eq. (28), are of $O(\varepsilon)$.

Furthermore, since $\nabla p_\alpha/n = \nabla h_\alpha - T_\alpha \nabla s_\alpha$ and using Eq. (15), the two drifts can be written as

$$u_{iD} = \varepsilon r\left(\frac{\nabla p_i}{n} + \nabla\phi + \frac{\nabla u^2}{2}\right)\cdot\frac{\nabla Y}{|\nabla Y|^2}; \tag{29a}$$

$$u_{eD} = -\varepsilon r\left(\frac{\nabla p_e}{n} - \nabla\phi\right)\cdot\frac{\nabla\psi}{|\nabla\psi|^2}. \tag{29b}$$

Thus the drift-driving mechanisms are the pressure gradient (diamagnetic drift), the electric field (electric drift), and the inertial drift (ions only).

IV. TRANSITION TO A SINGLE-FLUID

A. Introduction

In typical magnetically confined plasmas one expects "fully developed" flows where $m_i n\rho \mathbf{u}\cdot\nabla\mathbf{u}$, the inertia term in the equation of motion, is comparable to the pressure gradient.[21] This corresponds to flows comparable to the ion thermal speed $v_{ti} = (kT_i/m_i)^{1/2}$. It has further been argued[21] that such "fully developed" flows should be ascribed primarily to the electric drift, whereas other drifts, e.g., the diamagnetic drift, are relatively small. This gives rise to the MHD model (the "small Larmor radius" approximation[21]). The principle application of the MHD model has been the study of instabilities. For many decades it has been the foundational model for stability analysis,[22] where it has had many successes, both in linear and nonlinear time-dependent computations.

Large flows are an expected product of instabilities. However, even in supposedly quiescent states ($\partial/\partial t \sim 0$), large flows should be allowed for and, indeed, should be expected on several accounts. First, the startup phase generally involves large, unbalanced forces and significant instability. Even after the transient dynamics have passed and the plasma has resolved itself into a quiescent state, large flows may persist. That is, self-organization may leave behind self-generated flows. Second, the quiescent state is likely to be a quasiequilibrium in which low-level instabilities persist, regulating the profile, controlling β (ratio of thermal to magnetic pressure), and sustaining the self-generated flows. Finally, external driving methods such as neutral-beam injection may also serve to sustain large flows. Thus, even in equilibrium analyses, one should allow for large flows. The common MHD model has been adopted in several treatments of flowing equilibria.[1–5]

The two-fluid equilibrium model developed here is more general than MHD because it includes more terms in the Ohm's law. Since MHD has been in such wide use, it is important to establish the relationship between it and the two-fluid model. At first blush, it might seem that the transition would be effected simply by taking the limit $\varepsilon \to 0$. Unfortunately it is not so simple, mainly because in MHD the electric field and the associated electric drift v_E are ordered "large," i.e., MHD uses the ideal Ohm's law $\mathbf{E} + \mathbf{u}\times\mathbf{B} = 0$, which retains the electric drift. There is no inherent inconsistency in ordering the electric field as "large." However, it complicates the two-fluid to MHD transition. This difficulty is not eliminated by adopting a lower reference velocity.[23] A previous attempt to make the transition led to a $1/\varepsilon$ singularity in one treatment of the $\varepsilon \to 0$ transition which was only resolved by extraordinary means.[20] The purpose of this section is to address the two-fluid to MHD transition in a more detailed manner.

Before embarking on this task, it should be noted that the successes of MHD in modeling linear and nonlinear instabilities have been because the most extensive investigations have been for low β plasmas where the diamagnetic drift and Hall effect are small. However, significant differences emerge between the two-fluid and MHD models with increased β, as has been pointed out elsewhere.[24]

B. Reducing assumptions for nearby-fluids to single-fluid transition

Here the nearby-fluids equilibrium equations are reduced so as to arrive at the single-fluid equations. This reduced nearby-fluid (RNF) system is then compared to the MHD system as derived in Ref. 5. This reduction retains the electric field terms, recognizing the "large" electric field ordering in MHD. The objective of this exercise is to find if there are any differences between the RNF and standard MHD systems. Two differences are indeed found, one in the Bernoulli equation and one in the extended Grad-Shafranov (GS) equation.

The RNF system is found using three reducing assumptions: (1) most basically, the transition from two fluids to one is achieved by taking $\varepsilon \to 0$; (2) in order to compare with MHD, an exception is made for electric field terms which are retained even though they appear to $O(\varepsilon)$ in the dimensionless variable system used here; and (3) in the equation of motion, the MHD model does not distinguish between the electron and ion pressures, enthalpies and temperatures, which are combined in the RNF system. Observe first of all that assumption (1) causes $Y \to \psi$. This implies that in RNF (as in MHD), there is a single set of characteristic surfaces rather than two. Further, it implies that all surface functions, F, G, H_i, H_e, S_i, S_e, are functions of ψ only. As will be shown shortly, the electrostatic potential also emerges as a surface function $\phi(\psi)$, which is *not* the case in a two fluid.

C. Reduction procedure

The Ohm's law is represented by Eq. (24). In the RNF system the third term (Hall effect) is $O(\varepsilon)$ and is dropped in accordance with assumption (1). The second term contains both the electron pressure and electric field effects. It is reduced as follows. Combine the gradient of the electron Bernoulli Eq. (15b), and use the property of the enthalpy $\nabla h_e = \nabla p_e / n + T_e \nabla s_e$; neglecting the electron pressure leads to $(H_e' - T_e S_e')\nabla \psi = -\nabla \phi$. Thus a new surface function emerges, $\phi(\psi)$, so that quantity in square brackets in Eq. (24) becomes $-\phi'$. Thus in the RNF system, Ohm's law reduces to

$$u_\theta = (F/n) B_\theta + \varepsilon r \phi', \tag{30}$$

where F and ϕ are understood to be functions of ψ. In accordance with assumption (2), the electric field term $\varepsilon\phi'$ is retained.

The poloidal component of Ampere's law Eq. (20) gives the toroidal field. RNF discards terms of $O(\varepsilon)$ except those associated with the electric field. Combining Eqs. (20) and (30) gives

$$B_\theta = (G/r + \varepsilon r \phi' F)/(1 - F^2/n). \tag{31}$$

Again note the appearance of $\varepsilon\phi'$, representing the electric field. Observing the notational conversions (Ref. 5 → here, $I \to G$, $\Phi \to F$, $\Omega \to \varepsilon\phi'$, $\rho \to n$) Eq. (31) is identical to the MHD result, Eq. (5) of Ref. 5. The poloidal magnetic field is simply the poloidal part of Eq. (6a), in agreement with the MHD result.[5] The reduced poloidal flow velocity follows from Eq. (26) by letting $\varepsilon \to 0$, and recognizing $\mathbf{B}_p$ in the resulting expression $\mathbf{u}_p = (F/n)\mathbf{B}_p$. Combining the toroidal and poloidal flow parts gives

$$\mathbf{u} = (F/n)\mathbf{B} + \varepsilon r \phi' \hat{\boldsymbol{\theta}}. \tag{32}$$

With the notational conversions this RNF form is identical to the MHD result, Eq. (3) of Ref. 5.

Consider now the summed Bernoulli Eq. (25). Observe the conversions $H_e + H_i \to H(\psi)$, and $h_i + h_e \to h = [\gamma/(\gamma - 1)] n^\gamma S(\psi)$, both in view of assumption (3). The u^2 term is expanded using Eq. (32). Then the summed Bernoulli equation in the RNF system becomes

$$\frac{\gamma}{\gamma - 1} n^\gamma S + \frac{F^2}{2n^2} B^2 - \frac{r^2}{2}(\varepsilon\phi')^2 + \varepsilon r \phi' u_\theta = H. \tag{33}$$

Here the electric-field effect $\varepsilon\phi'$ appears twice. Compare this RNF result with the MHD from Eq. (6) of Ref. 5, using the aforementioned notational conversions. *Missing* from the MHD Bernoulli equation is the term $+r u_\theta \varepsilon \phi'$.

Consider now the extended GS equation, Eq. (23). Note that in the $\varepsilon \to 0$ limit, $[F(Y) - F(\psi)]/\varepsilon$ is reduced using the small ε Taylor expansion of $F(\psi + \varepsilon r u_\theta)$, becoming $F' r u_\theta$. Furthermore, observe the conversion $T_i S_i' + T_e S_e' \to n^\gamma S'/(\gamma - 1)$ in accordance with assumption (3). The second term in Eq. (23) can be expanded, and part of it grouped with the first term; in this step, recognize that $B_p = |\nabla \psi|/r$. Finally, in view of Eq. (32) recognize $(F/n)B_p^2 = \mathbf{u}_p \cdot \mathbf{B}_p$. This leads to the RNF form

$$\nabla \cdot \left[\left(1 - \frac{F^2}{n}\right)\frac{\nabla\psi}{r^2}\right] + \mathbf{u}\cdot\mathbf{B}F' + \frac{B_\theta}{r}G' + nH' - \frac{n^\gamma}{\gamma - 1}S' = 0. \tag{34}$$

The electric field effect $\varepsilon\phi'$ *does not* appear explicitly in the RNF equation. Compare the RNF result with the MHD form, Eq. (7) of Ref. 5, observing the aforementioned notational conversions. The MHD equation contains a *false* extra term $+r n u_\theta \varepsilon \phi'$.

In summary, the more general nearby-fluids system is reduced to a single-fluid (the RNF system) This is done by applying three assumptions which accommodate the nature of the MHD system, particularly the retention of the electric field effect. The resulting RNF system agrees with MHD except at two points. The MHD model is missing a term in the Bernoulli equation and contains a false extra term in the extended GS equation. Both the missing and false terms are related to the electric field. Evidently, even accounting for its "large" ordering of the electric field, MHD is not the single-fluid limit of the more general two-fluid system. Thus, the two-fluid system does not uniformly converge to MHD in the limit $\varepsilon \to 0$, even when electric fields are retained.

The RNF system can be further reduced to the familiar GS equation for *static* equilibria by eliminating the flow: F, $\mathbf{u} = 0$. Then Eq. (31) gives $rB_\theta = G(\psi)$. Furthermore, the last two terms in Eq. (34) become p'; note that the pressure is a surface function $p(\psi)$ in a static single-fluid.

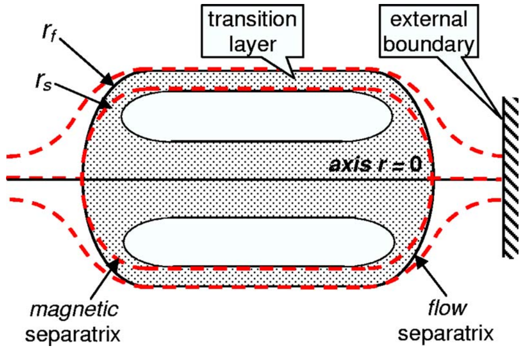

FIG. 1. Transition layer between magnetic and flow separatrices.

V. DISCUSSION

A. Summary

The nearby-fluids equilibrium system developed here achieves several things. It removes an unnecessary $1/\varepsilon$ singularity in the two-fluid model. It "softens" the otherwise stiff differential equation, which facilitates numerical computation. It helps clarify the transition from the two-fluid to a single-fluid. In so doing it exposes a seemingly unavoidable irregularity in the transition to the standard MHD model. The nearby-fluids system also becomes the platform for a straightforward to interpretation of a recent experiment, as in the companion paper.[18] In the following, several other issues illuminated by the nearby-fluids platform are discussed.

B. Purely rotational flow

In several previous works the assumption of purely rotational flow was adopted. In some cases this assumption is justified, such as in strongly rotating plasmas in mirror-like geometries.[25] In toroidal plasmas, however, it is a risky assumption. Pure rotation was assumed in a tokamak (Secs. 5–9 of Ref. 8) by setting $F(\psi)\rightarrow 0$ (our terminology). Further, in taking $\varepsilon \rightarrow 0$, leading to Eq. (34) of Ref. 8, the two-fluid effect disappears and the model becomes MHD, retaining the electric field. Thus the rotation is attributed entirely to a very large electric drift. In fact tokamaks routinely have $B_\theta \gg B_p$. It is more natural to identify the strong toroidal rotation primarily with *parallel* flow arising from $F\neq 0$. Forcing $F\rightarrow 0$ is a tortured assumption because it requires a very large electric potential of order $(1/\varepsilon)kT_e/e$.

C. Transition layer

The transition layer is the annular layer between the *magnetic* separatrix and the *flow* separatrix, as shown in Fig. 1 for a compact toroid configuration. The magnetic separatrix is shown as a dashed line (marked $r=r_s$) and the flow separatrix is shown as a solid line (marked $r=r_f$). The difference arises because $\psi=0$ defines the magnetic separatrix, while $Y=\psi+\varepsilon r u_\theta=0$ defines the flow separatrix. The thickness of this layer $\Delta_{\rm tr}$ can be found by expanding $\psi\approx B_z(r_s)r_s(r-r_s)$, leading to the expression (dimensionless units)

TABLE I. Transition layer.

Facility	$u_\theta(r_s)$ (km/s)	$B_z(r_s)$ (T)	$\Delta_{\rm tr}$ (cm)
TCS[a]	30	0.03	2.1
TS-3,4[b]	$\sim$0	0.03	$\sim$0
NSTX[c]	90	0.4	0.4

[a]Reference 26.
[b]Reference 27.
[c]Reference 28.

$$\Delta_{\rm tr} = -\varepsilon(u_\theta/B_z)_{r=r_s}. \tag{33'}$$

In dimensional units this is $\Delta_{\rm tr}=-u_\theta/\omega_{ci}$ evaluated at the separatrix; here $\omega_{ci}\equiv eB/m_i c$ is the ion cyclotron frequency. The transition layer is an inherently two-fluid effect because it vanishes for $\varepsilon\rightarrow 0$. There is *no* transition layer in the MHD model because it allows only a *single* set of surfaces. The transition layer only appears if there is rotational flow at the separatrix. In the typical case where the ion rotation is in the diamagnetic direction, u_θ/B_z is negative, the transition layer lies *inside* the flow separatrix, as in Fig. 1. (An exception to this is an FRC driven by rotating magnetic fields, in which the ions rotate in the *opposite* direction.) Some examples of the thickness of the transition layer are shown in Table I. More details of the transition layer are given for a specific FRC experiment in the companion paper.[18]

The existence of a transition layer has several important implications.

(1) In the transition layer the ion fluid lies on *closed* field lines and thus is *magnetically* confined. On the other hand, the electron fluid lies on *open* field lines and thus is *electrostatically* confined by the spontaneous ambipolar potential.
(2) The electrical nature of the external boundary condition (see Fig. 1) strongly affects rotation in the transition layer. A conducting boundary will produce shorting currents that alter the potential all along the open magnetic field lines, including in the transition layer. Thus the rotation in the transition layer depends on whether the end boundary has floating potential (dielectric) shorted potential (metal) or controlled potential (biased rings). The third option offers a practical way to control rotation and other behavior. Indeed, improved confinement might be achieved by modifying the rotation, just as it does in triggering the H-mode in tokamaks.
(3) The line-tying effect in the transition layer may profoundly affect stability. The transition layer actually has a substantial inner leg, the inside shaded region in Fig. 1. This inner leg can actually be quite broad; e.g., in TCS, its radius is about 40% of the magnetic separatrix radius. If line-tying serves to directly stabilize the affected "shaded" region, then global kink modes such as the tilt in FRCs might be suppressed. The inner boundary of the shaded region may act as a nearly rigid boundary on kink modes. The substantial transition layer in TCS and possibly other elongated FRCs may partly explain their resistance to the tilting instability. On the other hand,

present low-elongation FRCs such as in TS-3 (see Table I), which have no transition layer, are unstable to tilting. In summary, the transition layer profoundly alters the open-closed topology of the plasma structure, with possible effects on the rotational flow, stability, and confinement.

D. Parallel heat conductivity

As mentioned in Ref. 5, strong parallel heat conductivity tends to equalize the temperature on flux surfaces and, in view of $\mathbf{u}\cdot\nabla s=0$, causes the poloidal flow component to damp out. However, strong parallel conduction effectively makes $\gamma\rightarrow 1$, which gives, e.g., $T_i=s_i$. Since s_i is a surface function, this renders the temperature uniform on the respective surfaces. A related question concerns the possible enhancement of cross-field heat conduction. Since the Y=const surfaces stray from the ψ=const surfaces, one might expect an enhanced heat conduction *across* the surfaces. However, in axisymmetric plasmas, the canonical angular momentum of ion particles rP_θ is conserved. For an ion fluid, the fluid canonical angular momentum is rP_θ=Y, which is constant on a fluid drift surface since $\psi_i=\bar{\psi}_i(\mathrm{Y})$. Thus parallel ion heat conduction is exactly *along* the ion particle drift surfaces Y=const, and thus produces no cross-field thermal conduction.

E. Discontinuities

Discontinuities in the plasma or field structure may arise under certain conditions. The surface functions themselves may have discontinuities, e.g., a discontinuity in $S_i(Y)$ would give rise to a temperature discontinuity at a particular ion flow surface. Another more profound class of discontinuity may arise if the equations transition from elliptic to hyperbolic at some point. The threshold point has been investigated for the MHD model.[4,5,26] The appearance of the factor $(1-F^2/n)$ in Eqs. (30) and (31) is symptomatic of this possibility in the neighborhood of the point where the poloidal flow is comparable to the Alfvén speed. (In the dimensionless variables, flow at exactly the Alfvén speed occurs when $u_p=F/\sqrt{n}$.)

Recently, the threshold was investigated for the full two-fluid model.[17] This analysis considered the ion motion Eqs. (16) and (23), with the Bernoulli relation Eqs. (17) and (24) and led to a different and much simpler elliptic-to-hyperbolic threshold, namely, when the poloidal flow reaches the sound speed. However, the treatment of the ion equation of motion in isolation from the Ohm's law is problematical. The two-fluid system is essentially a pair of second-order equations for both Y and ψ. If, say, Y is eliminated in favor of ψ, the system is actually fourth order for ψ, with the highest order term $\sim\varepsilon^2(\Delta^*)^2\psi$. Since this term is multiplied by the small factor ε^2, the designations elliptic or hyperbolic might still be applied in the $O(1)$ equations, such as single-fluid MHD or nearby-fluids. The actual effect of the fourth order term will be to resolve (on a finer scale) any discontinuity arising in the $O(1)$ flow. This raises an interesting point. The fourth-order system for flowing equilibria is completely *dissipation free*. Thus, the fine-scale structure of the "discontinuity" is not the result of dissipative effects. Said another way, the discontinuity may not produce an entropy jump. This is markedly different from ordinary gas dynamics where the structure of the discontinuity is resolved by an entropy-generating dissipative effect, namely viscosity.

TABLE II. Example of an ion-inertia-length parameter.

	L(m)	n(m^{-3})	ε
FRC (TCS)	0.23	4.1×10^{19}	0.22
FRC (TS-3)	0.30	3×10^{19}	0.20
Spherical tokamak (NSTX)	1.4	3.5×10^{19}	0.04

Certainly, the resolution of discontinuities will require the inclusion of ε scale effects. Indeed, while the ion-inertia-length parameter based on the overall plasma size may be small, a "discontinuity" introduces phenomena with a much shorter length scale, possibly of order $L=\ell_i$. This is expected in the thin scrape-off-layer region, and possibly near critical rational surfaces such as the q=1 surface (q is the safety factor).

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APPENDIX A: ION-INERTIA-LENGTH PARAMETER IN EXPERIMENTS

In low-aspect-ratio plasmas, such as FRCs, spheromaks, and spherical tokamaks, it is convenient to set the system length scale $L=r_s$, the outboard separatrix radius. In a high-aspect ratio plasma such as a conventional tokamak, $L=a$, the minor radius. Table II shows the ion-inertia-length parameter ε for several examples of interest, two FRC examples, TCS-translation[27] and Tokyo Spheromak-3 (TS-3) FRC,[28] and a spherical tokamak, National Spherical Tokamak Experiment (NSTX).[29] In future "fusion-grade" plasmas, ε may be as small as a few percent. Note the importance of defining ε as a dimensionless parameter. In one instance it was defined as a dimensional quantity.[8] This led to confusion about which terms are small, i.e., O(ε), and which are not. For example (Ref. 8, Sec. I), the electric field appeared to $O(1)$.

APPENDIX B: DIMENSIONLESS VARIABLE SCHEME

A system of dimensionless variables was introduced previously.[16] It employs three reference scales: (1) global length scale L, e.g., the outboard separatrix radius; (2) magnetic field, B_R, e.g., the poloidal field at the outboard separatrix; and (3) density, n_R, e.g., the average density inside the separatrix. From these all other reference scales derive: velocity $V_{AR} \equiv B_R/(4\pi m_i n_R)^{1/2}$ (Alfvén speed); pressure $p_R \equiv B_R^2/4\pi$; temperature and enthalpy, $kT_R \equiv B_R^2/4\pi n_R$; electrical field $E_R = kT_R/eL = B_R^2/4\pi e n_R L$; magnetic flux, $B_R r_s^2$; and flow stream function, $n_R V_{AR} r_s^2$. The generalized vorticity introduced in the text is normalized by B_R.

APPENDIX C: PROOF OF TWO DISTINCT SETS OF SURFACES

Since the extended GS equation (23) gives the appearance of a single, second-order differential equation for ψ one might think that it might support only one set of characteristic surfaces instead of two. It is easily proved otherwise. The two sets of surfaces are defined by ψ_i=const (or equivalently,
Y=const) and ψ_e=const (or equivalently ψ=const). The nearby-fluids ordering, Eq. (19b) gives

$$\psi_i = \int_0^Y F(x')dx' + \varepsilon G(Y). \tag{C1}$$

Take the gradient and recognize that $Y \equiv \psi + \varepsilon r u_\theta$. Then

$$\nabla \psi_i = [F(Y) + \varepsilon G'(Y)] \nabla \psi + \varepsilon [F(Y) + \varepsilon G'(Y)] \nabla (r u_\theta). \tag{C2}$$

The gradient on the left-hand side is normal to the ion surfaces ψ_i=const (Y=const); the first term on the right-hand side is normal to the magnetic surfaces ψ=const (ψ_e=const). These two sets of surfaces are *not* equivalent unless the second term on the right-hand side vanishes. Thus, in the nearby-fluids model, the only instance where the ion and electron surfaces coincide is in the highly specialized case, $r u_\theta$=const.

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