Transient and quasisteady behavior with rotating magnetic field current drive

Loren C. Steinhauer
Redmond Plasma Physics Laboratory, University of Washington, 14700 NE 95th Street, Suite 100, Redmond, Washington 98052

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The time-dependent behavior of rotating magnetic field (RMF) current drive is investigated using a two-fluid model. The important new factor is the addition of transverse ion mobility in contrast to rigid-ion models of the past. The equations simplify conveniently, allowing the behavior on each surface (r=const) to be isolated, which permits a quadrature solution for the ion fluid rotation. A rapid transient phase leads to quasisteady behavior that evolves on the relatively slow diffusion timescale. The fast transient timescale is set by the ion inertia. Unless there is an ion momentum source to balance the electron drag on the ion fluid, there is no quasisteady current drive effect. Collisions with neutrals offer such a momentum source in some experiments, notably rotamaks and the Star Thrust Experiment. Other sources of ion momentum are essential for RMF current drive in hotter, fusion-relevant plasmas. The properties of the quasisteady state are found, including the self-consistent ion fluid rotation rate and radial electric field, and RMF corrections on the pressure balance. [DOI: 10.1063/1.1377613]

I. INTRODUCTION

Rotating magnetic fields (RMFs) that penetrate the plasma are a promising method by which to drive a steady current in magnetic fusion systems. RMF can be generated by simple antennas, operate at low frequency (<MHz), and may be very efficient because they drive current in the bulk electrons. Steady field-reversed configuration (FRC)-like plasmas have been formed and sustained by this method (see, e.g., Refs. 1–5). These experiments, commonly called ''rotamaks'' produced nearly spherical, relatively cold plasmas that were sustained as long as many msec. Currently two RMF driven facilities are in operation. The Star Thrust Experiment (STX) has somewhat higher RMF antenna power; it forms FRCs that are more elongated than in previous experiments.[6] Another facility, the Translation, Confinement and Sustainment (TCS) Experiment, has recently been converted to include RMF antennas; there RMF current drive will eventually be applied to a hot FRC initially created by a field-reversed theta pinch.[7]

The objective of this article is to investigate transient RMF phenomena on intermediate timescales that are associated with the ion fluid inertia. These are faster than the diffusion time but slower than timescales associated with the electron inertia (plasma frequency, electron cyclotron frequency). Intermediate frequencies are bracketed above and below by high- and low-frequency phenomena as shown in Table I, which shows conditions for the two active experiments as well as for a projected fusion plasma. (See Appendix A for a listing of plasma parameters for these examples.) There are seven or more orders of magnitude between the high- and low-frequency phenomena in which intermediate timescale phenomena can take place.

Previous theoretical analyses made various reducing assumptions that either exclude or strongly modify the intermediate timescale behavior, e.g., the assumption of steady state[8–10] or the rigid-ion assumption,[1,11–14] which have been the standard models of RMF. The approach here is a systematic treatment of the time-dependent two-fluid equations of motion for ion and electron fluids together with Maxwell's equations. The model allows variation on the slow, diffusive timescale, including slow evolution of the equilibrium quantities (density, pressure, ambient magnetic field). Since the transient penetration of the RMF into the plasma is on the diffusive timescale,[13–16] the model here allows for the slow evolution of the RMF amplitude as well. The model excludes high-frequency phenomena by assuming massless electrons. The model also only considers the fluid picture of RMF effects; it does not address essentially kinetic phenomena which may be important in weakly collisional plasmas.[17] The principle conclusion is that the ion fluid is highly mobile and quickly spins up to cancel the RMF current drive effect *unless* there is an ion momentum source, such as friction or an externally applied source such as a neutral beam.

An outline of this article with a preview of results is as follows. In Sec. II A the basic assumptions including how various quantities are either averaged or resolved in the azimuthal (θ) direction are explained. In Sec. II B the quiver (z) motion of both species in the RMF and the resulting quadratic terms in the Lorentz force are given. Introduced here is the important intermediate timescale associated with transverse ion inertia. In Sec. II C the transverse (r, θ) dynamics of both ions and electrons are examined using the Hall-magnetohydrodynamic (MHD) format. This leads to a 7×7 system of equations. A separation between quantities with intermediate and slow timescale variations leads to a great simplification in which the azimuthal equation of motion separates from the others and can be solved in quadrature form (Sec. II D). Section II E focuses on the quasisteady

TABLE I. Bounds on the intermediate frequency range.

Frequency	STX	TCS	Fusion
High frequency			
Plasma frequency, ω_p (rad/s)	1.3×10^{11}	8×10^{11}	1.4×10^{12}
Electron cyclotron ω_{ce} (rad/s)	1.8×10^{9}	2.6×10^{10}	5.3×10^{11}
Low frequency			
Classical diffusion $1/\tau_D$ (s^{-1})	230	100	1.8×10^{-3}

phase. Simple expressions are found for the radial force balance, slip frequency, radial force balance, and radial drift. Section II F contains a physical interpretation of what happens during the transient and quasisteady phases in terms of the azimuthal forces on each species. This includes a treatment of the RMF torque that is applied to the ion fluid indirectly through friction and Lorentz forces. In Sec. III the consequences of *no* ion momentum source, as would be the case in a hot plasma without externally applied sources, namely, that there is no RMF current drive are discussed. In Sec. IV some possible ion momentum sources are considered. In Sec. IV A simple friction against a stationary background is considered and the resulting ion fluid rotation, slip frequency, radial drift, and corrections to radial force balance are found. This is applied to the case of friction against neutrals. In Sec. IV B ion viscous friction as an ion momentum source is considered. There is a brief review of other possible momentum sources in Sec. IV C. The article is concluded in Sec. V with a summary and discussion of the results.

II. ANALYSIS

A. Basic assumptions and terminology

An idealized FRC equilibrium and RMF geometry are adopted as have been widely used. The equilibrium assumes an infinitely long plasma with no dependence on *z*, no toroidal field, and no poloidal flow. The idealized RMF assumes an infinitely long antenna (the wave vector is purely azimuthal). In this case the RMF produces no compressibility effect and can be described entirely using the axial component of the vector potential.[11,13,16] (The adopted gauge assumes zero scalar potential.)

The several variables are Fourier resolved into their azimuthal components $e^{-i\ell\theta}$. In the idealized geometry each variable has a Fourier expansion involving either even-ℓ modes alone, or odd-ℓ modes alone. The even-ℓ variables are the transverse (r, θ) components of the flow velocities for both ion and electron species, the radial electric field component, and the axial (z) magnetic field component. For these *even* variables, the lowest order mode, $\ell=0$, is axisymmetric. The odd-ℓ variables include the axial flow velocity component for both species, the axial electric field component, and the transverse magnetic field components. The approximation adopted here is to retain only the *lowest* order mode terms for each type of variable, i.e., the $\ell=0$ axisymmetric component of the *even* variable, and the $\ell=1$ component of the *odd* variables. In effect the even ℓ are ''surface averaged'' to select only their axisymmetric component. The RMF effect appears as quadratic terms involving the product of odd-ℓ variables; thus these products have even-ℓ symmetry and are treated in the same way as the even-ℓ variables, i.e., they too are surface averaged. The truncation of Fourier expansions in θ has been used in all RMF modeling, until a recent study[15] that included higher harmonics in the RMF ''potential'' A_z. There, the higher harmonics were found to be unimportant unless the RMF antenna structure itself generated a large harmonic content.

The terminology adopted in this article is as follows. (1) *Cyclotron frequencies* for the electrons and ions $\omega_{ce}=eB_z/m_ec$, $\omega_{ci}=eB_z/m_ic$ are based on the *ambient* magnetic field B_z, where m_e and m_i are the electron and ion masses, respectively, e is the magnitude of the ion (hydrogen) and electron charge, and c is the speed of light. This differs from several previous papers which based the cyclotron frequencies on the RMF field amplitude. (2) The $e^{i\omega t}$ factor in odd-ℓ quantities is understood. (3) The *slip frequency* ω' is the difference between the RMF frequency ω and the electron fluid rotation frequency:

$$\omega'=\omega-ue_\theta/r,$$

where $u_{e\theta}$ is the electron fluid rotational velocity. The *electron magnetization* in the rotating frame is $\omega'/\nu_{e\parallel}$ where $\nu_{e\parallel}$ is the *electron* collision frequency (with ions) associated with motion *parallel* to the ambient magnetic field. The anticipated limit of interest is $\omega'/\nu_{e\parallel}\ll1$, i.e., the electrons are weakly magnetized in the frame rotating with the RMF. (4) The *radial gradient* of the RMF is described using $\sigma\equiv(r/A_z)dA_z/dr$ where A_z is the component of the vector potential associated with the RMF. Therefore the radial component is $B_r=iA_z/r$. The azimuthal component $B_\theta=-dA_z/dr$ can thus be expressed in terms of the radial as

$$B_\theta=i\sigma B_r. \tag{1}$$

For fully penetrated RMF $\sigma=1$, while for partial penetration $|\sigma|>1$. Since σ is determined by the transient penetration of the RMF, it varies on the diffusive timescale. In general it is complex: the real part σ_R gives spatial *evanescence* of the RMF, and the imaginary part σ_1 one gives cyclic radial structure.

B. Quiver motion

An axial ''quiver motion'' of both ion and electron species arises because of the RMF field. This is derived in Appendix B using the terminology defined earlier. The quiver velocities are

$$m_eu_{ez}=\frac{\omega'(\omega_*-\omega')-i(m_e/m_i)\nu_{e\parallel}\omega_*}{(i\nu_{e\parallel}+\omega')(\omega_*-\omega')-(m_e/m_i)\omega'\nu_{e\parallel}}\,\frac{eA_z}{c}, \tag{2a}$$

$$m_iu_{iz}=\frac{-\omega'(\omega_*-\omega')+i\nu_{e\parallel}\omega_*}{(i\nu_{e\parallel}+\omega')(\omega_*-\omega')-(m_e/m_i)\omega'\nu_{e\parallel}}\,\frac{eA_z}{c}, \tag{2b}$$

where the diamagnetic frequency (an equilibrium quantity) is

$$\omega_*\equiv(u_{i\theta}-u_{e\theta})/r=-\frac{c}{4\pi en}\frac{\partial B_z}{r\partial r}; \tag{3}$$

TABLE II. Intermediate frequency.

	STX	TCS	Fusion
$\nu_{\rm int}$ (s^{-1})	1.3×10^8	3.6×10^7	2.2×10^{10}

the latter expression arises from the surface-averaged θ component of Ampere's law. Using the approximations m_e/m_i, and $\omega'\ll\omega_*\approx\omega$ (weakly magnetized electrons):

$$u_{ez}\approx-\frac{\omega'}{\nu_{e\parallel}}r\frac{eB_r}{m_ec},\tag{4}$$

$$u_{iz}\approx-ir\frac{eB_r}{m_ic}.\tag{5}$$

The former is equivalent to Eq. (7) of Ref. 9 for weak magnetization. The latter is consistent with Refs. 8 and 18. Note that the ion motion is nearly 90° out of phase (the i factor) with B_r and thus in phase with B_θ. The part of the ion quiver motion in phase with B_r, smaller than u_{ez} by a factor of $2m_e/m_i$, is ignored. As observed elsewhere the small mass of the electrons causes their quiver motion to be friction dominated; on the other hand, the large inertia of the ions makes their quiver motion inductive, or "reactive."[8,9,12] Note that the amplitude of the ion quiver motion is very likely *larger* than the electrons depending on the magnetization $\omega'/\nu_{e\parallel}$.

The quiver motion leads to quadratic terms in the transverse equations of motion, which represent the RMF effect on the equilibrium quantities. Here, it is convenient to introduce an intermediate frequency scale (to be justified later),

$$\nu_{\rm int}\equiv\frac{e^2|B_r|^2}{2m_em_ic^2\nu_{e\parallel}},\tag{6}$$

where $|B_r|$ denotes the *amplitude* of B_r (independent of θ). Physically $\nu_{\rm int}$ is a ratio of RMF-to-ion "inertia" forces. It depends only on quantities that vary on the diffusive timescale. Its values for experimental and projected fusion plasmas are shown in Table II. Comparing with Table I note that $\nu_{\rm int}$ is one or more orders of magnitude slower than the high frequencies and *five* or more orders of magnitude faster than the low frequency associated with classical diffusion. Even if the diffusion has a significant anomaly factor compared to the classical, say, 20, the diffusion rate is still much slower. Note further that the intermediate rate is significantly faster than even radial acoustic rates (acoustic speed/separatrix radius) which are 2.5×10^5, 1.3×10^6, and 5×10^6, rad/s for STX, TCS, and the fusion example, respectively.

In terms of the intermediate time constant $\nu_{\rm int}\propto|B_r|^2$ the quadratic terms are

$$u_{iz}B_\theta=\sigma_R\frac{m_ec}{e}r\nu_{\rm int}\nu_{e\parallel},\tag{7}$$

$$u_{iz}B_r=-2\frac{m_ec}{e}r\nu_{\rm int}\omega',\tag{8}$$

$$j_zB_\theta/c=-\sigma_1m_inr\nu_{\rm int}\omega',\tag{9}$$

$$j_zB_r/c=m_inr\nu_{\rm int}\omega'.\tag{10}$$

Here n is the electron and hydrogen ion density, and j_z is the axial current density. Weak collisionality $\omega'/\nu_{e\parallel}\ll1$ is assumed as before. Note that these are surface-averaged quantities (even-ℓ symmetry). Equations (9) and (10) anticipate the Hall-MHD format. This is equivalent to the two-fluid format: *Ohm's* law follows from the electron motion equation but uses the current rather than the electron fluid velocity; the *fluid* motion equation is the sum over the species.

C. Transverse motion

The r and θ components of the equation of motion and Ohm's law determine the species drifts and electric fields. Eliminate the quadratic terms using Eqs. (7)–(10); then all the remaining variables are axisymmetric. The surface-averaged r, θ equations of motion then become

$$m_i\frac{\partial u_{ir}}{\partial t}=-\frac{1}{n}\frac{\partial p}{\partial r}+\frac{j_\theta B_z}{cn}+\nu_{\rm int}r(m_i\sigma_1\omega'-m_e\sigma_R\nu_{e\parallel}),\tag{11}$$

$$m_i\frac{\partial u_{i\theta}}{\partial t}=-\frac{j_rB_z}{c}+m_ir\nu_{\rm int}\omega'+F_{i\theta},\tag{12}$$

where p is the pressure and j_r and j_θ are components of the current density. In the middle term on the right side of Eq. (12) $1-2m_e/m_i\rightarrow1$. The terms here are (left side) inertia and (right side) pressure gradient (r only), $j\times B$ force, and RMF force. The θ component includes $F_{i\theta}$, an unspecified azimuthal force on the ion fluid; this may arise from viscosity, friction with neutrals, or neutral beam injection. Note also that three convective inertia terms on the left side have been neglected. Dropping $m_iu_{ir}\partial u_{ir}/\partial r$ in Eq. (10) is warranted if the diffusion timescale is long. Dropping the centrifugal effect $-m_iu_{i\theta}^2/r$ in Eq. (11) is warranted if the ion fluid rotation is low enough not to significantly affect the equilibrium. Dropping $m_iu_{ir}\partial u_{i\theta}/\partial r$ in Eq. (12) is less obvious; this convective inertia effect is the "radial flow."[8] This effect is investigated in Appendix C where it is found to be usually negligible. The surface-averaged r, θ Ohm's laws are

$$0=E_r+\frac{1}{c}u_{i\theta}B_z-\eta_\perp j_r+\frac{1}{en}\left[\frac{\partial p_e}{\partial r}-\frac{j_\theta B_z}{c}\right]-\sigma_1\frac{m_i}{e}r\nu_{\rm int}\omega',\tag{13}$$

$$0=E_\theta-\frac{1}{c}u_{ir}B_z-\eta_\perp j_\theta+\frac{j_rB_z}{cen}-\frac{m_i}{e}r\nu_{\rm int}\omega'.\tag{14}$$

Here the perpendicular resistivity is $\eta_\perp=m_e\nu_{e\perp}/e^2n$ and $\nu_{e\perp}$ is the *electron* collision frequency associated with motions *perpendicular* to the magnetic field. The terms in Ohm's law are the electric field, $u\times B$ force, resistive force, diamagnetic effect (r component only), Hall effect, and RMF force. The left sides are zero since the electron inertia is negligible at intermediate frequencies.

The system of equations is completed by Ampere's law and an auxiliary equation for the slip frequency. The surface-averaged r, θ Ampere's laws are

$$j_r=0,\tag{15}$$

$$j_\theta = -\frac{c}{4\pi}\frac{\partial B_z}{\partial r}. \tag{16}$$

The displacement current is ignored, consistent with the neglect of high-frequency phenomena. Note that $\nabla\times\mathbf{B}$ has no radial component in the standard RMF geometry. Equation (15) implies that the radial motion is ambipolar once high-frequency effects are excluded. The slip frequency in terms of the ion fluid rotation is

$$\omega' = \omega + \omega_* - \frac{u_{i\theta}}{r}, \tag{17}$$

where the diamagnetic frequency is as given in Eq. (3). The system, Eqs. (11)–(17) can be viewed in the following way. The quantities varying on the diffusive time scale τ_D are the equilibrium variables, p, n, B_z, and ω_*, and the RMF quantities $|B_r|$, σ_R, and σ_1. Their evolution is governed by the time-dependent continuity equation, energy conservation law, and the z component of Faraday's law. The seven remaining variables that *may* exhibit intermediate timescale effects are $u_{ir}, j_r, E_r, u_{i\theta}, j_\theta, E_\theta$, and ω'. The separation of quantities into slow $(1/\tau_D)$ and fast $(\nu_{\rm int})$ is not assumed *a priori* but is evident from the solution that will be found shortly.

This system of equations can be simplified considerably. Eliminate the currents using Eqs. (15) and (16) and eliminate the slip frequency ω' in favor of $u_{i\theta}$ using Eq. (17). Then the surface-averaged azimuthal equation of motion, Eq. (12), becomes

$$\frac{\partial u_{i\theta}}{\partial t} + \nu_{\rm int} u_{i\theta} = \nu_{\rm int} r(\omega+\omega_*) + \frac{F_{i\theta}}{m_i}. \tag{18}$$

This is solved for $u_{i\theta}$ in Sec. II D. The radial equation of motion, Eq. (11), becomes

$$\frac{\partial u_{ir}}{\partial t} + \sigma_1 \nu_{\rm int} u_{i\theta} = \sigma_1 r \nu_{\rm int}(\omega+\omega_*) - \frac{1}{m_i n}\frac{\partial}{\partial r}\left(p + \frac{B_z^2}{8\pi}\right) - \sigma_R \frac{m_e}{m_i} r \nu_{\rm int}\nu_{e\parallel}. \tag{19}$$

This can be solved for u_{ir} by simple integration since $u_{i\theta}$ was already found using Eq. (18). All that remain are the two components of Ohm's law, Eqs. (13) and (14), which are purely algebraic,

$$E_r + \frac{1}{c}\left(B_z + \sigma_1 \nu_{\rm int}\frac{m_i c}{e}\right)u_{i\theta} = \frac{1}{c}B_z r\omega_* + \sigma_1\nu_{\rm int}\frac{m_i c}{e}(\omega+\omega_*) - \frac{1}{en}\frac{\partial p_e}{\partial r}, \tag{20}$$

$$E_\theta - \frac{1}{c}B_z u_{ir} + \frac{m_i}{e}\nu_{\rm int}u_{i\theta} = \frac{m_i}{e} r\nu_{\rm int}(\omega+\omega_*) + \frac{m_e}{e} r \nu_{e\perp}\omega_*. \tag{21}$$

Equation (20) gives the radial electric field ($u_{i\theta}$ having already been found), and Eq. (21) gives the azimuthal electric field ($u_{i\theta}$, and u_{ir} having already been found). The last term on the right side of Eq. (21) is the resistance effect.

D. Quadrature solution

Equation (18) is a linear first order differential equation for the ion fluid rotation and can be solved in quadrature form:

$$u_{i\theta} = u_{i\theta 0}\exp\left(-\int_0^t \nu_{\rm int}(t')dt'\right) - \int_0^t \left[\nu_{\rm int} r(\omega+\omega_*) + \frac{F_{i\theta}}{m_i}\right]\exp\left(-\int_{t'}^t \nu_{\rm int}(t'')dt''\right)dt',$$

where $u_{i\theta 0}$ is the initial $(t=0)$ value of $u_{i\theta}$. Note in the square brackets that $\nu_{\rm int}$, ω_*, and $F_{i\theta}$ are all functions of t'. Expand these in a Taylor series about the current time t; e.g., $\nu_{\rm int}(t') = \nu_{\rm int}(t) + (t'-t)(d\nu_{\rm int}/dt)_t + \ldots$. Introduce another time ''constant'' that reflects the rate that a combination of these quantities changes,

$$\nu_{\rm equ} \equiv \frac{d}{dt}\ln\left(\frac{\nu_{\rm int}}{F_{i\theta}}\right) = \mathcal{O}(1/\tau_D).$$

Then the rotational velocity is

$$u_{i\theta} = \left[r(\omega+\omega_*) + \frac{F_{i\theta}}{m_i\nu_{\rm int}}\right]\left(1 - \frac{\nu_{\rm equ}}{\nu_{\rm int}} + \mathcal{O}(\nu_{\rm equ}^2/\nu_{\rm int}^2)\right) + \exp\left(-\int_0^t \nu_{\rm int}(t')dt'\right)\left\{u_{i\theta 0} - \left[r(\omega+\omega_*) + \frac{F_{i\theta}}{m_i\nu_{\rm int}}\right] \times [1 + \mathcal{O}(\nu_{\rm equ}/\nu_{\rm int})]\right\}. \tag{22}$$

The first group of terms is the *quasisteady* part, and the second group decays on the fast timescale $1/\nu_{\rm int}$. Since this transient time is so short (see Tables I and II), the rotation very quickly assumes the quasisteady value, which then varies on the slow τ_D timescale. Likewise, the other quantities with intermediate timescale variation u_{ir}, E_r, and E_θ have a fast transient phase followed by quasisteady evolution.

E. Quasisteady behavior

The quasisteady ion fluid rotation $(t \gg 1/\nu_{\rm int})$ for slowly varying equilibrium quantities $\nu_{\rm equ} \ll \nu_{\rm int}$ from Eq. (22) is

$$u_{i\theta} = r(\omega+\omega_*) + F_{i\theta}/m_i\nu_{\rm int}. \tag{23}$$

The quasisteady radial ion flow follows by a simple integration of Eq. (19) and the quasisteady forms of the other variables follow from the algebraic relations, Eqs. (20) and (21). The surface-averaged *radial force balance* equation results from eliminating $u_{i\theta}$ in Eq. (19):

$$\frac{\partial}{\partial r}\left(p + \frac{B_z^2}{8\pi}\right) = rn\nu_{\rm int}(\sigma_1 m_i\omega' - \sigma_R m_e \nu_{e\parallel}), \tag{24}$$

where the RMF correction appears on the right side. The σ_1 term arising from the B_θ component has been noted in previous work.[19–21] Since $\nu_{\rm int} \propto |B_r|^2/8\pi$ it might be imagined that this is a pressure effect from the RMF. In fact it is not because it does not arise from a pressure gradient, i.e., $\nabla|B_r|^2/8\pi$. In fusion applications the corrections to the radial force balance should be very small, otherwise a large

RMF circulating power is implied. As will be seen shortly, these corrections are not small in current experiments. The slip frequency follows from Eqs. (17) and (23),

$$\omega' = -\frac{F_{i\theta}}{m_i \nu_{\rm int} r}. \tag{25}$$

The electron fluid rotation speed follows at once from $u_{e\theta} = r(\omega-\omega')$. The self-consistent *radial electric field* follows from Eqs. (20) and (23),

$$E_r = -\left(\frac{\omega_{ci}}{\nu_{\rm int}} + \sigma_1\right)\frac{F_{i\theta}}{e} - r\omega\frac{B_z}{c} - \frac{1}{en}\frac{\partial p_e}{\partial r}. \tag{26}$$

Define the *net radial drift* as the radial flow relative to the ambient magnetic field lines, $u_D = u_{ir} - cE_\theta/B_z$; this follows from Eqs. (21) and (23),

$$u_D = \frac{cF_{i\theta}}{eB_z} - \frac{\nu_{e\perp}}{\omega_{ce}} r\omega_*, \tag{27}$$

where the first term is the RMF drive effect and the second is the resistive diffusion.

It has been customary to suppose that one is free to choose all equilibrium properties. Among these is the ion fluid rotation speed, which is often conveniently set to zero. However, the results here show that the ion fluid rotation and the self-consistent electric field rapidly relax to a particular value set by the quasisteady properties, Eqs. (23) and (26). Since the transition to quasisteady state is so rapid, an arbitrary initial choice of the ion fluid rotation is therefore not meaningful.

Note that the RMF $(\omega, B_r, \sigma_R, \sigma_1, \nu_{\rm int})$ does not appear explicitly in the net radial drift, Eq. (27). This raises the question whether the RMF plays any role at all other than causing the electron fluid to rotate at a certain speed and adjusting the ion fluid rotation speed accordingly. Further, from Eq. (25) it appears that the slip frequency is unbounded if the RMF strength $|B_r|$ gets small, since $\nu_{\rm int} \propto |B_r|^2$. These questions are resolved by retaining terms of the order of $\omega'/\nu_{e\parallel}$ which is the electron magnetization in the frame rotating with the electron fluid. In the quiver velocity, Eq. (4), a factor of $1/(1+\omega'^2/\nu_{e\parallel}^2)$ was dropped; by restoring it, Eq. (25) becomes

$$\frac{\omega'}{1+\omega'^2/\nu_{ei\parallel}^2} = -\frac{F_{i\theta}}{m_i \nu_{\rm int} r}. \tag{28}$$

The left side has a maximum of $\nu_{e\parallel}/2$ at $\omega' = \nu_{e\parallel}$. Thus for a given momentum source there is a minimum value of the RMF for which a solution to Eq. (28) exists. If the RMF falls below this minimum, a quasisteady solution does not exist. The same correction factor $1/(1+\omega'^2/\nu_{e\parallel}^2)$ appears in each quadratic RMF term, Eqs. (7)–(10). All but one, Eq. (8), have the factor $\omega'/(1+\omega'^2/\nu_{e\parallel}^2)$ as in Eq. (28) so that the expressions for $u_{i\theta}$, E_r, and u_D, Eqs. (25)–(27), are unchanged. The only modification is in the RMF pressure term in Eq. (24) where a factor of $1/(1+\omega'^2/\nu_{e\parallel}^2)$ should be added. These corrections are usually unnecessary since $\omega'/\nu_{e\parallel} \ll 1$ generally holds in practice.

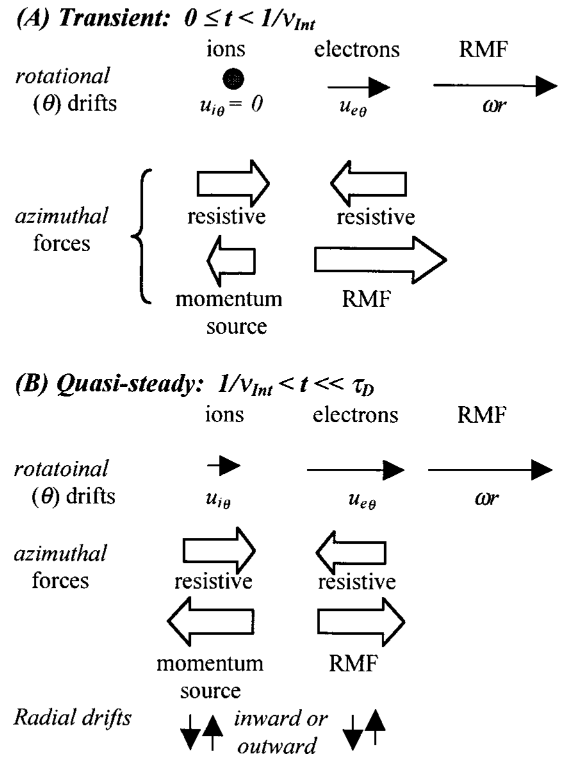

FIG. 1. Evolution of the RMF effect.

F. Interpretation of transient and quasisteady phases

Figure 1 illustrates the transient and quasisteady phases in terms of the forces and drifts occurring in each. In the initial state $(t=0)$ suppose the ion fluid is nonrotating and the slip frequency is larger than the electron fluid drift. In the example [Fig. 1(A)] the RMF force is large enough to exceed the drag on the electron fluid. During the *transient phase* the azimuthal force on the ion fluid is unbalanced and it responds by spinning up. In so doing it modifies the radial electric field so as to cause an exact equal spin up of the electron fluid. This happens without changing the current $\propto u_{i\theta} - u_{e\theta}$. The ion fluid spin up is inertial and lasts about an inertial time $\sim 1/\nu_{\rm int}$, hence the term transient phase. What the transient phase does is to bring the total azimuthal forces on the two species into equality: the momentum source+resistive force on the ion fluid becomes equal (but opposite in sign) to the RMF+resistive force on the electron fluid. This is the nature of $F\times B$ drifts, $u_{d_\alpha} = (c/q_\alpha)F_{\theta\alpha}/B_z$ where q_α and $F_{\theta\alpha}$ are the charge and azimuthal force on species α (ion, electron, respectively). Equal and opposite forces on the two species give equal (ambipolar) drifts.

The *quasisteady phase* follows. Here, the long-term effects of radial drifts come into play. This modifies the current $\propto u_{i\theta} - u_{e\theta}$, and causes the equilibrium to evolve on the diffusive timescale τ_D. The direction of the evolution depends on the *sign* of the azimuthal forces. If the force on the electron fluid is positive (ion fluid negative) as in Fig. 1(B), both will drift inward, tending to increase the internal magnetic flux. If the force on the electron fluid is negative (ion fluid positive), both will drift outward, causing a decay of the equilibrium. Throughout the quasisteady phase the equal magnitude of the forces on the species is maintained.

Since the transient spin up is rapid, an obvious question arises: Is the RMF adequate to supply the required torque? This question is addressed in Appendix D where it is shown that torque is transmitted through a *twist* of the RMF, and that the degree of twist to cause the predicted spin-up rate is quite modest.

Note that the timescale for the transient phase $1/\nu_{\rm int}$ (Table II) is shorter than the RMF time, $1/\omega$ (Table V). Thus the RMF force, Eqs. (7)–(10), may introduce higher harmonics not contained in this or in other models that truncate the Fourier expansion in θ after $\ell=1$. These missing details do not alter the claim that a quasisteady state is reached in a time exceedingly short compared with the diffusion time.

III. RMF CURRENT DRIVE *WITHOUT* AN ION MOMENTUM SOURCE

In the event that there are no appreciable momentum sources the current drive behavior is quite different. Figure 1 also illustrates this case. The transient phase brings the magnitude of the total forces on each species into equality. If there is no momentum source, then the only force on the ion fluid is resistive. Since the resistive forces are equal and opposite, the only way the forces on the ion and electron fluids can have equal magnitude is if the RMF force is zero. In this case the slip frequency is exactly zero and there is no current drive effect after an exceedingly short transient phase.

There is a slight qualification on the claim that the slip frequency falls exactly to zero. If the convective inertia is included (see Appendix C) the quasisteady ω' is nonzero even without a momentum source [Eq. (C4)]. However the RMF diffusion drift [Eq. (C5)] is only a small fraction of the resistive drift and can generally be neglected.

The conclusion that there is no steady RMF current drive without an ion momentum source confirms previous work.[10] In the steady state analysis of Ohnishi and Ishida,[8] a particle source or friction with neutrals allows a steady current drive effect; however the steady state is lost if both effects are missing [compare their Eqs. (17) and (24) and note trends in their Figs. 1 and 2]. A particle source acts as a momentum source (or sink) if the ion fluid is rotating and the injection particles have no rotation as injected. The steady state analysis of Hoffman reaches the same conclusion [see his Eq. (9)].[9] The simulations of Milroy,[14] which predicted steady FRCs in effect have a momentum source. These simulations assumed that the ion fluid rotation is continuously reset to zero, which is equivalent to an artificial momentum source. Therefore, the new conclusion here is *not* that the current drive without a momentum source turns off but that it turns off *extremely fast*, in $\Delta t\sim 1/\nu_{\rm int}$ rather than in the relatively long diffusion time τ_D.

An important question concerns the *rate* of ion spin up. Since the electrons are maintained at a fixed rotation $u_{e\theta}=r\omega$ by the RMF, changes in the current come from ion spin up or spin down. It has been suggested[4] that the ion fluid spin-up time arises from the mutual friction force, and since the ions have larger inertia than the electrons by the mass ratio, the spin-up time is $\tau=m_i/(m_e\nu_{e\perp})$, which is relatively slow. If so, the RMF would act as a *partial* current drive, significantly slowing down the decay of the configuration. However, the theory here shows that ion spin up arises from both the friction force *and* the Lorentz force (Appendix D). The RMF force [second term on the right side of Eq. (12)] gives a spin-up rate of $\partial(u_{i\theta}/r)/\partial t=\nu_{\rm int}\omega'$. Compare this with the spin-up rate needed in quasisteady state to keep up with evolution of the equilibrium, found by taking the derivative of Eq. (23), $\partial(u_{i\theta}/r)/\partial t=\partial\omega_*/\partial t\sim\nu_{\rm equ}\omega_*$ where $1/\nu_{\rm equ}$ is the nominal timescale for evolution of the equilibrium. Equating these gives the slip frequency which supplies the torque for spin up, $\omega'=\omega_*\nu_{\rm equ}/\nu_{\rm int}$.

IV. FRICTIONAL MOMENTUM SOURCES

A. Friction against a stationary background

Suppose the ion momentum source (or sink) arises from friction against a stationary background,

$$F_{i\theta}=-m_i\nu_{i0}u_{i\theta},$$

where ν_{i0} is the *ion* collision frequency against the background. In a simple fluid this would cause a simple exponential decay of the rotation with decay constant ν_{i0}. Then the azimuthal motion, Eq. (18), becomes

$$\frac{\partial u_{i\theta}}{\partial t}+(\nu_{\rm int}+\nu_{i0})u_{i\theta}=\nu_{\rm int}r(\omega+\omega_*).$$

For $\nu_{i0}\ll\nu_{\rm int}$ (verifiable *a posteriori*) and quasisteady state ($t>1/\nu_{\rm int}$) the ion fluid rotation, slip frequency, and RMF part of the radial drift [first term in Eq. (27)] are

$$u_{i\theta}=r(\omega+\omega_*),\quad \omega'=\frac{\nu_{i0}}{\nu_{\rm int}}(\omega+\omega_*),$$

$$(u_D)_{\rm RMF}=-\frac{\nu_{i0}}{\omega_{ci}}r(\omega+\omega_*).$$

Apply these results to steady state, i.e., allow the equilibrium to evolve on the slow τ_D timescale. This causes the diamagnetic frequency ω_* (a equilibrium property) to evolve to the state where the net radial drift is zero, i.e., the two terms in Eq. (27) add up to zero. It is convenient here to introduce a parameter that is nominally the ratio of friction forces on the ion fluid: friction with background/friction with electrons,

$$\chi\equiv m_i\nu_{i0}/m_3\nu_{e\perp}.$$

Then in terms of χ and the RMF frequency,

$$\omega_*=-\frac{\chi}{1+\chi}\omega, \tag{29}$$

$$\frac{u_{i\theta}}{r}=\frac{1}{1+\chi}\omega, \tag{30}$$

$$\omega'=\frac{m_e}{m_i}\frac{\nu_{e\perp}}{\nu_{\rm int}}\frac{\chi}{1+\chi}\omega. \tag{31}$$

In the limit $\chi\gg1$ (large ion friction with neutrals) the so-called ''rigid-ion'' assumption is approximately valid, i.e., the ion rotation is small compared with both the RMF and diamagnetic frequencies. These expression will be applied to a practical example shortly.

A comment is in order about the commonly used γ parameter, which is the nominal ratio of the RMF torque to the resistive friction torque assuming rigid ions.[21,22] This parameter is useful in experiments, such as STX, where the rigid-ion assumption is approximately valid. However γ is not meaningful in the absence of a significant ion momentum source.

These results can be used to evaluate corrections to the radial force balance. In the absence of RMF effects Eq. (24) integrates into the familiar pressure balance expression $p+B_z^2/8\pi=B_e^2/8\pi$, where $B_e=\text{const}$ is the external (vacuum) magnetic field. Both RMF and centrifugal effects represent corrections to this. The centrifugal term, $m_i n u_{i\theta}^2/r$, was left off the right side of Eq. (24). The magnitudes of these corrections are evaluated as follows: integrate Eq. (24) on $0\leqslant r\leqslant a$ and divide by the ''total'' pressure $B_e^2/8\pi$; then the corrections appear as fractions.

$$f_{\text{centr}}=\frac{8\pi}{B_e^2}\int_{r=0}^{a}\frac{m_i n u_{i\theta}^2}{r}dr,$$

$$f_{\text{RMF}}=\frac{8\pi}{B_e^2}\int_0^a rn\nu_{\text{int}}(\sigma_I m_i\omega'-\sigma_R m_e\nu_{e\parallel})dr.$$

Calculate the integrals as follows. Factor out the density by treating it as an average density. Estimate the field penetration parameters as $\sigma_R=-\sigma_1=a/\delta_{\text{RMF}}$ where δ_{RMF} is the penetration depth of the RMF. The minus sign on the ''twist'' σ_1 is consistent with the evolution equations of Ref. 13. Use the steady expressions for ω' and $u_{i\theta}$ in Eqs. (30) and (31). Finally, note that $\nu_{\text{int}}\propto|B_r|^2\propto\exp[2(r-a)/\delta_{\text{RMF}}]$ in view of the penetration depth. Then

$$f_{\text{centr}}=\left(\frac{a\omega}{\ell_i\omega_{ci0}}\frac{1}{1+\chi}\right)^2,$$

$$f_{\text{RMF}}=-\frac{a^2}{\ell_i^2\omega_{ci0}\omega_{ce0}}\left(2\nu_{e\perp}\omega\frac{\chi}{1+\chi}\frac{a}{\delta_{\text{RMF}}}+\nu_{\text{int}}\nu_{e\parallel}\right),$$

where ω_{ci0} and ω_{ce0} are based on B_e and ν_{int} here is the value at the edge $r=a$. Note that the centrifugal correction is positive and the RMF correction is negative.

Friction against neutrals is likely an important effect in the STX experiment[6] where the plasma is relatively cold and in close proximity to the wall. The ion collision frequency with neutrals is

$$\nu_{i0}=n_n\sigma_{i/n}(kT_i/m_i)^{1/2},$$

where n_n is the neutral density and $\sigma_{i/n}$ is the cross section. Typically $\sigma_{i/n}\sim5\times10^{-15}\,\text{cm}^2$ and is weakly dependent on temperature. The neutral density in STX has not been measured, so a range of neutral densities should be considered in making a comparison. However, measurements of the pressure and magnetic field profiles have been made.[6] These indicate that the total pressure $p+B_z^2/8\pi$ is *lower* on the outside ($r\sim a$) than on the axis ($r=0$) by ~20%–30%. This is indicative of a RMF correction f_{RMF} somewhat stronger than the centrifugal effect f_{centr}. Two examples are shown in Table III: the low and high neutral density cases correspond to ~83% and ~33% ionization, respectively. The predicted f_{RMF} is consistent with the −20% to −30% correction observed in STX. The predicted correction f_{centr} is very sensitive to neutral density; it is small only for quite high neutral density. It is unlikely that such a high density of neutrals is present in STX. It bears repeating that in fusion applications the corrections to the radial force balance should be small to avoid a large RMF circulating power.

TABLE III. STX example of ion friction with neutrals.

Neutral density, n_n (cm^{-3})	10^{12}	10^{13}
Anomaly $\nu_{e\perp}/(\nu_{e\perp})_{\text{classical}}$	20	20
Friction parameter, χ	2.2	22
Ion fluid rotation, $u_{i\theta}/r$ (rad/s)	$+6.9\times10^5$	$+9.6\times10^4$
Centrifugal correction, f_{centr}	+320%	6%
RMF correction f_{RME}	−23%	−30%

B. Viscous momentum source

The ion momentum source from ion viscosity is

$$F_{i\theta}=\frac{1}{nr^2}\frac{\partial}{\partial r}\left[\eta_1 r^3\frac{\partial}{\partial r}\left(\frac{u_{i\theta}}{r}\right)\right], \tag{32}$$

where $\eta_1\approx nkT_i/\omega_{ci}$ is the Braginskii shear viscosity coefficient (magnetized limit). The momentum source depends on the second order operator in Eq. (32). An estimate is made here rather than solving a differential equation. Suppose the length scale in the operator is the velocity ''shear length'' is L_{sh}; then the momentum source is

$$F_{i\theta}/m_i\sim-\frac{\eta_1}{m_i n}\frac{u_{i\theta}}{L_{\text{sh}}^2}.$$

The sign here corresponds to the case where the ion rotation is fastest (positive) at the edge and lags in the interior of the FRC. Substitute this into Eq. (27) and assume the steady case where $u_D=0$. Then steady state requires

$$\frac{u_{i\theta}}{r}=-\frac{m_e n}{\eta_1}L_{\text{sh}}^2\nu_{e\perp}\omega_*.$$

Table IV shows the required ion fluid rotation rate to account for steady state, where the velocity shear length is assumed to be comparable to the minor radius of the plasma. The required rotation rate in the STX case is moderate ($f_{\text{centr}}\sim11\%$) and is roughly consistent with observations. Thus the steady state in STX is approximately accounted for by the viscous friction effect. In the two other examples, the

TABLE IV. Rotation rates required for steady state.

	STX	TCS	Fusion
$\eta_1/m_e n$ (cm^2/s)	2.6×10^9	6.4×10^7	3.9×10^4
Velocity shear length, L_{sh} (cm)	5	4.5	25
Collision frequency $\nu_{e\perp}$ (s^{-1}) (anomaly factor=20)	8.2×10^6	1.2×10^8	1.9×10^5
Ion fluid rotation $u_{i\theta s}/r$ (rad/s)	1.3×10^5	8.5×10^6	2.5×10^8

TABLE V. Experimental and fusion parameters.

	STX	TCS	Fusion
Ion temperature, T_i (eV)	2	150	20 000
Electron temperature, T_e (eV)	50	100	15 000
Density, n (cm^{-3})	5×10^{12}	2×10^{14}	6×10^{14}
Separatrix radius, a (cm)	20	18	100
Confining coil radius, r_c (cm)	25	45	100
External magnetic field, B_e (G)	100	1500	30 000
RMF amplitude, $\|B_r\|$ (G)	25	50	50
RMF frequency, ω (rad/s)	$\sim2.2\times10^6$	$\sim1\times10^6$	$\sim8\times10^4$
Diamagnetic frequency, ω_* (rad/s)	-9.6×10^5	-2.3×10^5	-8.3×10^4
Classical diffusion timescale, $\tau_D=\pi a^2/c^2\eta_\perp$ (cgs)	4.3 ms	9.9 ms	560 s

rotation rates required for steady state are enormous. Evidently viscous friction is much too weak to produce steady state in TCS or the fusion example.

C. Other ion momentum sources

Other natural momentum sources might be considered. The gyroviscous force does not give an azimuthal source (in the simple RMF geometry), although it affects the radial pressure balance. The thermal force appears to produce an azimuthal source. Further investigation is needed to ascertain if any of these is appreciable.

An ion momentum source can also be generated by external means. Neutral beam injection can couple a significant momentum to the ion species. This can also be done by generating a second RMF field, as was proposed by Clemente.[20] If effective, this would involve a negligible complication to RMF current drive since the same antenna structure could be used to drive both the RMF signal for the electrons and the signal for the ions.

V. SUMMARY AND DISCUSSION

A full treatment of the ion (as well as of the electron) mobility shows the critical shortness of an ion spin-up phase. The duration of this transient phase is orders of magnitude shorter than the diffusion time. This contradicts earlier work that suggests the transient time may last as long as or longer than the diffusion time. Thus, effective RMF requires the quasisteady state to have adequate current drive. In fact, without an ion momentum source of some kind, no persistent current drive can exist. A momentum source (or sink) has been provided in current experiments by collisions with neutrals and possible ion viscosity. These appear to explain several features of these experiments. However, in fusion applications some other momentum source, such as neutral beam injection, is needed. A momentum sink would also be supplied if there is refueling.

The analysis here establishes a system of quasisteady equations that can be used to model the time-dependent RMF effects, including the penetration of the RMF into the plasma and the evolution of the equilibrium, while circumventing the very fast ion spin-up transients. Existing computational models integrate the *full* time-dependent equations; these computations must take exceedingly short time steps in view of the shortness of the inertial time $1/\nu_{\rm int}$. The analysis here also shows the inaccuracy of models that assume rigid ions: this assumption is doubtful even in current experiments which have a large population of neutrals; it is completely invalid in the case in which neutrals are absent. The analysis here may also enable other issues to be investigated, including the effect on thermal loss from the oscillation and the breaking of magnetic surfaces.

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APPENDIX A: RMF RELEVANT PARAMETERS IN FRC EXPERIMENTS

Plasma properties characteristic of two current RMF experiments[6,7] and projected fusion conditions are shown in Table V.

APPENDIX B: QUIVER MOTION OF ION AND ELECTRON SPECIES

The equation of motion for each species is

$$m_\alpha \frac{d_\alpha \mathbf{u}_\alpha}{dt} = -\frac{\nabla p_\alpha}{n} + q_\alpha\left(\mathbf{E}+\frac{1}{c}\mathbf{u}_\alpha\times\mathbf{B}\right)+\frac{\mathbf{R}_\alpha}{n}, \qquad \text{(B1)}$$

where $\alpha=i,e$ (ions, electrons) denotes the species; $\mathbf{u}_\alpha$, p_α, q_α, and $\mathbf{R}_\alpha$ are the velocity, pressure, charge, and friction force, respectively; n is the common particle density (quasineutrality); and the species-specific total derivative is $d_\alpha/dt=\partial/\partial t+\mathbf{u}_\alpha\cdot\nabla$. If interspecies friction provides the dominant friction force, then $\mathbf{R}_e=-\mathbf{R}_i=m_e n(\mathbf{u}_i-\mathbf{u}_e)\nu_e$.

Assume no equilibrium poloidal flow, i.e., the surface-averaged ($\ell=0$) part of $\mathbf{u}_{\alpha z}$ vanishes. The nonzero surface averaged velocities are $\mathbf{u}_{\alpha r}$ and $\mathbf{u}_{\alpha\theta}$; the latter (rotational drift) dominates the former (radial diffusion). Assume a cyclic variation of the $\ell=1$ quantities, e.g., $u_{\alpha z}=\hat{u}_{\alpha z}(r)\exp(i\ell\theta-i\omega t)$; hereafter the caret and the exponential factor will be understood. Then the total derivatives are

$$d_e/dt=-i(\omega-u_{e\theta}/r)\equiv-i\omega',$$

$$d_i/dt=-i(\omega-u_{i\theta}/r)\equiv-i(\omega'-\omega_*),$$

where ω_* is the diamagnetic frequency as defined in Eq. (3). Finally assume the ideal, infinitely long geometry ($\partial/\partial z = 0$). This implies that the vector potential has only $\ell = 1$ symmetry and only an axial component A_z. Then adopting the gauge $\phi = 0$ (ϕ = scalar potential), the electric and magnetic fields are $\mathbf{E} = i(\omega/c)A_z$, $\mathbf{B} = (i\hat{\mathbf{r}} - \sigma\hat{\boldsymbol{\theta}})A_z/r$, where the radial gradient parameter is $\sigma \equiv (r/A_z)dA_z/dr$.

Consider now the axial component of the equations of motion, Eq. (11), which govern the axial quiver motion of each species caused by the RMF. These can be expressed in a compact matrix format as

$$\begin{bmatrix} i\omega' - \nu_{e\parallel} & (m_e/m_i)\nu_{e\parallel} \\ \nu_{e\parallel} & i\omega' - i\omega_* - \nu_{e\parallel} \end{bmatrix} \begin{Bmatrix} m_e u_{ez} \\ m_i u_{iz} \end{Bmatrix} = i \begin{Bmatrix} \omega' \\ \omega' - \omega_* \end{Bmatrix} \frac{eA_z}{c}.$$

Here the $u_{\alpha r}B_\theta$ term in the Lorentz force was neglected as small compared to $u_{\alpha\theta}B_r$ since $u_{\alpha r} \ll u_{\alpha\theta}$; the pressure gradient vanishes at $\partial/\partial z = 0$; and the *parallel* collision frequency $\nu_{e\parallel}$ is distinguished since anomalous effects may cause anisotropy in the effective collisionality. Note that the axial equations of motion for the quiver motion have *decoupled* from the other components. Solving this matrix equation leads to the quiver velocities of each species, Eq. (2).

APPENDIX C: RADIAL CONVECTIVE INERTIA EFFECT

The radial convection inertia effect was included in the steady state model of Ohnishi and Ishida.[8] Their Eqs. (17) and (24) can be expressed in the form

$$u_{er} = \frac{\nu_{e\perp}}{\omega_{ce}}(u_{e\theta} - u_{i\theta}) - r\omega' \frac{\omega^2}{2\omega_{ce}\nu_{e\perp}}, \tag{C1}$$

$$u_{ir} = \frac{\nu_{e\perp}(u_{e\theta} - u_{i\theta})}{\omega_{ce}} \frac{1}{1+\kappa}, \tag{C2}$$

where no particle source and no friction with neutrals are assumed in order to be consistent with the assumption of no ion momentum source. Here the convective inertia effect $m_i u_{ir}\partial u_{i\theta}/\partial r$ is represented by the parameter

$$\kappa = \frac{1}{r\omega_{ci}} \frac{\partial(r u_{i\theta})}{\partial r}. \tag{C3}$$

This was neglected in Sec. II [Eq. (11)], which is equivalent to setting $\kappa = 0$. The two radial flows must be equal in view of quasineutrality [radial Ampere's law, Eq. (14)], which determines the steady state slip frequency,

$$\omega' = 2 \frac{u_{e\theta} - u_{i\theta}}{r} \frac{\nu_{e\perp}^2}{\omega^2} \frac{\kappa}{1+\kappa}. \tag{C4}$$

This is nonzero if the convective inertia (represented by κ) is accounted for. Then the RMF contribution to the radial flow, i.e., the second term in Eq. (C1), becomes

$$(u_r)_{\mathrm{RMF}} = \frac{\nu_{e\perp}(u_{e\theta} - u_{i\theta})}{\omega_{ce}} \frac{-\kappa}{1+\kappa}.$$

The first factor is identical to the resistivity contribution to radial flow, i.e., the first term in Eq. (C1). Thus the importance of the convective inertia effect depends on the sign and magnitude of κ.

The convective inertia parameter κ can be estimated for a rigid rotor profile with rigid rotation. A reasonable ''high'' value of rotation would arise if the ion fluid carried all the current, $u_{i\theta} = (1/enB_z)\partial p/\partial r$. Then $u_{i\theta}/r \approx -4K\omega_{ci}\ell_i^2/a^2$ = const, where $K \sim 1$ is the profile shape parameter, and $\ell_i = c/\omega_{pi}$ is the ion skin depth. Using this in Eq. (C3):

$$\kappa = -8K\ell_i^2/a^2. \tag{C5}$$

Note the familiar radial size parameter a/ℓ_i which is $\gg 1$ for typical FRCs; thus $|\kappa| \ll 1$, even in the rapidly rotating case in which the ion fluid carries the current. An exception is the STX experiment where $a/\ell_i \sim 1.4$; however, it is believed that the ion fluid rotation in STX is small, as discussed in Sec. IV.

Finally, if the ion fluid rotation is in the expected (positive) direction, as it would be if the ion fluid were spun up by the RMF, then κ, Eq. (C5), is positive. Therefore the RMF contribution to radial flow, Eq. (C5) is *inward*, i.e., the radial convective inertia *augments* the current drive, although weakly so.

APPENDIX D: TORQUE FOR ION FLUID SPIN UP BY RMF

The transient spin up is rapid, which raises questions of how much torque is required, and can it be transmitted by the RMF. First, a possible misapprehension needs to be avoided. The azimuthal force on the fluid in the Hall-MHD format, $m_i n r \nu_{\mathrm{int}}\omega'$ in Eq. (12) suggests a direct RMF force on the ion fluid, which is misleading. This is resolved in the two-fluid format, the total azimuthal forces on the two fluids are

$$(F_\theta)_{\mathrm{ion}} = eE_\theta - eu_rB_z/c - 2m_e n r\nu_{\mathrm{int}}\omega' - m_e\nu_{e\perp}(u_{i\theta} - u_{e\theta}),$$

$$(F_\theta)_{\mathrm{electron}} = -eE_\theta + eu_rB_z/c + m_i n r \nu_{\mathrm{int}}\omega' + m_e\nu_{e\perp}(u_{i\theta} - u_{e\theta}),$$

where $u_r = u_{er} = u_{ir}$ from quasineutrality. However when the forces on the two species are added, as in the Hall-MHD equation of motion, the Lorentz and friction forces cancel out and only the RMF force remains, which is predominantly the RMF force on the electrons. Thus the RMF force on the electron fluid is transmitted to the ion fluid through the Lorentz and friction forces.

Suppose that as the RMF penetrates into the plasma the fluid is spun up layer by layer by an amount $\Delta u_{i\theta} = fr\omega_*$, i.e., a fraction f of the diamagnetic drift speed. The penetration occurs on a timescale[13] of $\tau_{D\parallel} \approx a^2/16(\eta_\parallel/4\pi)$, so the incremental penetration distance in a time Δt is $\Delta r \approx (a/\tau_{D\parallel})\Delta t$, or

$$\Delta r \approx \frac{16}{a} \frac{m_e\nu_{e\parallel}}{4\pi e^2 n}\Delta t.$$

The rate of change of the angular momentum per unit length of plasma is

$$m_i n f r\omega_* 2\pi r\Delta r/\Delta t, \tag{D1}$$

which is balanced by the torque.

The torque imparted by the RMF can be analyzed using the Maxwell stress tensor, $\overleftrightarrow{\mathbf{\Pi}}_M=[\mathbf{BB}-(B^2/2)\overleftrightarrow{\mathbf{I}}]/4\pi$. The force on a volume element $d\tau$ is $d\mathbf{F}=\nabla\cdot\overleftrightarrow{\mathbf{\Pi}}_I d\tau$. The torque imparted by this force is $dT_z=\hat{\mathbf{z}}\cdot(\mathbf{r}\times d\mathbf{F})=r\hat{\boldsymbol{\theta}}\cdot\mathbf{dF}$. The torque per unit length is $T_z'=\int r\hat{\boldsymbol{\theta}}\cdot\nabla\cdot\overleftrightarrow{\mathbf{\Pi}}_I d\tau'$, where $d\tau'=rdrd\theta$ is the volume increment per unit length.

Consider a control volume of fixed radius r and apply Gauss's theorem, which converts the volume integral to a surface integral. Then the torque per unit length is

$$T_z'=r^2\int_0^{2\pi}(B_rB_\theta/4\pi)d\theta.$$

With Eq. (1) the torque is

$$T_z'=-\pi r^2\sigma_1|B_r|^2/4\pi. \tag{D2}$$

Combining Eqs. (D1), (D2), (6) gives the ''twist'' of the RMF to impart the required torque,

$$\sigma_1\approx-16f\omega_*/\nu_{\mathrm{int}}.$$

If $f=\frac{1}{2}$, the twists in the three examples are 3.4° (STX), 2.9° (TCS), and 0.017° (fusion).

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